Fattorizzazione LU, $R^T R$

- $Ax = b$

  $A = LU$ (senza pivoting) `>>[l,u,p]=lu(a)`

1) $(LU)x = b$
2) $L(\underbrace{Ux}_{y}) = b$

In sequenza
1) $Ly = b$ tr. inf.
2) $Ux = y$ tr. sup

- A simmetrica e definita positiva `>> r=chol(a)`

  $A = R^T R$ $(\mathrm{diag}(R^T) = \mathrm{diag}(R))$

  In generale $R_{ii} \neq 1$

$R =$ tr. sup $R^T =$ tr. inf

1) $(R^T R)x = b$
2) $R^T(\underbrace{Rx}_{y}) = b$

$\Rightarrow$

In sequenza
1) $R^T y = b$ tr inf.
2) $Rx = y$ tr sup

$$Ax = b \qquad A = LU$$

$$\begin{cases} Ly = b \\ Ux = y \end{cases} \qquad \underbrace{LU}_{A}x = b$$

CON PIVOT >>[l, u, p] = lu(a)

$p \neq Id$

$$PA = LU$$

$$Ax = b \qquad \underbrace{PA}_{LU}x = Pb$$

$$L\underbrace{Ux}_{y} = Pb$$

$$\begin{cases} Ly = Pb \\ Ux = y \end{cases}$$

Generalizzazione $PA = LU$

P: matrice di permutazione

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \overset{A}{\begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix}} = \begin{bmatrix} R_4 \\ R_1 \\ R_2 \\ R_3 \end{bmatrix}$$

Su ogni riga / colonna solo un elemento $= 1$, ... altri $= 0$

$P_{14} = 1$ $\quad R_4 \to R_1$

$P_{21} = 1$ $\quad R_1 \to R_2$

$P_{32} = 1$ $\quad R_2 \to R_3$

$P_{43} = 1$ $\quad R_3 \to R_4$

$P_{ij} = 1$ $\quad R_j \to R_i$

Scambio righe
$\downarrow$
Sistema equiv.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$m_{21} = \frac{1}{1} = 1$

$a_{21} = 1 - 1 = 0$

$a_{22} = 1 - 1 = 0$

$a_{23} = 2 - 1 = 1$

$m_{31} = 1$

$a_{31} = 1 - 1 = 0$

$a_{32} = 2 - 1 = 1$

$a_{33} = 2 - 1 = 1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$m_{32} = \frac{1}{0}$ STOP

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

2^a riga → 3^a riga e viceversa

MATLAB

$PA = LU$

$$A = \begin{bmatrix} 2 & 5 & -1 \\ 4 & 8 & 12 \\ 1 & -2 & 2 \end{bmatrix}$$

Scambio 1ª con 2ª

$$\begin{bmatrix} 4 & 8 & 12 \\ 2 & 5 & -1 \\ 1 & -2 & 2 \end{bmatrix}$$

$m_{21} = \frac{1}{2}$

$a_{21} = 0 \qquad a_{22} = 5 - \frac{1}{2} \cdot 8 = 1 \qquad a_{23} = -1 - \frac{1}{2} \cdot 12 = -7$

$m_{31} = \frac{1}{4}$

$a_{31} = 0 \qquad a_{32} = -2 - \frac{1}{4} \cdot 8 = -4 \qquad a_{33} = 2 - \frac{1}{4} \cdot 12 = -1$

$$\begin{bmatrix} 4 & 8 & 12 \\ 0 & 1 & -7 \\ 0 & -4 & -1 \end{bmatrix}$$

Scambio R_2 con R_3

$$\begin{bmatrix} 4 & 8 & 12 \\ 0 & -4 & -1 \\ 0 & 1 & -7 \end{bmatrix}$$

$$m_{32} = -\frac{1}{4}$$

$$a_{33} = -7 - \left(-\frac{1}{4}\right)(-1) = -7 - \frac{1}{4} = -\frac{29}{4}$$

$R_1 \leftrightarrow R_2$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{4} & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 4 & 8 & 12 \\ 0 & -4 & -1 \\ 0 & 0 & -\frac{29}{4} \end{bmatrix}}_{U} =$$

$R_2 \leftrightarrow R_3$

$$\underbrace{\begin{bmatrix} 4 & 8 & 12 \\ 1 & -2 & 2 \\ 2 & 5 & -1 \end{bmatrix}}_{LU}$$

$PA = LU$

$$\overset{P}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}} \overset{A}{\begin{bmatrix} 2 & 5 & -1 \\ 4 & 8 & 12 \\ 1 & -2 & 2 \end{bmatrix}} =$$

$$\begin{bmatrix} 4 & 8 & 12 \\ 1 & -2 & 2 \\ 2 & 5 & -1 \end{bmatrix}$$

$P_{12} = 1 \qquad R_2 \to R_1$

$P_{23} = 1 \qquad R_3 \to R_2$

$P_{31} = 1 \qquad R_1 \to R_3$