3D_DIRECTION COSINES & DIRECTION RATIOS

SYNOPSIS

1. Relation between direction ratios and direction coins:

i) Let (a,b,c) be direction ratios and (l_2, m_2, n_2) be direction cosine of a line. Then

$$\frac{1}{a}=\frac{m}{b}=\frac{n}{c}=\pm\frac{1}{\sqrt{a^2+b^2+c^2}} \Rightarrow l=\pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m=\pm\frac{b}{\sqrt{l^2+m^2+n^2}}, n=\pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

2. Direction ratios and direction cosines of a line segment:

i) The direction ratios of the line segment joining A (x_1, y_1, z_1) and B (x_2, y_2, z_2) may be taken as $(x_2-x_1, y_2-y_1, z_2-z_1)$ or $(x_1-x_2, y_1-y_2, z_1-z_2)$

ii) Direction cosines of line segment joining A(x_1, y_1, z_1) and B(x_2, y_2, z_2) are

$$\pm\left(\frac{x_2-x_1}{AB}, \frac{y_2-y_1}{AB}, \frac{z_2-z_1}{AB}\right)$$

iii) A line has two sets of d.c's. If (l, m, n) is one set then other set is *(-l,-m,-n)*

3. Co-ordinates of a point on directed line:

i) If *(l,m,n)* are the d.c's of $\overline{OP}$ where 'O' is the origin and OP=r then $P=(lr, mr, nr)$

Lagrange's identity:

$$\left(l_1^2+m_1^2+n_1^2\right)\left(l_2^2+m_2^2+n_2^2\right)-\left(l_1l_2+m_1m_2+n_1n_2\right)^2=\left(l_1m_2-l_2m_1\right)^2+\left(m_1n_2-m_2n_1\right)^2+\left(n_1l_2-n_2l_1\right)^2$$

4. Angle between two lines:

i) If θ is acute angle between two lines whose direction cosines (l_1, m_1, n_1) are and (l_2, m_2, n_2) then

a) $\cos\theta=\left|l_1l_2+m_1m_2+n_1n_2\right|$ b) $\sin\theta=\sqrt{\sum\left(l_1m_2-l_2m_1\right)^2}$

ii) If 'θ'is acute angle between the lines whose direction ratios (a_1, b_1, c_1) are (a_2, b_2, c_2) and respectively then $\cos\theta = \dfrac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$

iii) If (l_1, m_1, n_1) and (l_2, m_2, n_2) are direction cosines of two intersecting lines then the d.c's of the line bisecting angle between them are proportional to $(l_1 \pm l_2, m_1 \pm m_2, n_1 \pm n_2)$

iv) D.c's of angular bisectors are $\left.\begin{matrix} \dfrac{l_1 + l_2}{2\cos\theta/2}, \dfrac{m_1 + m_2}{2\cos\theta/2}, \dfrac{n_1 + n_2}{2\cos\theta/2} \\ \dfrac{l_1 - l_2}{2\sin\theta/2}, \dfrac{m_1 - m_2}{2\sin\theta/2}, \dfrac{n_1 + n_2}{2\sin\theta/2} \end{matrix}\right\}$ where θ is angle between the lines

5. Condition that line are perpendicular, parallel :

i) (l_1, m_1, n_1) and (l_2, m_2, n_2) are d.c's of two lines. Then

a) The lines are perpendicular if $l_1l_2 + m_1m_2 + n_1n_2 = 0$

b) The lines are parallel if $\dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$

ii) Let (a_1, b_1, c_1) and (a_2, b_2, c_2) be d.r's of two lines, Then

a) The lines are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

b)The lines are parallel if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

iii) If the d.c's (l, m, n) of two lines are connected by the relations $al + bm + cn = 0$ and $fmn + gnl + hlm = 0$, the the lines are

a) perpendicular if $\dfrac{f}{a} + \dfrac{g}{b} + \dfrac{h}{c} = 0$

b) parallel if $\sqrt{af} \pm \sqrt{bg} \pm \sqrt{ch} = 0$

iv) If the d.c 's (l, m, n) of two lines are connected by the relations $al + bm + cn = 0$ and $ul^2 + vm^2 + wn^2 = 0$, then the lines are

a) perpendicular if $\sum a^2(v + w) = 0$

b) parallel if $\dfrac{a^2}{u} + \dfrac{b^2}{v} + \dfrac{c^2}{w} = 0$

6. Areas :

i) If A (x_1, y_1, z_1),B $(x_2, y_2, z_2), C(x_3, y_3, z_3)$ are the vertices of triangle ABC then area of $\Delta ABC = \frac{1}{2}\left|\overline{AB} \times \overline{AC}\right|$

ii) If A (x_1, y_1, z_1),B $(x_2, y_2, z_2), C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ then

a) Area of parallelogram $ABCD = \frac{1}{2}\left|\overrightarrow{AC} \times \overrightarrow{BD}\right| = \left|\overrightarrow{AB} \times \overrightarrow{AD}\right|$

b) Area of plane quadrilateral $ABCD = \frac{1}{2}\left|\overrightarrow{AC} \times \overrightarrow{BD}\right|$

7. Some standard results :

i) D.c's of line equally inclined with coordinate axes are $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$

ii) a) Angle between any two diagonals of a cube is $\cos^{-1}\left(\frac{1}{3}\right)$

b) The angle between a diagonal of a cube and the diagonal of a face of the cube is $\cos^{-1}\sqrt{\frac{2}{3}}$

iii) If a variable line in two adjacent position has direction cosines

iv) $(l, m, n), (l+\delta l, m+\delta m, n+\delta l)$ and $\delta\theta$ is the angle between the two positions then $(\delta l)^2 + (\delta m)^2 + (\delta n)^2 = (\delta\theta)^2$

v) If a,b,c are the lengths of the sides of a rectangular parallelopiped then angle between any two diagonals is given by $\cos^{-}\left(\frac{a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}\right)$, (In numerator all the three terms not have the same sign)

vi) If a line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$$

Planes & Lines

1. Normal from of a plane :

i) If (l,m,n) are the direction cosines of normal to plane π and P is the $\perp^{er}$ distance from origin to the plane then the equation of plane is $lx+by+nz=p$

ii) The normal from of the plane representing by the equation $ax+by+cz+d=0$ is

a) If $d<0 \frac{a}{\sqrt{a^2+b^2+c^2}}x+\frac{b}{\sqrt{a^2+b^2+c^2}}y+\frac{c}{\sqrt{a^2+b^2+c^2}}z=\frac{-d}{\sqrt{a^2+b^2+c^2}}$

b) If $d>0 \frac{-a}{\sqrt{a^2+b^2+c^2}}x+\frac{-b}{\sqrt{a^2+b^2+c^2}}y+\frac{-c}{\sqrt{a^2+b^2+c^2}}z=\frac{d}{\sqrt{a^2+b^2+c^2}}$

2. Perpendicular distance from point to the plane :

i) The perpendicular distance from (x_1,y_1,z_1) to the plane $ax+by+cz=0$ is $\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$

ii) The perpendicular distance of the plane $ax+by+cz=0$ from the origin is $\frac{|d|}{\sqrt{a^2+b^2+c^2}}$

3. Intercept from of a plane :

i) If a plane cuts X-axis at A(a,0,0), Y-axis at B(0,b,0) and Z-axis at C(0,0,c) then a,b,c are called X-intercept, Y-intercept, Z-intercept of the plane.

ii) The equation of the plane in intercept from is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

iii) If $ax+by+cz+d=0$ is a plane if $a\neq 0, b\neq 0, c\neq 0$ then

X-intercept $=-\frac{d}{a}$, Y-intercept , $=-\frac{d}{b}$, Z-intercept $=-\frac{d}{c}$,

iv) The equation of the plane whose intercepts are K times the intercepts made by the plane $ax+by+cz+d=0$ on corresponding axes is $ax+by+cz+d=0$

4. Areas : i) Area of the triangle formed by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ with

a) X-axis , Y-axis is $\frac{1}{2}|ab|$ Sq. units b) Y-axis, Z-axis is $\frac{1}{2}|bc|$ Sq. units

c) Z-axis , X-axis is $\frac{1}{2}|ca|$ Sq. units

ii) If the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the co-ordinate axes in the points A,B,C. then the area of the triangle ABC is $\frac{1}{2}\sqrt{(ab)^2+(bc)^2+(ca)^2}$

5. Angle between Two Planes :

i) The angle between two planes is equal to the angle between the perpendiculars from the origin to the planes.

ii) If 'θ' is the angle between the planes $a_1x+b_1y+c_1z+d=0$ and $a_2x+b_2y+c_2z+d=0$ then $\cos\theta=\pm\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$

iii) If the above two planes are parallel then $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

iv) If the above two planes are perpendicular then $a_1a_2+b_1b_2+c_1c_2=0$

v) Angle between the line with d.c's (l_1,m_1,n_1) and the plane whose normal with d.c's (l_2,m_2,n_2) is θ then $\cos(90-\theta)=|l_1l_2+m_1m_2+n_1n_2|$

vi) If θ is angle between a line L and a plane π then the angle between L and normal to the plane π is $90\pm\theta$

6. Foot and image :

i) The foot of the perpendicular of the point p(x_1,y_1,z_1) on the plane $ax+by+cz+d=0$ is $Q(h,k,l)$ then $\frac{h-x_1}{a}=\frac{k-y_1}{b}=\frac{l-z_1}{c}=\frac{-(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$

ii) If $Q(h,k,l)$ is the image of the point p(x_1,y_1,z_1) w.r.to the plane $ax+by+cz+d=0$ then $\frac{h-x_1}{a}=\frac{k-y_1}{b}=\frac{l-z_1}{c}=\frac{-2(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$

iii) If 'd' is the distance from the origin and (l,m,n) are the dc's of the normal to the plane through the origin, then the foot of the per perpendicular is *(ld, md, nd)*

7. Equations of planes bisecting the angles between given planes :

i) Equations of two planes dissecting the angles between the planes $a_1x+b_1y+c_1z+d_1=0$ and; $a_2x+b_2y+c_2z+d_2=0$ are

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

ii) If $d_1, d_2 > 0$

Condition	Acute	Obtuse
$a_1a_2+b_1b_2+c_1c_2>0$	$-$	$+$
$a_1a_2+b_1b_2+c_1c_2<0$	$+$	$-$

iii) a) The Bisector planes are perpendicular to each other

b) Positive sign bisector is the bisector containing the origin

Some standard results:

i) If $p_1=a_1x+b_1y+c_1z+d_1=0$ and $p_2=a_2x+b_2y+c_2z+d_2=0$ are two intersecting planes then the plane passing through their line of intersection is $p_1+kp_2=0$ where k is any constant.

ii) The equation of plane which bisects the join of the points (x_1,y_2,z_1) and (x_2,y_2,z_2) at right angles is $\sum(x_1-x_2)\left\{x-\frac{1}{2}(x_1+x_2)\right\}=0$

iii) If a plane meets the coordinates axes in A, B ,C such that the centroid of the triangle ABC is the point (p,q.r) then the equation of the plane is $\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=3$

iv) Two systems of rectangular axes have the same origin. If a plane cuts them at distance a,b,c and a_1,b_1,c_1 respectively from the origin, then $a^{-2}+b^{-2}+c^{-2}=a_1^{-2}+b_1^{-2}+c_1^{-2}$

v) A variable plane is at a constant distance 'p' from the origin and meets the axes in A, B and C. The locus of the centroid of the triangle ABC is $x^{-2}+y^{-2}+z^{-2}=9p^{-2}$

vi) A variable plane is at a constant distance 'p' from the origin and meets the axes in A, B and C. The locus of the centroid of the tetrahedron OABC is $x^{-2}+y^{-2}+z^{-2}=16p^{-2}$

vii) A variable plane passes through a fixed point (α,β,γ) and meets the coordinates axes in A, B ,C. Then the locus of the point of intersection of the planes through A, B,C parallel to the coordinates planes is $\frac{\alpha}{x}+\frac{\beta}{y}+\frac{\gamma}{z}=1$

viii) A variable plane at a distance 'p' from the origin meets the axes in A, B and C. Through A, B and C planes are drawn parallel to the coordinate planes. Then the locus of their point of intersection is $x^{-2}+y^{-2}+z^{-2}=p^{-2}$

ix) A point P moves on the fixed plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. The plane through P, perpendicular to OP meets the coordinates axes in A, B and C . Then the locus of the point of intersection of planes through A,B ,C parallel to the coordinate planes is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}$

x) The planes $x=\pm a, y=\pm b$ and $z=\pm c$ form a rectangular parallelopiped

xi) A parallelopiped is formed by the planes drawn through the point (x_1,y_1,z_1) and (x_2,y_2,z_2) parallel to the coordinate planes. The length of a diagonal of the parallelepiped $=\sqrt{a^2+b^2+c^2}$ .Here $a=x_2-x_1, b=y_2-y_1, c=z_2-z_1$

3D- LINES

1) Symmetrical forms of a line

The equation of the line passing through the point (x_1,y_1,z_1) and (x_2,y_2,z_2) having dc's is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

2) vector form of a line :

Cartesian equation of a line passing through the point (x_1,y_1,z_1) and (x_2,y_2,z_2) having is

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

3) Angle between two lines:

If θ is the angle between the lines given by $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and

$\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ then $\cos\theta=\left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{\sum a_1^2}\sqrt{\sum a_2^2}}\right|$

i) a) If the lines are parallel then $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

b) If the lines are perpendicular then $a_1a_2+b_1b_2+c_1c_2=0$

ii) If θ is the acute angle between the lines $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ and the plane $ax+by+cz+d=0$ the $\sin\theta=\frac{|al+bm+cn|}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}$

iii) If the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ is parallel to the plane $ax+by+cz+d=0$ then $al+bm+cn=0$ (Normal to the plane is perpendicular to the line)

iv) If the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ perpendicular to the planes $ax+by+cz+d=0$ then $\frac{a}{l}=\frac{b}{m}=\frac{c}{n}$

v) D.C's of the line which make equal angles with coordinate axes are $\pm\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$ and the dr's of the line are $(1,1,1)$.

4. **Coplanar lines:** Two lines are said to be coplanar if they are either parallel or intersect.

5. **Non-Coplanar Lines:** Two lines are said to be non coplanar or skew lines if they are neither parallel nor intersecting.

6. **Condition for two lines to be coplanar:**

i) The line lies $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ in the plane $ax+by+cz+d=0$ if $ax_1+by_1+cz_1+d=0,\ al+bm+cn=0$.

ii) The lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1},\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ are coplanar $\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

7. Equation of a plane containing lines:

i) The equation of the plane containing the lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$ (OR) $\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

ii) If the lies $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$, $a_1x+b_1y+c_1z+d_1 = 0 = a_2x+b_2y+c_2z+d_2 = 0$ are coplanar then $\frac{a_1x+b_1y+c_1z+d}{a_1l+b_1m+c_1n} = \frac{a_2x+b_2y+c_2z+d_2}{a_2l+b_2m+c_2n}$

8. Skew lines:

Two straight lines are said to be skew lines if they are neither parallel nor intersecting. i.e the lines which do not lie in plane.

9. Shortest distance:

If L_1 and L_2 are skew lines then there is one and only one line perpendicular to both of the lines L_1 and L_2 which is called the line of shortest distance. If PQ is the line of shortest distance then the distance between P and Q is called distance between the skew lines.

i. The shortest distance between the skew lines $\bar{r} = \bar{a}_1 + \lambda \bar{b}_1, \bar{r} = \bar{a}_2 + \mu \bar{b}_2$ is

$$\frac{\left|\left(\overline{a_1} - \overline{a_2}\right).\left(\overline{b_1} \times \overline{b_2}\right)\right|}{\left|\left(\overline{b_1} \times \overline{b_2}\right)\right|} (or) \frac{\left|\left[\overline{a_1} - \overline{a_2}\,\overline{b_1}\,\overline{b_2}\right]\right|}{\left|\overline{b_1} \times \overline{b_2}\right|}$$

ii. If the above two lines are coplanar or intersecting then $\left[\overline{a_1} - \overline{a_2}\,\overline{b_1}\,\overline{b_2}\right] = 0$

iii. Shortest distance between the lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1} and \frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$. is

$$\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{\sum\left(b_1c_1-b_2c_1\right)^2}}$$

10. Distance between parallel lines:

The distance between the parallel lines $\bar{r}=\overline{a_1}+\lambda\bar{b},\bar{r}=\overline{a_2}+\mu\bar{b}$ is $\frac{\left|\bar{b}\times\left(\overline{a_1}-\overline{a_2}\right)\right|}{\left|\bar{b}\right|}$

i. To find the direction of a line with greatest slope:

Let $\pi_{1,}\pi_2$ be two planes intersecting in a line l_1 then the line of greatest slope in π_1 is the line lying in the plane π_1 and perpendicular to the line l_1.

ii .Let $\bar{a},\bar{b}$ be the vectors along the normals to the planes π_1 and π_2 respectively then the vector $\bar{a}\times\left(\bar{a}\times\bar{b}\right)$ will be along the line of greatest slope is π_1.