Approssimazione, rappresentazione, trasformate discrete (e Fourier)

Giovanni Naldi- ESP UNIMI

Image Representation, Image <-> function f

Computational Harmonic Analysis

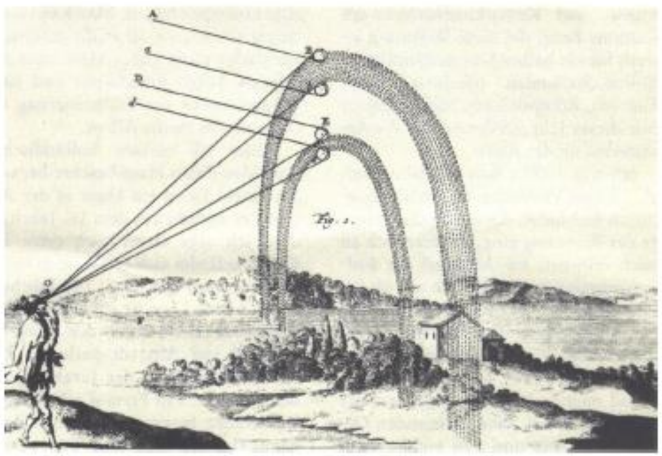
- Representation $f = \sum_k a_k \mathbf{b}_k$

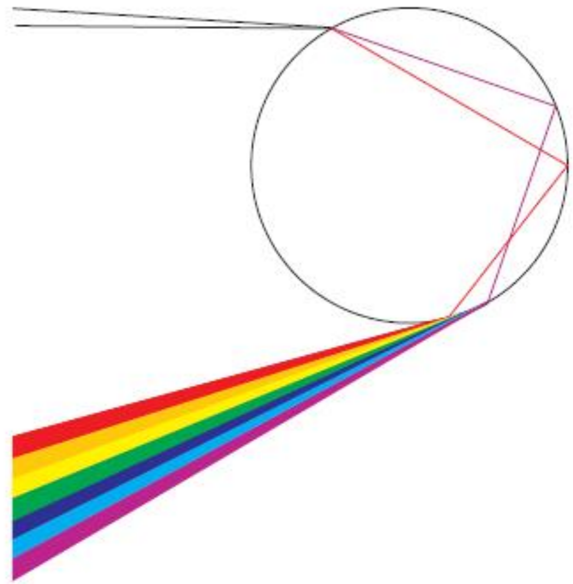
  coefficients basis, frame

- **Analysis** study f through *structure* of $\{a_k\}$
  $\{\mathbf{b}_k\}$ should *extract features* of interest

- **Approximation** $\widehat{f}_N$ uses just a few terms N
  exploit *sparsity* of $\{a_k\}$

From Rainbows to Spectras



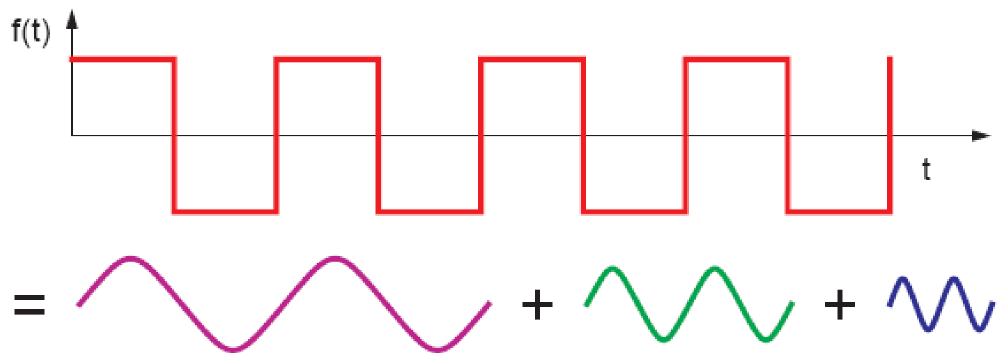


Von Freiberg, 1304: Primary and secondary rainbow

Newton and Goethe

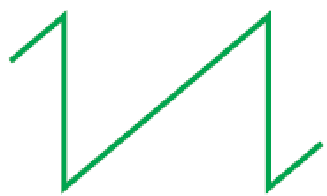
Signal Representations

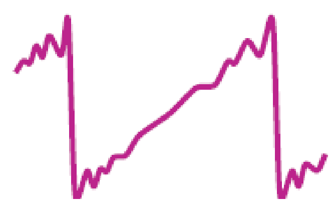
1807: Fourier upsets the French Academy....



Fourier Series: Harmonic series, frequency changes, f_0, $2f_0$, $3f_0$, ...

But... 1898: Gibbs' paper 1899: Gibbs' correction





Orthogonality, convergence, complexity

Expansion of functions in trigonometric series by

use of approximation by trigonometric functions was used earlier by

even earlier by

Jean Baptiste Joseph

Leonard Eul

. . . . . . but

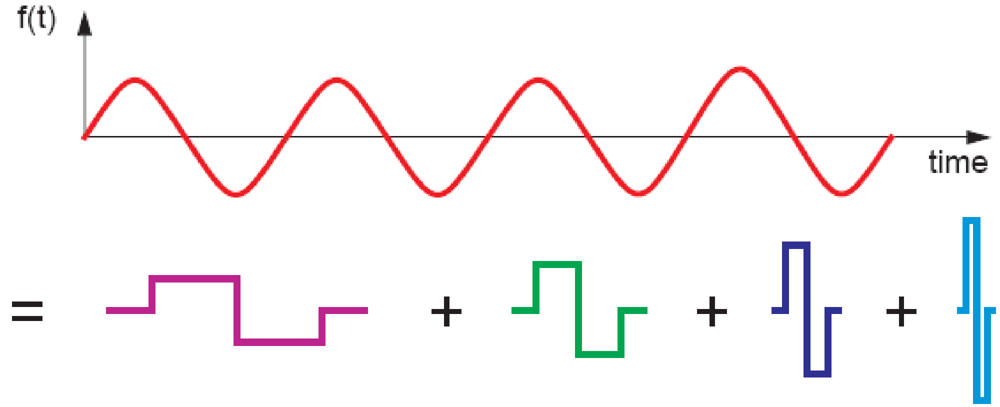
. . . . . . but

Daniel Bernuolli(1700 - 1783). “He showed that the movements of strings of musical instruments are composed of and infinite number of harmonic vibrations all superimposed on the string.” (late 1720th)

Building blocks of a signal

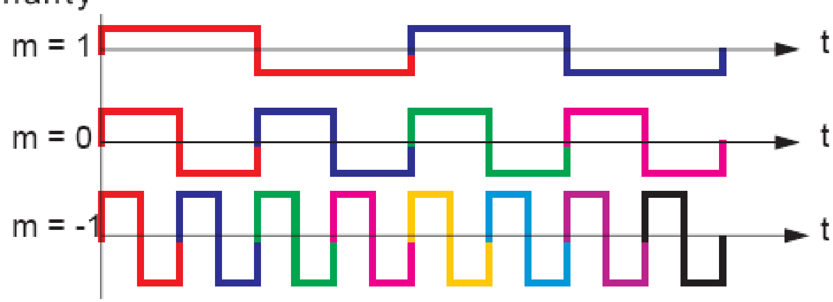
- Sampling of a signal:
  representation in standard basis
- Frequency description of the signal
  Fourier basis
- Wavelets an compromise

1910: Alfred Haar discovers the Haar wavelet "dual" to the Fourier construction



Haar series:

- Scale changes S_0, $2S_0$, $4S_0$, $8S_0$ ...
- orthogonality



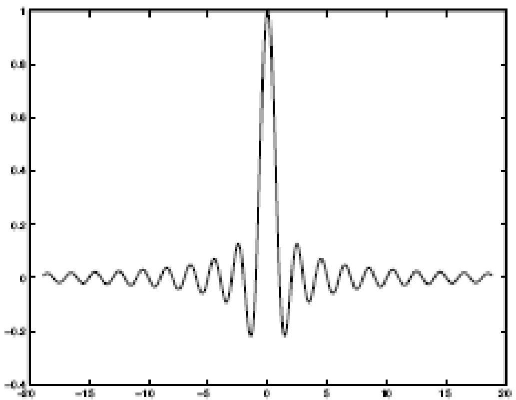
Theorem 1 (Shannon-48, Whittaker-35, Nyquist-28, Gabor-46)

If a function f(t) contains no frequencies higher than W cps, it is completely determined by giving its ordinates at a series of points spaced 1/(2W) seconds apart.

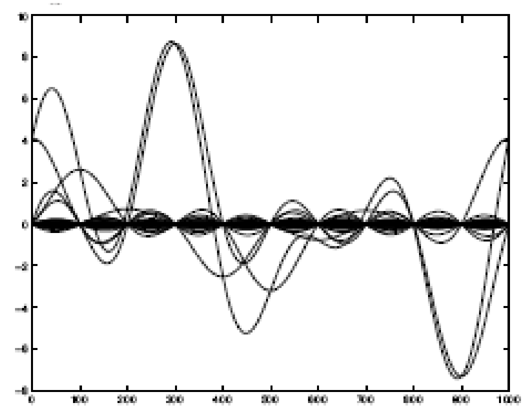
[if approx. T long, W wide, 2TW numbers specify the function]

It is a representation theorem:

- $\{\mathrm{sinc}(t-n)\}_{n \text{ in } Z}$, is an orthogonal basis for BL[$-\pi,\pi$]
- f(t) in BL[$-\pi,\pi$] can be written as $f(t)) = \sum f(n) \cdot \mathrm{sinc}(t-n)$



... slow...!



Representations, Bases and Frames

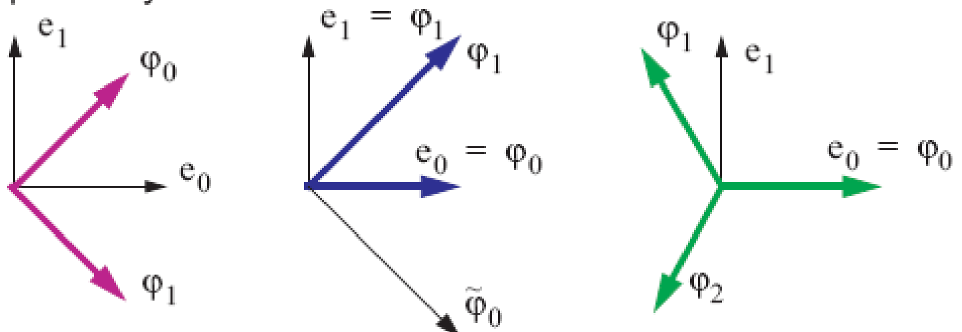
Ingredients:

- as set of vectors, or ''atoms'', $\{\varphi_n\}$
- an inner product, e.g. $\langle \varphi_n, f \rangle = \int (\varphi_n \cdot f)$
- a series expansion

$$f(t) = \sum_n \langle \varphi_n, f \rangle \cdot \varphi_n(t)$$

Many possibilities:

- orthonormal bases (e.g. Fourier series, wavelet series)
- biorthogonal bases
- overcomplete systems or frames



OB BOB UTF

Note: no transforms, uncountable

Approximations, aproximation...

The linear approximation method

Given an orthonormal basis $\{g_n\}$ for a space S and a signal

$$f = \sum_n \langle f, g_n \rangle \cdot g_n ,$$

the best linear approximation is given by the projection onto a fixed sub-space of size M (independent of f!)

$$f_M = \sum_{n \in J_M} \langle f, g_n \rangle \cdot g_n$$

The error (MSE) is thus

$$\varepsilon_M = \|f - f\|^2 = \sum_{n \notin J_M} \left| \langle f, g_n \rangle \right|^2$$

Ex: Truncated Fourier series
project onto first M vectors corresponding to largest expected inner products, typically LP

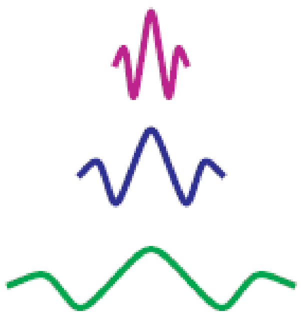
Replace (shift, modulation)

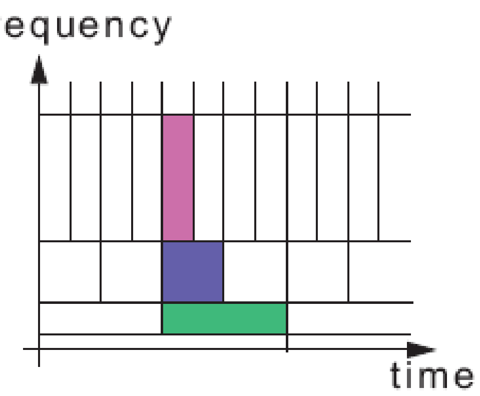
by (shift, scale)

or

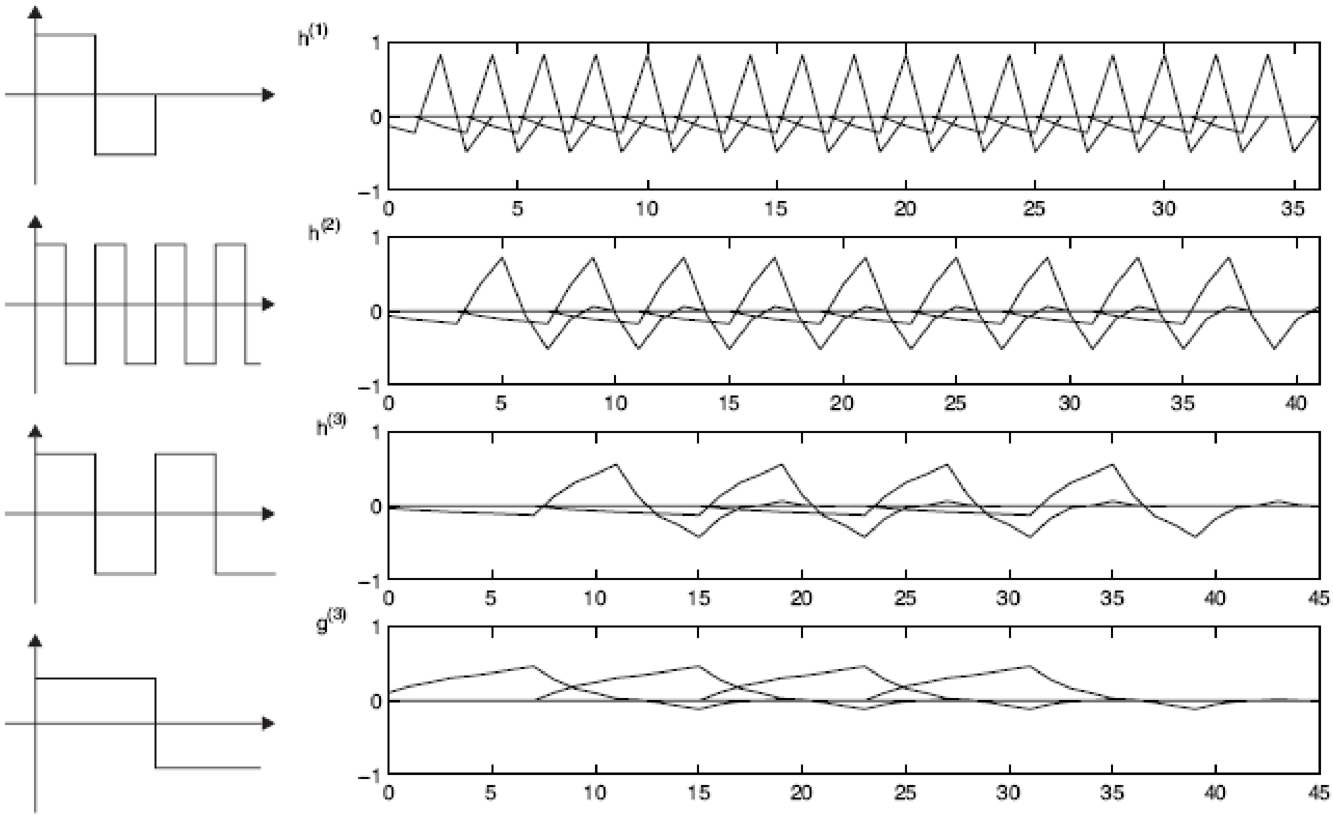
$$\Psi_{m,n}(t) = 2^{-m/2}\Psi\left(\frac{t-2^m n}{2^m}\right) \qquad n, m \in Z$$

then there exist “good” localized orthonormal bases, or wavelet bases





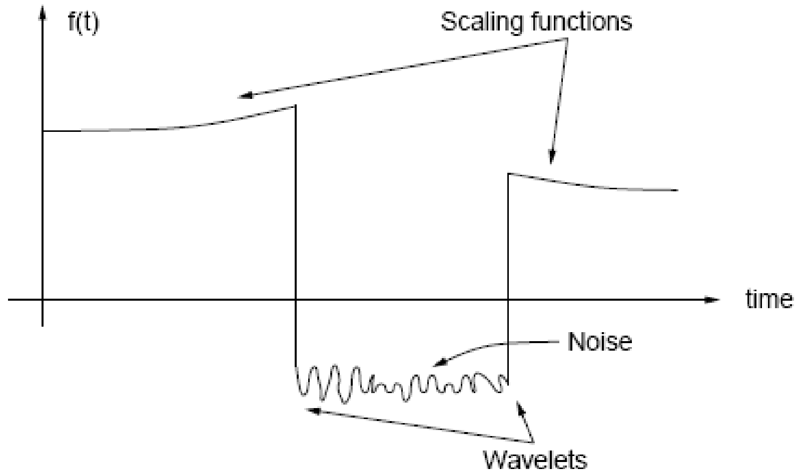
Examples of bases



Haar **Daubechies, D_2**

Wavelets and representation of piecewise smooth functions

Goal: efficient representation of signals like:

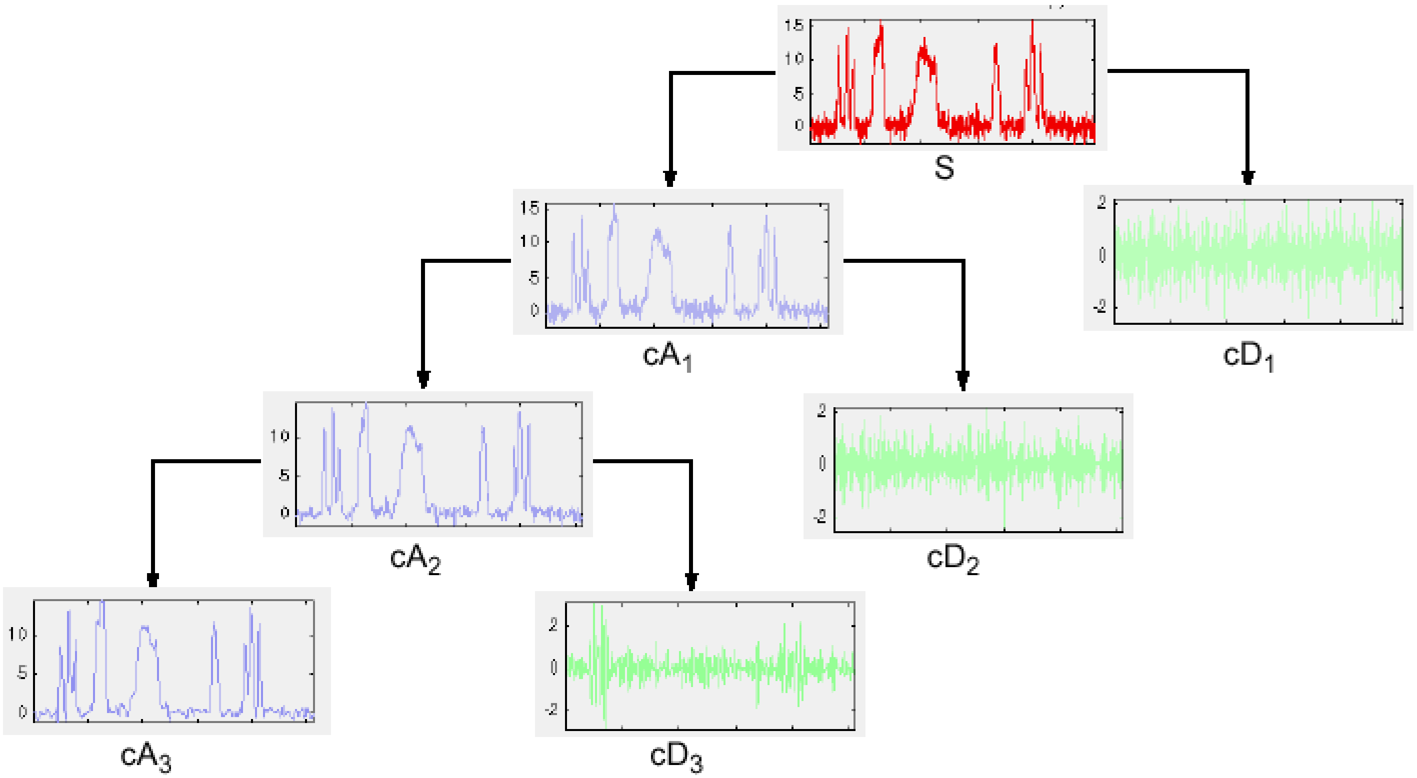


where:

- Wavelet act as singularity detectors
- Scaling functions catch smooth parts
- "Noise" is circularly symmetric

Note: Fourier gets all Gibbs-ed up!

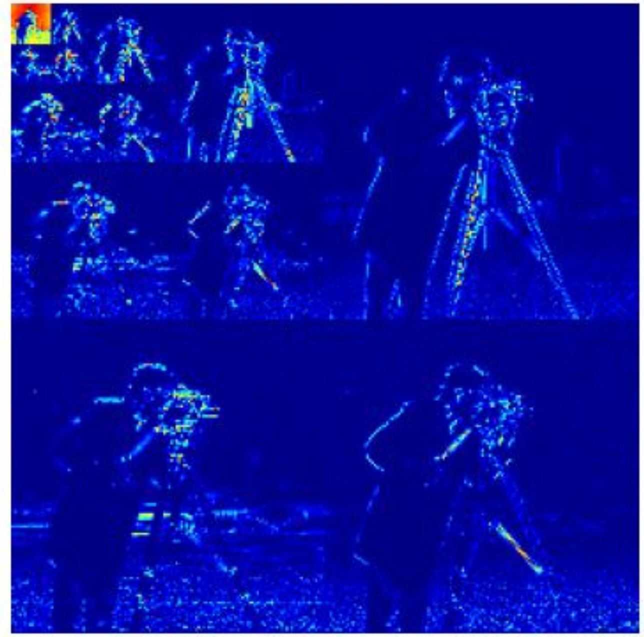
Examples of 1-D Wavelet Transform



2-D Wavelet Transform via Separable Filters

From Matlab Wavelet Toolbox Documentation

Wavelet Transform Sparsity



$$f = \sum_k a_k \mathbf{b}_k$$

- Many $a_k \approx 0$

  (blue)

Nonlinear Approximation

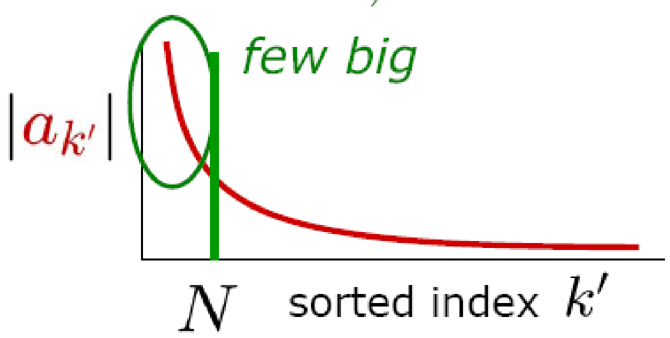
$$f \;=\; \sum_k a_k\, \mathbf{b}_k$$

- N-*term approximation*:

  use *largest* a_k *independently*

$$\widehat{f}_N \;:=\; \sum_{k'=1}^{N} a_{k'}\, \mathbf{b}_{k'}$$

- Greedy / *thresholding*

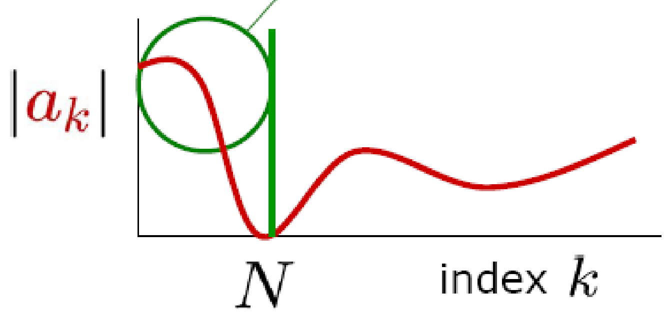


Linear Approximation

$$f \;=\; \sum_k a_k \,\mathbf{b}_k$$

- N-*term approximation*: use "*first*" a_k

$$\widetilde{f}_N \;:=\; \sum_{k=1}^{N} a_k \,\mathbf{b}_k$$



Error Approximation Rates

$$f = \sum_k a_k \mathbf{b}_k$$

$$\widehat{f}_N = \sum_{k'=1}^{N} a_{k'} \mathbf{b}_{k'}$$

$$\|f - \widehat{f}_N\|_2^2 < C\, N^{-\alpha} \qquad \text{as } N \to \infty$$

- Optimize asymptotic *error decay rate* α
- Nonlinear approximation works better than linear

Consideriamo un caso importante, V=spazio funzioni «regolari» $f : \mathbb{R} \rightarrow \mathbb{R}$ periodiche: $\exists\, T \neq 0$ *tale che* $f(x) = f(x+T) \quad \forall\, x \in \mathbb{R}$
(funzione T-periodica)

Possiamo considerare $T = 2\pi$, infatti se T $\neq 2\pi$ si definisce $g(x) = f\left(\frac{Tx}{2\pi}\right)$ che risulta 2π-periodica.

Definiamo il prodotto scalare per f,g$\in$ V (caso reale)

$$< f,g> = \int_0^{2\pi} f(x)\, g(x)\, dx$$

(possiamo selezionare, per comodità, un altro intervallo lungo un periodo, per esempio [-π, +π])

Come sottospazio scegliamo quello dei polinomi trigonometrici definiti come

$$S_n = \frac{a_0}{2} + \sum_{k=1}^{n} [a_k \cos(kx) + b_k \sin(kx)]$$

(utilizziamo la notazione più diffusa e derivante dalla normalizzazione)

Abbiamo:

$$< \sin(kx), \sin(mx) > = \begin{cases} 0 & k \neq m \\ \pi & k = m \end{cases} \quad ; k, m = 1,2, \dots n$$

$$< cos(kx), \cos(mx) > = \begin{cases} 2\pi, & k = m = 0 \\ \pi, & k = m \neq 0 \\ 0, & k \neq m \end{cases} \qquad ; k, m = 0,1,2, \dots n$$

$$< \sin(kx) , \cos(mx) > = 0, \qquad k = 1,2, \dots , n; \quad m = 0,1,2, \dots , n$$

e quindi un sistema ortogonale, dai risultati sulla migliore approssimazione in spazi con prodotti scalari si può ottenere il polinomio trigonometrico «ottimale» definendo

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \, cos(kx) \, dx \, , \qquad k = 0,1,2, \dots , n$$

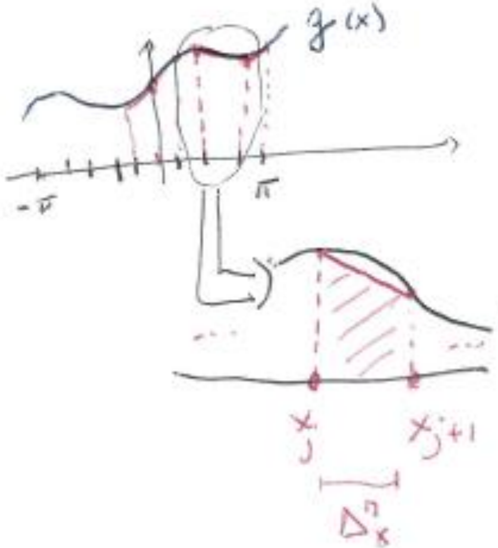
$$b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \, sin(kx) \, dx \, , \qquad k = 1,2, \dots , n$$

Nota Occorre formule di quadratura numerica per il calcolo di a_k, b_k. Per esempio:

$$\Delta_x^M = \frac{2\pi}{M} \quad (M > 1 \text{ intero}) \text{ punti (campionamento)}$$

$$x_j = -\pi + j\,\Delta_x^M \quad j = 0, \dots, M;$$

Formula dei trapezi composita



$$\int_{-\pi}^{\pi} g(x)\,dx \simeq \frac{g(x_0) + g(x_M)}{2}\Delta_x^M + \Delta_x^M \sum_{j=1}^{M-1} g(x_j)$$

Ma se g è 2π-periodica

$g(x_0) = g(x_M) \Rightarrow$

$$\int_{-\pi}^{\pi} g(x)\,dx \simeq \boxed{\Delta_x^M \sum_{j=0}^{M-1} g(x_j)}$$

1)

Calcolo numerico coefficienti a_k e b_k

Serie di Fourier: (f 2π-periodica, $\int_{-\pi}^{\pi} |f|^2(x)\, dx < +\infty$)

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

NOTA! Specificare la convergenza!

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx \quad ; \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx$$

supponiamo di conoscere un sistema ortogonale di vettori $\{\boldsymbol{e}_n\}_{n=1}^{\infty}$

Problema **(Decomposizione ortogonale)** *Dato un elemento* $\boldsymbol{u}$ *di* V *ci chiediamo se è possibile determinare una successione di coefficienti complessi* $\{\hat{u}_n\}_{n\in\mathbb{N}}$ *tali che*

$$\boldsymbol{u} = \sum_{n=1}^{+\infty} \hat{u}_n \boldsymbol{e}_n \quad \textit{cioè} \quad \lim_{N\uparrow+\infty} \left\| \boldsymbol{u} - \sum_{n=1}^{N} \hat{u}_n \boldsymbol{e}_n \right\| = 0.$$

Proposizione *La soluzione del problema, se esiste, è necessariamente data dai coefficienti*

$$\hat{u}_n := \frac{(\boldsymbol{u}, \boldsymbol{e}_n)}{\|\boldsymbol{e}_n\|^2}.$$

Definizione (Coefficienti di Fourier) *I coefficienti* $\hat{u}_n$ *si chiamano* coefficienti di Fourier *di* $\boldsymbol{u}$ *rispetto al sistema ortogonale* $\{\boldsymbol{e}_n\}_{n=1}^{\infty}$.

A questo punto il Problema si riduce ai due seguenti:

1. Trovare condizioni per cui la serie
$$\sum_{n=1}^{+\infty} \hat{u}_n \boldsymbol{e}_n \quad \text{converge in } V;$$

2. Trovare condizioni per cui la somma della serie coincide con $\boldsymbol{u}$.

Teorema *Supponiamo che* $\{\boldsymbol{u}_n\}_{n=1}^{+\infty}$ *sia un insieme di vettori ortogonali. Se* V *è* completo, *allora*

$$\textit{la serie } \sum_{n=1}^{+\infty} \boldsymbol{u}_n \textit{ converge in } V \quad \Leftrightarrow \quad \sum_{n=1}^{+\infty} |\boldsymbol{u}_n|^2 < +\infty.$$

Definizione (Spazi di Hilbert) *Uno spazio funzionale* V *dotato di prodotto scalare e completo si dice* spazio di Hilbert.

Definizione (Sistemi ortogonali completi) *Un sistema ortogonale* $\{\boldsymbol{e}_n\}_{n=1}^{\infty}$ *si dice* completo *in V se ogni elemento* $\boldsymbol{u}$ *di V che è ortogonale a ciascun* $\boldsymbol{e}_n$ *è* trascurabile, *cioè altri termini*

$$(\boldsymbol{u}, \boldsymbol{e}_n) = 0 \quad \forall\, n \in \mathbb{N} \quad \Rightarrow \quad \|\boldsymbol{u}\| = 0.$$

Teorema (Decomposizione e identità di Parseval) *Se V è uno spazio di Hilbert e il sistema ortogonale* $\{\boldsymbol{e}_n\}_{n=1}^{\infty}$ è completo *allora il problema di decomposizione ortogonale si può sempre risolvere. In particolare, per ogni* $\boldsymbol{u} \in V$ *si ha*

$$\boldsymbol{u} = \sum_{n=1}^{+\infty} \hat{u}_n \boldsymbol{e}_n \quad \text{in } V, \quad \|\boldsymbol{u}\|^2 = \sum_{n=1}^{+\infty} |\hat{u}_n|^2 \|\boldsymbol{e}_n\|^2,$$

dove $\hat{u}_n$ *sono i* coefficienti di Fourier *di* $\boldsymbol{u}$

Almeno tre aspetti

V= spazio funzionale opportuno (sui reali o sui complessi) con prodotto scalare e orma indotta

- Migliore approssimazione di f$\in$V nello spazio W finito dimensionale dei polinomi trigonometrici:

$$S_n = \frac{a_0}{2} + \sum_{k=1}^{n} [a_k \cos(kx) + b_k \sin(kx)]$$

- Ridotta della serie (attenzione occorre specificare in che senso ho convergenza!)

$$S_n = \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(kx) + b_k \sin(kx)]$$

- Relazione con trasformata continua

Aspetti analoghi per il caso discreto (numerico)

Punto di vista teoria approssimazione

`Alcune somme ...`

Consideriamo la seguente somma (polinomio trigonometrico) che potrebbe essere anche pensata come somma parziale di una serie, con $M > 1$ e $b_0 = 0$,

$$S(x) = \sum_{k=0}^{M} [a_k \cos(kx) + b_k \sin(kx)]$$

Partendo dall'identità di Eulero (i unità immaginaria) $e^{ix} = \cos(x) + i \sin x$, vogliamo scrivere una somma equivalente

$$S(x) = \sum_{k=-M}^{M} c_k e^{ikx}$$

con $c_k \in \mathbb{C}$ coefficienti opportuni.

Esempio. con `a=[1 2 -2 1]`, `b=[0 4 1 2]`,

$$S(x) = 1 + 2\cos(x) + 4\sin(x) - 2\cos(2x) + \sin(2x) + \cos(3x) + 2\sin(3x).$$

Abbiamo,

$$\sum_{k=-M}^{M} c_k e^{ikx} = \sum_{k=-M}^{-1} c_k e^{ikx} + c_0 + \sum_{k=1}^{M} c_k e^{ikx} =$$
$$= \sum_{k=1}^{M} c_{-k} e^{-ikx} + c_0 + \sum_{k=1}^{M} c_k e^{ikx},$$

quindi

$$\sum_{k=-M}^{M} c_k e^{ikx} = \sum_{k=1}^{M} c_{-k}[\cos(kx) - i\sin(kx)] + c_0 +$$
$$+ \sum_{k=1}^{M} c_k[\cos(kx) + i\sin(kx)] =$$
$$= \sum_{k=1}^{M} (c_{-k} + c_k)\cos(kx) + i(c_k - c_{-k})\sin(kx),$$

da cui

$$c_0 = a_0, \quad c_k + c_{-k} = a_k, \quad i(c_k - c_{-k}) = b_k \; k = 1, 2, \ldots, M;$$
$$\Downarrow$$
$$c_k = \frac{a_k - ib_k}{2}, \; c_{-k} = \frac{a_k + ib_k}{2}.$$

Consideriamo funzioni $f : \mathbb{R} \to \mathbb{C}$ che siano 2π-periodiche (se di periodo $T > 0$, $T \neq 2\pi$ occorre un semplice cambiamento di variabile). Sia $N > 1$ e $x_j = 2\pi j/N$, $j = 0, 1, \ldots, (N-1)$ e definiamo il (pseudo-) prodotto scalare,

$$< f, g >_N = \frac{1}{N} \sum_{j=0}^{N-1} f(x_j) \bar{g}(x_j),$$

dove $\bar{g}$ indica il complesso coniugato di g.

Nota. Se consideriamo le funzioni definite su tutto l'intervallo $[0, 2\pi]$ non abbiamo un prodotto scalare perchè $< f, f >_N = 0 \not\Rightarrow f \equiv 0$, mentre se restringiamo le funzioni alla griglia definita dai punti x_j abbiamo un prodotto scalare: considereremo il caso discreto.

Sia $E_k(x) = exp(ikx)$, $k = 0, 1, ..., N-1$, abbiamo che

$$< E_k, E_k >_N = \frac{1}{N} \sum_{j=0}^{N-1} e^{2\pi jki/N} e^{-2\pi jki/N} = \frac{1}{N} \sum_{j=0}^{N-1} 1 = 1,$$

e per $k \neq p$

$$< E_k, E_p >_N = \frac{1}{N} \sum_{j=0}^{N-1} e^{2\pi jki/N} e^{-2\pi jpi/N} =$$

$$= \frac{1}{N} \sum_{j=0}^{N-1} \left[e^{\frac{2\pi(k-p)i}{N}} \right]^j$$

Nota. Prodotto scalare Complesso se consideriamo funzioni di griglia, altrimenti prodotto pseudo-scalare perché $<f,f>_N = 0$ non implica f nulla ma solo f(xj)=0 per ogni xj.

Posto

$$\lambda = e^{\frac{2\pi(k-p)i}{N}}$$

si ha che $\lambda \neq 1$ e

$$< E_k, E_p >_N = \frac{1}{N} \sum_{j=0}^{N-1} \lambda^j = \frac{1}{N} \frac{\lambda^N - 1}{\lambda - 1}$$

ma $\lambda^N = exp(2\pi(k-p)i) = 1$, quindi $< E_k, E_p >_N = 0$.
Abbiamo quindi un sistema ortonormale

$$\{E_k(x)\}_{k=0}^{N-1}$$

e per ogni funzione di griglia f,

$$f(x) = \sum_{k=0}^{N-1} < f, E_k >_N \; E_k(x)$$

Nota. Se scegliessi un sottoinsieme delle funzioni esponenziali E_k potrei generare un sottospazio lineare W e ottenere analoga approssimazione nel senso dei minimi quadrati.

$\{f_n\}$ spazio fisico

$\{c_k\}$ spazio frequenze

Algoritmo ingenuo: costo $O(N^2)$

Algoritmo rapido FFT: costo $O(N \log N)$
Fast **F**ourier **T**ransform

Fast Fourier transform: brief history

Gauss (1805, 1866). Analyzed periodic motion of asteroid Ceres.

Runge–König (1924). Laid theoretical groundwork.

Danielson–Lanczos (1942). Efficient algorithm, x-ray crystallography.

Cooley–Tukey (1965). Detect nuclear tests in Soviet Union and track submarines. Rediscovered and popularized FFT.

An Algorithm for the Machine Calculation of Complex Fourier Series

By James W. Cooley and John W. Tukey

An efficient method for the calculation of the interactions of a 2^m factorial experiment was introduced by Yates and is widely known by his name. The generalization to 3^m was given by Box et al. [1]. Good [2] generalized these methods and gave elegant algorithms for which one class of applications is the calculation of Fourier series. In their full generality, Good's methods are applicable to certain problems in which one must multiply an N-vector by an $N \times N$ matrix which can be factored into m sparse matrices, where m is proportional to $\log N$. This results in a procedure requiring a number of operations proportional to $N \log N$ rather than N^2.

Importance not fully realized until emergence of digital computers.

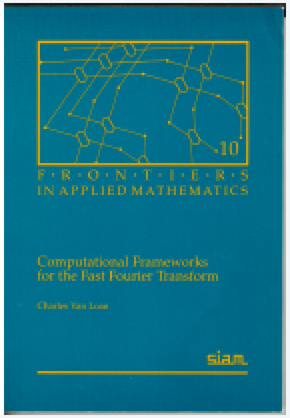
Adattato da slide di: Algorithm design / Jon Kleinberg, Ěva Tardos

Fast Fourier transform: applications

Applications.

- Optics, acoustics, quantum physics, telecommunications, radar, control systems, signal processing, speech recognition, data compression, image processing, seismology, mass spectrometry, …
- Digital media. [DVD, JPEG, MP3, H.264]
- Medical diagnostics. [MRI, CT, PET scans, ultrasound]
- Numerical solutions to Poisson's equation.
- Integer and polynomial multiplication.
- Shor's quantum factoring algorithm.
- …

> *" The FFT is one of the truly great computational developments of [the 20th] century. It has changed the face of science and engineering so much that it is not an exaggeration to say that life as we know it would be very different without the FFT. "*
>
> *— Charles van Loan*



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