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DIFFERENTIAL EQUATIONS

KEY CONCEPTS

I. Differential Equations of First order and First degree Definitions :

(a) An equation that involves independent and dependent variables and the derivatives of the dependent variables is called a Differential Equation.

(b) A differential equation is said to be ordinary, if the differential coefficients have reference to a single independent variable only and it is said to be Partial if there are two or more independent variables. We are concerned with ordinary differential equations only. e.g. $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$ is a partial differential equation.

(c) Finding the unknown function is called Solving or Integrating the differential equation. The solution of the differential equation is also called its Primitive, because the differential equation can be regarded as a relation derived from it.

(d) The order of a differential equation is the order of highest differential coefficient occuring in it.

(e) The degree of a differential equation which can be written as a polynomial in the derivatives is the degree of the derivative of the highest order occuring in it, after it has been expressed in a form free from radicals and fractions so far as derivatives are concerned, thus the differential equation :

$f(x, y)\left[\frac{d^m y}{dx^m}\right]^p+\phi(x,y)\left[\frac{d^{m-1}(y)}{dx^{m-1}}\right]^q+.....=0$ is order m and degree p.

Note that in the differential equation $e^{y'''}-xy''+y=0$ order is three but degree doesn't apply.

2. Formation of a Differential Equation :

If an equation in independent and dopondent variables having some arbitrary some arbitrary constant is given a differential equation is obtained as follows :

(a) Differentiate the given equation w.r.t. the independent variable (say x) as many times as the number of arbitrary constants in it.

(b) Eliminate the arbitrary constants.

The eliminate is the required differential equation. Consider forming a differential equation for $y^2=4a(x+b)$ where a and b are arbitrary constant.

NOTE : A differential equation represents a family of curves all satisfying some common properties. This can be considered as the geometrical interpretation of the differential equation.

3. **General and Particular Solutions:**

The solution of a differential equation which contains a number of independent arbitrary constants equal to the order of the differential equation is called the General Solution (Complete Integral or Complete Primitive). A solution obtainable from the general solution by giving particular values to the constants is called a Particular Solution.

Note that the general solution of a differential equation of the n^{th} order contains 'n' and only 'n' independent arbitrary constants. The arbitrary constants in the solution of a differential equation are said to be independent, when it is impossible to deduce from the solution an equivalent relation containing fewer arbitrary constants A, B in the equation $y = Ae^B.e^x = Ce^x$. Similarly the solution $y = A\sin x + B\cos(x+C)$ appears to contain three arbitrary constants, but they are really equivalent to two only

4. **Elementary Types of First Order and First Degree Differential Equations :**

TYPE-1:

Variables Separable : If the differential equation can be expressed as; $f(x)dx + g(y)dy = 0$ then this is said to be variables separable type. A general solution of this is given by $\int f(x)dx + \int g(y)dy = c$; where c is the arbitrary constant. Consider the example $\left(\frac{dy}{dx}\right) = e^{x-y} + x^2.e^{-y}$.

Note : Sometimes transformation to the polar co-ordinates facilitates separation of variables. In this connection it is convenient to remember the following differentials.

If $x = r\cos\theta$; $y = r\sin\theta$ then,

1. $x\,dx + y\,dy = r\,dr$
2. $dx^2 + dy^2 = dr^2 + r^2 d\theta^2$
3. $xdy - ydx = r^2 d\theta$

If $x = r\sec\theta$ and $y = r\tan\theta$ then $xdx - ydy = r\,dr$ and $x\,dy - y\,dx = r^2\sec\theta d\theta$

TYPE-2 :

$$\frac{dy}{dx} = f(ax+by+c),\ b \neq 0$$

To solve this, substitute $t = ax + by + c$. Then the equation reduces to separable type in the variable t and x which can be solved.

Consider the example $(x+y)^2 \frac{dy}{dx} = a^2$

TYPE-3 :

Homogeneous Equations : A differential equation of the form $\frac{dy}{dx}=\frac{f(x,y)}{\phi(x,y)}$, where $f(x,y)$ *and* $\phi(x,y)$ are homogeneous functions of *x* and *y*, and of the same degree, is called Homogeneous. This equation may also be reduced to the form $\frac{dy}{dx}=g\left(\frac{x}{y}\right)$ and is solved by putting $y=\upsilon x$ so that the dependent variable *y* is changed to another variable υ, where υ is some unknown function, the differential equation is transformed to an equation with variables separable. Consider $\frac{dy}{dx}+\frac{y(x+y)}{x^2}=0$.

TYPE – 4 :

Equations Reducible to the Homogeneous Form : If $\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$; where

$a_1b_2-a_2b_1\neq 0,\ i.e.\ \frac{a_1}{b_1}\neq\frac{a_2}{b_2}$ then the substitution $x=u+h,\ y=v+k$ transform this equation to a homogeneous type in the new variables *u* and *v* where *h* and *k* are arbitrary constants to be chosen so as to make the given equation homogeneous which can be solved by the method as given in **Type – 3.** If :

(a) $a_1b_2-a_2b_1=0$, then a substitution $u=a_1x+b_1y$ transforms the differential equation to an equation with variables separable, and

(b) $b_1+a_2=0$, then a simple cross multiplication and substituting $d(xy)$ for $xdy+ydx$ and integrating term by term yields the result easily.

Consider $\frac{dy}{dx}=\frac{x-2y+5}{2x+y-1};\ \frac{dy}{dx}=\frac{2x+3y-1}{4x+6y-5}$ and $\frac{dy}{dx}=\frac{2x-y+1}{6x-5y+4}$

(c) In an equation of the form : $yf(xy)dx+xg(xy)dy=0$ the variables can be separated by the substitution $xy=\upsilon$.

Note :

1. The function $f(x,y)$ is said to be a homogeneous function of degree *n* if for any real number $t(\neq 0)$, we have $f(tx,ty)=t^nf(x,y)$

For e.g. $f(x,y)=ax^{2/3}+hx^{1/3}.y^{1/3}+by^{2/3}$ is a homogeneous function of degree $\frac{2}{3}$

2. A differential equation of the form $\frac{dy}{dx}=f(x,y)$ is homogeneous if $f(x,y)$ is a homogeneous function of degree zero i.e. $f(tx,ty)=t^0f(x,y)=f(x,y)$. The function f does not depend on *x* and *y* separately but only on their ratio $\frac{y}{x}$ *or* $\frac{x}{y}$.

Linear Differential Equations : A differential equation is said to be linear if the dependent variable and its differential coefficients occur in the first degree only and are not multiplied together.

The nth order linear differential equation is of the form;

$a_0(x)\frac{d^n y}{dx^n}+a_1(x)\frac{d^{n-1}}{dx^{n-1}}+.....+a_n(x).y=\phi(x).$ where $a_0(x),a_1(x),.....a_n(x)$ are called the coefficient of the differential equation.

Note : A linear differential equation is always of the first degree but every differential equation of the first degree need not be linear e.g. the differential equation

$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3+y^2=0$ is not linear, though its degree is 1.

TYPE-5:

Linear Differential Equations of First Order : The most general form of a linear differential equations of first order is $\frac{dy}{dx}+Py=Q$, where P and Q are functions of x .

To solve such an equation multiply both sides by $e^{\int P\,dx}$.

Note:

1. The factor $e^{\int P\,dx}$ on multiplying by which the left hand side of the differential equation becomes the differential coefficient of some function of *x* and *y*, is called integrating factor of the differential equation popularly abbreviated as I. F.

2. It is very important to remember that on multiplying by the integrating factor, the left hand side becomes the derivative of the product of *y* and the LF.

3. Sometimes a given differential equation becomes linear if we take *y* as the independent variable and *x* as the dependent variable. e.g., the equation;

$(x+y+1)\frac{dy}{dx}=y^2+3$ can be written as $\left(y^2+3\right)\frac{dx}{dy}=x+y+1$ which is a linear differential equation.

TYPE-6 :

Equations Reducible to Linear Form : The equation $\frac{dy}{dx}+py+Q.y^n$ where P and Q functions of *x*, is reducible to the linear form by dividing it by y^n and then substituting $y^{-n+1}=Z$. Its solution can be obtained as in **Type-5.** Consider the example

$\left(x^3y^2+xy\right)dx=dy$

The equation $\frac{dy}{dx}+Py=Q.y^n$ is called **Bernouli's Equation.**

TYPE - 7

5. TRAJECTORIES :

Suppose we are given the family of plane curves. $\Phi(x,y,a)=0$ depending on a single parameter a.

A curve making at each of its points a fixed angle α with the curve of the family passing through that point is called an **isogonal trajectory** of that family; if in particular $\alpha=\frac{\pi}{2}$, then it is called an **orthogonal trajectory.**

Orthogonal trajectory : We set up the differential equation of the given family of curves. Let it be of the form $F(x,y,y')=0$

The differential equation of the orthogonal trajectories is of the form $F\left(x,y,-\frac{1}{y'}\right)=0$

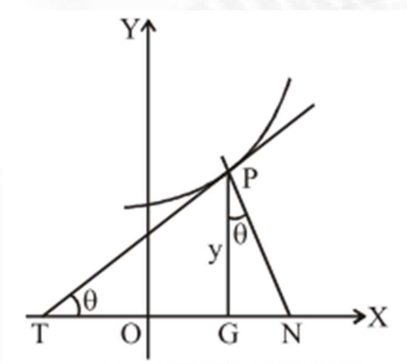
The general integral of this equation $\Phi_1(x,\ y,\ C)=0$ gives the family of orthogonal trajectories.

FINDING EQUATION OF A CURVE WHOSE GEOMETRICAL PROPERTIES ARE GIVEN

The following properties of a curve are sometimes very useful in determining the equation of a curve. Using these properties, first the differential equation of the curve is formed and then this differential equations is solved to get the equation of the curve.

Let the tangent and normal at a point P(x, y) on the curve $y=f(x)$, meet the X-axis at T and N respectively. If G is the foot the ordinate at P, then TG and GN are called the Cartesian sub-tangent and subnormal, while the lengths PT and PN are called the lengths of the tangent and normal respectively.

If PT make an angle θ with X-axis, then $\tan\theta=\frac{dy}{dx}$. From the figure we can find that :



$\Rightarrow$ Subtangent = TG = $y\cot\theta=\frac{y}{\left(\frac{dy}{dx}\right)}$ $\Rightarrow$ Subnormal = G = $y\tan\theta=y\frac{dy}{dx}$

$\Rightarrow$ Length of the tangent = PT = $y\operatorname{cosec}\theta=y\sqrt{1+\cot^2\theta}=\frac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}}$

$\Rightarrow$ Length of the normal = PN = $y\sec\theta=y\sqrt{1+\tan^2\theta}=y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$\Rightarrow$ Equation of tangent at $P(x,\ y)\equiv Y-y=\frac{dy}{dx}(X-x)$

$\Rightarrow$ Equation of normal at $P(x, y) \equiv Y - y = -\frac{dy}{dx}(X - x)$

$\Rightarrow$ Length of radius vector $OP = \sqrt{x^2 + y^2}$

The following examples will illustrate the concept of forming and solving the differential equations of the curves whose geometrical properties are given.

Note : Following exact differentials must be remembered :

1. $xdy + ydx = d(xy)$
2. $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$
3. $\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$
4. $\frac{x\,dy + y\,dx}{xy} = d(\ln xy)$
5. $\frac{dx + dy}{x + y} = d(\ln(x + y))$
6. $\frac{x\,dy - y\,dx}{xy} = d\left(\ln\frac{y}{x}\right)$
7. $\frac{y\,dx - x\,dy}{xy} = d\left(\ln\frac{x}{y}\right)$
8. $\frac{x\,dy - y\,dx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$
9. $\frac{y\,dx - x\,dy}{x^2 + y^2} = d\left(\tan^{-1}\frac{x}{y}\right)$
10. $\frac{x\,dx + y\,dy}{x^2 + y^2} = d\left(\ln\sqrt{x^2 + y^2}\right)$
11. $d\left(-\frac{1}{xy}\right) = \frac{x\,dy + y\,dx}{x^2y^2}$
12. $d\left(\frac{e^x}{y}\right) = \frac{ye^x\,dx - e^x\,dy}{y^2}$
13. $d\left(\frac{e^y}{x}\right) = \frac{xe^y\,dy - e^y\,dx}{x^2}$

Applications of First – Order Differential Equations

Type -1 : Growth and Decay Problems

Let N(t) denote the amount of substance (or population) that is either growing or decaying. If we assume that dN/dt, the time rate of change of this amount of substance, is proportional to the amount of substance present, then dN/dt = kN, or $\frac{dN}{dt} - kN = 0$ ….(a)

Where k is the constant of proportionality. We are assuming that N(t) is a differentiable, hence continuous, function of time.

Type -2 : Temperature Problems

Newton's law of cooling, which is equally applicable to heating, states that the time rate of change of the temperature of a body is proportional to the temperature difference between the body and its surrounding medium. Let T denote the temperature of the body and let T_m denote the temperature of the surrounding medium. Then the time rate of change temperature of the body is $\frac{dT}{dt}$, and Newton's law of cooling can be formulated as $\frac{dT}{dt} = -k\left(T - T_m\right)$ *or as* $\frac{dT}{dt} + kT = kT_m....(a)$ where k is a positive constant of proportionality. Once k is chosen positive, the minus sign is required is Newton's law to make $\frac{dT}{dt}$ negative in a cooling process, when T is greater than T_m, and positive in a heating process, when T is less than T_m.

Type – 3 : Dilution Problems

Consider a tank which initially holds V_0, lit. of brine that contains a Ib of salt. Another brine solution, containing b 1b of salt per liter, is poured into the tank at the rate of e lit/min while, simultaneously, the well-stirred solution leaves the tank at the rate of f lit/min. The problem is to find the amount of salt in the tank at any time 1.

Let Q denote the amount of salt in the tank at any time. The time rate of change of Q, dQ/dt, equals the rate at which salt enters the tank minus the rate at which salt leaves the tank. Salt enters the tank at the rate of be Ib/min. To determine the rate at which salt leaves the tank, we first calculate the volume of brine in the tank at any time t, which is the initial volume V_0 plus the volume of brine added et minus the volume of brine removed ft. Thus, the volume of brine at any time is

$V_0 + et - ft \quad ....(\text{i})$

The concentration of salt in the tank at any time is $Q/\left(V_0 + et - ft\right)$ from which it follows that salt leaves the tank at the rate of

$$f\left(\frac{Q}{V_0 + et - ft}\right) lb/\min$$

Thus, $\frac{dQ}{dt} = be - f\left(\frac{Q}{V_0 + et - ft}\right)$.

EXERCISE - I

SINGLE CORRECT ANSWER I TYPE QUESTIONS

Order and Degree of Differential Equations :

1. The order of the differential equation of all tangent lines to the parabola $y = x^2$ is
   A) 1 B) 2 C) 3 D) 4

2. The order of the differential equation whose general solution is given by **[Adv. 1998]**
   $y = (c_1 + c_2)\cos(x + c_3) - c_4 e^{x+c_5}$, where c_1, c_2, c_3, c_4, c_5, are arbitrary constants, is
   A) 5 B) 4 C) 3 D) 2

3. The Differential Equations representing the family of curves $y^2 = 2c\left(x + \sqrt{c}\right)$ where 'c' is positive parameters, is of
   A) order 1 B) order 2 C) degree 3 D) degree 4

4. If p and q are order and degree of differential equation
   $y^2\left(\frac{d^2y}{dx^2}\right)^2 + 3x\left(\frac{dy}{dx}\right)^{\frac{1}{3}} + x^2y^2 = \sin x$, then
   A) p > q B) $\frac{p}{q} = \frac{1}{2}$ C) p = q D) p < q

5. The difference between degree and order of a differential equation that represents the family of curves given by $y^2 = a\left(x + \frac{\sqrt{a}}{2}\right), a > 0$ is….. **[Main 2021]**
   A) 1 B) 2 C) 4 D) 5

PRACTICE QUESTIONS

6. For the differential equation whose solution is
   $y = c_1\cos(x + c_2) - c_3e^{(-x+c_4)} + c_5\sin x$ where $c_{1,}\, c_2, c_3, c_4, c_5$ are arbitrary constants, is of
   A) order 3 B) order 5 C) degree 1 D) degree 3

7. The order, degree of the differential equation satisfying the relation
   $\sqrt{1+x^2} + \sqrt{1+y^2} = \lambda\left(x\sqrt{1+y^2} - y\sqrt{1+x^2}\right)$ is
   A) 1, 1 B) 2, 1 C) 3, 2 D) 0, 1

8. The order, degree of the differential equation of all circles in the 1st quadrant which touch the coordinate axes is
   A) 1, 2 B) 2, 1 C) 3, 2 D) 4, 3

9. The order and degree of the differential equation $e^{\frac{d^3y}{dx^3}} - x\frac{d^2y}{dx^2} + y = 0$
   A) 3, 1 B) 3, 2 C) 3 D) Not defined

Formation of Differential Equation :

10. The differential equation of the family of circles passing through the points (0, 2) and (0, -2) is: **[Main 2022]**

A) $2xy\frac{dy}{dx}+\left(x^2-y^2+4\right)=0$ B) $2xy\frac{dy}{dx}+\left(x^2+y^2-4\right)=0$

C) $2xy\frac{dy}{dx}+\left(y^2-x^2+4\right)=0$ D) $2xy\frac{dy}{dx}-\left(x^2-y^2+4\right)=0$

11. A differential equation representing the family of parabolas with axis parallel to y-axis and whose length of latus rectum is the distance of the point (2, -3) form the line $3x+4y=5$, is given by: **[Main Aug.27,2021 (II)]**

A) $10\frac{d^2y}{dx^2}=11$ B) $11\frac{d^2x}{dy^2}=10$ C) $10\frac{d^2x}{dy^2}=11$ D) $11\frac{d^2y}{dx^2}=10$

12. The differential equation satisfied by the system of parabolas $y^2=4a\left(x+a\right)$ is **[Main July.20,2021 (I)]**

A) $y\left(\frac{dy}{dx}\right)^2-2x\left(\frac{dy}{dx}\right)+y=0$ B) $y\left(\frac{dy}{dx}\right)^2-2x\left(\frac{dy}{dx}\right)-y=0$

C) $y\left(\frac{dy}{dx}\right)+2x\left(\frac{dy}{dx}\right)-y=0$ D) $y\left(\frac{dy}{dx}\right)^2+2x\left(\frac{dy}{dx}\right)-y=0$

13. The D.E. of all non horizontal lines in palne is

A) $\frac{d^2y}{dx^2}=0$ B) $\frac{d^2x}{dy^2}=0$ C) $\frac{dy}{dx}=0$ D) $\frac{dx}{dy}=0$

PRACTICE QUESTIONS

14. The differential equation of the family of curves, $x^2=4b\left(y+b\right), b\in R,$ is **[Main Jan.8,2020 (II)]**

A) $x\left(y'\right)^2=x+2yy'$ B) $x\left(y'\right)^2=2yy'-x$

C) $xy''=y'$ D) $x\left(y'\right)^2=x-2yy'$

15. If $x^2+y^2=1$, then **[Mai 2000]**

A) $yy''-2\left(y'\right)^2+1=0$ B) $yy''+\left(y'\right)^2+1=0$

C) $yy''+\left(y'\right)^2-1=0$ D) $yy''+2\left(y'\right)^2+1=0$

16. For the curve $C:\left(x^2+y^2-3\right)+\left(x^2-y^2-1\right)^5=0$, the value of $3y'-y^3y''$, at the point $\left(\alpha,\alpha\right), \alpha<0$, on C, is equal to **[Main 2022]**

A) 16 B) 14 C) 26 D) 34

Solutions of the Differential Equations :

(a) Inspection method of solving D.E :

17. If $\cos x\frac{dy}{dx}-y\sin x=6x,\left(0<x<\frac{\pi}{2}\right)$ and $y\left(\frac{\pi}{3}\right)=0$, then $y\left(\frac{\pi}{6}\right)$ is equal to

A) $\frac{\pi^2}{2\sqrt{3}}$ B) $-\frac{\pi^2}{2}$ C) $-\frac{\pi^2}{2\sqrt{3}}$ D) $-\frac{\pi^2}{4\sqrt{3}}$ **[Main 2019]**

18. If $(2+\sin x)\frac{dy}{dx}+(y+1)\cos x=0$ and $y(0)=1$, then $y\left(\frac{\pi}{2}\right)$ is equal to:**[M 2017]**

A) $\frac{4}{3}$ B) $\frac{1}{3}$ C) $-\frac{2}{3}$ D) $-\frac{1}{3}$

19. If a curve $y=f(x)$ passes through the point $(1,-1)$ and satisfies the differential equation, $y(1+xy)dx=x\,dy$, then $f\left(-\frac{1}{2}\right)$ is equal : **[Main 2016]**

A) $\frac{2}{5}$ B) $\frac{4}{5}$ C) $-\frac{2}{5}$ D) $-\frac{4}{5}$

20. The solution of the differential equation $ydx-\left(x+2y^2\right)dy=0$ is $x=f(y)$. If $f(-1)=1$, then $f(1)$ is equal to:

A) 4 B) 3 C) 1 D) 2

21. Let the solution curve of the differential equation $xdy=\left(\sqrt{x^2+y^2}+y\right)dx, x>0,$ interested the line $x=1$ at $y=0$ and the line $x=2$ at $y=\alpha$. Then the value of α is : **[Main July 29, 2022(I)]**

A) $\frac{1}{2}$ B) $\frac{3}{2}$ C) $-\frac{3}{2}$ D) $\frac{5}{2}$

22. Let $x=x(y)$ be the solution of the differential equation **[Main June 28, 2022(II)]**
$2ye^{x/y^2}dx+\left(y^2-4xe^{x/y^2}\right)dy=0$ such that $x(1)=0$. Then, $x(e)$ is equal to

A) $e\log_e(2)$ B) $-e\log_e(2)$ C) $e^2\log_e(2)$ D) $-e^2\log_e(2)$

23. Let $y=y(x)$ be the solution curve of the differential equation

$$\sin\left(2x^2\right)\log_e\left(\tan x^2\right)dy+\left(4xy-4\sqrt{2}x\sin\left(x^2-\frac{\pi}{4}\right)\right)dx=0,\ 0<x<\sqrt{\frac{\pi}{2}},$$

which passes through the point $\left(\sqrt{\frac{\pi}{6}},1\right)$. Then $\left|y\left(\sqrt{\frac{\pi}{3}}\right)\right|$ is equal to **[Main 2022(I)]**

A) 1 B) 11 C) 2 D) 22

PRACTICE QUESTIONS

24. Solution of the differential equation $\left\{\frac{1}{x}-\frac{y^2}{(x-y)^2}\right\}dx+\left\{\frac{x^2}{(x-y)^2}-\frac{1}{y}\right\}dy=0$ is

(A) $\ln\left|\frac{x}{y}\right|+\frac{xy}{(x-y)}=c$ (B) $\ln|xy|+\frac{xy}{(x-y)}=c$

(C) $\frac{xy}{(x-y)}=ce^{x/y}$ (D) $\frac{xy}{(x-y)}=ce^{xy}$

25. The solution of differential equation $xdx+ydy+\frac{xdy-ydx}{x^2+y^2}=0$

A) $y=\tan\left(\frac{c-x-y}{2}\right)$ B) $y=x\tan\left(\frac{c-x^2-y^2}{2}\right)$

C) $y=x\tan\left(\frac{c+x^2+y^2}{2}\right)$ D) $y=x\tan\left(\frac{c+x^2-y^2}{2}\right)$

26. The solution curves of the differential equation
$(xdx+ydy)\sqrt{x^2+y^2}=(xdy-ydx)\left(\sqrt{1-x^2-y^2}\right)$ are

A) circles of radius 1, passing through the origin

B) circles of radius $\frac{1}{2}$,passing through the origin

C) circles not passing through origin

D) solution curve is not a circle

27. The solution of differential equation $(x^5+x+2x^2y)\,dy+(3x^4y-y)\,dx=0$ is

(A) $x^4y+xy^2+y=cx$ (B) $x^4y^2+xy+y=cx$

(C) $x^4y+x^2y^2+xy=c$ (D) $x^4y-xy^2-y=cx$

28. The solution of the differential equation $\frac{\sqrt{x}dx+\sqrt{y}dy}{\sqrt{x}dx-\sqrt{y}dy}=\sqrt{\frac{y^3}{x^3}}$ is given by

A) $\frac{3}{2}\log\left(\frac{y}{x}\right)+\log\left|\frac{x^{3/2}+y^{3/2}}{x^{3/2}}\right|+\tan^{-1}\left(\frac{y}{x}\right)^{3/2}+c=0$

B) $\frac{2}{3}\log\left(\frac{y}{x}\right)+\log\left|\frac{x^{3/2}+y^{3/2}}{x^{3/2}}\right|+\tan^{-1}\frac{y}{x}+c=0$

C) $\frac{2}{3}\left(\frac{y}{x}\right)+\log\left(\frac{x+y}{x}\right)+\tan^{-1}\left(\frac{y^{3/2}}{x^{3/2}}\right)+c=0$

D) None of these

29. The solution of the differential equation
$\left(x^2\sin^3y-y^2\cos x\right)dx+\left(x^3\cos y\sin^2y-2y\sin x\right)dy=0$ is

A) $x^3\sin^3y=3y^2\sin x+C$ B) $x^3\sin^3y=3y^2\sin x=C$

C) $x^3\sin^3y=y^3\sin x=C$ D) $2x^2\sin y+y^2\sin x=C$

30. Solution of the differential equation $2y\sin x\frac{dy}{dx}=2\sin x\cos x-y^2\cos x$ satisfying $y(\pi/2)=1$ is given by

A) $y^2=\sin x$ B) $y=\sin^2x$ C) $y^2=\cos x+1$ D) $y^2\sin x=4\cos^2x$

31. Solution of the differential equation : $\frac{dy}{dx}=\frac{3x^2y^4+2xy}{x^2-2x^3y^3}$ is

(A) $x^2y^2+\frac{x^2}{y}=c$ (B) $x^3y^2+\frac{x^2}{y}=c$

(C) $x^3y^2+\frac{y^2}{x}=c$ (D) $x^2y^3+\frac{x^2}{y}=c$

32. An equation of the curve satisfying $xdy-ydx=\sqrt{x^2-y^2}dx$ and $y(1)=0$ is

A) $y=x^2\log|\sin x|$ B) $y=x\sin(\log|x|)$ C) $y^2=x(x-1)^2$ D) $y=2x^2(x-1)$

33. Solution of $\frac{x+y\frac{dy}{dx}}{y-x\frac{dy}{dx}}=x^2+2y^2+\frac{y^4}{x^2}$

A) $\frac{y}{x}-\frac{1}{2(x^2+y^2)}=c$ B) $\frac{y}{x}+\frac{1}{2(x^2+y^2)}=c$

C) $\frac{x}{y}+\frac{1}{2(x^2+y^2)}=c$ D) $\frac{x}{y}-\frac{1}{2(x^2+y^2)}=c$

34. Solution of $\frac{y+\sin x\cos^2(xy)}{\cos^2(xy)}dx+\left(\frac{x}{\cos^2(xy)}+\sin y\right)dy=0$

A) $\tan(xy)+\cos x+\cos y=c$ B) $\tan(xy)-\cos x-\cos y=c$

C) $\tan(xy)+\cos x-\cos y=c$ D) $\tan(xy)-\cos x+\cos y=c$

35. The solution of $\left(y\left(1+x^{-1}\right)+\sin y\right)dx+\left(x+\log x+x\cos y\right)dy=0$ is

A) $\left(1+y^{-1}\sin y\right)+x^{-1}\log x=C$ B) $\left(y+\sin y\right)+xy\log x=C$

C) $xy+y\log x+x\sin y=C$ D) None of these

(b) Varaiable separable form :

36. The general solution of the differential equation $\sqrt{1+x^2+y^2+x^2y^2}+xy\frac{dy}{dx}=0$ is : (where C is a constant of integration) **[Main Sep.06, 2020(I)]**

A) $\sqrt{1+y^2}+\sqrt{1+x^2}=\frac{1}{2}\log_e\left(\frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}-1}\right)+C$

B) $\sqrt{1+y^2}-\sqrt{1+x^2}=\frac{1}{2}\log_e\left(\frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}-1}\right)+C$

C) $\sqrt{1+y^2}+\sqrt{1+x^2}=\frac{1}{2}\log_e\left(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right)+C$

D) $\sqrt{1+y^2}-\sqrt{1+x^2}=\frac{1}{2}\log_e\left(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right)+C$

37. The solution of the differential equation $\frac{dy}{dx}-ky=0,\ y(0)=1$, approach zero when $x\to\infty$ if

A) K = 0 B) k > 0 C) k < 0 D) none of these

38. If $\frac{dy}{dx}=\frac{2^{x+y}-2^{x}}{2^{y}},y\left(0\right)=1,$ then $y\left(1\right)$ is equal to : **[Main Aug.31,2021(I)]**

A) $\log_2\left(2+e\right)$ B) $\log_2\left(1+e\right)$ C) $\log_2\left(2e\right)$ D) $\log_2\left(1+e^2\right)$

39. If $\frac{dy}{dx}=\frac{x^2y+2^y.2^x}{2^x+2^{x+y}\log_e 2},y\left(0\right)=0,$

then for y=1, the value of x lies in the interval **[Main Aug.31,2021(II)]**

A) $\left(1,2\right)$ B) $\left(\frac{1}{2},1\right]$ C) $\left(2,3\right)$ D) $\left(0,\frac{1}{2}\right]$

40. Which of the following is true for $y\left(x\right)$ that satisfies the differential equation

$\frac{dy}{dx}=xy-1+x-y;y\left(0\right)=0:$ **[Main March 17, 2021(I)]**

A) $y\left(1\right)=e^{\frac{1}{2}}-e^{\frac{1}{2}}$ B) $y\left(1\right)=1$

C) $y\left(1\right)=e^{\frac{1}{2}}-1$ D) $y\left(1\right)=e^{-\frac{1}{2}}-1$

41. The solution curve of the differential equation, $\left(1+e^{-x}\right)\left(1+y^2\right)\frac{dy}{dx}=y^2$, which passes through the point $\left(0,1\right)$ is **[Main 2020]**

A) $y^2+1=y\left(\log_e\left(\frac{1+e^{-x}}{2}\right)+2\right)$ B) $y^2+1=y\left(\log_e\left(\frac{1+e^{x}}{2}\right)+2\right)$

C) $y^2=1+y\log_e\left(\frac{1+e^{x}}{2}\right)$ D) $y^2=1+y\log_e\left(\frac{1+e^{-x}}{2}\right)$

42. If $y=y\left(x\right)$ satisfies the differential equation

$8\sqrt{x}\left(\sqrt{9+\sqrt{x}}\right)dy=\left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)^{-1}dx,x>0$ and $y\left(0\right)=\sqrt{7}$, then

$y\left(256\right)=$ **[Adv.2018]**

A) 3 B) 9 C) 16 D) 80

43. If the solution of differential equation $\frac{dy}{dx}=\frac{ax+3}{2y+f}$ represents a circle of non zero radius then

A) $a=2\,;9+4f^2>4c$ B) $a=-2\,;9+4f^2<4c$

C) $a=2\,;9+4f^2<4c$ D) $a=-2;9+4f^2>4c$

PRACTICE QUESTIONS

44. If $y = y(x)$ is the solution of the differential equation
$\left(1+e^{2x}\right)\frac{dy}{dx}+2\left(1+y^2\right)e^x=0$ and $y(0)=0,$ then $6\left(y'(0)+\left(y\left(\log_e\sqrt{3}\right)\right)^2\right)$ is equal to **[Main June 29, 2022 (II)]**
A) 2 B) -2 C) -4 D) -1

45. If $\frac{dy}{dx}+\frac{2^{x-y}\left(2^y-1\right)}{2^x-1}=0, x, y>0, y(1)=1,$
then $y(2)$ is equal to **[Main June 27, 2022 (I)]**
A) $2+\log_2 3$ B) $2+\log_3 2$ C) $2-\log_3 3$ D) $2-\log_2 3$

46. Solution to the differential equation $\frac{x+\frac{x^3}{3!}+\frac{x^5}{5!}+........}{1+\frac{x^2}{2!}+\frac{x^4}{4!}+......}=\frac{dx-dy}{dx+dy}$ is
A) $2y\, e^{2x} = C.e^{2x} + 1$ B) $2y\, e^{2x} = C.e^{2x} - 1$
C) $y\, e^{2x} = C.e^{2x} + 2$ D) $2x\, e^{2y} = C.e^{x} - 1$

47. An equation of the curve in which subnormal varies as the square of the ordinate is (k is constant of proportionality)
A) $y = Ae^{kx}$ B) $y = e^{kx}$ C) $y^2/2+kx=A$ D) $y^2+kx^2=A$

48. The solution of $\frac{dy}{dx}=\frac{ax+b}{cy+d}$ represents a parabola if
A) $a=0, c=0$ B) $a=1, b=2$ C) $a=0, c\neq 0$ D) $a=1, c=1$

49. The equation of the curve passing through $(3,9)$ which satisfies $dy/dx = x+1/x^2$ is
A) $6xy=3x^2-6x+29$ B) $6xy=3x^2-29x+6$
C) $6xy=3x^3+29x-6$ D) None of these

(c) Differential Equations reducible to variable separable form :

50. Let $y = y_1(x)$ and $y = y_2(x)$ be two distinct solutions of the differential equation $\frac{dy}{dx}=x+y,$ with $y_1(0)=0$ and $y_2(0)=1$ respectively. Then, the number of points of intersection of $y = y_1(x)$ and $y = y_2(x)$ is **[July 27, 2022 (I)]**
A) 0 B) 1 C) 2 D) 3

(d) Homogeneous Differential Equation :

51. A curve passes through the point $\left(1,\frac{\pi}{6}\right)$. Let the slope of the curve at each point (x,y) be $\frac{y}{x}+\sec\left(\frac{y}{x}\right), x>0$. Then the equation of the curve is **[Adv.2013]**

A) $\sin\left(\frac{y}{x}\right)=\log x+\frac{1}{2}$ B) $\operatorname{cosec}\left(\frac{y}{x}\right)=\log x+2$

C) $\sec\left(\frac{2y}{x}\right)=\log x+2$ D) $\cos\left(\frac{2y}{x}\right)=\log x+\frac{1}{2}$

52. If a curve $y-f(x)$, passing through the point (1, 2), is the solution of the differential equation, $2x^2dy=\left(2xy+y^2\right)dx$, then $f\left(\frac{1}{2}\right)$ is equal to : **[Main 2020]**

A) $\frac{1}{1+\log_e 2}$ B) $\frac{1}{1-\log_e 2}$ C) $1+\log_e 2$ D) $\frac{-1}{1+\log_e 2}$

53. Solution of $\left(xe^{\frac{y}{x}}-y\mathrm{Sin}\frac{y}{x}\right)dx+x\mathrm{Sin}\frac{y}{x}dy=0$

A) $\log x=c+\frac{1}{2}e^{\frac{-y}{x}}\left(\mathrm{Sin}\frac{y}{x}+\mathrm{Cos}\frac{y}{x}\right)$ B) $\log x=c+\frac{1}{2}e^{\frac{y}{x}}\left(\mathrm{Sin}\frac{y}{x}-\mathrm{Cos}\frac{y}{x}\right)$

C) $\log x=c+\frac{1}{2}e^{\frac{y}{x}}\left(\mathrm{Sin}\frac{y}{x}+\mathrm{Cos}\frac{y}{x}\right)$ D) $\log x=c+\frac{1}{2}e^{\frac{-y}{x}}\left(\mathrm{Sin}\frac{y}{x}-\mathrm{Cos}\frac{y}{x}\right)$

54. The solution of the differential equation $y^2dx+\left(x^2-xy+y^2\right)dy=0$ is

(A) $\tan^{-1}\left(\frac{x}{y}\right)+\ln y+C=0$ (B) $2\tan^{-1}\left(\frac{x}{y}\right)+\ln x+C=0$

(C) $\ln\left(y+\sqrt{x^2+y^2}\right)+\ln y+C=0$ (D) $\ln\left(x+\sqrt{x^2+y^2}\right)+C=0$

(Where C is arbitrary constant)

55. Let the solution curve $y=y(x)$ of the differential equation $\left[\frac{x}{\sqrt{x^2-y^2}}+e^{\frac{y}{x}}\right]x\frac{dy}{dx}=x+\left[\frac{x}{\sqrt{x^2-y^2}}+e^{\frac{y}{x}}\right]y$ pass through the points $(1,0)$ and $(2\alpha,\alpha),\alpha>0$. Then α is equal to **[Main June 28,2022 (I)]**

A) $\frac{1}{2}\exp\left(\frac{\pi}{6}+\sqrt{e}-1\right)$ B) $\frac{1}{2}\exp\left(\frac{\pi}{3}+\sqrt{e}-1\right)$

C) $\exp\left(\frac{\pi}{6}+\sqrt{e}+1\right)$ D) $2\exp\left(\frac{\pi}{3}+\sqrt{e}-1\right)$

PRACTICE QUESTIONS

56. Solution of the equation $xdy = \left(y + x\dfrac{f(y/x)}{f(y/x)} \right)dx$

A) $f\left(\dfrac{x}{y}\right) = cy$ B) $f\left(\dfrac{y}{x}\right) = cx$ C) $f\left(\dfrac{y}{x}\right) = cxy$ D) $f\left(\dfrac{y}{x}\right) = 0$

57. If for the differential equation $y^1 = \dfrac{y}{x} + \phi\left(\dfrac{x}{y}\right)$ the general solution is $y = \dfrac{x}{\log|Cx|}$ then $\phi(x/y)$ is given by

A) $-x^2/y^2$ B) y^2/x^2 C) x^2/y^2 D) $-y^2/x^2$

58. Let the solution curve of the differential equation $x\dfrac{dy}{dx} - y = \sqrt{y^2 + 16x^2}, y(1) = 3$ be $y = y(x)$. Then $y(2)$ is equal to : **[M-2022]**

A) 15 B) 11 C) 13 D) 17

(e) Equations reducible to homogeneous equation :

59. The general solution of the differential equation $\left(x - y^2\right)dx + y\left(5x + y^2\right)dy = 0$ is **[Main July 25,2022(I)]**

A) $\left(y^2 + x\right)^4 = C\left|\left(y^2 + 2x\right)^3\right|$ B) $\left(y^2 + 2x\right)^4 = C\left|\left(y^2 + x\right)^3\right|$

C) $\left|\left(y^2 + x\right)^3\right| = C\left(2y^2 + x\right)^4$ D) $\left|\left(y^2 + 2x\right)^3\right| = C\left(2y^2 + x\right)^4$

60. Solution of D.E $\dfrac{dy}{dx} = \dfrac{x + 2y - 1}{x + 2y + 1}$ is

A) $y - x + \dfrac{2}{3}\ln(3x + 6y - 1) = C$ B) $y + x + \dfrac{2}{3}\ln(3x - 6y - 1) = C$

C) $y^2 + x - \dfrac{2}{3}\ln\left(3x^2 - 6y - 1\right) = C$ D) $y^2 - x - \dfrac{2}{3}\ln\left(3x^2 - 6y^2 + 1\right) = C$

61. Solution of D.E $\dfrac{dy}{dx} = \dfrac{x + 2y - 3}{2x + y - 3}$ is

A) $x + y - 2 = C^2(x - y)^3$ B) $x - y - 7 = C(x + y)^4$

C) $(x + y)^2 - 7 = C(x + 2y)^3$ D) $x + y - 9 = C(x + 2y)^3$

(f) Lagrange's linear differential equation :

62. The general solution of the differential equation, $\sin 2x\left(\dfrac{dy}{dx} - \sqrt{\tan x}\right) - y = 0$, is : **[Main 2014]**

A) $y\sqrt{\tan x} = x + c$ B) $y\sqrt{\cot x} = \tan x + c$

C) $y\sqrt{\tan x} = \cot x + c$ D) $y\sqrt{\cot x} = x + c$

63. Let the solution curve $y = y(x)$ of the differential equation $(1+e^{2x})\left(\frac{dy}{dx}+y\right)=1$ pass through the point $\left(0,\frac{\pi}{2}\right)$. Then, $\lim\limits_{x\to\infty} e^x y(x)$ is equal to **[Main July 29, 2022(I)]**

A) $\frac{\pi}{4}$ B) $\frac{3\pi}{4}$ C) $\frac{\pi}{2}$ D) $\frac{3\pi}{2}$

64. Let $y = y(x)$ be the solution curve of the differential equation $\frac{dy}{dx}+\left(\frac{2x^2-11x+13}{x^3+6x^2+11x+6}\right)y=\frac{(x+3)}{x+1}, x>-1$, which passes through the point $(0,1)$. Then $y(1)$ is equal to **[Main July 29, 2022 (II)]**

A) $\frac{1}{2}$ B) $\frac{3}{2}$ C) $\frac{5}{2}$ D) $\frac{7}{2}$

65. Let $y = y(x)$ be the solution curve of the differential equation $\frac{dy}{dx}+\frac{1}{x^2-1}y=\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}}, x>1$ passing through the point $\left(2,\sqrt{\frac{1}{3}}\right)$. Then $\sqrt{7}y(8)$ is equal to **[Main July 28, 2022 (II)]**

A) $11+6\log_e 3$ B) 9 C) $12-2\log_e 3$ D) $19-6\log_e 3$

66. $y = y(x), x\in\left(0,\frac{\pi}{2}\right)$ be the solution curve of the differential equation $(\sin^2 2x)+\frac{dy}{dx}+(8\sin^2 2x+2\sin 4x)y=$ **[Main July 28, 2022 (I)]**

$2e^{-4x}(2\sin 2x+\cos 2x)$, with $y\left(\frac{\pi}{4}\right)3^{-\pi}$, then $y\left(\frac{\pi}{6}\right)$ is equal to :

A) $\frac{2}{\sqrt{3}}e^{-2\pi/3}$ B) $\frac{2}{\sqrt{3}}e^{2\pi/3}$ C) $\frac{1}{\sqrt{3}}e^{-2\pi/3}$ D) $\frac{1}{\sqrt{3}}e^{2\pi/3}$

67. For $x\in\mathbb{R}$, let the function $y(x)$ be the solution of the differential equation $\frac{dy}{dx}+12y=\cos\left(\frac{\pi}{12}x\right), y(0)=0$. Then, which of the following statements is/are TRUE? **[Adv. 2022]**

A) $y(x)$ is an increasing function

B) $y(x)$ is a decreasing function

C) There exists a real number β such that the line $y=\beta$ intersects the curve $y = y(x)$ at infinitely many points

D) $y(x)$ is a periodic function

68. Let $y=y(x)$ be the solution of the differential equation $\frac{dy}{dx}+2y=f(x)$, where $f(x)=\begin{cases}1, & x\in[0,1]\\ 0, & \text{otherwise}\end{cases}$. If $y(0)=0$, then $y\left(\frac{3}{2}\right)$ is **[Main 2018]**

A) $\frac{e^2-1}{2e^3}$ B) $\frac{e^2-1}{e^3}$ C) $\frac{1}{2e}$ D) $\frac{e^2+1}{2e^2}$

69. The function $y=f(x)$ is the solution of the differential equation $\frac{dy}{dx}+\frac{xy}{x^2-1}=\frac{x^4+2x}{\sqrt{1-x^2}}$ in $(-1,1)$ satisfying $f(0)=0$. Then $\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x)d(x)$ is

A) $\frac{\pi}{3}-\frac{\sqrt{3}}{2}$ B) $\frac{\pi}{3}-\frac{\sqrt{3}}{4}$ C) $\frac{\pi}{6}-\frac{\sqrt{3}}{4}$ D) $\frac{\pi}{6}-\frac{\sqrt{3}}{2}$ **[Adv.2014]**

70. Let $y=y(x)$ be the solution of the differential equation $\frac{dy}{dx}+\frac{\sqrt{2}y}{2\cos^4 x-\cos 2x}=xe^{\tan^{-1}\left(\sqrt{2}\cot 2x\right)}, 0<x<\frac{\pi}{2}$ with $y\left(\frac{\pi}{4}\right)=\frac{\pi^2}{32}$. If $y\left(\frac{\pi}{3}\right)=\frac{\pi^2}{18}e^{-\tan^{-1}(\alpha)}$, then the value of $3\alpha^2$ is equal to … **[Main June 29,2022(I)]**

A) 2 **B) 1** **C) 3** **D) 5**

PRACTICE QUESTIONS

71. Let $y(x)$ be the solution of the differential equation $(x\log x)\frac{dy}{dx}+y=2x\log x,(x\geq 1)$. Then $y(e)$ is equal to :

A) 2 B) 2e C) e D) 0

72. If $\frac{dy}{dx}+2y\tan x=\sin x, 0<x<\frac{\pi}{2}$ and $y\left(\frac{\pi}{3}\right)=0$, then the maximum value $y(x)$ is :

A) $\frac{1}{8}$ B) $\frac{3}{4}$ C) $\frac{1}{4}$ D) $\frac{3}{8}$

73. A solution of $y=2x\left(\frac{dy}{dx}\right)+x^2\left(\frac{dy}{dx}\right)^4$ is

A) $y=2c^{1/2}x^{1/4}+c$ B) $y=2\sqrt{c}x^2+c^2$ C) $y=2\sqrt{c}(x+1)$ D) $y=2\sqrt{cx}+c^2$

74. Let $f:(0,\infty)\to\mathbb{R}$ be a differentiable function such that $f'(x)=2-\frac{f(x)}{x}$ for all $x\in(0,\infty)$ and $f(1)\neq 1$. Then **[Adv. 2016]**

A) $\lim\limits_{x\to 0+} f'\left(\frac{1}{x}\right)=1$ B) $\lim\limits_{x\to 0+} xf\left(\frac{1}{x}\right)=2$

C) $\lim\limits_{x\to 0+} x^2 f'(x)=0$ D) $|f(x)|\le 2$ for all $x\in(0,2)$

75. The solution of $(x+2y^3)(dy/dx)=y$ is (where c is arbitrary constant)

A) $x=y^3+cy$ B) $x=y^2-cy$ C) $y=x^3-cy$ D) $y=x^3+cy$

76. If $y_1(x)$ is a solution of the differential equation $\frac{dy}{dx}+f(x)y=0$, then a solution of differential equation $\frac{dy}{dx}+f(x)y=r(x)$ is

A) $\frac{1}{y(x)}\int y_1(x)dx$ B) $y_1(x)\int\frac{r(x)}{y_1(x)}dx$

C) $\int r(x)y_1(x)dx$ D) none of these

77. The general solution of $x\frac{dy}{dx}+(\log x)y=x^{1-\frac{1}{2}\log x}$ is

A) $y=x^{1-\frac{1}{2}\log x}+cx^{-\frac{1}{2}\log x}$ B) $y.x^{\frac{1}{2}\log x}=x^{\frac{1}{2}\log x}+c$

C) $y=e^{\frac{(\log x)^2}{2}}(x+c)$ D) $y=e^{\frac{1}{2}(\log x)^2}(x^{1-\frac{1}{2}(\log x)}-x^{-\frac{1}{2}(\log x)})+c$

Bernoulli's Equation (Reducible to linear equations) :

78. If $y=y(x)$ is the solution curve of the differential equation $x^2dy+\left(y-\frac{1}{x}\right)dx=0; x>0$ and $y(1)=1$, then $y\left(\frac{1}{2}\right)$ is equal to : **[Main 2021]**

A) $\frac{3}{2}-\frac{1}{\sqrt{e}}$ B) $3+\frac{1}{\sqrt{e}}$ C) $3+e$ D) $3-e$

79. The solution of the differential equation $\frac{dy}{dx}+\frac{y}{2}\sec x=\frac{\tan x}{2y}$, then $0\le x<\frac{\pi}{2}$, and $y(0)=1$, is given by : **[Main 2016]**

A) $y^2=1+\frac{x}{\sec x+\tan x}$ B) $y=1+\frac{x}{\sec x+\tan x}$

C) $y=1-\frac{x}{\sec x+\tan x}$ D) $y^2=1-\frac{x}{\sec x+\tan x}$

80. The equation to the curve which is such that portion of the axis of x cutoff between the origin and the tangent at any point is proportional to the ordinate of that point is (constant of proportionality is K)

A) x=y(C-K log y) B) log x = Ky^2+C

C) x^2=y(C- K log y) D) None of these

81. The equation of the curve which is passing through (1, 1) and whose differential equation is $\frac{dy}{dx}+\frac{y}{x}=y^3$ is

(A) $2x^2y^2 - xy^2 = 1$ (B) $2xy^2 + x^2y^2 = 1$

(C) $2x^2y^2 + xy^2 = 1$ (D) $2xy^2 - x^2y^2 = 1$

PRACTICE QUESTIONS

82. The solution of $\frac{dy}{dx}+x(x+y)=x^3(x+y)^3-1, y(0)=1;$ is given by $2(1+x^2)-\frac{1}{(x+y)^2}=f(x)$ where $f(x)=$

A) e^{-x} B) e^{-x^2} C) e^{x} D) e^{x^2}

83. Which of the following transformation reduces the differential equation $\frac{dz}{dx}+\frac{z}{x}\log z=\frac{z}{x^2}(\log z)^2$ to the form $P(x)u=Q(x)$

A) $u=\log z$ B) $u=e^z$ C) $u=(\log z)^{-1}$ D) $u=(\log z)^2$

84. The general solution of the differential equation $\frac{dy}{dx}=y\tan x-y^2\sec x$ is

A) $\tan x=(c+\sec x)y$ B) $\sec y=(c+\tan y)x$

C) $\sec x=(c+\tan x)y$ D) None of these

85. The solution of the differential equation $2x^3ydy+(1-y^2)(x^2y^2+y^2-1)dx=0$ [Where c is a constant]

(A) $x^2y^2=(cx+1)(1-y^2)$ (B) $x^2y^2=(cx+1)(1+y^2)$

(C) $x^2y^2=(cx-1)(1-y^2)$ (D) none of these

Miscellaneous methods on solving D.E :

86. If the solution curve of the differential equation $\frac{dy}{dx}=\frac{x+y-2}{x-y}$ passes through the point $(2,1)$ and $(k+1,2), k>0$ then **[Main July 29, 2022 (II)]**

A) $2\tan^{-1}\left(\frac{1}{k}\right)=\log_e(k^2+1)$ B) $\tan^{-1}\left(\frac{1}{k}\right)=\log_e(k^2+1)$

C) $2\tan^{-1}\left(\frac{1}{k+1}\right)=\log_e(k^2+2k+2)$ D) $2\tan^{-1}\left(\frac{1}{k}\right)=\log_e\left(\frac{k^2+1}{k^2}\right)$

87. Solution of $\left(\frac{x+y-1}{x+y-2}\right)\frac{dy}{dx}=\left(\frac{x+y+1}{x+y+2}\right)$, given that y = 1 when x = 1 is

(A) $\ln\left|\frac{(x-y)^2-2}{2}\right|=2(x+y)$ (B) $\ln\left|\frac{(x+y)^2-2}{2}\right|=2(x-y)$

(C) $\ln\left|\frac{(x-y)^2+2}{2}\right|=2(x+y)$ (D) $\ln\left|\frac{(x+y)^2+2}{2}\right|=2(x-y)$

88. The solution of $y_2 - 7y_1 + 12y = 0$ is

A) $y = C_1e^{3x} + C_2e^{4x}$ B) $y = C_1xe^{3x} + C_2e^{4x}$ C) $y = C_1e^{3x} + C_2xe^{4x}$ D) None of these

89. The solution of the differential equation $y_1y_3 = 3y_2^2$ is

A) $x = A_1y^2 + A_2y + A_3$

B) $x = A_1y + A_2$

C) $x = A_1y^2 + A_2y$

D) none of these

90. The solution $y(x)$ of the differential equation $\frac{d^2y}{dx^2} = \sin 3x + e^x + x^2$ when $y_1(0) = 1$ and $y(0) = 0$ is

A) $-\frac{\sin 3x}{9} + e^x + \frac{x^4}{12} + \frac{1}{3}x - 1$

B) $-\frac{\sin 3x}{9} + e^x + \frac{x^4}{12} + \frac{1}{3}x$

C) $-\frac{\cos 3x}{3} + e^x + \frac{x^4}{12} + \frac{1}{3}x + 1$

D) None of these

91. The solution of $\frac{dy}{dx}\sqrt{1+x+y} = x + y - 1$ is

A) $2\left[\sqrt{1+x+y} + \frac{1}{3}\log\left|\sqrt{1+x+y} - 1\right| - \frac{4}{3}\log\left|\sqrt{1+x+y} + 2\right|\right] = x + c$

B) $2\left[\sqrt{1+x+y} + \frac{1}{3}\log\left|\sqrt{1+x+y}\right| - \frac{4}{3}\log\left|\sqrt{1+x+y} + 2\right|\right] = x + c$

C) $2\left[\sqrt{1+x+y} + \frac{1}{3}\log\left|\sqrt{1+x+y}\right| - \frac{4}{3}\log\left|\sqrt{1+x+y}\right|\right] = x + c$

D) $\left[\sqrt{1+x+y} + \frac{1}{3}\log\left|\sqrt{1+x+y}\right| - \frac{4}{3}\log\left|\sqrt{1+x+y}\right|\right] = x + c$

Applications of Differential Equations :

a) Geometric Application of D.E. :

92. Let a smooth curve $y = f(x)$ be such that the slope of the tangent at any point (x, y) on it is directly proportional to $\left(\frac{-y}{x}\right)$. If the curve passes through the point $(1,2)$ and $(8,1)$, then $\left|y\left(\frac{1}{8}\right)\right|$ is equal to **[Main July 25, 2022(II)]**

A) $2\log_e^2$ B) 4 C) 1 D) $4\log_e^2$

93. A curve 'C' passes through (2, 0) and the slope at (x, y) as $\frac{(x+1)^2 + (y-3)}{x+1}$. Find the equation of the curve. Find the area bounded by curve and x-axis in fourth quadrant ______ sq.units **[Main 2004]**

A) $\frac{4}{3}$ B) $\frac{8}{3}$ C) $\frac{6}{3}$ D) $\frac{5}{3}$

94. A particle is moving in the xy-plane along a curve C passing through the point (3, 3). The tangent to the curve C at the point P meets the x-axis at Q. If the y-axis bisects the segment PQ,
then C is a parabola with **[Main June 24, 2022 (II)]**

A) length of latus return 3
B) Length of latus return 6
C) focus $\left(\frac{4}{3},0\right)$
D) focus $\left(0,\frac{3}{4}\right)$

95. The slope of normal at any point $(x,y), x>0, y>0$ on the curve $y=y(x)$ is given by $\frac{x^2}{xy-x^2y^2-1}$. If the curve passes through the point $(1,1)$, then $e.y(e)$ is equal to

[Main June 24, 2022(II)]

A) $\frac{1-\tan(1)}{1+\tan(1)}$
B) $\tan(1)$
C) 1
D) $\frac{1+\tan(1)}{1-\tan(1)}$

96. Let a curve $y=y(x)$ pass through the point (3, 3) and the area of the region under this curve, above the x-axis and between the abscissae 3 and $x(>3)$ be $\left(\frac{y}{x}\right)^3$. If this curve also passes through the point $\left(\alpha, 6\sqrt{10}\right)$ in the first quadrant, then α is equal to

[Main July 26, 2022(I)]

A) 5
B) 8
C) 6
D) 4

97. Suppose $y=y(x)$ be the solution curve to the differential equation $\frac{dy}{dx}-y=2-e^{-x}$ such that $\lim\limits_{x\to\infty} y(x)$ is finite. If a and b are respectively the x- and y- intercepts of the tangent to the curve at $x=0$, then the value of $a-4b$ is equal to …

[Main July 26, 2022(II)]

A) 1
B) 2
C) 3
D) 4

98. The curve for which the x-intercept of the tangent drawn at any point P on the curve is three times the x-coordinate of the point P, is
(A) $xy = c$
(B) $xy^2 = c$
(C) $xy^3 = c$
(D) $x^2y = c$

PRACTICE QUESTIONS

99. A curve is such that the mid point of the portion of the tangent intercepted between the point where the tangent is drawn and the point where the tangent meets y-axis, lies on the line $y = x$. If the curve passes through (1, 0), then the curve is
(A) $2y = x^2 - x$
(B) $y = x^2 - x$
(C) $y = x - x^2$
(D) $y = 2(x - x^2)$

100. A function $y=f(x)$ has a second order derivative $f''=6(x-1)$. If its graph passes through the point(2,1) and at that point the tangent to the graph is $y=3x-5$, then the function is
A) $(x-1)^2$
B) $(x-1)^3$
C) $(x+1)^2$
D) $(x+1)^3$

101. The normal to a curve at $P(x,y)$ meets the $x-axis$ at G. If the distance of G from the origin is twice the abscissa of P, then the curve is
A) a parabola B) a circle C) a hyperbola D) an ellipse

102. The curves satisfying the differential equation $\left(1-x^2\right)y^1+xy=ax$ are
A) ellipses and hyperbolas B) ellipses and parabola
C) ellipses and straight lines D) circles and ellipses

103. If for a curve ratio of the distance between the normal at any of its points and the origin to the distance between the same normal and the point (a, b) is equal to the constant k (k >0, k ≠1), then the curve is a
(A) circle (B) parabola
(C) ellipse (D) hyperbola

104. The differential equation of all circles in a plane must be
A) $y_3(1+y_1^2)-3y_1y_2^2=0$ B) of order 3 and degree 3
C) of order 3 and degree 2 D) $y_3(1-y_1^2)-3y_1y_2^2=0$

105. The equation of the curve not passing through the origin and having the portion of the tangent included between the coordinate axes is bisected at the point of contact is
A) a parabola B) an ellipse or a straight line
C) a circle or an ellipse D) a hyperbola or a straight line

106. The curve $y=f(x)$ is such that the area of the trapezium formed by the coordinate axes, ordinate of an arbitrary point and the tangent at this point equals half the square of its abscissa. The equation of the curve can be
A) $y=cx^2\pm x$ B) $y=cx^2\pm 1$
C) $y=cx^2\pm x^2$ D) $y=cx^2\pm x^2\pm 1$

107. The family of curves passing through (0,0) and satisfying the differential equation $\frac{y_2}{y_1}=1$ $\left(where\ \ y_n=d^ny/dx^n\right)$ is
A) $y=k$ B) $y=kx$ C) $y=k(e^x+1)$ D) $y=k(e^x-1)$

b) Orthogonal Trajectory :

108. The orthogonal trajectories of the family of curves $a^{n-1}y=x^n$ are given by
A) $x^n+n^2y=$constant B) $ny^2+x^2=$constant
C) $n^2x+y^n=$constant D) $n^2x-y^n=$constant

109. The orthogonal trajectories of the curves $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$
A) $\left(x+y\frac{dy}{dx}\right)\left(x-y\frac{dx}{dy}\right)=a^2-b^2$ B) $\left(x^2+y^2\frac{dy}{dx}\right)\left(x^2-y^2\frac{dx}{dy}\right)=a^2-b^2$
C) $\left(x-y\frac{dy}{dx}\right)\left(x+y\frac{dx}{dy}\right)^2=\left(a^2-b^2\right)^2$ D) $\left(x+y\frac{dy}{dx}\right)\left(x-y\frac{dx}{dy}\right)^2=\left(a^2-b^2\right)^2$

110. The orthogonal trajectories of the circles $x^2+y^2-ay=0$ where ‘a’ is a parameter.
A) $x^2+y^2=cx$ B) $x+y=cx$ C) $x^2+y^2=cy$ D) $x^2-y^2=cy$
Where ‘c’ is orthogonal constant.

111. Let γ be the curve which passes (0, 1) and intersects each curve of the family $y = cx^2$ orthogonally. Then γ also passes through the point

A) $\left(\sqrt{2},0\right)$ B) $\left(0,\sqrt{2}\right)$ C) (1, 1) D) (-1, 1)

PRACTICE QUESTIONS

112. The orthogonal trajectories of the one parameter family of curves $y^2 = 4k(k+x)$ where k is an arbitrary constant is

A) $x^2 = 2c(c+y)$ B) $y = 4c(c+x)$ C) $x^2 = 4c(c+y)$ D) $x^2 = 4c(c+x)$

113. Let c be an arbitrary non-zero constant. Then the orthogonal family of curves to the family $y(1-cx) = 1+cx$ is

A) $3y - y^3 + 3x^2 =$ constant B) $3y + y^3 - 3x^2 =$constant

C) $3y - y^3 - 3x^2 =$ constant D) $3y + y^3 + 3x^2 =$constant

114. The orthogonal trajectory of the cardioids $r = a(1-\cos\theta)$

A) $r = a'(1+\cos\theta)$ B) $r = a'(1-\cos\theta)$ C) $r = a'(2\cos\theta/2)$ D) $r = a'(2\sin\theta/2)$

Where a' is orthogonal constant

c) Statistical Application of D.E :

115. If the population of a country doubles in 50 years in how many years will it become thrice the original, assume the rate of increase is proportional to the number of inhabitants

A) 75 B) $50\log_2 3$ C) $50\log_3 2$ D) 100

116. The population P(t) at time t of a certain mouse species satisfies the differential equation, $\frac{dp(t)}{dt} = 0.5\,p(t) - 450$. If $p(0) = 850$, then the time at which the population becomes zero is

A) $\frac{1}{2}\ln 18$ B) $\ln 18$ C) $2\ln 18$ D) $\ln 9$

117. Let the population of rabbits surviving at a time t be governed by the differential equation $\frac{dp(t)}{dt} = \frac{1}{2}p(t) - 200$. If $p(0) = 100$, the $p(t)$ equals :

A) $600 - 500e^{t/2}$ B) $400 - 300e^{-t/2}$

C) $400 - 300e^{t/2}$ D) $300 - 200e^{-t/2}$

PRACTICE QUESTIONS

118. The population of a certain country is known to increase at a rate proportional to the number of people presently living in the country. If after two years the population has doubled, and after three years the population is 20,000 estimate the number of people initially living in the country. $\left(\sqrt{2} = 1.4142\right)$

A) $\cong 7091$ B) $\cong 7071$ C) $\cong 7081$ D) $\cong 8081$

119. Five mice in a stable population of 500 are intentionally infected with a contagious disease to test a theory of epidemic spread that postulates the rate of change in the infected population is proportional to the product of the number of mice who have the disease with the number that are disease free. Assuming the theory is correct, how long will it take half the population to contract the disease ?

A) t = 0.009 1/k time units B) t = 0.015 1/k time units

C) t = 0.020 1/k time units D) t = 0.031 1/k time units

d) Physical Application of D.E :

120. At present, a firm is manufacturing 2000 items. It is estimated that the rate of change of production P w.r.t. additional number of workers x is given by $\frac{dP}{dx}=100-12\sqrt{x}$. If the firm employs 25 more workers, then the new level of production of item is :
A) 3500 B) 4500 C) 2500 D) 3000

121. A 50-lit. tank initially contains 10 lit. of fresh water. At t = 0, a brine solution containing 1 lb of salt per gallon is poured into the tank at the rate of 4 lit/min, while the well-stirred mixture leaves the tank at the rate of 2 lit./min. Find (a) the amount of time required for overflow to occur and (b) the amount of salt in the tank at the moment of overflow.
A) 20 min, 48 lb B) 30 min, 49 lb C) 60 min, 78 lb D) 70 min, 98 lb

122. A tank initially holds 100 lit. of a brine solution containing 20 lb of salt. At t = 0, fresh water is poured into the tank at the rate of 5 lit./min, while the well-stirred mixture leaves he tank at the same rate. Find amount of salt in the tank at any time t.
A) $20e^{-t/20}$ B) $20e^{-t/20}$ C) $20e^{t/30}$ D) $40e^{-t/20}$

PRACTICE QUESTIONS

123. A tank initially holds 100 lit. of a brine solution containing 1 lb of salt. At t = 0 another brine solution containing 1 lb of slat per liter is poured into the tank at the rate of 3 lit./min, while the well-stirred mixture leaves the tank at the same rate. Find (a) the amount of salt in the tank at any time t and (b) the time at which the mixture in the tank contains 2 lb of salt.
A) $-99e^{0.o3t}+100$, $-\frac{1}{0.03}\ell n\frac{98}{99}$min B) $-100e^{0.o3t}+100$, $-\frac{1}{0.07}\ell n\frac{98}{99}$min
C) $-101e^{0.o3t}+100$, $-\frac{1}{0.77}\ell n\frac{98}{99}$min D) $-101e^{0.o3t}+100$, $-\frac{1}{0.77}\ell n\frac{97}{100}$min

124. A metal bar at a temperature of 100^0F is placed in a room at a constant temperature of 0^0F. If after 20 minutes the temperature of the bar is 50^0F, then the time it will take the bar to reach a temperature of 25^0F is
A) 39.6 min. B) 40 min. C) 38 min. D) 41 min.

125. A certain radioactive material is known to decay at a rate proportional to the amount present. If initially these is 50 Kg. of the material present and after two hours it is observed that the material has lost 10 percent of its originals mass, find (a) and expression for the mass of the material remaining at any time t, (b) the mass of the material remaining at any time t, (b) the mass of the material after four hours, and (c) the time at which the material has decayed to one half of its initial mass.
A) $N=50e^{-1/2\ln .9t}$, $N=50e^{-2\ln .9}$ and $\ln 1/2/(-1/2\ln .9).hr.$
B) $N=59e^{-1/2\ln .9t}$, $N=59e^{-2\ln .9}$ and $\ln 1/2/(-1/2\ln .9).hr.$
C) $N=60e^{-1/2\ln .9t}$, $N=60e^{-2\ln .9}$ and $\ln 1/2/(-1/2\ln .9).hr.$
D) $N=70e^{-1/2\ln .9t}$, $N=50e^{-2\ln .9}$ and $\ln 1/7/(-1/2\ln .9).hr.$

KEY SHEET

1.	A	**2.**	C	**3.**	A	**4.**	D	**5.**	B
6.	A	**7.**	A	**8.**	A	**9.**	C	**10.**	A
11.	D	**12.**	D	**13.**	B	**14.**	A	**15.**	B
16.	A	**17.**	C	**18.**	B	**19.**	B	**20.**	B
21.	B	**22.**	D	**23.**	A	**24.**	A	**25.**	B
26.	B	**27.**	A	**28.**	D	**29.**	A	**30.**	A
31.	B	**32.**	B	**33.**	A	**34.**	B	**35.**	C
36.	A	**37.**	C	**38.**	B	**39.**	A	**40.**	D
41.	C	**42.**	A	**43.**	D	**44.**	C	**45.**	D
46.	B	**47.**	A	**48.**	C	**49.**	C	**50.**	A
51.	A	**52.**	A	**53.**	A	**54.**	A	**55.**	A
56.	B	**57.**	D	**58.**	A	**59.**	A	**60.**	A
61.	C	**62.**	D	**63.**	B	**64.**	B	**65.**	D
66.	A	**67.**	C	**68.**	A	69.	B	70.	B
71.	A	**72.**	A	**73.**	D	74.	A	75.	A
76.	B	**77.**	A	**78.**	D	79.	D	80.	A
81.	D	**82.**	D	**83.**	C	84.	C	85.	C
86.	A	**87.**	B	**88.**	A	89.	A	90.	A
91.	A	**92.**	B	**93.**	A	94.	A	95.	D
96.	C	**97.**	C	**98.**	B	99.	C	100	B
101	C	**102**	A	**103**	A	104.	A	105.	D
106	A	**107**	D	**108**	B	109.	A	110.	A
111	A	**112**	C	**113**	A	114.	B	115.	C
116	C	**117**	B	**118**	A	119.	A	120.	A
121	A	**122**	A	**123**	A	124.	A	125.	A

HINTS & SOLUTIONS HINTS & SOLUTIONS

1. The parametric form of the given equation is $x = t, y = t^2$. The equation of any tangent at t is $2xt = y + t^2$, Differentiating we get $2t = y_1 \left(= \frac{dy}{dx} \right)$ putting this value in the above equation, we have $2x\frac{y_1}{2} = y + \left(\frac{y_1}{2} \right)^2 \Rightarrow 4xy_1 = 4y + y_1^2$

   The order of this equation is 1

   Hence (A) is the correct answer

2. The given solution of differential equation is

   $y = (c_1 + c_2)\cos(x + c_3) - c_4e^{x+c_5} \quad = (c_1 + c_2)\cos(x + c_3) - c_4e^{c_5 . e^x}$

   $= A\cos(x + c_3) - Be^x$

   [Here, $c_1 + c_2 = A, c_4e^{c_5} = B$]

   Hence in the solution there are actually three arbitrary contains and hence this differential equation should be of order 3.

3. $y^2 = 2c\left(x + \sqrt{c}\right)$ ….(1)

   D.O.B.S. wrt to 'x' we get

   $2yy' = 2c \ (or) \ c = yy'$ …..(2)

   Put (2) in (1), we get

   $y^2 = 2yy_1\left(x + \sqrt{yy_1}\right) \Rightarrow \left(y^2 - 2yy_1\right)^2 = 4y^3y_1^3$

   So degree (1), Degree (3)

4. $y^2\left(\frac{d^2y}{dx^2}\right)^2 + x^2y^2 - \sin x = -3x\left(\frac{dy}{dx}\right)^{\frac{1}{3}}$

   $\left(y^2\left(\frac{d^2y}{dx^2}\right)^2 + x^2y^2 - \sin x\right)^3 = -9x^3\left(\frac{dy}{dx}\right)$

   Here order = 2 = p

   Degree = 6 = q

   $\therefore \ p < q$

5. Given that $y^2 = a\left(x + \frac{\sqrt{a}}{2}\right)$

   Differentiating it w.r.t. x

$$2y.\frac{dy}{dx}=a$$

From (i) and (ii) $y^2=2y\frac{dy}{dx}\left(x+\frac{1}{2}\sqrt{2y\frac{dy}{dx}}\right)$

$y-2x\frac{dy}{dx}=\frac{dy}{dx}\sqrt{2y\frac{dy}{dx}}\Rightarrow\left(y-2x\frac{dy}{dx}\right)^2=2y\left(\frac{dy}{dx}\right)^3\Rightarrow$ Order 1 and degree 3. So, difference = 2

6. $y=c_1\cos\left(x+c_2\right)-c_3\left(e^{-x+c_4}\right)+c_5{}^{\sin x}$

$y=c_1\left(\cos x\cos c_2-\sin x\sin c_2\right)-c_3\left(e^{-x}.e^{c_4}\right)+c_5\sin x$

$\Rightarrow y=\left(c_1\cos c_2\right)\cos x-\left(c_1\sin c_2-c_5\right)\sin x-\left(c_3e^{c_4}\right)e^{-x}$

$\Rightarrow y=\lambda_1\cos x-\lambda_2\sin x-\lambda_3e^{-x}$

Climinating this constants, we get $y_3+y_2+y_1+y_1=0$

So degree (1) and order (3)

7. From the given equation $\frac{\sqrt{1+x^2}+\sqrt{1+y^2}}{x\sqrt{1+y^2}-y\sqrt{1+x^2}}=\lambda$

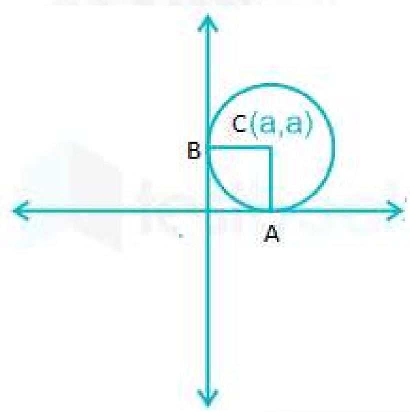
D.O.B.S. with 'x'

$$\left(\frac{2x}{\sqrt{1+x^2}}+\frac{2yy'}{\sqrt{1+y^2}}\right)(Dr)-(Nr)\left[x\frac{2yy'}{\sqrt{1+y^2}}+\sqrt{1+y^2}-y\frac{2x}{\sqrt{1+x^2}}-\sqrt{1+x^2}\,y'\right]=0$$

By observing above equation we can conclude order 1,

By observing above equation we can conclude degree 1

8.



Circle $\left(x-a\right)^2+\left(y-a\right)^2=a^2$ where 'a' arbitrary constant.

Order 1, Degree 2

9. Order (3), degree not defined

10. Since, circle passing through (0, -2) and (0, 2)

$\therefore$ Equation of circle is $x^2+(y^2-4)+\lambda x=0,\ (y\in R)$

Divide by x we get $\frac{x^2+(y^2-4)}{x}+\lambda=0$

Differentiating with respect to x

$$\frac{x\left[2x+2y.\frac{dy}{dx}\right]-\left[x^2+y^2-4\right].1}{x^2}=0 \Rightarrow 2xy.\frac{dy}{dx}+(x^2-y^2+4)=0$$

11. Since, the length of latusrectum is the distance of the point (2, -3) from the line $3x+4y=5$.

$$4a=\frac{|3(2)+4(-3)-5|}{5}=\frac{11}{5}$$

$$\therefore (x-h)^2=\frac{11}{5}(y-k) \qquad \text{....(i)}$$

Differentiate (i) w.r.t. 'x', we get $2(x-h)=\frac{11}{5}\frac{dy}{dx}$

Again differentiate, w.r.t. x we get $2=\frac{11}{5}\frac{d^2y}{dx^2} \Rightarrow \frac{11d^2y}{dx^2}=10$

12. Consider the given system of parabolas $y^2=4ax+4a^2$ .....(i)

Differentiate both sides with respect to *x* we get, $2yy'=4a \Rightarrow a=\frac{yy'}{2}$

Substitute the value of *a* from (ii) in equation (i), we get

$$y^2=\frac{4.yy'}{2}.x+\frac{4y^2(y')^2}{4} \Rightarrow y\left(\frac{dy}{dx}\right)^2+2x\left(\frac{dy}{dx}\right)-y=0$$

13. The general equation of all non-horizontal lines in *xy*-plane is $ax+by=1, a\neq 0$

$$\Rightarrow \frac{d^2x}{dy^x}=0$$

14. Since, $x^2=4b(y+b)$

$x^2=4by+4b^2$

$2x=4by^2 \Rightarrow b=\frac{x}{2y'}$

So, differential equation is $x^2=\frac{2x}{y'}.y+\left(\frac{x}{y}\right)^2$; $x(y')^2=2yy'+x$

15. Given $x^2+y^2=1$, Differentiating w.r.t. *x,* we get

$x+yy'=0$

Again differentiating w.r.t. *x,* $\Rightarrow$ $1+(y')^2+yy''=0$

16. Given circle $C{:}x^2+y^2-3+(x^2-y^2-1)^5=0$

Which satisfies the point (α,α), $2\alpha^2-3+(-1)^5=0 \Rightarrow \alpha=\sqrt{2}$

Now, differentiate w.r.t. x

$2x+2y.y'+5(x^2-y^2-1)^4\ (2x-2yy')=0$ .....(i)

So, point is $\left(\sqrt{2},\sqrt{2}\right)$

$\sqrt{2}+\sqrt{2}y'5(-1)^4\left(\sqrt{2}-\sqrt{2}y'\right)=0 \Rightarrow y'=\dfrac{3}{2}$ ....(ii)

Diff.(i) w.r.t. x

$$1+(y')^2+yy''+20\left(x^2-y^2-1\right)^3(x-yy')^2.2+5\left(x^2-y^2-1\right)^4\left(1-(y')^2-yy''\right)=0$$

At $\left(\sqrt{2},\sqrt{2}\right)$ and $y'=\dfrac{3}{2}$

Now, take $\left(1+\dfrac{9}{4}\right)+\sqrt{2}y''-40\left(\sqrt{2}-\sqrt{2}.\dfrac{3}{2}\right)^2+5(1)\left(1-\dfrac{9}{4}-\sqrt{2}y''\right)=0 \Rightarrow 4\sqrt{2}y''=-33$

Therefore, $3y^2-y^3y''=\dfrac{9}{2}+\dfrac{23}{2}=16$

17. $\cos x dy-(\sin x)ydx=6x\,dx \Rightarrow \int d\left(y\cos x\right)=\int 6x\,dx \Rightarrow y\cos x=3x^2+C$ ... (1)

Given, $y\left(\dfrac{\pi}{3}\right)=0$

Putting $x=\dfrac{\pi}{3}$ and $y=0$ in eq. (1) , we get $(10)\times\left(\dfrac{1}{2}\right)=\dfrac{3\pi^2}{9}+C \Rightarrow C=\dfrac{-\pi^2}{3}$

So, from (1) $y\cos x=3x^2-\dfrac{\pi^2}{3}$

Now, put $x=\dfrac{\pi}{6}$ in the above equation, $y\dfrac{\sqrt{3}}{2}=\dfrac{3\pi^2}{36}-\dfrac{\pi^2}{3} \Rightarrow \dfrac{\sqrt{3}}{2}=\dfrac{-3\pi^2}{12} \Rightarrow y=\dfrac{-\pi^2}{2\sqrt{3}}$

18. We have $(2+\sin x)\dfrac{dy}{dx}+(y+1)\cos x=0 \Rightarrow \dfrac{d}{dx}(2+\sin x)(y+1)=0$

On integrating, we get

$(2+\sin x)(y+1)=C;\ At\ x=0, y=1$ we have

$(2+\sin 0)(1+1)=C \Rightarrow C=4 \Rightarrow y+1=\dfrac{4}{2+\sin x}$

$y=\dfrac{4}{2+\sin x}-1$

Now $y\left(\dfrac{\pi}{2}\right)=\dfrac{4}{2+\sin\dfrac{\pi}{2}}-1=\dfrac{4}{3}-1=\dfrac{1}{3}$

19. $y(1+xy)dx=xdy$

$\dfrac{xdy-ydx}{y^2}=xdx \Rightarrow \int -d\left(\dfrac{x}{y}\right)=\int xdx$

$-\dfrac{x}{y}=\dfrac{x^2}{2}+C\ as\ y(1)=-1 \Rightarrow C=\dfrac{1}{2}$

Hence, $y=\frac{-2x}{x^2+1} \Rightarrow f\left(\frac{-1}{2}\right)=\frac{4}{5}$

20. Given differential equation is

$$ydx-\left(x+2y^2\right)dy=0 \Rightarrow ydx-xdy-2y^2dy=0 \Rightarrow \frac{ydx-xdy}{y^2}=2dy \Rightarrow d\left(\frac{x}{y}\right)=2dy$$

Using $f(-1)=1$, we get $c=1 \Rightarrow \frac{x}{y}=2y+1$

Put $y=1$, we get $f(1)=3$

21. We are given that the differential equation is

$$xdy=\left(\sqrt{x^2+y^2}+y\right)dx;\ x>0 \Rightarrow xdy-ydx=\sqrt{x^2+y^2}\,dx \Rightarrow \frac{xdy-ydx}{x^2}=\sqrt{1+\frac{y^2}{x^2}}.\frac{dx}{x}$$

Now divide by 'x' on both sides, we get

$$\frac{d(y/x)}{\sqrt{1+\left(\frac{y}{x}\right)^2}}=\frac{dx}{x} \Rightarrow \ln\left(\frac{y}{x}+\sqrt{\left(\frac{y}{x}\right)^2}\right)=\ln x+\ln c$$

$$\Rightarrow \frac{y+\sqrt{y^2+x^2}}{x}=cx \Rightarrow y+\sqrt{y^2+x^2}=cx^2$$

Now we are given that $x=1,\ y=0 \Rightarrow 0+1=c \Rightarrow c=1 \Rightarrow$ Curve is $y+\sqrt{x^2+y^2}=x^2$

Now at $x=2,\ y=\alpha \Rightarrow \alpha+\sqrt{4+\alpha^2}=4 \Rightarrow 4+\alpha^2=16+\alpha^2-8\alpha \Rightarrow \alpha=\frac{3}{2}$

22. We are given that $2ye^{x/y^2}dx+(y^2-4xe^{x/y^2})dy=0 \Rightarrow 2ye^{x/y^2}[ydx-2xdy]+y^2dy=0$

Multiply Multiply 'y' in numerator and denominator

$$2e^{x/y^2}\left[\frac{y^2dx-x.(2y)dy}{y}\right]+y^2dy=0$$

Now divide by y^3 on both sides $2e^{x/y^2}\left[\frac{y^2dx-x.(2y)dy}{y^4}\right]+\frac{1}{y}dy=0$

$$2e^{x/y^2}d\left(\frac{x}{y^2}\right)+\frac{1}{y}dy=0$$

Now integrating $\int 2e^{\frac{x}{y^2}}d\left(\frac{x}{y^2}\right)+\int\frac{1}{y}dy=0 \Rightarrow 2e^{x/y^2}+\ell ny+c=0$

$\because$ (0, 1) lies on curve

$\therefore\ 2e^0+\ell n1+c=0 \Rightarrow c=-2$

$\therefore$ Required curve : $\boxed{2e^{x/y^2} + \ell ny - 2 = 0}$

Now for x(e)$\Rightarrow 2e^{\frac{x}{e^2}} + \ln e - 2 = 0 \Rightarrow x - e^2 \log_e 2$

Now for $x(e)$

$$2e^{x/e^2} + \ln e - 2 = 0 \Rightarrow x = -e^2 \log_e 2$$

$$2e^{\frac{x}{e^2}} + \ln e - 2 = 0 \Rightarrow x = -e^2 \log_e 2$$

23. $$\sin\left(2x^2\right)\ln\left(\tan x^2\right)dy + \left(4xy - 4\sqrt{2}x\sin\left(x^2 - \frac{\pi}{4}\right)\right)dx = 0$$

$$\ln\left(\tan x^2\right)dy + \frac{4xydx}{\sin(2x^2)} - \frac{4\sqrt{2}x\sin\left(x^2 - \frac{\pi}{4}\right)}{\sin\left(2x^2\right)}dx = 0$$

$$d\left(y.\ln(\tan x^2)\right) - 4\sqrt{2}x\frac{(\sin x^2 - \cos x^2)}{\sqrt{2 - 2\sin x^2 \cos x^2}}dx = 0$$

$$d\left(y\ln(\tan x^2)\right) - \frac{4x(\sin x^2 - \cos x^2)}{(\sin x^2 + \cos x^2) - 1}dx = 0 = \int d\left(y\ln\left(\tan x^2\right)\right)2\int\frac{dt}{t^2 - 1}\int 0$$

$$= y\ln\left(\tan x^2\right) + 2.\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| = c$$

$$y\ln\left(\tan x^2\right) + \ln\left(\frac{\sin x^2 + \cos x^2 - 1}{\sin x^2 + \cos x^2 + 1}\right) = c$$

Put $y = 1$ *and* $x = \sqrt{\frac{x}{6}}$

$$1\ln\left(\frac{1}{\sqrt{3}}\right) + \ln\frac{\left(\frac{1}{2} + \frac{\sqrt{3}}{2} - 1\right)}{\left(\frac{1}{2} + \frac{\sqrt{3}}{2} + 1\right)} = c$$

Now $\ln\left(\frac{1}{\sqrt{3}}\right) + \ln\left(\frac{\sqrt{3} - 1}{\sqrt{3} + 3}\right); y\left(\ln\sqrt{3}\right) = \ln\left(\frac{1}{\sqrt{3}}\right) \Rightarrow y = -1 \Rightarrow |y| = 1$

24. The given equation can be written as

$$\left(\frac{dx}{x} - \frac{dy}{y}\right) + \frac{(x^2dy - y^2dx)}{(x-y)^2} = 0 \Rightarrow \left(\frac{dx}{x} - \frac{dy}{y}\right) + \frac{\left(\frac{dy}{y^2} - \frac{dx}{x^2}\right)}{\left(\frac{1}{y} - \frac{1}{x}\right)^2} = 0 \Rightarrow \left(\frac{dx}{x} - \frac{dy}{y}\right) + \frac{\frac{dy}{y^2} - \frac{dx}{x^2}}{\left(\frac{1}{x} - \frac{1}{y}\right)^2} = 0$$

Integrating, we get $\ln|x| - \ln|y| - \frac{1}{\left(\frac{1}{x} - \frac{1}{y}\right)} = c$ or $\ln\left|\frac{x}{y}\right| - \frac{xy}{(y-x)} = c = \ln\left|\frac{x}{y}\right| + \frac{xy}{x-y} = c$

25. Given equation $\frac{1}{2}d\left(x^2+y^2\right)+d\left(\tan^{-1}\frac{y}{x}\right)=0 \Rightarrow\ y=x\tan\left(\frac{c-x^2-y^2}{2}\right)$

26. $x=r\cos\theta$, $x=r\cos\theta$

$$\frac{dr}{\sqrt{1-r^2}}=d\theta \Rightarrow \sin^{-1}r=\theta+\alpha$$

$$r=\sin\left(\theta+\alpha\right)\Rightarrow x^2+y^2-x\sin\alpha-y\cos\alpha=0$$

$$\therefore \text{radius}=\frac{1}{2}$$

27. Divide both sides by x^2

$$x^3dy+y.3x^2dx+\frac{xdy-ydx}{x^2}+2ydy=0\Rightarrow\ d(x^3y)+d\left(\frac{y}{x}\right)+d(y^2)=0$$

$\Rightarrow$ $x^4y + xy^2 + y = cx$

28. We have $\frac{\sqrt{x}dx+\sqrt{y}dy}{\sqrt{x}dx-\sqrt{y}dy}=\sqrt{\frac{y^3}{x^3}}\Rightarrow\frac{d(x^{3/2})+d(y^{3/2})}{d(x^{3/2})-d(y^{3/2)})}=\frac{y^{3/2}}{x^{3/2}}$

$\Rightarrow\frac{du+dv}{du-dv}=\frac{v}{u}$, where $u=x^{3/2}$ and $v=y^{3/2}$

$$\Rightarrow udu+udv=vdu-vdv\Rightarrow udu+vdv=vdu-udv$$

$$\Rightarrow\frac{udu+vdv}{u^2+v^2}=\frac{vdu-udv}{u^2+v^2}\Rightarrow\frac{d(u^2+v^2)}{u^2+v^2}=-2d\tan^{-1}\left(\frac{v}{u}\right)+c$$

On integrating, we get $\log(u^2+v^2)=-2\tan^{-1}\left(\frac{v}{u}\right)+c\Rightarrow\frac{1}{2}\log(x^3+y^3)+\tan^{-1}\left(\frac{y}{x}\right)^{3/2}=\frac{c}{2}$

29. $\left(x^2\sin^2-y^2\cos x\right)dx+\left(x^3\cos y\sin^2 y-2y\sin x\right)dy=0$

$$\frac{dy}{dx}=\frac{y^2\cos x-x^2\sin^3 y}{x^3\cos y\sin^2-2y\sin x}$$

$$\Rightarrow\left(x^3\cos y\sin^2 y-2y\sin x\right)dy=\left(y^2\cos x-x^2\sin^3 y\right)dx=0$$

$$\Rightarrow d\left(\frac{x^3}{3}\sin^3 y\right)-d\left(y^2\sin x\right)=0\Rightarrow\frac{x^3}{3}\sin^3 y-y^2\sin x=c$$

30. The given equation can be written as $2y\sin x\frac{dy}{dx}+y^2\cos x=\sin 2x$

$$\Rightarrow\frac{d}{dx}\left(y^2\sin x\right)=\sin 2x\Rightarrow y^2\sin x=\left(-1/2\right)\cos 2x+C.$$

So $\left(y\left(\pi/2\right)\right)^2\sin\left(\pi/2\right)=\left(-1/2\right)\cos\left(2\pi/2\right)+C\Rightarrow C=1/2.$

Hence $y^2 \sin x = (1/2)(1-\cos 2x) = \sin^2 x \Rightarrow y^2 = \sin x$

31.

$$x^2 dy - 2x^3 y^3 dy = 3x^2 y^4 dx + 2xydx$$

$$\Rightarrow x^2 dy - 2xydx = 3x^2 y^4 dx + 2x^3 y^3 dy$$

$$\Rightarrow \frac{2xydx - x^2 dy}{y^2} + 3x^2 y^2 dx + 2x^3 ydy = 0$$

$$\Rightarrow d\left(\frac{x^2}{y}\right) + d\left(x^3 y^2\right) = 0$$

$$\Rightarrow \frac{x^2}{y} + x^3 y^2 = C$$

32. The equation can be written as $x^2 \dfrac{xdy - ydx}{x^2} = x\sqrt{1-(y/x)^2}\,dx$

$$\Rightarrow \frac{d(y/x)}{\sqrt{1-(y/x)^2}} = \frac{dx}{x} \Rightarrow \sin^{-1} y/x = \log|x| + \text{const}$$

Since $y(1) = 0$ so const =0. Hence $y = x\sin(\log|x|)$.

33. $$\frac{xdx + ydy}{ydx - xdy} = \frac{x^4 + 2x^2 y^2 + y^4}{x^2}$$

$$\frac{2xdx + 2ydy}{2\left(x^2 + y^2\right)^2} = -d\left(\frac{y}{x}\right)$$

$$\frac{xdx + ydy}{\left(x^2 + y^2\right)^2} = \frac{ydx - xdy}{x^2}$$

$$\Rightarrow \quad \frac{-1}{x^2 + y^2} = -2\frac{y}{x} + c$$

$$\Rightarrow \quad 2\frac{y}{x} - \frac{1}{x^2 + y^2} = c$$

34. $$\frac{y\,dx + x\,dy}{\cos^2(xy)} + \sin xdx + \sin ydy = 0$$

$$\int \frac{d(xy)}{\cos^2(xy)} + \int \sin x\,dx + \int \sin y\,dy = 0$$

$$\tan(xy) - \cos x - \cos y = c$$

35. The given equation can be written as $y\left(1 + x^{-1}\right)dx + (x + \log x)dy + \sin ydx + x\cos ydy = 0$

$$\Rightarrow d\left(y(x + \log x)\right) + d(x\sin y) = 0 \Rightarrow y(x + \log x) + x\sin y = C$$

36. $$\sqrt{1+x^2}.\sqrt{1+y^2} = -xy\frac{dy}{dx}$$

$$\int \frac{\sqrt{1+x^2}}{x}dx = -\int \frac{y}{\sqrt{1+y^2}}dy$$

Let $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta \Rightarrow \int \frac{\sec^3\theta d\theta}{\tan\theta} = -\int \frac{2y}{2\sqrt{1+y^2}}dy$

$$\Rightarrow \int \frac{\sin^2\theta + \cos^2\theta}{\sin\theta.\cos^2\theta}d\theta = -\sqrt{1+y^2} \Rightarrow \int \left(\tan\theta.\sec\theta + \cos ec\theta\right)d\theta = -\sqrt{1+y^2}$$

$$\Rightarrow \sec\theta + \log_e\left|\cos ec\theta - \cot\theta\right| = -\sqrt{1+y^2} + C$$

$$\therefore \sqrt{1+x^2} + \log_e\left|\frac{\sqrt{1+x^2}-1}{x}\right| = -\sqrt{1+y^2} + C \Rightarrow \sqrt{1+y^2} + \sqrt{1+x^2} = \frac{1}{2}\ln\left|\frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}-1}\right| + C$$

37. $\frac{dy}{dx} - Ky = 0, \frac{dy}{y} = Kdx$

$\ln y = Kx + c$

$At\ x = 0, y = 1 \qquad \therefore C = 0$

Now, $\ln y = Kx$

$y = e^{Kx}$

$\lim\limits_{x\to\infty} y = \lim\limits_{x\to\infty} e^{Kx} = 0$

$\therefore K < 0$

38. Given, $\frac{dy}{dx} = \frac{2^x 2^y - 2^x}{2^y} \Rightarrow \frac{dy}{dx} = \frac{2^x 2^y - 2^x}{2^y} \Rightarrow 2^y\frac{dy}{dx} = 2^x\left(2^y - 1\right)$

$$\Rightarrow \int \frac{2^y}{2^y - 1}dy = \int 2^x\, dx; \frac{\log_e\left(2^y - 1\right)}{\log_e 2} = \frac{2^x}{\log_e 2} + C \Rightarrow \log_2\left(2^y - 1\right) = 2^x\log_2 e + C$$

$\because y(0) = 1 \Rightarrow 0 = \log_2 e + C$

$C = -\log_2 e \Rightarrow \log_2\left(2^y - 1\right) = \left(2^x - 1\right)\log_2 e$

Put $x = 1, \log_2(2^y - 1) = \log_2 e$

$2^y = e + 1$

39. $\frac{dy}{dx} = \frac{2^x\left(y + 2^y\right)}{2^x\left(1 + 2^y\log_e 2\right)}$

$$\int \frac{\left(1 + 2^y\log_e 2\right)}{\left(y + 2^y\right)}dy = \int dx \Rightarrow \log_e\left|y + 2^y\right| = x + c$$

Put $x = 0, y = 0 \Rightarrow c = 0 \Rightarrow x = \log_e\left|y + 2^y\right| \Rightarrow Put\ y = 1, x = \log_e 3$

$\because\ 3 \in \left(e, e^2\right) \Rightarrow x \in (1, 2)$

40. $\frac{dy}{dx} = xy - 1 + x - y \Rightarrow \frac{dy}{dx} = (x-1)(y+1) \Rightarrow \frac{dy}{y+1} = (x-1)dx \Rightarrow \ln(y+1) = \frac{x^2}{2} - x + C$

$\because\ y(0) = 0 \Rightarrow C = 0$

Therefore, $y(x) = e^{\left(\frac{x^2}{2} - x\right)} - 1$

$y(1) = e^{\frac{-1}{2}} - 1$

Differentiating (ii) with respect to x, we get $(y'\sin x + y\cos x + \sin y, y'')(\cos x - x\sin y)$

$y'' = \frac{(-\sin x - \sin y - x\cos y y'')(y\sin x - \cos y)}{(\cos x - x\sin y)^2} \Rightarrow y''(0) = \frac{\pi(1) - 1}{1} = \pi - 1$

41. $\int \left(\frac{y^2+1}{y^2}\right) dy = \int \frac{e^x dx}{e^x + 1} \Rightarrow y - \frac{1}{y} = \log_e \left|e^x + 1\right| + c$

$\because$ Passes through (0, 1)

$\therefore c = -\log_e 2 \Rightarrow y^2 - 1 = y\log_e\left(\frac{e^x + 1}{2}\right) \Rightarrow y^2 = 1 + y\log_e\left(\frac{e^x+1}{2}\right)$

42. Given differential equation can be written as

$$\int dy = \int \frac{1}{\left(\sqrt{4 + \sqrt{9 + \sqrt{x}}}\right)\left(\sqrt{9 + \sqrt{x}}\right) 8\sqrt{x}} dx$$

Putting $\sqrt{4 + \sqrt{9 + \sqrt{x}}} = t$, we get

$$\frac{1}{2\sqrt{4 + \sqrt{9 + \sqrt{x}}} \,.\, 2\sqrt{9 + \sqrt{x}} \,.\, 2\sqrt{x}} dx = dt$$

$\therefore \int dy = \int dt \Rightarrow y = t + c \Rightarrow y = \sqrt{4 + \sqrt{9 + \sqrt{x}}} + c$

Now, $y(0) = \sqrt{7} \Rightarrow c = 0$

$\therefore y = \sqrt{4 + \sqrt{9 + \sqrt{x}}} \Rightarrow y(256) = 3$

43. $(2y + f)dy = (ax + 3)dx$ on integration

$y^2 + fy + c = \frac{ax^2}{2} + 3x \quad \Rightarrow \quad -\frac{ax^2}{2} + y^2 + fy - 3x + c = 0$

$\Rightarrow \qquad a = -2,\ 9 + 4f^2 - 4c > 0.$

44. Given $(1 + e^{2x})\frac{dy}{dx} + 2(1 + y^2)e^x = 0$

$$\frac{dy}{1+y^2}+\frac{2e^x}{1+e^{2x}}dx=0 \qquad \ldots.(i)$$

On integration $\tan^{-1}y+2\tan^{-1}e^x=c$

$\because\ y(0)=0$

So, $C=\frac{\pi}{2}$ $\qquad \tan^{-1}y+2\tan^{-1}e^x=\frac{\pi}{4}$

From equation (i), $\left(\frac{dy}{dx}\right)_{x=0}=\frac{-2}{2}=-1$

arg $y\left(\ln\sqrt{3}\right)=-\frac{1}{\sqrt{3}}$

$$6\left[y'(0)+\left(y(\ln\sqrt{3})^2\right)\right]=6\left[-1+\frac{1}{3}\right]=-4$$

45. $\frac{dy}{dx}+\frac{2^{x-y}(2^y-1)}{2^x-1}=0,\ x,y>0,\ y(1)=1$

$\frac{dy}{dx}=-\frac{2^x(2^y-1)}{2^y(2^x-1)};\int\frac{2^y}{2^y-1}dy=-\int\frac{2^x}{2^x-1}dx$

Let $2^y-1=t,\ 2^x-1=p$

Differentiate w.r.t to $y\ \&\ x$ respectively,

$2^y\ln 2=\frac{dt}{dy},\ 2^x\ln 2=\frac{dt}{dx};\ 2^y dy=\frac{dt}{\ell n2},2^x dx\frac{dp}{\ell n2}$

Then, $\frac{1}{\ell n^2}\int\frac{dt}{\ell}=\frac{-1}{\ell n^2}\int\frac{dp}{p}+c$

$\frac{1}{ln2}\int\frac{2^y\ln 2}{\ell 2^y-1}dy=\frac{-1}{ln^2}\int\frac{2^x\ln 2}{2^x-1}dx$

$\frac{1}{ln2}\ln\left|2^y-1\right|=\frac{-1}{\ln 2}\ell n\left|2^x-1\right|+C$

At $x=1,\ y=1$

Put these values in equation (i), we get C = 0

$\ln\left|2^y-1\right|+\ln\left|2^x-1\right|=0$

$(2^x-1)(2^y-1)=1$

$2^y-1=\frac{1}{2^x-1}$

Put x = 2 in above equation

$2^y=\frac{1}{3}+1=\frac{4}{3}$

$y=\log_2\frac{4}{3}=\log_2 4-\log_2 3=2-\log_2 3$

46. Applying C and D, we get

$\frac{dy}{dx}=\frac{e^{-x}}{e^{x}}=e^{-2x} \Rightarrow 2y=-e^{-2x}+C$

or $2y\,e^{2x}=C.e^{2x}-1$.

47. According to the given condition $y\frac{dy}{dx}=ky^2 \Rightarrow \frac{dy}{y}=kdx$ (variables separable equation)

$\Rightarrow \log|y|=kx+C \Rightarrow |y|=Be^{kx} \Rightarrow y=Ae^{kx}$ where $A=\pm B$ and k is the constant of proportionality

48. The given equation is with separable variables so $(cy+d)dy=(ax+b)dx$. Integrating we have $\frac{cy^2}{2}+dy+K=\frac{ax^2}{2}+bx, K$ being the constant of integration. The last equation represents a parabola if $c=0, a\neq 0$ or $a=0, c\neq 0$.

49. The given differential equation has variable separable so integrating, we have $y=x^2/2-1/x+C$. This will pass through $(3,9)$ if $9=9/2-1/3+C \Rightarrow C=29/6$. Hence the required equation is $6xy=3x^3+29x-6$

50. Let $x+y=t \Rightarrow 1+\frac{dy}{dx}=\frac{dt}{dx}$ or $\frac{dt}{t+1}=d \Rightarrow \ln|t+1|=x+c$

$\Rightarrow |t+1|=e^x+c$

$|x+y+1|=e^x+c$

Now, $y_1(0)=0 \Rightarrow y_1(x)$ is given by $|x+y+1|=e^x$ and $y_2(0)=1 \Rightarrow y_2(x)$ is given by $|x+y+1|=2e^x$

$\therefore\ y_2 \neq y_1$

$\therefore$ Number of points of intersection of y_1 and y_2 is zero.

51. $\frac{dy}{dx}=\frac{y}{x}+\sec\frac{y}{x}$

Put $y=vx$, $\therefore \frac{dy}{dx}=v+x\frac{dv}{dx}$

$\Rightarrow x\frac{dv}{dx}=\sec v \Rightarrow \int \cos v\,dv=\int\frac{dx}{x} \Rightarrow \sin v=\log x+c$

$\Rightarrow \sin\frac{y}{x}=\log x+c \qquad (\because\ x>0)$

Since, it passes through $\left(1,\frac{\pi}{6}\right) \Rightarrow c=\frac{1}{2}$

Hence, $\sin\frac{y}{x}=\log x+\frac{1}{2}$

52. $\frac{dy}{dx}=\frac{2xy+y^2}{2x^2}$

It is homogenous differential equation

$\therefore\ Put\ \ y=vx \Rightarrow\ v+x\frac{dv}{dx}=v+\frac{v^2}{2} \Rightarrow \int 2\frac{dv}{v^2}=\int\frac{dx}{x}$

$\Rightarrow \frac{-2}{v}=\log_e x+c \Rightarrow \frac{-2x}{y}=\log_e x+c$

Put $x=1,\ y=2$, we get $c=-1 \Rightarrow \frac{-2x}{y}=\log_e x-1$

Hence, put $x=\frac{1}{2} \Rightarrow y=\frac{1}{1+\log_e 2}$

53. Put y =vx $\Rightarrow \int\frac{dx}{x}+\int e^{-v}\text{Sin}v dv=c \Rightarrow \int e^{-v}\sin v dv=-\frac{e^{-v}}{2}(\text{Sin}v+\text{Cos}v)$

$\Rightarrow\ \ \log x=c+\frac{1}{2}e^{\frac{-y}{x}}\left(\text{Sin}\frac{y}{x}+\text{Cos}\frac{y}{x}\right)$

54. $\frac{dx}{dy}+\frac{x^2}{y^2}-\frac{x}{y}+1=0$

Put $x=vy$ where v is a function of 'y' $\Rightarrow \frac{dx}{dy}=v+y\frac{dv}{dy}$

$\therefore v+y\frac{dv}{dy}+v^2-v+1=0\ \Rightarrow -y\frac{dv}{dy}=\left(1+v^2\right)\ \Rightarrow \frac{dv}{v^2+1}=-\frac{dy}{y}$

$\tan^{-1}v+C=-\ln y\ \Rightarrow \tan^{-1}\left(\frac{x}{y}\right)+\ln y+C=0$

Where C is arbitrary constant

55. Given differential equation is

$$\left(\frac{x}{\sqrt{x^2-y^2}}+e^{\frac{y}{x}}\right)x\frac{dy}{dx}=x+\left(\frac{x}{\sqrt{x^2-y^2}}+e^{\frac{y}{x}}\right)y$$

$$\left(\frac{1}{\sqrt{\frac{1-y^2}{x^2}}}+e^{\frac{y}{x}}\right)\frac{dy}{dx}=1+\left(\frac{1}{\sqrt{\frac{1-y^2}{x^2}}}+e^{\frac{y}{x}}\right)\frac{y}{x}$$

Dividing both side by x^2

Put $y=vx$ then, $\frac{dy}{dx}=v+x\frac{dv}{dx}$

So, $\left(\frac{1}{\sqrt{1-v^2}}+ev\right)\left(v7x\frac{dy}{dx}\right)=1+\left(\frac{1}{\sqrt{1-v^2}}+e^v\right)v$

$x\left(\frac{1}{\sqrt{1-v^2}}+e^v\right)\frac{dy}{dx}=1$

Take integral both side,$\Rightarrow e^{\frac{y}{x}}+\sin^{-1}\left(\frac{y}{x}\right)=\ln x+C$

In passes through (1, 0)

$1+0=0+C \Rightarrow C=1$

It passes through $(2\alpha, \alpha)$

$e^{\frac{1}{2}}+\sin^{-1}\frac{1}{2}=\ln 2\alpha+1 \Rightarrow \ln 2\alpha=\sqrt{e}+\frac{\pi}{6}-1 \Rightarrow 2\alpha=e^{\left(\sqrt{e}+\frac{\pi}{6}-1\right)}$

$\Rightarrow \alpha=\frac{1}{2}e^{\left(\frac{\pi}{6}+\sqrt{e}-1\right)}$

56. We have, $xdy=\left(y+\frac{xf(y/x)}{f(y/x)}\right)dx \Rightarrow \frac{dy}{dx}=\frac{y}{x}+\frac{f(y/x)}{f'(y/x)}$ which is homogenous

Putting $y=vx \Rightarrow \frac{dy}{dx}=v+x+\frac{dv}{dx}$,we obtain

$v+x\frac{dv}{dx}=v+\frac{f(v)}{f'(v)} \Rightarrow \frac{f(v)}{f'(v)}dv=\frac{dx}{x}$

Integrating, we get log $f(v)=\log x+\log c \Rightarrow \log f(v)=\log cx \Rightarrow f\left(\frac{y}{x}\right)=cx$

57. Putting $v=y/x$ so that $\frac{xdv}{dx}+v=\frac{dy}{dx}$, we have

$\frac{xdv}{dx}+v=v+\phi(1/v) \Rightarrow \frac{dv}{\phi(1/v)}=\frac{dx}{x} \Rightarrow \log|Cx|=\int\frac{dv}{\phi(1/v)}$ (being constant of integration.)

But $y=\frac{x}{\log|Cx|}$ is the general solution so

$\frac{x}{y}=\frac{1}{v}=\int\frac{dv}{\phi(1/v)} \Rightarrow \phi(1/v)=-1/v^2 \Rightarrow \phi(x/y)=-y^2/x^2.$

58. Let $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

Now, given differentiable equation is

$$\frac{dy}{dx}=\frac{y}{x}+\frac{1}{x}\sqrt{y^2+16x^2}\Rightarrow x\frac{dv}{dx}=\sqrt{v^2+16}$$

$$\Rightarrow\int\frac{dv}{\sqrt{v^2+16}}=\int\frac{dx}{x}\Rightarrow\ell n\left|v+\sqrt{v^2+16}\right|=\ell nx+\ell nC$$

$$\left|v+\sqrt{v^2+16}\right|=CX\frac{y}{x}+\frac{\sqrt{y^2+16}}{x}=Cx$$

$$\Rightarrow\ y+\sqrt{y^2+16x^2}=Cx^2;\ As\ y(1)=3\Rightarrow C=8\Rightarrow\ y(2)=15$$

59. $\left(x-y^2\right)dx+y\left(5x+y^2\right)dy=0$

$$\frac{dy}{dx}=\frac{y^2-x}{y\left(5x+y^2\right)}.\quad \text{Let}\ \ y^2=t$$

$$\frac{2ydy}{dx}=2\left(\frac{y^2-x}{5x+y^2}\right)\Rightarrow\frac{dt}{dx}=2\left(\frac{t-x}{5x+t}\right)\qquad [t=kx]$$

$$\Rightarrow\ k+x\frac{dk}{dx}=2\left(\frac{kx-x}{5x+kx}\right)\Rightarrow x\frac{dk}{dx}=-\frac{\left(k^2+3k+2\right)}{k+5}$$

$$\int\frac{(5+k)}{(k+1)(k+2)}dk=\int-\frac{dx}{x};\int\left(\frac{4}{k+1}-\frac{3}{k+2}\right)dk=-\int\frac{dx}{x}$$

$$4\ln(k+1)-3\ln(k+2)=-\ln x+\ln c$$

$$\frac{(k+1)^4}{(k+2)^3}=-\ln\ x+\ln\ c$$

$$c\left(y^2+2x\right)^3=\left(y^2+x\right)^4$$

60. Note that h, k do not exist in this case which can reduce this D.E. to homogeneous form. Thus, we use the substitution

$$x+2y=v\Rightarrow 1+2\frac{dy}{dx}=\frac{dv}{dx}\qquad \text{….(i)}$$

Substitute (i) in the given equation, we get $\frac{1}{3}\left(1+\frac{4}{3v-1}\right)dv=dx$ then integrate and

replace v by x + 2y we get . $y-x+\frac{2}{3}\ln\left(3x+6y-1\right)=C$

61. Here $a_1b_2-a_2b_1\neq 0$ put $x=X+h,\ y=Y+k\ \Rightarrow\ dx=dX,\ dy=dY$

$\therefore$ Given D,E. becomes changed to $\frac{dY}{dX}=\frac{X+2Y+\left(h+2k-3\right)}{2X+Y+\left(2h+k-3\right)}$ ….(i)

Choose h and k so that $h+2k-3=0$; $2h+k-3=0$

Solving we get h = 1 and k = 1.

$$\frac{dY}{dX}=\frac{X+2Y}{2X+Y}$$

Which is homogeneous equation and proceed we get

$x+y-2=C^2\left(x-y\right)^3$

$X=x-1, Y=y-1$

62. Given, $\sin 2x\left(\frac{dy}{dx}-\sqrt{\tan x}\right)-y=0$ or $\frac{dy}{dx}=\frac{y}{\sin 2x}+\sqrt{\tan x}$

Or, $\frac{dy}{dx}-y\operatorname{cosec}2x=\sqrt{\tan x}$ ….(i)

Which is LDE I.F = $e^{\log\left(\sqrt{\tan x}\right)^{-1}}=\sqrt{\cot}x$

$\therefore$ General solution of eq. (i) is $\boxed{\therefore\ y\sqrt{\cot x}=x+c}$

63. $\frac{dy}{dx}+y=\frac{1}{1+e^{2x}}$ is linear diff.eqn.

$\therefore$ integrating factor is $e^{\int 1dx}=e^x$

So solution is $y.e^x=\tan^{-1}\left(e^x\right)+c$

$\because$ curve is passing through $\left(0,\frac{\pi}{2}\right)\Rightarrow c=\frac{\pi}{4}$

$\therefore$ Required particular solution is $y.e^x=\tan^{-1}\left(e^x\right)+\frac{\pi}{4}$

Now $\lim\limits_{x\to\infty}\left(y.e^x\right)=\lim\limits_{x\to\infty}\left(\tan^{-1}\left(e^x\right)+\frac{\pi}{4}\right)=\frac{3\pi}{4}$

64. Given differential equation is $\frac{dy}{dx}+\left(\frac{2x^2+11x+13}{x^3+6x^2+11x+6}\right)y=\frac{x+3}{x+1}$

$\Rightarrow I.F=e^{\int p(x)dx}$ …..(i)

$$\int p(x)dx=\int\frac{\left(2x^2+11x+13\right)dx}{(x+1)(x+2)(x+3)}$$

Using partial fraction

$$\frac{2x^2+11x+13}{(x+1)(x+2)(x+3)}=\frac{P}{x+1}+\frac{Q}{x+2}+\frac{R}{x+3}$$

$P=\frac{4}{2}=2$

Q = 1 ; R = -1

$\because\int p(x)dx=p\ln(x+1)+Q\ln(x+2)+R\ln(x+3)=\ln\left(\frac{(x+1)^2(x+2)}{x+3}\right)$

From (i) I.F = $e^{\int p(x)dx} = \dfrac{(x+1)^2(x+2)}{(x+3)}$

Solution y (IF) = $\int Q.(IF)dx$

$$y\left(\frac{(x+1)^2(x+2)}{x+3}\right) = \int\left(\frac{x+3}{x+1}\right)\frac{(x+1)^2(x+2)}{(x+3)}dx$$

$$y\left(\frac{(x+1)^2(x+2)}{x+3}\right) = \frac{x^3}{3}+\frac{3x^2}{2}+2x+C$$

Satisfy (0, 1) then C = $\dfrac{2}{3}$ Now put $x=1 \Rightarrow y(1)=\dfrac{3}{2}$

65. $\dfrac{dy}{dx}+\dfrac{1}{x^2-1}y=\left(\dfrac{x-1}{x+1}\right)^{\frac{1}{2}}$,

$$\frac{dy}{dx}+Py=Q$$

If $= e^{\int\frac{1}{x^2-1}dx} = e^{\frac{1}{2}\ell n\left(\frac{x-1}{x+1}\right)} = e^{\int Pdx} = \left(\dfrac{x-1}{x+1}\right)^{\frac{1}{2}}$

$$y\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}} = \int\left(\frac{x-1}{x+1}\right)^{1}dx = x-2\log_e|x+1|+C$$

Since, curve passes through $\left(2,\dfrac{1}{\sqrt{3}}\right) \Rightarrow C=2\log_e 3-\dfrac{5}{3}; at\ x=8, \sqrt{7}y(8)=19-6\log_e 3$

66. Given differential equation we write as $\dfrac{dy}{dx}+(8+4\cot 2x)y=\dfrac{2e^{-4}x}{\sin^2 2x}(2\sin x+\cos 2x)$ ….(i)

Now equation(i) is linear differential equation

Now I.F = $e^{\int(8+4\cos 2x)dx} = e^{8x+2\ln(\sin 2x)} = e^{8x}.\sin^2 2x$

$\therefore$ solution is given by

$$y\left(e^{8x}.\sin^2 2x\right) = \int 2e^{4x}(2\sin 2x+\cos 2x)dx + C = e^{4x}.\sin 2x + C$$

We have given that 0020 $y\left(\dfrac{\pi}{4}\right)=e^{-\pi} \Rightarrow C=0 \Rightarrow y=\dfrac{e^{-4x}}{\sin 2x}$

Now $y\left(\dfrac{\pi}{6}\right)=\dfrac{e^{-4\frac{\pi}{6}}}{\sin\left(2.\dfrac{\pi}{6}\right)}=\dfrac{2}{\sqrt{3}}e^{-\frac{2\pi}{3}}$

67. $\dfrac{dy}{dx}+12y=\cos\left(\dfrac{\pi x}{12}\right)$

$I.F=e^{12\int dx=e^{12x}}$

Solutions is : $y.e^{12x}=\int e^{12x}.\cos\left(\dfrac{\pi x}{12}\right)dx+C$

$\Rightarrow\ y.e^{12x}=\dfrac{2^{12x}}{12^2+\left(\dfrac{\pi}{12}\right)^2}\left[12\cos\dfrac{\pi x}{12}+\dfrac{\pi}{12}\sin\left(\dfrac{\pi x}{12}\right)\right]+c$

Put $x=0, y=0$ we get $C=\dfrac{12}{12^2+\left(\dfrac{\pi}{12}\right)^2}$

So $y=\dfrac{1}{\lambda}\left[\dfrac{12\cos\left(\dfrac{\pi x}{12}\right)+\dfrac{\pi}{12}\sin\left(\dfrac{\pi x}{12}\right)-12e^{-12x}}{f_1(x)}\right]$ $\left[Let\ \dfrac{1}{\lambda}=\dfrac{1}{\left(\dfrac{\pi}{12}\right)^2+12^2}\right]$

$\dfrac{dy}{dx}=\dfrac{1}{\lambda}\left[\dfrac{-\pi\sin\left(\dfrac{\pi x}{12}\right)+\dfrac{\pi^2}{12^2}\cos\dfrac{\pi x}{12}+144e^{-12x}}{f_2(x)}\right]$

When x is large then $12e^{-12x}$ tends to zero.

But $f_2(x)$ varies in $\left[-\sqrt{\pi^2+\left(\dfrac{\pi}{12}\right)^4}.\sqrt{\pi^2+\left(\dfrac{\pi}{12}\right)^4}\right]$

Hence $\dfrac{dy}{dx}$ is changing its sign.

So, *y(x)* is non-monotonic for all real number.

Also when x is very large then again $-12e^{-12x}$ is almost zero but $f_1(x)$ is periodic, so there exist some β for which

$y=\beta$ intersect $y=y(x)$ at infinitely many points.

68. When $x\in[0,1]$, then $\dfrac{dy}{dx}+2y=1\Rightarrow\ y=\dfrac{1}{2}+C_1e^{-2x}$

$\because y(0)=0\Rightarrow y(x)=\dfrac{1}{2}-\dfrac{1}{2}e^{-2x}$

Here, $y(1)=\dfrac{1}{2}-\dfrac{1}{2}e^{-2}=\dfrac{e^2-1}{2e^2}$

When $x \notin [0, 1]$, then $\frac{dy}{dx} + 2y = 0 \Rightarrow y = c_2 e^{-2x}$

$\because y(1) = \frac{e^2 - 1}{2} \Rightarrow \frac{e^2 - 1}{2} = c^2 e^{-2} \Rightarrow C_2 = \frac{e^2 - 1}{2}$

$\therefore\ y(1) = \left(\frac{e^2 - 1}{2}\right) e^{-2x} \Rightarrow u\left(\frac{3}{2}\right) = \frac{e^2 - 1}{2e^3}$

69. $\frac{dy}{dx} - \frac{x}{1 - x^2} y = \frac{x^4 + 2x}{\sqrt{1 - x^2}}$

I.F. $e^{\int \frac{-x}{1-x^2} dx} = e^{\frac{+1}{2}\log\left(1 - x^2\right)} = \sqrt{1 - x^2}$

Hence, solution is given by

$y\sqrt{1 - x^2} = \int \sqrt{1 - x^2} . \frac{x^4 + 2x}{\sqrt{1 - x^2}} dx$

$y\sqrt{1 - x^2} = \frac{x^5}{5} + x^2 + c$

$f(0) = 0 \Rightarrow At\ x = 0,\ y = 0 \Rightarrow c = 0$

$\therefore\ y\sqrt{1 - x^2} = \frac{x^5}{5} + x^2 \Rightarrow y = f(x) = \frac{\frac{x^5}{5} + x^2}{\sqrt{1 - x^2}},\ \therefore I = \int_{\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \frac{\frac{x^5}{5} + x^2}{\sqrt{1 - x^2}} dx$

$= 2\int_0^{\frac{\sqrt{3}}{2}} \frac{x^2}{\sqrt{1 - x^2}} dx \qquad \left(\because \frac{x^5}{\sqrt{1 - x^2}} is\ odd\right)$

Put $x = \sin\theta = dx = \cos\theta\ d\theta$

$\therefore\ I = \int_0^{\frac{\pi}{2}} \sin^2\theta\ d\theta = \int_0^{\frac{\pi}{3}} \left(1 - \cos 2\theta\right) d\theta = \left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\frac{\pi}{3}} = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$

70. $\frac{dy}{dx} + \frac{\sqrt{2}}{2\cos^4 x - \cos 2x} y = xe^{\tan^{-1}(\sqrt{2}\cot 2x)}$

Here given differential equation is in the form of $\frac{dy}{dx} + py = Q$

$\int \frac{dx}{2\cos^4 x - \cos 2x} \left\{\because P = \int \frac{dx}{2\cos^4 x - \cos 2x} = \int \frac{dx}{2\cos^4 x + \sin^4 x} = \int \frac{\cos ec^4 x\ dx}{1 + \cot^4 x}\right.$

$= -\int \frac{t^2 + 1}{t^4 + 1} dt = -\int \frac{\left(1 + \frac{1}{t^2}\right)}{\left(t - \frac{1}{t}\right)^2 + 2} dt = \frac{-1}{\sqrt{2}} \tan^{-1}\left(\frac{t - \frac{1}{t}}{\sqrt{2}}\right)$

$\cot x = t$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\cos 2x\right) \qquad \therefore IF=e^{-\tan^{-1}\left(\sqrt{2}\cot 2x\right)}$$

$$ye^{-\tan^{-1}(\sqrt{2}\cot 2x)}=\int x dx;\ \ ye^{-\tan^{-1}(\sqrt{2}\cot 2x)}=\frac{x^2}{2}+c$$

$$y\left(\frac{\pi}{4}\right)-\frac{\pi^2}{32}+c \Rightarrow c=0;\ y=\frac{x^2}{2}e^{\tan^{-1}(\sqrt{2}\cot 2x)}$$

$$y\left(\frac{\pi}{3}\right)=\frac{\pi^2}{18}e^{\tan^{-1}\left(\sqrt{2}\cot\frac{2\pi}{3}\right)}=\frac{\pi^2}{18}e^{\tan^{-1}\left(\sqrt{\frac{2}{3}}\right)}$$

$$\alpha=\sqrt{\frac{2}{3}}\Rightarrow 3\alpha^2=2$$

71. Given, $\frac{dy}{dx}+\left(\frac{1}{x\log x}\right)y=2$

$$I.F.=e^{\int\frac{1}{x\log x}dx}=e^{\log(\log x)}=\log x$$

$$y.\log x=\int 2\log x dx+c$$

$$y.\log x=2\left[x\log x-x\right]+c$$

Put $x=1,\ y.0=-2+c$

c = 2

Put $x=e$

$$y\log e=2e(\log e-1)+c$$

$$y(e)=c=2$$

72. Given differential equation is $\frac{dy}{dx}+2y\tan x=\sin x$

So, it is linear equation.

I.F = $e^{\int P dx}=e^{2\int\tan x dx}=e^{2\ln|\sec x|}=\sec^2 x$

Solution of differential equation is,

$$y\times I.F=\int(Q\times I.F)dx+c$$

$$y\times\sec^2 x=\int(\sin x\times\sec^2 x)dx+c$$

$$y\sec^2 x=\int\frac{\sin x}{\cos^2 x}.\,dx+c$$

Let $\cos x=t$

$$\sin x dx=-dt$$

$y\sec^2 x = -\int \frac{dt}{t^2} + c;\ y\sec^2 x = \frac{-t^{-2+1}}{-2+1} + c$

$y\sec^2 x = \frac{1}{\cos x} + c$

$y\sec^2 x = \sec x + c$ …..(i)

Put $x = \frac{\pi}{3}$ & $y = 0$

$y\left(\frac{\pi}{3}\right)\left(\sec\left(\frac{\pi}{3}\right)\right)^2 = \sec\left(\frac{\pi}{3}\right) + c$

$0 = 2 + c$

$c = -2$

from (i)

$y\sec^2 x = \sec x - 2$

$y = \cos x - 2\cos^2 x$

Differentiate w.r.t x both sides,

$y' = -\sin x + 4\cos x \sin x$

$y' = 0$

$4\cos x \sin x - \sin x = 0$

$\sin x(4\cos x - 1) = 0$

Take, $\sin x = 0, 4\cos x - 1 = 0$

$x = 0, \cos x = \frac{1}{4}$

Put $x = 0$ in the value of y

$y = \cos x - 2\cos^2 x = \frac{1}{4} - 2\times\frac{1}{16} = \frac{1}{4} - \frac{1}{8} = \frac{2-1}{8} = \frac{1}{8}$

Therefore, maximum value is $\frac{1}{8}$

73. Writing $p = \frac{dy}{dx}$ and differentiating w.r.t.x, we have

$p = 2p + 2x\frac{dp}{dx} + 2xp^4 + 4p^3x^2\frac{dp}{dx} \Rightarrow 0 = p\left(1 + 2xp^3\right) + 2x\frac{dp}{dx}\left(1 + 2p^3x\right)$

$\Rightarrow p + 2x\frac{dp}{dx} = 0 \Rightarrow 2\frac{dp}{p} = -\frac{dx}{x} \Rightarrow 2\log p + \log x = \text{const} \Rightarrow p^2x = c$ or $p = \sqrt{\frac{c}{x}}$

Substituting this value in the given equation, we get $y = 2\sqrt{cx} + c^2$

74. $f'(x) = 2 - \frac{f(x)}{x} \Rightarrow f'(x) + \frac{1}{x} f(x) = 2$

IF = $e^{\log x} = x, \ \therefore f(x).\, x = \int 2x\, dx = x^2 + c \Rightarrow f(x) = x + \frac{c}{x}, c \neq 0\ as\ f(x) \neq 1$

(a) $\lim_{x \to 0^+} f'\left(\frac{1}{x}\right) = \lim_{x \to 0^+} \left(1 - cx^2\right) = 1$

(b) $\lim_{x \to 0^+} xf\left(\frac{1}{x}\right) = \lim_{x \to 0^+} x\left(\frac{1}{x} + cx\right) \lim_{x \to 0^+} 1 + cx^2 = 1$

(c) $\lim_{x \to 0^+} x^2 f' x = \lim_{x \to 0^+} x^2\left(1 - \frac{c}{x^2}\right) = \lim_{x \to 0^+} x^2 - c = -c$

(d) For $c \neq 0$, $f(x)$ is unbounded as $0 < x < 2 \Rightarrow \frac{c}{2} < \frac{c}{x} < \infty \Rightarrow \frac{c}{2} < x + \frac{c}{x} < \infty$

75. $\frac{dx}{dy} - \frac{1}{y} x = 2y^2$

This is linear equation taking y as independent variable.

Here, I.F. $= e^{-\int \frac{1}{y} dy} = e^{-\log y} = \frac{1}{y}$

$\therefore$ solution is $x\frac{1}{y} = \int \frac{1}{y} 2y^2 dy + c \Rightarrow \frac{x}{y} = y^2 + c \Rightarrow x = y^3 + cy$

76. i) $\frac{dy_1}{dx} + \int (x) y_1 = 0 \Rightarrow f(x) = \frac{-1}{y_1} \frac{dy_1}{dx}$

ii) $\frac{dy}{dx} - \frac{1}{y_1} \frac{dy_1}{dx} . y = r(x)$

$e^{-\int \frac{1}{y_1} \frac{dy}{dx} dx} = e^{-\int \frac{dy_1}{y_1}} = \frac{1}{y_1}$

$\frac{d}{dx}\left(\frac{y}{y_1}\right) = \frac{r(x)}{y_1} \Rightarrow \frac{y}{y_1} = \int \frac{r(x)dx + c}{y_1}$

$y = y_1 \int \frac{r(x)dx}{y_1} + cy_1$

77. $\frac{dy}{dx} + \frac{\log x}{x} y = x^{-\frac{1}{2}} \log x$

$I.F = e^{\int \frac{\log x}{x} dx} = e^{\frac{(\log x)^2}{2}} = \left(e^{\log x}\right)^{\frac{\log x}{2}} = x^{\frac{1}{2} \log x}$

$G.S is x^{\frac{1}{2}\log x}.y=\int dx$

$y\,x^{\frac{1}{2}\log x}=x+c$

78. Given differential equation ;

$$x^2dy+\left(y-\frac{1}{x}\right)dx=0;\ x>0, y(1)=1\Rightarrow\ x^2\frac{dy}{dx}+y-\frac{1}{x}=0$$

$$\Rightarrow\frac{dy}{dx}+\frac{y}{x^2}=\frac{1}{x^3};\ I.F.\ e^{\int\frac{1}{x^2}dx}\ e^{\frac{1}{x}}\quad\therefore\ ye^{\frac{1}{x}}=\int\frac{1}{x^3}.\ e^{\frac{1}{x}}$$

$$ye^{\frac{1}{x}}=e^{\frac{1}{x}}\left(1+\frac{1}{x}\right)+C$$

$$1.e^{-1}=e^{-1}(2)+C$$

$$C=-e^{-1}=\frac{1}{e};\ ye^{-\frac{1}{x}}=e^{\frac{1}{x}}\left(1+\frac{1}{x}\right)-\frac{1}{e}$$

$$y=1+\frac{1}{x}-\frac{e^{1/x}}{e};\ y\left(\frac{1}{2}\right)=3-\frac{1}{e}\times e^2;\ y\left(\frac{1}{2}\right)=3-e$$

79. $\dfrac{dy}{dx}+\dfrac{y}{2}\sec x=\dfrac{\tan x}{2y}$

$$2y\frac{dy}{dx}+y^2\sec x=\tan x$$

Put $y^2=t\Rightarrow 2y\dfrac{dy}{dx}=\dfrac{dt}{dx}$

$$\frac{dt}{dx}+t\sec x=\tan x$$

$$I.f=e^{\int\sec x dx=e^{\ln(\sec x+\tan x)}}=\sec x+\tan x\Rightarrow\ t.(\sec x+\tan x)=\int\tan x.(\sec x+\tan x)dx+c$$

$$\Rightarrow\ t.(\sec x+\tan x)=\int\sec x.\tan x\ dx+\int\tan^2 x dx+c$$

$$\Rightarrow\ t.(\sec x+\tan x x=)\sec x+\tan x-x+c$$

Put $x=0\ \Rightarrow y=1$

$\therefore\ \boxed{c=0}$

$$\Rightarrow\ (\sec x+\tan x)=\sec x+\tan x-x\Rightarrow\ y^2=1-\frac{x}{\sec x+\tan x}$$

80. $x-y\dfrac{dx}{dy}=ky$

81. $\dfrac{dy}{dx}+\dfrac{y}{x}=y^3\ \Rightarrow\ \dfrac{1}{y^3}.\dfrac{dy}{dx}+\dfrac{1}{x^2}=1$

Put t = $\frac{1}{y^2} \Rightarrow \frac{dt}{dx} - \frac{2t}{x} = -2$

$\Rightarrow$ Solution is t. $\frac{1}{x^2} = -2\int \frac{1}{x^2}dx \Rightarrow 2xy^2 + cx^2y^2 = 1$

Since this curve passes through (1, 1) $\Rightarrow$ c = 1.

82. Put $x+y=Y$ then equation given becomes $\frac{dY}{dx}+Y=x^3Y^3$.

$\Rightarrow \frac{1}{Y^3}\frac{dY}{dx}+\frac{x}{Y^2}=x^3$ *putting* $z=\frac{1}{Y^2}$ *makes it* $\frac{dz}{dx}-2xz=-2x^3$

$\Rightarrow z=2+2x^2+ce^{x^2} \Rightarrow \frac{1}{(x+y)^2}=2+2x^2+ce^{x^2}$ *putting* $x=0, y=1$ *gives* $c=-1$

83. Given equ. Can be written as $\frac{dz}{z(\ln z)^2}+\frac{dx}{x(\ln z)}=\frac{dx}{x^2}$

Put $u=\frac{1}{\ln z}$

$\frac{du}{dx}=-\frac{1}{(\ln z)^2}.\frac{1}{z}.\frac{dz}{dx}$

$\therefore -\frac{du}{dx}+\frac{1}{x}.u=\frac{1}{x^2}$

$I.F=e^{\int -\frac{1}{x}dx}=e^{\ln\frac{1}{x}}=\frac{1}{x}$

$\frac{u}{x}=-\int\frac{1}{x^3}dx=+\frac{1}{4x^4}+c$

84. We have $\frac{dy}{dx}=y\tan x-y^2\sec x \Rightarrow \frac{1}{y^2}\frac{dy}{dx}-\frac{1}{y}\tan x=-\sec x$

Putting $\frac{1}{y}=v \Rightarrow \frac{-1}{y^2}\frac{dy}{dx}=\frac{dv}{dx}$, we obtain

$\frac{dV}{dx}+\tan x.v=\sec x$ which is linear

$I.F=e^{\int \tan x dx}=e^{\log\sec x}=\sec x$

$\therefore$ The solution is $v\sec x=\int \sec^2 x\,dx+c \Rightarrow \frac{1}{y}\sec x=\tan x+c$

$\Rightarrow \sec x=y(c+\tan x)$

85. $\frac{2y}{\left(1-y^2\right)^2}\cdot\frac{dy}{dx}+\frac{y^2}{1-y^2}\frac{1}{x}=\frac{1}{x^3}$

Put $\frac{y^2}{1-y^2}=t\Rightarrow\frac{2y}{\left(1-y^2\right)^2}\frac{dy}{dx}=\frac{dt}{dx}\Rightarrow\frac{dt}{dx}+\frac{t}{x}=\frac{1}{x^3}\Rightarrow t\cdot x=\int\frac{1}{x^2}dx+c$

$\Rightarrow\ x^2y^2=(cx-1)\left(1-y^2\right)$.

86. Given differential equations is $\frac{dy}{dx}=\frac{x+y-2}{x-y}=\frac{(x-1)+(y-1)}{(x-1)-(y-1)}$

Put $x-1=h,\ y-1=k$

$\frac{dy}{dx}=\frac{h+k}{h-k};\ k=Vh\frac{dk}{dh}=V+h\frac{dV}{dh}$

$V+X\frac{dV}{dh}=\frac{1+V}{1-V}h\frac{dV}{dh}=\frac{V^2+1}{1-V}$

$\int\frac{1-V}{1+V^2}dV=\int\frac{dh}{h};\int\frac{dV}{1+V^2}-\frac{1}{2}\int\frac{2VdV}{1+V^2}=\int\frac{dh}{h}$

$\tan^{-1}V-\frac{1}{2}\ln\left(1+V^2\right)=\ln h+c$

$\tan^{-1}\left(\frac{k}{h}\right)-\frac{1}{2}\ln\left(1+\frac{k^2}{h^2}\right)=\ln\ h+c$

Replace h & k by $(x-1)\&(y-1)$ respectively

$\tan^{-1}\left(\frac{y-1}{x-1}\right)-\frac{1}{2}\ln\left(1+\frac{(y-1)^2}{(x-1)^2}\right)=\ln(x-1)+c$

Satisfy (2, 1),

$0-\frac{1}{2}\ln 1=\ln 1+c\quad\therefore c=0$

Satisfy (k + 1, 2),

Therefore, $\tan^{-1}\left(\frac{1}{k}\right)-\frac{1}{2}\ln\left(1+\frac{1}{k^2}\right)=\ln k$

$2\tan^{-1}\left(\frac{1}{k}\right)=\ln\left(\frac{1+k^2}{k^2}\right)+2\ln k$

$2\tan^{-1}\left(\frac{1}{k}\right)=\ln(1+k^2)$

87. $\because\quad\left(\frac{x+y-1}{x+y-2}\right)\frac{dy}{dx}=\left(\frac{x+y+1}{x+y+2}\right)$

Put x + y = v

$\therefore \quad \left(\frac{v-1}{v-2}\right)\left(\frac{dv}{dx}-1\right)=\left(\frac{v+1}{v+2}\right) \Rightarrow \quad \frac{dv}{dx}-1=\frac{(v+1)(v-2)}{(v-1)(v+2)}=\frac{v^2-v-2}{v^2+v-2}$

or $\quad \frac{dv}{dx}=\frac{2v^2-4}{(v^2-v-2)}$

Given that y = 1, when x = 1

$\therefore \quad 0+\frac{1}{2}\ln 2=c$

$(y-x)+\frac{1}{2}\left|\frac{\ln(x+y)^2-2}{2}\right|=0 \quad$ or $\quad \ln\left|\frac{(x+y)^2-2}{2}\right|=2(x-y)$

88. The given equation can be written as $\left(\frac{d}{dx}-3\right)\left(\frac{dy}{dx}-4y\right)=0$ …… (i)

If $\frac{dy}{dx}-4y=u$ then (I) reduces to $\frac{du}{dx}-3u=0$

$\Rightarrow \frac{du}{u}=3dx \Rightarrow u=C_1e^{3x}$. Therefore, we have $\frac{dy}{dx}-4y=C_1e^{3x}$ which is a linear equation whose

$I.F.$ is e^{-4x}. So $\frac{d}{dx}\left(ye^{-4x}\right)=C_1e^{-x}$

$\Rightarrow ye^{-4x}=-C_1e^{-x}+C_2 \Rightarrow y=C_1e^{3x}+C_2e^{4x}$

89. $y_1y_3=3y_2^2$

$\frac{y_3}{y_2}=3\frac{y_2}{y_1} \Rightarrow \ln y_2=3\ln y_1+\ln c$

$y_2=cy_1^3 \quad \Rightarrow \frac{y_2}{y_1^2}=cy_1$

$-\frac{1}{y_1}=cy+c_2 \quad \Rightarrow \frac{dx}{dy}=-cy-c_2$

$x=-\frac{cy^2}{2}-c_2y+c_3$

$\therefore\ x=A_1y^2+A_2y+A_3$

90. Integrating the given differential equation, we have $\frac{dy}{dx}=-\frac{\cos 3x}{3}+e^x+\frac{x^3}{3}+C_1$

but $y_1(0)=1$ so $1=-\frac{1}{3}+1+C_1 \Rightarrow C_1=\frac{1}{3}$

Again integrating, we get $y=-\frac{\sin 3x}{9}+e^x+\frac{x^4}{12}+\frac{1}{3}x+C_2$

but $y(0)=0$ so $0=1+C_2 \Rightarrow C_2=-1.$ Thus $y=-\frac{\sin 3x}{9}+e^x+\frac{x^4}{12}+\frac{1}{3}x-1.$

91. Putting $\sqrt{1+x+y}=v$, we have,

$x+y-1=v^2-2 \Rightarrow 1+\frac{dy}{dx}=2v\frac{dv}{dx}$

Then the given equation transforms to

$\left(2v\frac{dv}{dx}-1\right)v=v^2-2 \Rightarrow \frac{dv}{dx}=\frac{v^2+v-2}{2v^2}$

$\Rightarrow \int \frac{2v^2}{v^2+v-2}dv=\int dx \Rightarrow 2\int\left[1+\frac{1}{3(v-1)}-\frac{4}{3(v+2)}\right]dv=\int dx$

$\Rightarrow\ 2\left[v+\frac{1}{3}\log|v-1|-\frac{4}{3}\log|v+2|\right]=x+c$

Where $v=\sqrt{1+x+y}$

92. According to question

$\frac{dy}{dx}\propto\frac{-y}{x}\Rightarrow\frac{dy}{dx}=\frac{-Ky}{x}\Rightarrow\int\frac{1}{y}dy=-K\int\frac{1}{x}dx\Rightarrow \log_e y=-K\log_e x+\log_e c\Rightarrow y=cx^{-K}$

Since it passes through (1, 2)

$\therefore\ 2=c(1)^{-K}\ \ \Rightarrow c=2\ \ \therefore\ y=2x^{-K}$

Now, it also passes through (8, 1)

$\therefore\ 1=2.8^{-K}\ \Rightarrow 8^K=2\Rightarrow(2^3)^K=2^1$

$3K=1\ \Rightarrow\ K=\frac{1}{3}\ \ \therefore\ y=2x^{-1/3}$

Put $x=\frac{1}{8}\ \Rightarrow\ y=2(2^{-3})^{-1/3}=4$

93. Given $\frac{dy}{dx}=\frac{(x+1)^2+(y-3)}{(x+1)}=(x-1)+\frac{y}{(x+1)}-\frac{3}{(x+1)}$

$=\frac{dy}{dx}-\frac{y}{(x+1)}=(x+1)-\frac{3}{(x+1)}$

Which is a linear Differential Equation

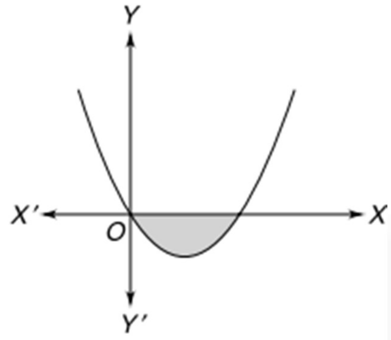
$Thus,\ IF=e^{-\int\frac{dx}{x+1}}=e^{-\log(x+1)}=\frac{1}{(x+1)}$

Therefore, the solution is $\frac{y}{x+1}=\int\left(1-\frac{3}{(x+1)^2}\right)dx\Rightarrow\frac{y}{y+1}=x+\frac{3}{x+1}+c$

Put x = 2 and y = 0, then c = -3

Hence, the equation of the curve is

$\frac{y}{x+1}=x+\frac{3}{X+1}-3 \Rightarrow y=x^2+x+3-3x-3=x^2-2x$



Hence, the required area $=\int_0^2\left[0-\left(x^2-2x\right)\right]dx=\left(x^2-\frac{x^3}{3}\right)_0^2=\left(4-\frac{8}{3}\right)=\frac{4}{3}sq.units$

94. Let the moving point be $P(x,y)$ and slope of tangent be *m*.

Equation of tangent PQ is $Y-y=m(X-x)$

Put $Y=0 \Rightarrow X=x-\frac{y}{m} \Rightarrow ;Q\left(x-\frac{y}{m},0\right)$

Mid-point of PQ lies on *y*-axis which bisects the line PQ.

$\frac{x-\frac{y}{m}+x}{2}=0 \Rightarrow m=\frac{y}{2.x}$

Here, $m=\frac{dy}{dx} \Rightarrow 2\frac{dy}{y}=\frac{dx}{x}$

Now, integrate w.r.t. '*x*'.

$2\ln y=\ln x+\ln k$

$y^2=kx$ ….(i)

Satisfy (3, 3) in equation (i) $\Rightarrow k=3$

Curve $c \Rightarrow y^2=3x$, Length of L.R. $=4a=3$

95. Given slope of normal $\frac{-dx}{dy}=\frac{x^2}{xy-x^2y^2-1}$

$x^2y^2dx+dx-xy\,dx=x^2dy;\ x^2y^2dx+dx=x^2dy+xy\,dx\ x^2y^2dx+dx=x(xdy+ydx)$

$x^2y^2dx+dx=xd(xy);\ \frac{dx}{x}=\frac{d(xy)}{1+x^2y^2}$

Take integral both sides,

$\ln kx=\tan^{-1}(xy)$ ….(i)

Satisfy $y(1)-1\ for\ x=1,\ \ln k=\frac{\pi}{2} \Rightarrow k=e^{\frac{\pi}{4}}$

Put the value of link in eq. (i),

$$\frac{\pi}{4}+\ln x=\tan^{-1}(xy)\Rightarrow xy=\tan\left(\frac{\pi}{4}+\ln x\right)\Rightarrow xy=\left(\frac{1+\tan(\ln x)}{1-\tan(\ln x)}\right) \quad \text{....(ii)}$$

Satisfy $x=e$ in (ii)

Therefore, $ey(e)=\dfrac{1+\tan 1}{1-\tan 1}$

96. Area of given curve $y=y(x)$ *is* $\left(\dfrac{y}{x}\right)^3$

According to question, $\displaystyle\int_3^x y(x)dx=\left(\frac{y}{x}\right)^3$

Take derivative both sides,

$$x.\left(\frac{y}{x}\right)=3\left(\frac{y}{x}\right)^2.\frac{d}{dx}\left(\frac{y}{x}\right);\ x=3\left(\frac{y}{x}\right)\frac{d}{dx}\left(\frac{y}{x}\right)$$

$$\int xdx=3\int\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right);\ \frac{x^2}{2}=\frac{3}{2}\left(\frac{y}{x}\right)^2+c$$

Here, curve passes through (3, 3)

$$\frac{9}{2}=\frac{3}{2}\left(\frac{3}{2}\right)^2+c;\ \frac{9}{2}=\frac{3}{2}+c$$

$c=3$

So, $\dfrac{x^2}{2}=\dfrac{3}{2}\left(\dfrac{y}{x}\right)^2+3$

Satisfy $\left(\alpha,6\sqrt{10}\right)$ in above equation.

$$\frac{\alpha^2}{2}=\frac{3}{2}\left(\frac{6\sqrt{10}}{\alpha}\right)^2+3;\ \frac{\alpha^2}{2}=\frac{3}{2}\times\frac{36\times10}{\alpha^2}+3$$

$$\frac{\alpha^2}{2}=\frac{540+3\alpha^2}{\alpha^2}$$

$$\alpha^4-6\alpha^2-1080=0$$

$$\alpha^4-36\alpha^2+30\alpha-1080=0$$

$$\left(\alpha^2-36\right)\left(\alpha^2+30\right)=0$$

$$\alpha^2=36,\ \alpha^2=-30$$

$$\alpha=\pm6$$

α lies in first quadrant then $\alpha=6$

97. Given differential equation is $\frac{dy}{dx} - y = 2 - e^{-x}$

$I.F = e^{-\int dx} = e^{-x}$

Solution of D.E = $y.e^{-x} = \int\left(2e^{-x} - e^{-2x}\right)dx \Rightarrow y = -2 + \frac{e^{-x}}{2} + C.e^{x}$

Here, $\lim_{x\to\infty}\left(-2 + \frac{e^{-x}}{2} + C.e^{x}\right) \to \textit{finite}$

Function is possible only when $c = 0$

$\therefore y = y(x) = -2 + \frac{e^{-x}}{2}$

Differentiate w.r.t. $x, \frac{dy}{dx} = -\frac{1}{2}e^{-x}$

$\left.\frac{dy}{dx}\right|_{x=0} = -\frac{1}{2} = m,\ y(0) = -2 + \frac{1}{2} = \frac{-3}{2}$

So, equation of tangent $= y + \frac{3}{2} = -\frac{1}{2}(x-0) \Rightarrow x + 2y = -3 \Rightarrow \frac{x}{-3} + \frac{y}{\frac{-3}{2}} = 1$

$a = -3, b = \frac{-3}{2}$

Required expression, $a - 4b = -3 + 6 = 3$

98. $x - y\frac{dx}{dy} = 3x \Rightarrow \frac{dx}{x} + 2\frac{dy}{y} = 0 \Rightarrow \ln(xy^2) = k \Rightarrow xy^2 = c.$

99. The point on y-axis is $\left(0,\ y - x\frac{dy}{dx}\right)$.

According to given condition,

$\frac{x}{2} = y - \frac{x}{2}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = 2\frac{y}{x} - 1$

putting $\frac{y}{x} = v$ we get

$x\frac{dv}{dx} = v - 1 \Rightarrow \ln\left|\frac{y}{x} - 1\right| = \ln|x| + c \Rightarrow 1 - \frac{y}{x} = x$ (as $f(1) = 0$).

100. Since $f''(x) = 6(x-1) \Rightarrow f'(x) = 3(x-1)^2 + c$ (integrating) ----(i)

Also, at the point (2,1), the tangent to the graph is $y = 3x - 5$ and slope of the tangent = $3 \Rightarrow f'(2) = 3$

$3(2-1)^2 + c = 3$ [from Eq (i)$\Rightarrow 3 + c = 3 \Rightarrow c = 0$

From Eq (i) we have

$f'(x)=3(x-1)^2 \Rightarrow f(x)=(x-1)^2+k$ (Integrating) ----(ii)

$\therefore 1=(2-1)^2+k \Rightarrow k=0$

Hence the equation of the function is $f(x)=(x-1)^3$.

101. The slope of the tangent = $\frac{dy}{dx}$

$\therefore$ the slope of the normal $=-\frac{dx}{dy}$

$\therefore$ the equation of the normal is $Y-y=-\frac{dx}{dy}(X-x)$

This meets the x-axis $(Y=0)$, where

$-y=\frac{-dx}{dy}(X-x) \Rightarrow X=x+y\frac{dy}{dx}$

$\therefore$ G is $\left(x+y\frac{dy}{dx},0\right)$

$\therefore$ $OG=2x \Rightarrow x+y\frac{dy}{dx}=2x$

$\Rightarrow y\frac{dy}{dx}=x \Rightarrow ydy=xdx$ [variable separable integrating, we get]

$\frac{y^2}{2}=\frac{x^2}{2}+cons\tan t \Rightarrow x^2-y^2=c$, which is a hyperbola

102. The given equation is linear in y and can be written as $\frac{dy}{dx}+\frac{x}{1-x^2}y=\frac{ax}{1-x^2}$

Its integrating factor is $e^{\int\frac{x}{1-x^2}dx}=e^{-(1/2)\log\left(1-x^2\right)}=\frac{1}{\sqrt{1-x^2}}$ if $-1<x<1$ and if $x^2>1$ then

$$I.F.=\frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(y\frac{1}{\sqrt{1-x^2}}\right)=\frac{ax}{\left(1-x^2\right)^{3/2}}=-\frac{1}{2}a\frac{-2x}{\left(1-x^2\right)^{3/2}} \Rightarrow y\frac{1}{\sqrt{1-x^2}}=\frac{a}{\sqrt{1-x^2}}+C \Rightarrow y=a+C\sqrt{1-x^2}$$

$$\Rightarrow (y-a)^2=C^2\left(1-x^2\right) \Rightarrow (y-a)^2+C^2x^2=C^2$$

Thus if $-1<x<1$ the given equation represents an ellipse. If $x^2>1$ then the solution is of the form $-(y-a)^2+C^2x^2=C^2$ which represents a hyperbola.

103. Let y = f(x) be the curve and let P (x, y) be any point on the curve. The equation of the normal at P(x, y) to the given curve is

$$Y-y=-\frac{1}{\frac{dy}{dx}}(X-x)$$

The distance of (i) from the origin is

$$d_1=\left|\frac{y+\frac{x}{\frac{dy}{dx}}}{\sqrt{1+\frac{1}{(dy/dx)^2}}}\right|=\left|\frac{x+y\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right| \qquad d_2=\left|\frac{(a-x)+(b-y)\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right|$$

Now $d_1=kd_2$

On integrating, we get $\frac{x^2}{2}+\frac{y^2}{2}=\pm k\left[-\frac{(a-x)^2}{2}-\frac{(b-y)^2}{2}\right]+C$

104. Equation of circle is $x^2+y^2+2gx+2fy+c=0$

Differentiating two times w.r.t. x

we have $f=-\frac{1+yy_2+y_1^2}{y_2}$

Again differentiating

$$\frac{y_2\left[2y_1y_2+y_1y_2+yy_3\right]-\left[y_3(1+y_1^2+yy_2)\right]}{y_2^2}=0$$

$$\Rightarrow \quad 3y_1\,yy_1^2+yy_2y_3-y_3-y_3y_1^2-yy_2y_3=0 \Rightarrow \quad 3y_1\,y_2^2=y_3\left[1+y_1^2\right]$$

105. Equation of line passing through P(x_1,y_1)

Y-y_1 $=\frac{dy}{dx}$(X-x_1), x-int $=x_1-\frac{y_1}{m}$, y-int $=y_1-x_1$m,

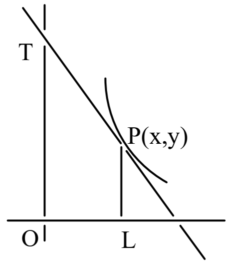
according to condition

$$\left(x-\left(x-y\frac{dx}{dy}\right)\right)^2+y^2=\left(y-\left(y-x\frac{dy}{dx}\right)\right)^2+x^2$$

$$\Rightarrow\left(x\frac{dy}{dx}-y\right)\left(x\frac{dy}{dx}+y\right)=0$$

i.e., y = cx or xy = c

106. Let $P(x,y)$ be any point on the curve. Length of intercept on y-axis by any tangent at P is



$OT = y - x\frac{dy}{dx}$

$\therefore$ Area of trapezium $OLPTO = \frac{1}{2}(PL + OT)OL = \frac{1}{2}\left(y + y - x\frac{dy}{dx}\right)x = \frac{1}{2}\left(2y - x\frac{dy}{dx}\right)x$

According to question

Area of trapezium $OLPTO = \frac{1}{2}x^2$

$i.e. \frac{1}{2}\left(2y - x\frac{dy}{dx}\right)x = \pm\frac{1}{2}x^2 \Rightarrow 2y - x\frac{dy}{dx} = \pm x$ or $\frac{dy}{dx} - \frac{2y}{x} = \pm 1$

Which is linear differential equation and $I.F = e^{-2\ln x} = \frac{1}{x^2}$

$\therefore$ The solution is $\frac{y}{x^2} = \int \pm\frac{1}{x^2}dx + c = \pm\frac{1}{x} + c$

$\therefore y = \pm x + cx^2$ or $y = cx^2 \pm x$, where c is an arbitrary constant

107. $\frac{dp}{dx} = P\left(where\ p = \frac{dy}{dx}\right)$

ln $P = x + c \Rightarrow p = e^{x+c}$

$\frac{dy}{dx} = ke^x$

$y = ke^x + \lambda$

Satisfying (0, 0), So $\lambda = -k$

$y = k\left(e^x - 1\right)$

108. Differentiating, we have $a^{n-1}\frac{dy}{dx} = nx^{n-1} \Rightarrow a^{n-1} = nx^{n-1}\frac{dx}{dy}$

Putting this value in the given equation, we have $nx^{n-1}\frac{dx}{dy}y = x^n$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, we have $ny = -x\frac{dx}{dy}$

$\Rightarrow nydy + xdx = 0 \Rightarrow ny^2 + x^2 =$ constant. Which is the required family of orthogonal trajectories.

109. Let $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ …..(i)

Differentiating (i) w.r.t. x, we have $\frac{x}{a^2+\lambda}+\frac{y}{b^2+\lambda}\frac{dy}{dx}=0$ ....(ii)

From (i) and (ii) we have to eliminate λ.

Now (ii) gives, $\lambda=-\left(\frac{b^2x+a^2y\frac{dy}{dx}}{x+y\frac{dy}{dx}}\right)$

$$a^2+\lambda=\frac{\left(a^2-b^2\right)x}{x+y\frac{dy}{dx}};\ b^2+\lambda=-\frac{\left(a^2-b^2\right)y\frac{dy}{dx}}{x+y\frac{dy}{dx}}$$

Substituting these values in (i), we get $\left(x-y\frac{dx}{dy}\right)\left(x+y\frac{dy}{dx}\right)=a^2-b^2$ ....(iii)

Is the differential equation of the given family of curves.

Replacing $\frac{dy}{dx}$ to $-\frac{dx}{dy}$ in (iii), we obtain $\left(x+y\frac{dy}{dx}\right)\left(x-y\frac{dx}{dy}\right)=a^2-b^2$ ....(iv)

Which is the same as (iii). Thus we see that the family (i) is self orthogonal. i.e., every member of the family (i) cuts every other member of the same family orthogonally.

110. $x^2+y^2-ay=0$. Differentiating w.r.t. x, we get $2x+2y\frac{dy}{dx}-a\frac{dy}{dx}=0$

$$\Rightarrow 2x+2y\frac{dy}{dx}-\frac{x^2+y^2}{y}\frac{dy}{dx}=0$$

$\Rightarrow 2x+\frac{y^2-x^2}{y}\frac{dy}{dx}=0$. This is the differential equation of the given circles.

$\therefore$ The equation of orthogonal trajectories is $2x+\frac{y^2-x^2}{y}.\left(-\frac{dx}{dy}\right)=0$

$$\left(Putting-\frac{dx}{dy}in\ place\ of\ \frac{dy}{dx}\right)$$

$$\Rightarrow 2xydy+\left(x^2-y^2\right)dx=0$$

It is a homogeneous equation

Put $y=\upsilon x,\ then\ \frac{dy}{dx}=\upsilon+x\frac{d\upsilon}{dx}\Rightarrow 2x.\upsilon x.\left(\upsilon+x\frac{d\upsilon}{dx}\right)+x^2-\upsilon^2x^2=0$

Or $\Rightarrow 2\upsilon\left(\upsilon+x\frac{d\upsilon}{dx}\right)+1-\upsilon^2=0\ \Rightarrow 1+\upsilon^2+2\upsilon x\frac{d\upsilon}{dx}=0\Rightarrow\frac{dx}{x}+\frac{2\upsilon}{1+\upsilon^2}d\upsilon=0$

Integrating, we get $\ell nx+\ell n\left(1+\upsilon^2\right)=\ell nc$

$$\Rightarrow x\left(1+\upsilon^2\right)=c\Rightarrow x\left(1+\frac{y^2}{x^2}\right)=c,\ i.e.,\ x^2+y^2=cx$$

Which is the required family of orthogonal trajectories.

111. $y=cx^2\Rightarrow c=\frac{y}{x^2}\Rightarrow\frac{dy}{dx}=c.2x$

We need to remove arbitrary constant c

$\Rightarrow \frac{dy}{dx} by \frac{-dx}{dy} \Rightarrow \frac{-dx}{dy} = \frac{dy}{x} \Rightarrow 2ydy + xdx = 0$

$\Rightarrow \frac{2y^2}{2} + \frac{x^2}{2} = \frac{c^2}{2} \Rightarrow 2y^2 + x^2 = c^2$ which is required equation of y.

Since, γ passes through (0, 1)

$2\times 1^2 + 0^2 = c^2 \Rightarrow c^2 = 2$

So, $\boxed{2y^2 + x^2 = 2}$

Now, $x = \sqrt{2},\ y = 0$ satisfies above equation of y.

Option (a) is correct.

$x = 0,\ y = \sqrt{2}$ doesnot satisfy the equation of y.

$x = 1,\ y = 1$ doesnot satisfy the equation of y.

$x = -1,\ y = 1$ doesnot satisfy the equation of y.

Option (b), (c), (d) are not correct.

112. $y^2 = 4k(k+x)$ ….(i)$\Rightarrow 2y\frac{dy}{dx} = 4k \Rightarrow \frac{y}{2}\frac{dy}{dx} = k$

To remove arbitrary constant k

Put k is equation (i)

$\Rightarrow y^2 = 4\frac{y}{2}\left(\frac{dy}{dx}\right)\left(\frac{y}{2}\frac{dy}{dx} + x\right) \Rightarrow y^2 = 2y\frac{dy}{dx}\left(\frac{y}{2}\frac{dy}{dx} + x\right)$

$\Rightarrow y = 2\frac{dy}{dx}\left(\frac{y}{2}\frac{dy}{dx} + x\right) \Rightarrow y = y = \left(\frac{dy}{dx}\right)^2 + 2xc\left(\frac{dy}{dx}\right)$ …..(iii)

To get orthogonal trajectory, replace $\frac{dy}{dx} by \frac{-dx}{dy}$

$\Rightarrow y + 2x\frac{dx}{dy} = y\left(\frac{dx}{dy}\right)^2 \Rightarrow \frac{ydy}{dx} + 2x = y\left(\frac{dx}{dy}\right) \Rightarrow \boxed{y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} = y}$ ….(iv)

Equation (iv) is same as equation (iii)

Given equation is self orthogonal family of curves.

The orthogonal trajectory will be

$\boxed{y^2 = 4c(c+x)}$ which c is orthogonal constant

Option (D) is correct

113. $y(1-cx) = 1 + cx \Rightarrow y - ycx = 1 + cx \Rightarrow y - 1 = ycx + cx$

$\Rightarrow y - 1 = c(xy + x) \Rightarrow \frac{y-1}{xy+x} = c \Rightarrow \frac{y-1}{(y+1)x} = c$

Differentiating both side w.r.t. x

$\Rightarrow \frac{d}{dx}\left(\frac{y-1}{(y+1)x}\right) = 0 \Rightarrow \frac{x(y+1)\frac{dy}{dx} - \left(y-1\left[x\frac{dy}{dx} + (y+1)\right]\right)}{x^2(y+1)^2} = 0$

$\Rightarrow x(y+1)\frac{dy}{dx}-(y-1)\left[x\frac{dy}{dx}+y+1\right]=0 \Rightarrow x(y+1)\frac{dy}{dx}-x(y-1)\frac{dy}{dx}-\left(y^2-1\right)=0$

$\Rightarrow x\frac{dy}{dx}(y+1-y-1)-\left(y^2-1\right)=0 \Rightarrow 2x\frac{dy}{dx}-y^2+1=0$

To get orthogonal trajectories, we replace $\frac{dy}{dx} by \frac{-dx}{dy}$

$2x\left(\frac{-dx}{dy}\right)-y^2+1=0 \Rightarrow \frac{-2xdx}{dy}=y^2-1$

$\Rightarrow 2xdx=\left(y^2-1\right)dy \Rightarrow 2\int xdx=\int\left(y^2-1\right)dy$

$\Rightarrow -2\frac{x^2}{2}=\frac{y^3}{3}-y+c_1 \Rightarrow -x^2=\frac{y^3}{3}-y+c_1$

$\Rightarrow -3x^2=y^3-3y+3c_1 \Rightarrow \boxed{3y-y^3-3x^2=\text{constant}}$

Option (C) is correct

114. Differentiating $r=a(1-\cos\theta)$ with respect to θ, we get $\frac{dr}{d\theta}=a\sin\theta$

Eliminating a from (i) and (ii), we obtain

$\frac{dr}{d\theta}.\frac{1}{r}=\frac{\sin\theta}{1-\cos\theta}=\cot\frac{\theta}{2}$ which is the differential equation of the given family.

Replacing $\frac{dr}{d\theta}$ $by -r^2\frac{d\theta}{dr}$, we obtain

$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right)=\cot\frac{\theta}{2} or \frac{dr}{r}+\tan\frac{\theta}{2}d\theta=0$

As the differential equation of orthogonal trajectories. It can be rewritten as

$\frac{dr}{r}=-\frac{(\sin\theta/2)d\theta}{\cos\theta/2}$

Integrating, $\log r=2\log\cos\theta/2+\log c$

$r=c\cos^2\theta/2=\frac{1}{2}c(1+\cos\theta) \; or \; r=a'(1+\cos\theta)$

Which is the required orthogonal trajectory.

115. P – Population, y – population after ‘t’ years

$\frac{dy}{dt}\alpha y \Rightarrow \frac{dy}{dt}=ky \Rightarrow \int\frac{dy}{y}=\int kdt \Rightarrow \log y=kt+c$

$t=0\,\&\,y=p \Rightarrow \log p=0+c$

$t=0\,\&\,y=2p \Rightarrow \log 2p=50k+\log p \Rightarrow \log\left(\frac{2p}{p}\right)=50k \Rightarrow k=\frac{\log(2)}{50}$

$t=?\&\,y=3p \Rightarrow kt=\log y-c$

$\left(\frac{\log 2}{50}\right)t=\log 3p-\log p$

$t=\frac{\log 3}{(\log 2/50)}=50\log_2 3$

116. Given differential equation is linear in t. $\therefore I.F=e^{\int -(0.5)dt}=e^{-0.5t}$

$P(t).\left(e^{-0.5t}\right)=\int(-450)e^{-0.5t}dt=(-450)e^{-0.5t}/(-0.5)+c \Rightarrow P(t)e^{-0.5t}=900e^{-0.5t}+c$

$P(0)=850 \Rightarrow 850=900+c=-50, \quad \therefore P(t)=900-50e^{0.5t}$

If $P(t)=0$ then $50e^{0.5t}=900 \Rightarrow 0.5t=\log_e 18 \Rightarrow t=2\log_e 18$

117. $\dfrac{dp(t)}{dt}-\dfrac{1}{2}p(t)=-200$ is a linear differential equation. $I.F=e^{\int -\frac{1}{2}dt}=e^{-t/2}$

It's a solution is $p(t)e^{-t/2}=\int -200e^{t/2}dt \Rightarrow p(t)e^{-t/2}=400e^{-t/2}+c$

$p(0)=100 \Rightarrow 100=400+c \Rightarrow c=-300$

$p(t)=e^{-t/2}=400\,e^{-t/2}-300 \Rightarrow p(t)=400-300e^{t/2}$

118. Let N denote the number of people living in the country at any time t, and let N_0 denote the number of people initially living in the country. Then, from equation (a)

$\dfrac{dN}{dt}-kN=0$ which has the solution $N=ce^{kt}$ ….(i)

At $t=0,\ N=N_0$; hence, it follows from (i) that $N_0=ce^{k(0)}$, *or* that $c=N_0$. Thus, $N=N_0e^{kt}$ …..(ii)

At $t=2,\ N=2N_0$; substituting these values into (ii), we have $2N_0=N_0e^{2k}$ from which $k=\dfrac{1}{2}\ell n2$ substituting this value into (i) gives $N=N_0e^{t/2\ \ln 2}$

At $t=3$, N = 20,000. Substituting these values into (iii), we obtain 20,000 = $=N_0e^{3/2\ \ln 2} \Rightarrow N_0=20{,}000/2\sqrt{2}\approx 7071$

119. Let N(t) denote the number of mice with the disease at time t. We are given that N(0) = 5 and it follows that 500 – N(t) is the number of mice without the disease at time t. The theory predicts that $\dfrac{dN}{dt}=KN(500-N)$ ….(i)

Where k is a constant of proportionality. This equation is different from equation (i) because the rate of change is no longer proportional to just the number of mice who have the disease. Equation (i) has the differential form $\dfrac{dN}{N(500-N)}=kdt=0$ ….(ii)

Which is separable. Using partial fraction decomposition, we have

$$\frac{1}{N(500-N)}=\frac{1/500}{N}+\frac{1/500}{500-N}$$

Hence (ii) may be rewritten as $\dfrac{1}{500}=\left(\dfrac{1}{N}+\dfrac{1}{500-N}\right)dN-kdt=0$

Its solution is $\dfrac{1}{500}=\int\left(\dfrac{1}{N}+\dfrac{1}{500-N}\right)dN-\int kdt=c$ or

$$\frac{1}{500}\left(\ell n|N|-\ell n|500-N|\right)-kt=c$$

Which may be rewritten as $\ell n\left|\dfrac{N}{500-N}\right|=500(c+kt)$

$\dfrac{N}{500-N}=ce^{500(c+kt)}$ …..(iii)

But $e^{500(c+kt)} = e^{500}e^{kt}$. Setting $c_1 = e^{500c}$, we can write (iii) as

$\frac{N}{500-N} = c_1 e^{500kt}$ ….(iv)

At t = 0, N = 5. Substituting these values into (iv),

We find $\frac{5}{495} = c_1 e^{500k(0)} = c_1$

So $c_1 = 1/99$ and (iv) becomes $\frac{N}{500-N} = \frac{1}{99}e^{500kt}$ ….(v)

We could solve (v) for N, but this is not necessary. We seek a value of t when N = 250, one-half the population. Substituting N = 250 into (v) and solving for t, we obtain $1 = \frac{1}{99}e^{500kt} \ell n 99 = 500kt$ or t = 0.009 1/k time units. Without additional information, we cannot obtain a numerical value for the constant of proportionality k or be more definitive about t.

120. Given that $\frac{dP}{dx} = 100 - 12\sqrt{x} \Rightarrow dP = \left(100 - 12\sqrt{x}\right)dx \Rightarrow \int dP = \int \left(100 - 12\sqrt{x}\right) dx$

$\Rightarrow P = 100x - 8x^{3/2} + c$. Given that P = 2000 when $x = 0 \Rightarrow 2000 = c$

$\therefore P = 100x - 8x^{3/2} + 2000$

If $x = 25$ then $P = 100(25) - 8(25)^{3/2} + 2000 = 2500 - 1000 + 2000 = 3500$

121. (a) Here a = 0, b = 1, e = 4, f = 2, and $V_0 = 10$. The volume of brine in the tank at any time *t* is given in equation (a) as $V_0 + et - ft = 10 + 2t$. We require *t* when 10 + 2*t* = 50, hence, t = 20 min.

(b) For the equation $\frac{dQ}{dt} + \frac{2}{10+2t}Q = 4$

This is a linear equation; its solution is $Q = \frac{40t + 4t^2 + c}{10+2t}$ ….(i)

At t = 0, Q = a = 0. Substituting these values into (i), we find that c = 0. We required Q at the moment of overflow, which from part (a) is t = 20. Thus

$Q = \frac{40(20) + 4(40)^2}{10 + 2(20)} = 48\ 1b$

122. Here, $V_0 = 100, a = 20, b = 0, and\ e = f = 5$,

Equation (b) $\frac{dQ}{dt} + \frac{1}{20}Q = 0$

The solution of this linear equation is $Q = ce^{-t/20}$ ….(i)

At $t = 0$, we are given Q = a = 20. Substituting these values into (i), we find that c = 20, so that (i) can be rewritten as $Q = 20e^{-t/20}$. Note that as $t \to \infty, Q \to 0$ as it should, since only fresh water is being added.

123. Here $V_0 = 100, a = 1, b = 1, and\ c = f = 3;$

Equation (b) $\frac{dQ}{dt} + 0.03Q = 3$

The solution to this linear differential equation is $Q = ce^{-0.o3t} + 100$ ….(i)

At t = 0, Q = a = 1, Substituting these values into (i),

We find $1 = ce^0 + 100, \ or \ c = -99$.

Then (i) can be rewritten as $Q = -99e^{0.o3t} + 100$ …(ii)

(b) we require t when Q = 2, Substituting Q = 2 into (ii), we obtain

$= -99e^{-0.o3t} + 100 \ or \ e^{-0.o3t} = \dfrac{98}{99}$

From which $t = -\dfrac{1}{0.03}\ell n\dfrac{98}{99}\min$

124. Use equation (a) with $T_m = 0$; the medium here is the room which is being held at a constant temperature of $0^0 F$.

Thus we have $\dfrac{dT}{dt} + kT = 0$ whose solution is $T = ce^{-kt}$ …(i)

Since T =100 at t = 0 (the temperature of the bar is initially $100^0 F$), it follows (i) that $100 = ce^{-k(0)} \ or \ 100 = c$. Substituting this value into (i), we obtain $T = 100e^{-kt}$ …(ii)

At t = 20, we are given that T = 50; hence, from (2),

$50 = 100e^{-20k}$ from which $k = -\dfrac{1}{20}\ell n\dfrac{50}{100}$

Substituting this value into (ii), we obtain the temperature of the bar at any time t as

$T = 100e^{\ln 1/2t}$ …..(iii)

We require t when T = 25. Substituting T = 25 into (iii), we have $25 = 100e^{1/20 \ \ln 1/2 \ t}$

Solving, we find that t = 39.6 min.

125. (A) Let N denote the amount of material present at time t. Then, from equ. (a)

$\dfrac{dN}{dt} - kN = 0$

This differential equation is separable and linear, its solution is $N = ce^{kt}$ ….(i)

At t = 0, we are given that N = 50. Therefore, from (i), $50 = ce^{k(0)}, \ or \ c = 50$

Thus, $N = 50e^{kt}$ ….(ii)

At t = 2, 10 percent of the original mass of 50 mg. or 5 mg, has decayed. Hence, at $t = 2$, N = 50 – 5 = 45. Substituting these values into (ii) and solving for k, we have $45 = 50e^{2k}$ or $k = \dfrac{1}{2}\ell n\dfrac{45}{50}$.

Substituting this value into (ii), we obtain the amount of mass present at any time t as

$N = 50e^{-1/2\ln .9t}$ ….(iii)

Where t is measured in hours.

(B) We require N at t = 4. Substituting t = 4 into (iii) and then solving for N, we find $N = 50e^{-2\ln .9}$

(C) We require t when $N = 50/2 = 25$. Substituting N = 25 into (iii) and solving for t, we fid $25 = 50e^{-1/2\ln .9t} \Rightarrow t = \ln 1/2/(-1/2\ln .9).hr.$