A constructive proof of the minimal entropy theorem

Master's Thesis

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August 2021

ETH zürich

A thesis supervised by
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Contents

0 Abstract

This Master's Thesis aims to prove the minimal entropy theorem established by Gerard Besson, Gilles Courtois and Sylvester Gallot (BCG) in the year 1995 using the methods established in the proof of the weaker version of the theorem. It does so by using Gromov hyperbolicity to construct the border of the universal covers, on which a measure is defined. Using this measure, the natural map can be derived. The desired properties are revealed by studying the Jacobian of the natural map.

1 Introduction

1.1 Prologue

In the year 1995, Gerard Besson, Gilles Courtois and Sylvester Gallot (BCG) proved an inequality for the volume entropy of locally symmetric spaces of negative curvature, which in turn provided a new, simpler proof of the rigidity theorem of George Mostow (1968), which states that compact hyperbolic manifolds in more than two dimensions are determined by their fundamental group except for isometry.

Their work was contained in a paper [BCG95] published in that same year in which they linked the topological quantities of entropy and volume to the geometric notion of a metric. The two main statements proved in [BCG95] are the ones reflected in Subsection 1.2 below.

A year later [BCG96] they gave a more constructive proof of a slightly weaker statement in which they added a curvature assumption to M.

The constructive proof of the statement below, Theorem 1.2, was lectured by Professor Urs Lang in a Graduate Course at ETH Zurich in the fall of 2020. [Lan21].

1.2 The statements

Theorem 1.1. [BCG 1995] [BCG95] First off, let $n \geq 3$. Next, let N be a compact, connected, locally symmetric, n-dimensional, Riemannian manifold with negative curvature, and let M be *any* compact, connected, n-dimensional Riemannian manifold. Additionally, assume that there exists a continuous map $f : M \to N$ nonzero degree. We then have

(1) $h^n(M)\mathrm{vol}(M) \geq h^n(N)\mathrm{vol}(N)$.

(2) The equality case occurs if and only if f is homotopic to a Riemannian covering, that is, the manifolds are locally isometric.

Theorem 1.2. Again, let $n \geq 3$. Next, let N be a compact, connected, locally symmetric, n-dimensional, Riemannian manifold with negative curvature, and let M be any compact, connected, n-dimensional Riemannian manifold. Assume that there exists a homotopy equivalence between M and N which we denote $f : M \to N$. We then have

(1) $h^n(M)\mathrm{vol}(M) \geq h^n(N)\mathrm{vol}(N)$.

(2) If $h(M) = h(N)$ and $\mathrm{vol}(M) = \mathrm{vol}(N)$, then the manifolds are isometric.

This corresponds to the statement in [BCG96] up to the fact that there it is assumed that M has negative curvature.

1.3 Roadmap and strategy

This thesis wishes to further develop the proof in [BCG96] for it to prove the more general Theorem 1.1 of the first paper mentioned, [BCG95].

In Section 2 we give an introduction to Gromov hyperbolic spaces consisting of the various equivalent definitions and the stability theorem.

Section 3 addresses the Patterson-Sullivan method on Gromov hyperbolic spaces as described in [Coo93]. Thereafter, Section 4 deals with the conceptual reversal of the Patterson-Sullivan method, the barycenter method.

Subsequently, we construct the so-called natural map using the two preceding sections. The natural map will be the key element in the proof of the main statement and we set out to study its properties in Section 6. Finally, in Section 7 we prove the main statement.

This thesis contains an Appendix A related to the equivalent definitions of Gromov hyperbolic spaces; that appendix is intended as supplemental material underpinning and extending the content of Section 2.

1.4 Acknowledgements

For many students, a master's thesis is their first piece of work of that size. That can be intimidating. As a matter of fact, that *is* intimidating.

However, it is not nearly as intimidating as it is exciting. Let us hope that the fun persists and the end product is readable as well as enjoyable.

I would like to thank Prof. Lang for his time and assistance during the process of writing this thesis – he managed not only to guide me through the material content, but also to stoke my genuine interest in Riemannian geometry beyond the scope of this Master's Thesis.

2 Gromov hyperbolic spaces

2.1 Preliminary definitions

Definition 2.1. A **path** in a metric space (X, d) is a continuous map: $\sigma : I \to X$ where $I = [a, b] \subseteq \mathbb{R}$ is a closed interval. We call $\sigma(a)$ the **starting point** and $\sigma(b)$ the **endpoint** of the path.

Definition 2.2. A path from x to y is said to be a **geodesic** if it is of length $d(x, y)$ and of constant speed. Consequently, a metric space is said to be a **geodesic metric space** if for every distinct pair of points $x \neq y$ there exists a geodesic from x to y.

We will denote open balls of radius r by $B(\cdot, r)$. Not only a point x can be the center of a ball, but also sets can be the center of a ball. In that case, we mean all points with distance less than r from the set. We sometimes call this the **r-neighbourhood** of a set.

Definition 2.3. The **Hausdorff distance** between two subsets $C, D \subset X$ of a metric space is defined as follows:

$$d_H(C, D) := \inf_{r>0} \{C \subset B(D, r), D \subset B(C, r)\}.$$

The Hausdorff distance can be thought of as the smallest number, so that each set is contained in the neighbourhood of said size of the other set. We can thus also describe the Hausdorff distance as follows:

$$\max \big\{ \sup_{x \in C} \{d(x, D)\}, \sup_{y \in D} \{d(C, y)\} \big\}.$$

2.2 Gromov hyperbolic spaces

Definition 2.4. Let (X, d) be a metric space and points $x, y, p \in X$. The **Gromov product with respect to** p is defined as

$$(x, y)_p := \frac{1}{2}\Big(d(x, p) + d(y, p) - d(x, y)\Big).$$

We call the point p the base point of the Gromov product and sometimes even the base point of the whole space X.

Definition 2.5. A metric space is said to be **δ-hyperbolic** for a $\delta > 0$, or just **hyperbolic**, if the Gromov product satisfies the following inequality for all $x, y, p \in X$:

$$(x, y)_p \geq \min \big\{(x, z)_p, (y, z)_p\big\} - \delta.$$

Another form of this property is as follows:

$$d(x,z) + d(y,p) \leq \max\{d(x,y) + d(z,p), d(x,p) + d(y,z)\} + 2\delta.$$

There are multiple, more geometric definitions of δ-hyperbolicity that are all equivalent. We discuss the equivalences in Appendix A. A more general approach can be found in [V̈05].

2.3 Maps in Gromov hyperbolic spaces

Lemma 2.6. We can limit the Gromov product of a pair of points by the distance that a geodesic segment has to the base point. Note that if the base point lies on the geodesic segment, their Gromov product is zero.

- For any geodesic metric space X, pick a base point p and let τ be a geodesic segment connecting the points x and y. Then we have $(x,y)_p \leq d(p, im(\tau))$.
- Let $x, y \in X$ be distinct points in a δ-hyperbolic space, and let p and τ be as before. Then, we have $(x,y)_p \leq d(p, im(\tau)) \leq (x,y)_p + 4\delta$.

Proof. For the first statement, pick $t \in im(\tau)$ so that $d(p,t) = d(p, im(\tau))$. Then, by the triangle inequality, we have $d(p,t) \geq d(p,x) - d(x,t)$ and $d(p,t) \geq d(p,y) - d(y,t)$ and thus, taking the average, we have

$$d(p,t) \geq \frac{1}{2}(d(p,y) + d(p,x) - d(x,t) - d(t,y)) = (x,y)_p,$$

which gives us $d(p, im(\tau)) = (x,y)_p$.

For the second statement, we only need to show the inequality on the right, as the inequality on the left has already been shown. Let τ_1, τ_2, τ_3 be segments connecting x, y and p, where τ_1 connects p and x, $\tau_2 = \tau$ and τ_3 connects p and y. Let c_1 be the point of distance $(x,y)_p$ from p on τ_1, c_3 the point of distance $(x,y)_p$ from p on τ_3 and lastly let c_2 be the point of distance $(y,p)_x$ from x on τ. Then we have

$$d(p, c_3) \leq d(p, c_1) + d(c_1, c_2). \tag{2.1}$$

By the definition of c_1, we have $d(p, c_1) = (x,y)_p$ and by slimness of geodesic triangles (see Appendix A), we have $d(c_1, c_3) \leq 4\delta$. So substituting into Equation (2.1) yields the statement since we establish

$$d(p, im(\tau)) \leq d(p, c_3) \leq (x,y)_p + 4\delta.$$

□

Definition 2.7. A map $f : (X, d) \to (Y, d)$ that a priori need not be continuous is said to be a **quasi-isometric embedding** if there exist constants $C \geq 0$ and $\lambda \geq 1$, so that for all x_1 and x_2 in X we have

$$\lambda^{-1} d(x_1, x_2) - C \leq d(f(x_1), f(x_2)) \leq \lambda d(x_1, x_2) + C.$$

If $f : (X, d) \to (Y, d)$ is a quasi-isometric embedding and moreover there exists a constant $M > 0$ so that for all $y \in Y$ there exists an $x \in X$ satisfying $d(y, f(x)) \leq M$, we call f a (λ, C)**-quasi-isometry**.

In the same way that a geodesic can be thought of as an isometric image of a bounded interval, the quasi-isometric image of a bounded interval is called a **quasi-geodesic**.

An important property of geodesic hyperbolic spaces is that quasi-geodesics lie at a finite distance to geodesics. In order to prove this so-called stability property, we will require some lemmas. We lean on [Lan19], section 5.3. for the rest of this subsection.

Lemma 2.8. Let X be a geodesic metric space, $\tau : [a, b] \to X$ a (λ, c)-quasi-geodesic with $\lambda \geq 1$ and $b - a \geq c/\lambda$. Then, there exists a $(\lambda, 4c)$-quasi-geodesic $\pi : [a, b] \to X$ with the same start and end points, which is 2λ-Lipschitz and lies $4c$-close at every point, that is $d(\tau(s), \pi(s)) < 4c$ for all $s \in (a, b)$.

Proof. Choose a subdivision $a = t_0 < t_1 < ... < t_k = b$ of equal length $l = |t_i - t_{i-1}| \in [c/\lambda, 2c/\lambda]$. Define π_j to be a geodesic segment from $\tau(t_{j-1})$ to $\tau(t_j)$ and π the concatenation of all the π_j's. We then have

$$d(\pi(t_{i-1}), \pi(t_i)) \leq \lambda l + c \leq \min\{2\lambda l, 3c\}.$$

Thus, π is $\min\{2\lambda l, 3c\}$-Lipschitz. For $s, s' \in [a, b]$ we can choose $t, t' \in \{t_0, ..., t_k\}$, so that on one hand $|t - t'| \geq |s - s'|$ and on the other hand $|s - t| + |s' - t'| \leq l$. This yields $d(\pi(s), \pi(t)) + d(\pi(s'), \pi(t')) \leq 3c$, which means

$$\begin{aligned} d(\pi(s), \pi(s')) &\geq d(\pi(t), \pi(t')) - 3c \\ &\geq \lambda^{-1}|t - t'| - 4c \\ &\geq \lambda^{-1}|s - s'| - 4c. \end{aligned}$$

Now, let $s \in [a, b]$ and pick a closest $t \in \{t_0, ..., t_k\}$. We then get $d(\tau(t), \tau(s)) \leq \lambda C/2 \leq 2c$ and $d(\pi(s), \pi(t)) \leq 3C/2$. Finally, this gives

$$d(\tau(s), \pi(s)) \leq 7c/2 < 4c.$$

$\square$

Lemma 2.9. Let now X be a hyperbolic geodesic space. Then, there exist an $s > 0$ and map $u : [0, \infty) \to [0, \infty)$ that is unbounded and non-decreasing and satisfies the following property: For each segment π connecting $x' \in X$ and $y' \in X$ with length at least s and points $x, y \in X$ satisfying $d(x, x') \leq d(x, z')$ as well as $d(y, y') \leq d(y, z')$ for all $z' \in [x', y']$, we have

$$d(x, y) \geq u\left(d(x, x') + d(y, y')\right).$$

One can think of this as follows: If there are two points x and y which, informally speaking, lie closer to the ends than to any other point on a segment connecting the points x' and y', then we can bound their distance from below.

Proof. Pick a $\delta > 0$, so that all geodesic triangles are δ-slim as defined in Appendix A. Pick a number $s > 6\delta$ and let π be the geodesic segment connecting x' and y' with $L(\pi) > s$ as in the statement. Additionally, also pick x and y as required in the statement.

Next, pick a point $z' \in im(\pi)$ satisfying $d(x', z') > 2\delta$ and $d(y', z') > 4\delta$. By the assumptions made in the statement, every point on a segment between x and x' is of distance greater than δ from z', and every point on a segment between y and y' is of distance greater than 2δ from z'.

Thus, there exists a point w on the segment between x and y' so that $d(z', w) \leq \delta$, and likewise there exists a point z on the segment between x and y such that $d(w, z) < \delta$.

From the above it follows that $d(x, x') \leq d(x, z) + 2\delta$ and $d(y, y') \leq d(y, z) + 2\delta$. So putting it all together, we have

$$d(x, y) = d(x, z) + d(y, z) \geq d(x, x') + d(y, y') - 4\delta$$

and we are done. □

Theorem 2.10. Stability theorem. [Lan19] For all $\delta > 0$, $\lambda \geq 1$ and $c \geq 0$ there exists a constant $D > 0$, which depends on δ, λ, and c, so that the following holds: Let X be a δ-hyperbolic geodesic metric space and $\tau : [a, b] \to X$ a (λ, c)-quasi-geodesic and let σ be a unit speed geodesic connecting the starting and ending points of τ. Then we have

$$d_H(im(\tau), im(\sigma)) < D.$$

Proof. We can assume by Lemma 2.8 that τ is λ-Lipschitz. If not, we can find a suitable $(\lambda, 4c)$-quasi-geodesic that we also denote by τ.

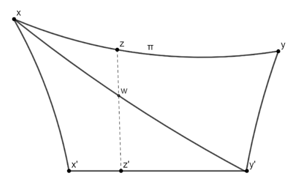

Figure 1: A visualisation of Lemma 2.9.

Suppose the function $f : [a, b] \to \mathbb{R}$ defined by $f(t) := d(\tau(t), \sigma)$ is bounded by D'. Additionally, suppose that any $w \in im(\sigma)$ satisfies $d(w, im(\tau) \leq 2D'$. Then, by Lemma 2.9, we would be done.

Choose a $h > 0$, so that $u(2h) > 2\lambda^2 s$ and recall that the numbers s and h depend only on δ. Suppose now that there exists a subinterval $[t_0, t] \subset [a, b]$, so that $f(t_0) = h = f(t)$ and $f(t') \geq h$ for all $t' \in [t_0, t]$ and $l := L(\tau|_{[t_0,t]}) > 0$. By the Lipschitzness of τ we have

$$l \leq \lambda|t - t_0| \leq \lambda^2 \left(d(\tau(t_0), \tau(t)) + c\right).$$

Now, choose a subdivision $t_0 \leq t_1 \leq ... \leq t_k = t$ so that $L(\tau|_{[t_{i-1},t_i]}) \leq 2\lambda^2 s$ for $i = 1, ..., k$. Then, there exists a subsegment of σ from x_0 to x_k, so that we have $d(\tau(t_0), x_0) = h = d(\tau(t_k), x_k)$. For $k > 1$ we pick points $x_1, ..., x_{k-1}$, so that each point x_i lies on the segment connecting x_{i-1} and x_k and is the closest point to $\tau(t_i)$.

Since we have $d(\tau(t_{i-1}), \tau(t_i)) \leq 2\lambda^2 s < u(2h)$, by Lemma 2.9, we have $d(x_{i-1}, x_i) < s$ and thus, $l \leq \lambda^2(ks + 2h + c)$.

By minimality of k, we obtain $l \geq 2(k - 1)\lambda^2 s$. We can write

$$2(k - 1)\lambda^2 s \leq l \leq \lambda^2(ks + 2h + c).$$

Eliminating the k gives $l \leq 2\lambda^2(s+2h+c) =: C$. Setting $D' = h+C/2$ yields $f(t) \leq D'$ for all $t \in [a, b]$ as desired.

There are points x and y in the subsegments connecting $\tau(a)$ and w as well as in the subsegments connecting w and $\tau(b)$ that satisfy $d(\tau(t), x) = d(\tau(t), y) = f(t) \leq D'$. Because of $d(x, y) \leq 2D'$ we have $d(w, \tau(t)) \leq \min\{d(w, x), d(w, y)\} + D' \leq 2D'$.

$\square$

Theorem 2.11. Let $f : X \to Y$ be a (λ, c)-quasi-isometric embedding between two geodesic metric spaces. If Y is δ-hyperbolic, then X is δ'-hyperbolic as well.

Proof. Let I be a bounded interval in $\mathbb{R}$ and $g : I \to X$ a geodesic segment. Then the map $f \circ g$ is a (λ, c)-quasi-geodesic. We denote the geodesic segments connecting the start point and end point of $F \circ g$ by h. Next, applying Theorem 2.10, we can bound the Hausdorff distance between the two paths by a constant H: $d_H(F \circ g, h) < H$.

Let now Δ be a geodesic triangle in X with sides g_i, $i = 1, 2, 3$. Similarly, let h_i be a geodesic triangle in Y. By δ-slimness, as reflected in Appendix A, we have

$$d\Big(y, \big(im(h_1) \cup im(h_2)\big)\Big) \leq 4\delta$$
$$d\Big(y', \big(im(F \circ g_1) \cup im(F \circ g_2)\big)\Big) \leq 4\delta + 2H$$

for all $y \in im(h_3)$ and all $y' \in im(F \circ g_3)$.

Thus, for all $x \in im(g_3)$ we have $d\big(x, \big(im(g_1) \cup im(g_2)\big)\big) \leq \lambda(4\delta + 2H + c) = \delta'$. As all triangles are δ'-slim, the space is hyperbolic, as it is described in Appendix A. $\square$

2.4 The boundary of a hyperbolic space

The boundary of a Gromov hyperbolic space is a way to conceptualize the infinite. For the use-case, we will restrict ourselves to the case in which X is proper, geodesic and Gromov hyperbolic with hyperbolicity constant δ as defined in Definition 2.5. This allows us to define the Gromov product on the boundary in a more elegant, but nonetheless equivalent way.

Definition 2.12. A sequence of points $(x_i)_i$ in a metric space X is said to **converge to infinity** if the Gromov product diverges, that is, $\lim_{i,j\to\infty}(x_i, x_j)_p = \infty$. Converging to infinity is a property that is independent of basepoints as we have $|(x, x')_p - (x, x')_{p'}| \leq d(p, p')$, meaning that a change of basepoint only affects the limit by a constant.

Definition 2.13. Two sequences in X, $(x_i)_i$ and $(z_j)_j$, which both converge to infinity, are said to be **equivalent** if they satisfy

$$\lim_{i\to\infty}(x_i, z_i)_p = \infty.$$

The set of equivalence classes is called the **boundary at infinity** of X and is denoted ∂X.

In order to check that this is indeed an equivalence relation, we denote that $(x_i, x_i)_p \to \infty$ as well as $(x_i, z_j)p \to \infty$ is satisfied if and only if $(z_j, x_i)_p \to \infty$. Lastly, suppose that $(x_i, z_j)_p \to \infty$ and $(z_j, w_k)_p \to \infty$, then we have

$$\begin{aligned}(x_i, w_k)_p &= \frac{1}{2}\big(d(x_i, p) + d(w_k, p) - d(x_i, w_k)\big) \\ &\geq \frac{1}{2}\big(d(x_i, p) + d(w_k, p) - d(x_i, z_j) - d(z_j, w_k)\big)\end{aligned}$$

and as $d(x_i, z_j)$ and $d(z_j, w_k)$ are finite, so is $d(x_i, w_k)$, and thus the whole term diverges.

As any sequence has a limit point now, the space $X \cup \partial X$ is compact.

Definition 2.14. We call a geodesic with domain $[0, \infty) \subset \mathbb{R}$ a **ray**. Note that some literature denotes the image instead of the map of the geodesic as a ray.

In a proper, Gromov hyperbolic metric space, an a priori different notion of the boundary is given by geodesic rays.

Definition 2.15. Let there be any two rays σ and σ' that perhaps have different starting points. We call the rays **equivalent** if there exists a $C > 0$, so that we have $d(\sigma(t), \sigma'(t)) \leq C$ for all $t \geq 0$. The set of equivalence classes of rays is then called the **geodesic boundary**, $\partial^g X$.

Suppose two rays ρ and ρ' start at the same point p. Additionally, suppose that for a given constant $C > 0$ there exist arbitrarily large t, so that $d(\rho(t), \rho'(t)) \leq C$. This means that the geodesic triangle given by $[p, \rho(t), \rho'(t)]$ is 4δ-slim, as reflected in Appendix A. This holds for all $t > 0$. Thus, geodesic rays starting from the same points and, due to $d(\rho(t), \rho'(t)) \leq C$, belonging to the same boundary point, are at most 4δ apart.

Lemma 2.16. Triangles with corners on $X \cup \partial X$ are also slim.

As soon as we have established a suitable metric on the boundary, we will be able to state that the space $X \cup \partial X$ is also Gromov hyperbolic. However, for now, that is not true.

Proof. Suppose that $a, b, c \in X \cup \partial^g X$; we want to show that the geodesic triangle $[a, b, c]$ is slim. If all of a, b, c lie in X, the triangle is 4δ slim as discussed in Appendix A.

Suppose now that $c \in \partial^g X$ and $a, b \in X$. Let us denote a geodesic ray starting at b and belonging to the boundary point c by ρ_b. We then define the geodesic segment $\sigma_t : [0, l] \to X$ as the segment starting at a and ending at $\rho_b(t)$. Then, the geodesic triangle $[a, b, \rho_b(t)]$ is 4δ slim. Using compactness of $X \cup \partial X$ and applying the Arzelà-Ascoli theorem, there is a subsequence of $(\sigma_t)_t$ so that its limit σ forms a 4δ-slim geodesic triangle $[a, b, c]$. Since any geodesic starting at a and belonging to the boundary point c lies at a finite distance from σ, we can state that any geodesic triangle $[a, b, c]$ is 8δ-slim. If two, or even three, of the points a, b, c lie on the boundary, we proceed the same way. $\square$

Proposition 2.17. If X is a proper, geodesic, δ-hyperbolic space, then $\partial^g X = \partial X$.

Proof. For an element of the geodesic boundary ∂^g, choose a ray $\sigma : [0, \infty) \to X$. We define the boundary point $[\sigma(n)_{n\in\mathbb{N}}] \in \partial X$. We now want to show that this is an injective and surjective correspondence between the two boundaries. Suppose that $\sigma \sim \rho$ under the equivalence of $\partial^g X$, then we study

$$\begin{aligned}(\sigma(n), \rho(n))_p &= \frac{1}{2}\big(d(\sigma(n), p) + d(p, \rho(n)) - d(\sigma(n), \rho(n))\big)\\ &= n - \frac{d(\sigma(n), \rho(n))}{2} \geq n - 8\delta/2 \to \infty.\end{aligned}$$

The last inequality is due to Lemma 2.16. Thus, the sequences $\sigma(n)$ and $\rho(n)$ are equivalent and therefore they are the same point in the (non-geodesic) boundary.

For surjectivity, it suffices to find a ray for a representative of a boundary point. Let $(x_i)_i$ be in the equivalence class of $\xi \in \partial X$. Then, we define a ray $\sigma_n : [0, n] \to X$ as a geodesic segment connecting x_0 and x_n. By properness, a subsequence of $(\sigma_n)_n$ converges to a ray $\sigma : [0, \infty] \to X$. We then have $(\sigma(n))_n \sim (x_i)_i$ which establishes surjectivity. $\square$

From now on, we will simply refer to the boundary of a Gromov hyperbolic space by ∂X because for our use-case the space X is complete and geodesic, and the two definitions of the boundary agree.

Lemma 2.18. [Gro87](Prop 2.1) Let X be a proper, geodesic, δ-hyperbolic space and $\eta \neq \xi \in \partial X$. Then, there exists a geodesic $c : \mathbb{R} \to X$, so that $c(t) \to \eta$ for $t \to -\infty$ as well as $c(t) \to \xi$ for $t \to \infty$.

Also, if $x \in X$ and $\xi \in \partial X$, then there exists a ray c with $c(0) = x$ and $c(t) \to \eta$ for $t \to \infty$.

Proof. We begin with the second statement. Let z_i be a sequence in X so that $z_i \to \eta$, and define c_i to be a unit speed geodesic segment connecting p and z_i. By the Arzelà-Ascoli theorem, the sequence $(c_i)_i$ has a converging subsequence to a map c that is a geodesic $c : [0, \infty) \to X$ and $c(t) \to \xi$ for $t \to \infty$.

For the first statement, pick a basepoint $p \in X$ and use the second statement to define c_1 and c_2 as follows:

$$\begin{aligned} c_1(0) &= p & \qquad c_1(t) &\to \eta,\ t \to \infty \\ c_2(0) &= p & \qquad c_2(t) &\to \xi,\ t \to \infty. \end{aligned}$$

Next, we define $a_i = c_1(t_i)$ and $b_i = c_2(t_i)$. As we have $\eta \neq \xi$, we have $\liminf_{i\to\infty}(a_i, b_i)_p < \infty$, so, up to subsequences, the sequence $(a_i, b_i)_p$ converges. Set now $l_i := \frac{1}{2}d(a_i, b_i)$ and $s_i : [-l_i, l_i] \to X$ the geodesic segment connecting a_i and b_i.

As X is δ-hyperbolic, all geodesic triangles are 4δ-slim, so in particular, the triangle connecting the points p, a_i and b_i is as well. The midpoint of the segment s_i, denoted by $s_i(0)$, satisfies $d(p, s_i(o)) \leq (a_i, b_i)_p + 4\delta$. By local compactness again, up to a subsequence, we have the limit $s_i(0) \to p$. Lastly, by the Arzelà-Ascoli theorem again, and up to a subsequence, $s_i \to c : \mathbb{R} \to X$ with the claimed properties. $\square$

2.5 Busemann functions

Definition 2.19. To every ray σ satisfying $\sigma(0) = p$, we assign the **Busemann function** $B_p(\sigma, x)$ **rooted at p**, which is defined as follows:

$$B_p(\sigma, x) := \lim_{t\to\infty} \big(d(x, \sigma(t)) - t\big) \in \mathbb{R}.$$

Even though the point p itself does not directly appear in the definition, it helps to clarify situations when the starting points of rays are of importance. In particular we have $B_p(\sigma, p) = 0$.

Lemma 2.20. The Busemann function of a ray ρ is 1-Lipschitz.

Proof. We note first that for all $s \geq 0$ we have $B_p(\rho, \rho(s)) = -s$. By the triangle inequality, we also have $|d(x, \rho(t) - d(y, \rho(t))| \leq d(x, y)$, so for all $t \geq 0$ we get

$$|B_p(\rho, x) - B_p(\rho, y)| \leq d(x, y).$$

This means that the Busemann function is 1-Lipschitz. $\square$

Lemma 2.21. Let X be a proper, geodesic, δ-hyperbolic space. Let ρ be a ray starting at $p \in M$, and let $q \in M$. Then, there exists a ray σ so that $\sigma(0) = q$ and $B_p(\rho, \sigma(s)) = B_p(\rho, q) - s$. Additionally, the Busemann functions satisfy $B_q(\sigma, x) \geq B_p(\rho, x) - B_p(\rho, q)$ for all $x \in M$.

Proof. Let $t \geq 0$ and define $\sigma_t : [0, d(q, \rho(t)) \to X$ to be a unit speed geodesic from q to $\rho(t)$. Fix an $s \geq 0$ and by 1-Lipschitzness we have

$$B_p(\rho, q) - s \leq B_p(\rho, \sigma_t(s)) \leq B_p(\rho, \rho(t)) + d(\sigma_t(s), \rho(t) = -t + d(q, \rho(t)) - s.$$

For $t \to \infty$, the term $-t + d(q, \rho(t)) - s$ tends to $B_p(\rho, q) - s$ for $t \to \infty$ and thus, we have $B_p(\rho, \sigma_t(s)) \to B_p(\rho, q) - s$. By properness, there exists a sequence $t_i \to \infty$, so that the geodesics σ_{t_i} converge to a ray σ with the desired properties.

For the last statement, pick a point $x \in X$. Then, we have

$$d(x, \sigma(s)) \geq B_p(\rho, x) - B_p(\rho, \sigma(s)) - s = B_p(\rho, x) - B_p(\rho, q).$$

So for all $s \geq 0$, the Busemann function of σ satisfies $B_q(\sigma, x) \geq B_p(\rho, x) - B_p(\rho, q)$ as claimed. We can think of this ray σ as the curve of steepest descent for the function $B_p(\rho, \cdot)$. □

2.6 The Gromov product on the boundary

As geodesic rays with a fixed origin and belonging to a boundary element need not be unique, defining the Gromov product on the boundary of a Gromov hyperbolic space requires some work.

Definition 2.22. Let X be a δ-hyperbolic space and pick a base point $p \in X$. Let $\xi, \eta \in \partial X$. Next, pick two rays σ, ρ belonging to the boundary elements ξ and η and set

$$(\xi, \eta)_p = \inf \liminf_{t \to \infty} (\sigma(t), \rho(t))_p,$$

where we take the infimum over all suitable rays. We can define the Gromov product on $X \cup \partial X$ in the same way, notably

$$(x, \xi)_p := \inf \liminf_{t \to \infty} (x, \sigma(t))_p,$$

where the infimum is taken over geodesic rays σ belonging to the boundary element ξ.

Lemma 2.23. Let X be a Gromov hyperbolic space and suppose that we have a fixed base point $p \in X$ as well as boundary elements $\xi, \eta, \zeta \in \partial X$ and rays σ, ρ starting at the base point and belonging to ξ and η. We then have

1. $(\xi, \eta)_p \leq \liminf_{t \to \infty} (\sigma(t), \rho(t))_P \leq \limsup_{t \to \infty} (\sigma(t), \rho(t))_p \leq (\xi, \eta)_p + 2\delta$.
2. $(\xi, \eta)_p \geq \min\{(\xi, \zeta)_p, (\zeta, \eta)_p\} - \delta$.

Proof. We refer to [BS07], chapter 2.2, for a precise proof of the lemma. □

Definition 2.24. For a set Z, a function $m : Z \times Z \to Z$ satisfying the following properties is called a **quasi-metric**:

1. $m(z_1, z_2) \geq 0$ and $m(z_1, z_2) = 0$ if and only if $z_1 = z_2$.
2. $m(z_1, z_2) = m(z_2, z_1)$ for all $z_i \in Z$.
3. $m(z_1, z_3) \leq K \max\{m(z_1, z_2), m(z_2, z_3)\}$ for all $z_i \in Z$ and a fixed number $K \geq 1$.

We note that for a given (normal) metric d, the p-th power of the metric d^p is not always a metric, however it is always a 2^p-quasi-metric for $p > 1$. In particular, d is always a 2-quasi-metric.

Lemma 2.25. The map $m : \partial X \times \partial X \to \partial X$ given by $m(\xi, \eta) = e^{-(\xi,\eta)_p}$ is a K-quasi-metric with $K = e^{\delta}$.

Proof. The three properties required as stated in Definition 2.24 can be easily checked. □

The so-called chain construction can turn a quasi-metric into a metric: Suppose that we have a quasi-metric m on a set Z. We then define $d : Z \times Z \to Z$ by

$$d(z, z') := \inf \sum_i m(z_i, m_{i+1}),$$

where we take the infimum over all finite sequences z_i with $z_0 = z$ and $z_{k+1} = z'$.

In the case that we have a quasi-metric with constant $K \leq 2$, this procedure yields a metric d. We refer to chapter 2.2 of [BS07] for a detailed proof. The assumption on K is necessary, as otherwise there might be elements $z \neq z'$ with $d(z, z') = 0$.

Applying this construction, we can obtain a metric d on the boundary ∂X of a proper, geodesic δ-hyperbolic space X which is visual in the sense that there exist constants c_1 and c_2 so that we have

$$c_1 e^{-(\xi,\eta)_p} \leq d(\xi, \eta) \leq c_2 e^{-(\xi,\eta)_p} \tag{2.2}$$

for all distinct boundary elements ξ and η. The metric also satisfies other properties, namely it is bi-Lipschitz equivalent with respect to a change of base point. We again refer to [BS07], chapter 2.2, for more detailed explanations.

The metric allows us to define a topology on the boundary that will make it possible to extend homeomorphisms on X to homeomorphisms on $X \cup \partial X$.

Theorem 2.26. [CDP90] Let X and Y be two hyperbolic geodesic metric spaces and suppose that $f : X \to Y$ is a (λ, c)-quasi-isometry, which maps a ray ρ in X to a quasi-geodesic ray $f \circ \rho$ in Y. Then, there exists an extension $f : X \cup \partial Y \to Y \cup \partial Y$, which we will also denote by f.

Proof. Let p be a basepoint in X and $q := f(p)$ be a basepoint in Y. Then the Gromov products of points on the boundary diverge for $s, t \to \infty$

$$\big(\rho(s), \rho(t)\big)_p \to \infty.$$

We have $d(p, \rho|_{[s,t]}) \to \infty$, again for $s, t \to \infty$ and, as f is a quasi-isometry, hence we also have

$$d(q, f \circ \rho|_{[s,t]}) \to \infty.$$

Let us denote the geodesic segment connecting $f(\rho(s))$ and $f(\rho(t))$ by $\pi_{s,t}$. Then, by Theorem 2.10, the stability statement, we have

$$|d(q, f \circ \rho|_{[s,t]}) - d(q, \pi_{s,t})| \leq C.$$

As the first term of the left-hand side equation diverges but the total is bounded, the second term, that is $d(q, \pi_{s,t})$, must also diverge for $s, t \to \infty$. This in turn implies $(f \circ \rho(s), f \circ \rho(t))_q \to \infty$. Thus, the path $f \circ \rho$ tends to a point on the boundary ∂Y. As it lies finitely close to π, it tends to the same boundary point as the geodesic ray π tends to, which defines a map on the boundary as desired. $\square$

In particular, this also applies to isometries γ of suitable spaces. Isometries $X \to X$ extend to homeomorphisms $X \cup \partial X \to X \cup \partial X$.

Lemma 2.27. Let X be a proper, Gromov hyperbolic metric space with boundary ∂X. Next, let ρ and ρ' both be rays, with potentially different starting points p, p', belonging to the same boundary element $\eta \in \partial X$. Then, there exists a constant C depending solely on the hyperbolicity factor δ, so that for all $x_1, x_2 \in X$ we have

$$|\,\big(B_p(\rho, x_1) - B_p(\rho, x_2)\big) - \big(B_{p'}(\rho', x_1) - B_{p'}(\rho', x_2)\big)\,| \leq C.$$

Proof. This proof relies on the contents of page 209 of [Gro87]. We compute as follows:

$$\begin{aligned}&(B_p(\rho, x_1) - B_p(\rho, x_2) - B_{p'}(\rho', x_1) + B_{p'}(\rho', x_2))\\ &= \lim_{t\to\infty} (d(x_1, \rho(t)) - t) - (d(x_2, \rho(t)) - t) - (d(x_1, \rho'(t) - t) + (d(x_2, \rho'(t)) - t).\end{aligned}$$

By crossing out the t, we obtain

$$
\begin{aligned}
&= \lim_{t\to\infty} (d(x_1,\rho(t)) - d(x_2,\rho(t)) - (d(x_1,\rho'(t)) - d(x_2,\rho'(t))\\
&= \lim_{t\to\infty} 2\left(x_1,x_2)_{\rho(t)} - d(x_1,x_2) - 2(x_1,x_2)_{\rho'(t)} + d(x_1,x_2\right)\\
&= 2\lim_{t\to\infty}\left((x_1,x_2)_{\rho(t)} - (x_1,x_2)_{\rho'(t)}\right) \leq \lim_{t\to\infty} d(\rho(t),\rho'(t)) \leq 2\delta.
\end{aligned}
$$

Here we used the fact that $(a,b)_p - (a,b)_q \leq d(p,q)$ and $\rho \sim \rho'$ imply $d(\rho(t),\rho'(t)) \leq \delta$ by Lemma 2.16. The circumstance that the rays have different starting points can be circumvented by applying Lemma 2.18.

□

Lemma 2.28. Let X be a δ-hyperbolic space. Suppose we have a ray ρ that belongs to the boundary element $\xi \in \partial X$. Then, there exists a neighbourhood $V \subset \partial X$, so that for all rays σ belonging to $\eta \in V \subset \partial X$, we have

$$B_p(\rho,x_1) - B_p(\rho,x_2) - \big(B_p(\sigma,x_1) - B_p(\sigma,x_2)\big) \leq C(\delta)$$

for any two points $x_1,x_2 \in X$ and the constant C depending only on δ.

Proof. We compute

$$
\begin{aligned}
&\lim_{t\to\infty} \big(d(\rho(t),x_1) - t\big) - \big(d(\rho(t),x_2) - t\big) - \Big(\big(d(\sigma(t),x_1) - t\big) - \big(d(\sigma(t),x_2) - t\big)\Big)\\
&= \lim_{t\to\infty} d(\rho(t),x_1) - d(\sigma(t),x_1) - \big(d(\rho(t),x_2) - d(\sigma(t),x_2)\big)\\
&= \lim_{t\to\infty} 2\left((x_1,x_2)_{\rho(t)} - \frac{1}{2}d(x_1,x_2) - (x_1,x_2)_{\sigma(t)} + \frac{1}{2}d(x_1,x_2)\right)\\
&= \lim_{t\to\infty} (x_1,x_2)_{\rho(t)} - (x_1,x_2)_{\sigma(t)} \leq C.
\end{aligned}
$$

The last inequality holds true due to the properties of the Gromov product on the boundary. □

Lemma 2.29. Let X be a δ-hyperbolic space. Suppose we have a ray ρ that belongs to the boundary element $\xi \in \partial X$. Then, there exists a neighbourhood $V \subset X \cup \partial X$, so that for all $y \in V \cap X$, we have

$$B_p(\rho,x_1) - B_p(\rho,x_2) - \big(d(y,x_1) - d(y,x_2)\big) \leq C(\delta)$$

for any two points $x_1,x_2 \in X$ and the constant C depending only on δ.

Proof. Pick points on rays corresponding to boundary points in the neighbourhood as defined in Lemma 2.28. By 1-Lipschitzness of the Busemann functions, we are done:

$$\lim_{t\to\infty} 2\left((x_1, x_2)_{\rho(t)} - (x_1, x_2)_y\right).$$

□

3 The Patterson-Sullivan method on Gromov spaces

3.1 Entropy and growth

Definition 3.1. Let M be a Riemannian manifold with universal cover X and suppose $x \in X$. The set of all isometries on X corresponding to an element in the fundamental group of M form a subgroup denoted Γ of the group of all isometries on X. We call Γ the group of deck transformations. Next, let $\Theta \subset X$ be the orbit of x under the group action of the isometries Γ. We define the intersection between open balls in X and the orbit Θ as follows:

$$n_\Theta(x,r) := \#B(x,r) \cap \Theta \subset X.$$

Due to proper discontinuity, this set is at most countably infinite for large r's.

Definition 3.2. We define the **critical exponent** as

$$e(\Gamma) := \limsup_{r\to\infty} \left(\frac{1}{r} \ln\{n_\Theta(r)\}\right).$$

Lemma 3.3. The limits

$$\lim_{r\to\infty} \frac{1}{r} \ln\Big(\mathrm{vol}\left(B(x,r)\right)\Big) = \lim_{r\to\infty} \frac{1}{r} \ln\left(n_\Theta(x,r)\right)$$

exist and are independent of base points. For a setup as above, that is, a Riemannian manifold M with universal cover X and isometry group Γ of deck transformations, we call this quantity the **entropy of the manifold** $h(M)$. This definition is further explored in [Man91].

Proof. Pick an $a > 0$ so that for all $x' \in X$, we have $d(x', \Theta) \leq a$. Next, pick an $r > a$ and $t > 0$. We see

$$d\left(\Theta \cap B(x, r+t), B(x, r-a)\right) < t + a.$$

Thus, for any x' as before, the following holds

$$d(\Theta \cap B(x, r+t), x') < t + 2a.$$

From $n_\Theta(x', t+2a) \leq n_\Theta(x, t+3a)$ we follow

$$n_\Theta(x, r+t) \leq n_\Theta(x,r) n_\Theta(x, t+3a).$$

Suppose now that $kt < r \leq (k+1)t$ for some $k \geq 0$. Then we have

$$n_\Theta(x,r) \leq n_\Theta(x, t+kt) \leq n_\Theta(x,t) n_\Theta(x, t+3a)^k,$$

as well as

$$\ln\left(n_\Theta(x,r)\right) \le \ln\left(n_\Theta(x,t)\right) + \frac{r}{t}\ln\left(n_\Theta(x,t+3a)\right).$$

This then yields

$$\limsup_{r\to\infty}\frac{1}{r}\ln\left(n_\Theta(x,r)\right) \le \frac{1}{t}\ln\left(n_\Theta(x,t+3a)\right)$$

for all $t > 0$. Thus, the existence of the limit follows.

Lastly, one can check the following inequality for some constants $b > 0, C > 1$

$$C^{-1}n_\Theta(x,r-b) \le \mathrm{vol}\left(B(x,r)\right) \le Cn_\Theta(x,r+b).$$

□

For the sake of an example, suppose that M is a Riemannian manifold, X is its universal cover and Γ is the group of deck transformations. Additionally, suppose that the sectional curvature $\sec = \kappa \le 0$ is a constant. Then, by [Lan21], chapter 6.1, we have $\mathrm{vol}(B(x,r)) = V_{n,\kappa}(r)$, yielding

$$e(\Gamma) = \lim_{r\to\infty}\frac{1}{r}\ln\left(\omega_{n-1}\int_0^r \mathrm{sn}_\kappa^{n-1}(t)dt\right),$$

which gives $(n-1)\sqrt{-\kappa}$.

Definition 3.4. Next, we define the **Poincaré series** for a given orbit Θ as

$$P_{\Theta,s}(x) = \sum_{y\in\Theta} e^{-sd(x,y)}.$$

Proposition 3.5. The Poincaré series converges for $s > e(\Gamma)$ and diverges for $s \le e(\Gamma)$.

Proof. Let $r > 0$. Pick numbers $0 < r_1 < ... < r_{k-1} < r_k = r$ and note that there are exactly $n_{i+1} - n_i$ points of distance $t \in (r_i, r_{i+1}]$ from x, where $n_i = n_\Theta(x, r_i)$. We then have

$$\sum_{i=1}^{k}(n_{i+1}-n_i)e^{-sr_i} = \sum_{i=1}^{k} n_{i+1}(e^{-sr_i} - e^{sr_{i+1}}) + n_{k+1}e^{-sr_{k+1}},$$

and the respective partial sum of $P_{\Theta,s}(x)$ can be written as follows:

$$\sum_{y\in\Theta\cap B(x,r)} e^{-sd(x,y)} = s\int_0^r n_\Theta(x,t)s^{-st}dt + n_\Theta(x,r)e^{-sr}.$$

Now, if $s > e(\Gamma)$, define $\epsilon = \frac{1}{2}(s - e(\Gamma))$. Then, for sufficiently large t, we have $\frac{1}{t}\ln(n_\Theta(x,t)) \le e(\Gamma) + \epsilon = s - \epsilon$. Further, we have $n_\Theta(x,r)e^{-st} \le e^{-st}$ and the series converges.

Suppose now $0 < s \leq e(\Gamma)$, then by Lemma 3.3, there exists an $a > 0$, so that we have

$$e(\Gamma) \leq \frac{1}{1-3a} \ln(n_\Theta(x,t))$$

for $t > 3a$. Thus, we follow

$$n_\Theta(x,t)e^{-st} \geq n_\Theta(x,t)e^{-e(\Gamma)t} \geq e^{-3ae(\Gamma)},$$

and the series diverges. Finally, for $s = 0$, we have $P_{\Theta,0}(x) = \#\Gamma = \infty$. $\square$

We now define the function j_γ on the border of a proper, geodesic, δ-hyperbolic space X, which we discussed in Section 2. Let $x \in X$ and $\rho : \mathbb{R} \to X$ be a ray starting at $p \in X$ with $\rho(t) \to \infty$ for $t \to \infty$ and $\eta \in \partial X$. Let γ be an isometry of X with extension to its boundary, also denoted $\gamma : X \cup \partial X \to X \cup \partial X$, which by Theorem 2.26 is a homeomorphism. Then, we define the map

$$j_\gamma(\rho) := e^{B_p(\rho,x) - B_p(\rho,\gamma^{-1}x)}.$$

Often enough, we will suppose that the ray starts at $\rho(0) = x$, so that we have

$$j_\gamma(\rho) = e^{0 - B_p(\rho,\gamma^{-1}x)} = e^{-B_p(\rho,\gamma^{-1}x)} \in \mathbb{R}_{\geq 0}.$$

Hence, if we suppose that rays start at a fixed base point, we can view the function j_γ as going from $\partial X$to $\mathbb{R}$.

3.2 Dilatation

We know that isometries on a Gromov hyperbolic space can be extended to homeomorphisms on the boundary, which we denoted by the same sign: $\gamma : X \cup \partial X \to X \cup \partial X$.

Definition 3.6. Given that the extension of an isometry to the boundary is *not* an isometry, we want to measure by how much it fails to be. For that, let $\eta_1, \eta_2 \in \partial X$ and γ be an extended isometry as mentioned above. We define the **dilatation of** γ as follows:

$$\mathrm{dil}_\gamma(\eta_1,\eta_2) = \frac{d_p(\gamma\eta_1,\gamma\eta_2)}{d_p(\eta_1,\eta_2)}.$$

Proposition 3.7. Let $\xi \in \partial X$. Suppose the same setting as above. Then, there exists a constant $C \geq 1$, which only depends on δ and there exists also a neighbourhood $V \ni \xi$, so that

$$C^{-1} j_\gamma(\xi) \leq \mathrm{dil}(\eta_1,\eta_2) \leq C j_\gamma(\xi)$$

for all $\eta_1 \neq \eta_2 \in V$.

Proof. Recall the property of the visual metrics displayed in Equation (2.2):

$$\begin{aligned}
\lambda^{-1}e^{-(x_1,x_2)_p} &\leq d_p(\eta_1,\eta_2) \leq \lambda e^{-(x_1,x_2)_p} \\
\lambda^{-1}e^{-(x_1,x_2)_p} &\leq d_p(\eta_1,\eta_2) \leq \lambda e^{-(x_1,x_2)_p} \\
\lambda^{-1}e^{-(\gamma x_1,\gamma x_2)_p} &\leq d_p(\gamma\eta_1,\gamma\eta_2) \leq \lambda e^{-(\gamma x_1,\gamma x_2)_p} \\
\lambda^{-1}e^{(x_1,x_2)_p-(\gamma x_1,\gamma x_2)_p} &\leq \operatorname{dil}_\gamma(\eta_1,\eta_2) \leq \lambda e^{(x_1,x_2)_p-(\gamma x_1,\gamma x_2)_p}.
\end{aligned}$$

As we have $(x_1,x_2)_p - (\gamma x_1,\gamma x_2)_p = \frac{1}{2}d(p,x_1) + d(p,x_2) - d(p,\gamma x_1) - d(p,\gamma x_2)$, by Lemma 2.29, we conclude

$$B_p(\xi\gamma^{-1}p) - d(p,x_1) + d(p,\gamma x_1) \leq C'.$$

So now, for $C = \lambda^2 e^{C'}$ we have $C^{-1}j_\gamma(\xi) \leq \operatorname{dil}(\eta_1,\eta_2) \leq Cj_\gamma(\xi)$. □

3.3 Measures on the boundary

Let us first clarify some notation. Let μ be a measure on a space X and let $\psi : X \to Y$ be a continuous and measurable map. We define the **pushforward measure** as follows: For a measurable function $h : Y \to \mathbb{R}$, we set $\psi_*\mu(h) := \mu(h \circ \psi)$. In particular, for measurable homeomorphisms and isomorphisms, say $\gamma : X \to X$, we have $\gamma_*\mu(f) = \mu(f \circ \gamma)$. In integral notation, that is

$$\begin{aligned}
\int_X f(x)d\gamma_*\mu(x) &= \int_X f(\gamma(x))d\mu(x) \\
&= \int_X f(x)d\mu(\gamma^{-1}x).
\end{aligned}$$

Definition 3.8. Let Γ be a group of isometries of X. Let $D \geq 0$ be a real number and μ a measure on ∂X with finite, nonzero total mass without atoms. The measure μ is said to be **Γ-quasi-conformal** if the pushforward measures $\gamma_*\mu$ are absolutely continuous with respect to each other, meaning that $\mu(A) = 0 \iff \gamma_*\mu(A) = 0$, and additionally, there exists a real constant $C \geq 1$, so that we have

$$C^{-1}j_\gamma^D \leq \frac{d\gamma_*\mu}{d\mu} \leq Cj_\gamma^D$$

μ-almost everywhere and for all $\gamma \in \Gamma$. Observe that this is an inequality of functions, meaning that we require this inequality to be true for each element in the domain of the respective functions, that is, the boundary ∂X.

Lemma 3.9. Suppose that we have a measure μ with finite nonzero total mass, that means $0 < \mu(\partial X) < \infty$, without atoms. For a group of isometries Γ, the measure μ

is Γ-quasi-conformal if and only if it satisfies the following property: There exists a number $C \geq 1$, so that for all $\xi \in \partial X$ and all rays in the equivalence class of ξ, we can find a neighbourhood V in ∂X containing ξ and satisfying

$$C^{-1} j_\gamma(\xi)^D \mu(A) \leq \mu(\gamma A) \leq C j_\gamma(\xi)^D \mu(A)$$

for all measurable sets $A \subset V$ and the constant $D \geq 0$.

Definition 3.10. A group action of a group Γ of isometries on the space X is called **properly discontinuous** if for all compact sets $K \subseteq X$ there are only finitely many elements $\gamma \in \Gamma$, so that the intersection $\gamma K \cap K$ is nonempty.

Definition 3.11. Let X now additionally be a Gromov-hyperbolic metric space and let $\Theta \subset X$ be an orbit of Γ. We define the **limit set** of Γ as the set of limit points of Θ on the border ∂X and denote it by Λ. The limit set is a Γ-invariant set.

Isometry groups on a δ-hyperbolic space have first been studied by Gromov in [Gro81] and specifically [Gro87], chapter 8. He proved that all isometries correspond to one of three types related to their behaviour on the boundary ∂X.

Definition 3.12. Let X be a δ-hyperbolic, complete, simply connected Riemannian manifold X with base point $p \in X$. The most important consequence of this assumption is Lemma 2.18, which allows us to connect any two points on $X \cup \partial X$ with a geodesic. Lastly, suppose Γ is a group of isometries.

- $\gamma \in \Gamma$ is called **elliptic** if for $x \in X$ the sequence $(\gamma^n x)_{n\in\mathbb{Z}}$ is bounded for one and thus any $x \in X$.

- $\gamma \in \Gamma$ is called **parabolic** if for $x \in X$ the sequence $(\gamma^n x)_{n\in\mathbb{Z}}$ has a unique limit point $\eta \in \partial X$. That means that we have $d(p, \gamma^{a_n} x) \to \infty$ for any subsequence a_n and, additionally, we have $\gamma^{a_n} x \to \eta$. Parabolic elements are called proper if $a_n = n$ can be chosen.

- $\gamma \in \Gamma$ is called **hyperbolic** if for $x \in X$ the sequence $(\gamma^n x)_{n\in\mathbb{Z}}$ has a two-point limit set: $\eta := \lim_{n\to-\infty} \gamma^n(x)$ and $\xi := \lim_{n\to\infty} \gamma^n(x)$. Observe that the limit points η and ξ are independent of the point x.

Gromov shows in [Gro87], section 8.1F, that any isometry is of one of the three types. He does that by showing that the product of two non-elliptic or non-parabolic isometries will always be hyperbolic.

We now want to explore how Γ-invariant sets on ∂X translate to Γ-invariant sets on X and vice versa. For that, suppose that $A \subset \partial X$ is a Γ-invariant set. We define the set $r(A)$ as follows:

$$r(A) := \bigcup_{a,b \in A} im(r_{a,b}),$$

where $r_{a,b} : \mathbb{R} \to X$ denotes the geodesic with $\lim_{t\to-\infty} r_{a,b}(t) = a$ and $\lim_{t\to\infty} r_{a,b}(t) = b$. Observe that $\gamma(x) \in im(\gamma(r_{a,b})) = im(r_{\gamma(a),\gamma(b)})$. By Γ-invariance of A, this set lies in $r(A)$ as well. This then means that the orbit of x is contained in $r(A)$. As the set of all limit points of $r(A)$ is exactly A, this means that Λ, the set of limit points of the orbit of x, is contained in A. In particular, every closed, nonempty subset of the boundary that is Γ-invariant contains Λ. The intersection of all these sets is then equal to Λ, showing that the limit set of an orbit of a group action is unique and well defined.

Recall the definitions involved in the notion of the Hausdorff measures. In particular, the diameter of a (subset of a) metric space is defined as the supremum over all distances between points inside that set. We then defined the Hausdorff premeasure using ϵ-coverings, which are coverings using sets with diameter less than or equal to ϵ,

$$\mathcal{H}^D_\epsilon(A) := \inf \left\{ \sum_i \mathrm{diam}(U_i)^D, U_i \text{ is an } \epsilon\text{-covering of } A \right\}$$

and

$$\mathcal{H}^D(A) := \lim_{\epsilon\to 0} \mathcal{H}^D_\epsilon(A).$$

The Hausdorff measure is a suitable prototype for a measure on the boundary.

Proposition 3.13. Let $E \subset \partial X$ be a Γ-invariant Borel set with nonzero, finite D-dimensional Hausdorff measure. Then, the D-dimensional Hausdorff measure on ∂X defines a Γ-quasi-conformal measure on ∂X of dimension D.

Proof. Observe that $\frac{\mathrm{diam}(\gamma U)}{\mathrm{diam}(U)} \geq \mathrm{dil}_\gamma$. Then, due to Lemma 3.7, for any $\xi \in \partial X$, we can find a neighbourhood $\xi \in V \subset \partial X$ so that for all subsets $U \subset V$

$$C^{-1} j_\gamma(\xi)\mathrm{diam}(U) \leq \mathrm{diam}(\gamma U) \leq C j_\gamma(\xi)\mathrm{diam}(U)$$

with respect to the visual metric d_p. By definition, the premeasure $\mathcal{H}^D_\epsilon$ is quasi-conformal and therefore also the measure $\mathcal{H}^D$ itself. □

3.4 The existence of a Γ-quasi-conformal measure

In this subsection we will prove the existence of a Γ-quasi-conformal measure. The dimension of this measure will be studied in the next subsection.

Lemma 3.14. If μ is a Γ-quasi-conformal measure with support contained in the limit set Λ, then its support is equal to Λ.

Proof. The support of a measure is defined to be the set of all elements $\eta \in \partial X$ for which there exists an open set $U_\eta \subseteq \partial X$ containing η and satisfying $\mu(U_\eta) \neq 0$. So if η lies in the support of μ, then there exists an open set $B \subset \partial X$, which by continuity is a preimage of another open set $\gamma^{-1}B =: A$, so that $\mu(B) > 0$. By Γ-quasi-conformality, $\gamma_*\mu(B) = \mu(A) > 0$. Thus, the support is a Γ-invariant set, meaning that η is contained in the support of the measure if and only if $\gamma(\eta)$ is contained in the support of the measure. By Theorem 3.12, the support is either empty or Λ. □

Theorem 3.15. If $e(\Gamma)$ is finite, then there exists a Γ-quasi-conformal measure on ∂X with support equal to the limit set Λ.

Proof. Let us assume that the Poincaré series diverges for $s = e(\Gamma)$. For a real number $s > e(\Gamma)$, let us consider the probability measure given by

$$\mu_{x,s} := \frac{1}{P_s(x)} \sum_{y \in \Theta} e^{-sd(x,y)} \delta_y$$

where δ denotes the Dirac measure. Given that $X \cup \partial X$ is compact, there exists a sequence $(s_i)_i \to e(\Gamma)$ with $s_i > e(\Gamma)$, so that the measures μ_{x,s_i} converge weakly to a measure μ_x almost everywhere.

Let us now show that the support of μ_x is contained in Λ. For that, consider a continuous function $f : X \cup \partial X \to \mathbb{R}$ satisfying $\operatorname{supp}(f) \cap \operatorname{supp}(\Lambda) = \emptyset$. Then, due to the group action being properly discontinuous, there are only finitely many points of the orbit Θ contained in the support of f. That is, $\#\{y \in \Theta : f(y) \neq 0\} < \infty$. As we assumed that the Poincaré series diverges for s, we have $P_{s_i} \to \infty$ for $i \to \infty$ and thus we also have $\mu_x(f) = \lim \mu_{x,s_i}(f) = 0$.

In order to show that μ_x is quasi-conformal, let first ρ be a ray whose border point is ξ. Again we set j_γ as before, meaning that due to Lemma 2.29 there exists a neighbourhood $V \subset X \cup \partial X$ and a constant C, so that for all $v \in V \cap X$ we have

$$C^{-1} j_\gamma(\xi) \leq e^{d(x,v) - d(x,\gamma^{-1}v)} \leq C j_\gamma(\xi). \tag{3.1}$$

Let now $f : X \cup \partial X \to \mathbb{R}$ be a continuous function whose support is contained in V. We then compute:

$$\mu_{x,s}(f) = \frac{1}{P_s(x)} \sum_{y \in \Theta} e^{-sd(x,y)} f(y) = \frac{1}{P_s(x)} \sum_{y \in \Theta \cap V} e^{-sd(x,y)} f(y);$$

and then, the pushforward as follows:

$$\begin{aligned}
\gamma_* \mu_{x,s}(f) &= \mu_{x,s}(f \circ \gamma) \\
&= \frac{1}{P_s(x)} \sum_{\gamma y \in V \cap \Theta} e^{-sd(x,y)} f(y) \\
&= \frac{1}{P_s(x)} \sum_{y \in V \cap \Theta} e^{-sd(x,\gamma^{-1} y)} f(y) \\
&= \frac{1}{P_s(x)} \sum_{y \in V \cap \Theta} e^{-s(d(x,y) - d(x,\gamma^{-1} y)} e^{-sd(x,y)} f(y) \\
&= \left(\sum_{y \in V \cap \Theta} e^{-s(d(x,y) - d(x,\gamma^{-1} y)} \right) \mu_{x,s}(f).
\end{aligned}$$

Now, using Equation (3.1), we obtain

$$C^{-s} j_\gamma(\xi)^s \mu_{x,s}(f) \leq \gamma_* \mu_{x,s}(f) \leq C^s j_\gamma(\xi)^s \mu_{x,s}(f).$$

By going to the limit for s_i, we have shown that the measure μ_x is quasi-conformal. By Lemma 3.14, the support of the measure is not only contained in Λ, but also equal to Λ. □

We want to note here that we have constructed a measure that is conformal with respect to different base points and quasi-conformal with respect to the group action. Between base points p and p' we have

$$\frac{d\mu_p}{d\mu_{p'}}(\xi) = \left(e^{B_{p'}(\xi,p)} \right)^D$$

and

$$C^{-1} j_\gamma^D \leq \frac{d\gamma_* \mu}{d\mu} \leq C j_\gamma^D,$$

as seen in Definition 3.8.

3.5 Shadows

Definition 3.16. We now want to define sets of certain sizes on the boundary. We will not use the (arguably simpler) notion of balls but will instead use the so-called

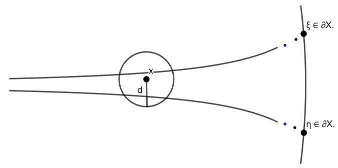

Figure 2: The shadow of a point of given size can be thought of as all the rays passing through the ball as seen here.

shadows. We define the **shadow** $\mathcal{O}(x,d)$ of size d of the point $x \in X$ as the set of all boundary points in whose equivalence class there is a ray starting at a fixed base point p and passing d-close to x.

$$\begin{aligned}\mathcal{O}(x,d) := \{\xi \in \partial X : &\text{ if } \rho(0) = p \text{ and } \rho \to \xi, \\ &\text{there exists } s \geq 0 : d(\rho(s), x) < d\}.\end{aligned}$$

Lemma 3.17. There exists a constant $C > 0$, which depends only on δ, so that for $x = \gamma^{-1}p$, we have

$$C^{-1}e^{d(x,p)-2d} \leq j_\gamma(\xi) \leq Ce^{d(x,p)}$$

for all $\xi \in \mathcal{O}(x,d)$.

Proof. Let us pick a suitable point $\rho(s)$ as base point of rays. Then, for $\xi \in \mathcal{O}(x,d)$ we have

$$j_\gamma(\xi) = e^{B_{\rho(s)}(\xi,p) - B_{\rho(s)}(\xi,x)},$$

and we note that $B_{\rho(s)}(\xi,p) \geq d(x,p) - d$ by Lemma 2.29 and $B_{\rho(s)}(\xi,x) \leq d$ so that we have

$$j_\gamma(\xi) \geq e^{d(x,p)-d-d} = e^{d(x,p)-2d},$$

which establishes one side of the proof. For the other side, pick p as the base point of the rays and use 1-Lipschitzness.

$\square$

Lemma 3.18. For all $\epsilon > 0$, there exists a real number $d_0 \geq 0$, so that for all $d \geq d_0$, the set $\partial X \setminus \gamma\left(\mathcal{O}(\gamma^{-1}p, d)\right)$ is of diameter at most ϵ for any isometry γ.

Proof. Let $\eta, \xi \in \partial X \setminus \gamma\left(\mathcal{O}(\gamma^{-1}p, d)\right)$. We want to show that they are finitely close with respect to the visual metric. There exist geodesic rays starting at γp and going to their respective boundary points. These rays do not intersect $B(p,d) \subset X$. By the property established in Equation (2.2), we have

$$d_p(\eta, \xi) \leq K e^{-(u,x)_p},$$

where u lies on the ray belonging to η and v on the ray belonging to ξ. By δ-hyperbolicity, we compute

$$\begin{aligned}(u,v)_p &\geq \min\{(u,\gamma p)_p, (v,\gamma p)_p\} - \delta \\ &\geq d - 5\delta,\end{aligned}$$

where the second statement is by Lemma 2.6. This then yields $d_p(\eta,\xi) \leq K e^{5\delta} e^{-d}$, which proves the statement, as we can pick d so that $K e^{5\delta} e^{-d} < \epsilon$. $\square$

The next proposition bounds the measures of the shadows and is a first step towards our main goal of understanding the dimension of measures on the boundary.

Proposition 3.19. Let μ be a Γ-quasi-conformal measure of dimension D on ∂X. Then, there exists a $C \geq 1$ as well as a $d_0 \geq 0$ so that for all $d \geq d_0$ and all $\gamma \in \Gamma$ we have

$$C^{-1} r^d \leq \mu\left(\mathcal{O}(x,d)\right) \leq C r^d e^{2dD},$$

where we set $x = \gamma^{-1}p$ and $r = e^{d(\gamma^{-1}p,p)}$.

Proof. Recall that we assume that the measure μ has no atoms. Pick an $m < M = \mu(\partial X)$. By compactness of X, we can also pick an $\epsilon > 0$, so that all subsets of diameter less than or equal to ϵ have measure less than or equal to m. By Lemma 3.18, there exists a d_0, so that we have the following

$$0 \neq \mu\left(\partial X \setminus \gamma(\mathcal{O}(x,d)\right) < m$$

for all $d \geq d_0$. In particular, that means that for such d's, we have

$$M - m < \mu\left(\gamma(\mathcal{O}(x,d)\right) < M. \tag{3.2}$$

Now, by Lemma 3.17, we have

$$C_1^{-1} e^{d(x,p)-2d} \leq j_\gamma(\xi) \leq C_1 e^{d(x,p)}$$

for all $\xi \in \mathcal{O}(x,d)$. By inserting this inequality into the definition of quasi-conformality, as seen in Definition 3.8, we obtain the equation

$$C_1^{-1} e^{Dd(x,p)} \leq \frac{\mu(\mathcal{O}(x,d))}{\mu(\gamma\mathcal{O}(x,d))} \leq C_1 e^{-Dd(x,p)} e^{2Dd}. \tag{3.3}$$

Now, termwise multiplying Equation (3.3) by Equation (3.2), we obtain

$$\frac{C_1^{-1} e^{Dd(x,p)}}{M} \leq \mu(\mathcal{O}(x,d)) \leq \frac{C_1 e^{-Dd(x,p)+2Dd}}{M-m},$$

which establishes the statement. □

Lemma 3.20. Let Θ be an orbit of the group action of a properly discontinuous group of isometries $\Gamma < \mathrm{Isom}(X)$ and let $d > 0$. Then, there exists an integer $N \in \mathbb{N}$, so that for all $\xi \in \partial X$ and all integers k, we have

$$\#\{x \in \Theta : \xi \in \mathcal{O}(x,d) \text{ and } d(p,x) \in (k-1,k]\} \leq N.$$

That means that the number of all orbit points at a fixed distance in whose shadow ξ lies is finite. We prove the following statement: A boundary element ξ does not lie in infinitely many orbit points' shadow at a given distance from the base point.

Proof. Suppose $x, x' \in \Theta$ and let $\xi \in \mathcal{O}(x,d) \cap \mathcal{O}(x',d)$ so that $d(p,x), d(p,x') \in (k-1,k]$. We now want to show that $d(x,x') < 4d+1$. For that, pick a geodesic ray ρ belonging to ξ and starting at p. By the very definition of the shadow, there exists a point $\rho(s)$ which satisfies $d(\rho(s),x) \leq d$. Using the triangle inequality, we obtain

$$d(x,p) - d(x,\rho(s)) \leq d(\rho(s),p) \leq d(x,p) + d(x,\rho(s)).$$

Thus, we have $d(\rho(s),p) \in (k-1-d, k+d]$. By the same means, there exists a point $\rho(s')$ on the same ray ρ satisfying $d(\rho(s'),x') \leq d$ and thus, also satisfying $d(\rho(s'),p,) \in (k-1-d,k+d]$. As the ray is a geodesic, we have $d(\rho(s),\rho(s')) = |s-s'|$ where we suppose without loss of generality that $|s-s'| = s-s'$. Hence, we have $s - s' \leq (k+d) - (k-1-d) = 2d+1$ and subsequently

$$\begin{aligned} d(x,x') &\leq d(x,\rho(s)) + d(\rho(s),\rho(s')) + d(\rho(s'),x') \\ &\leq d + 2d + 1 + d = 4d+1. \end{aligned}$$

Because the group Γ acts properly discontinuously on X, the closed ball in X centered at x of radius $4d+1$ only contains finitely many points of the orbit Θ. That finite number is denoted N and we are done. □

Proposition 3.21. Let μ be a Γ-quasi-conformal measure of dimension D on ∂X, the boundary of a Gromov hyperbolic space X. Let $\Theta \subset X$ be an orbit of the properly discontinuous action of Γ on X. Recall from Definition 3.1, the quantity $n_\Theta(r) := \#\{x \in \Theta : d(x,p) < r\}$. We claim that there exists a constant C so that $n_\Theta(r) \leq C^{Dr}$.

Proof. Let us denote by E_k the set of all $\gamma \in \Gamma$ satisfying $d(p, \gamma^{-1}p) \in (k-1, k]$. We define the cardinality of E_k by $v_k := \#E_k$. By Proposition 3.19, we can find real numbers, so that for all k and all $\gamma \in E_K$

$$e^{-Dk} r^d \leq C_1 \mu\left(\mathcal{O}(\gamma^{-1}p, d)\right).$$

However, we also have

$$\sum_{\gamma \in E_k} \mu(\mathcal{O}(\gamma^{-1}p, d)) \leq C_2 \mu\left(\bigcup_{\gamma \in E_k} \mathcal{O}(\gamma^{-1}p, d)\right).$$

Due to Lemma 3.20, the number of γ's in E_k for which $\xi \in \mathcal{O}(\gamma^{-1}p, d)$ is bounded from above by a constant that does not depend either on the boundary element ξ or on the integer k. From the two equations above, we follow $v_k \leq C_1 C_2 \mu(\partial X) e^{Dk} := C_3 e^{Dk}$. Now, we can decompose the quantity $n_\Theta(k)$ as follows

$$n_\Theta(k) = \phi_0 + v_1 + ... + v_k \leq C_4 e^{Dk},$$

and thus $n_\Theta(r) \leq C_5 e^{Dr}$, establishing the statement. □

Recall the definition $e(\Gamma) := \limsup \frac{1}{r} \ln(n_\Theta(r))$. This then yields the following corollary.

Corollary 3.22. Any Γ-quasi-conformal measure of dimension D on ∂X satisfies $D \geq e(\Gamma)$.

Recall Proposition 3.13 that established a prototype of a measure on the boundary. We can now conclude.

Corollary 3.23. If B is a Γ-invariant subset of ∂X and $D \geq 0$ a is real number for which the D-dimensional Hausdorff measure is of nonzero finite mass, then we have $D \geq e(\Gamma)$.

4 Barycenters

Let N be a compact, connected, n-dimensional Riemannian manifold with negative curvature and let Y be the universal cover of N. Let μ be a probability measure on ∂Y without atoms, that is, $\mu(\partial Y) = 1$ and satisfying $\mu(\{\eta\}) = 0$ for all $\eta \in \partial Y$. Let us denote by B_q the Busemann functions of rays starting at $q \in Y$. These elaborations rely in equal parts on Appendix A of [BCG95] and section 8.5 of [Lan21]. Further insight on spaces of nonpositive curvature can be found in chapter 2 of [Jos97].

Proposition 4.1. Suppose the measure μ is as described above. Then, the equation

$$\int_{\partial Y} dB_q(\theta, y) d\mu(\theta) = 0$$

implicitly defines a unique point $y \in Y$, which we call the **barycenter** of μ. It is realized as the argument minimum of the derivative over all Busemann functions starting at a given point. The term barycenter stems from the notion of a center of weight of the measure, described by the integrals of Busemann functions.

Proof. We want to find the critical point of the map $\mathcal{B} : Y \to \mathbb{R}$ defined by

$$\mathcal{B}(y) := \int_{\partial Y} B_q(\theta, y) d\mu(\theta).$$

Due to the negative curvature of N and Y, the map $y \mapsto B_q(\theta, y)$ is convex along geodesics. [Lan21] We define the set $A_c := \{y \in Y \cup \partial Y : \mathcal{B}(y) \leq c\} \setminus \{y\}$. Then, for all elements of the boundary $\eta \in A_c \cap \partial Y$ there exists a ray ρ starting at a base point q' and belonging to η contained in A_c.

We define the sublevel sets $A_c := \{y \in Y : \mathcal{B}(y) \leq c\}$. Note that these sets are convex and contain the base point q. Recall the definition of the Busemann functions $B_q(\rho, \sigma(s)) = \lim_{t\to\infty} d(\sigma(s), \rho(t)) - t$. We compute

$$\begin{aligned}
d(\sigma(s), \rho(t)) - t &= d(\sigma(s), \rho(t)) - d(q, \rho(t)) \\
&= -2\left(\frac{1}{2}\big(-d(\sigma(s), \rho(t)\big) + d(q, \rho(t)) + d(q, \sigma(s))\right) + d(q, \sigma(s)) \\
&= s - 2(\sigma(s), \rho(t))_q \geq s - 2(\eta, \xi)_p.
\end{aligned}$$

Using the visual metric property from Equation (2.2), we obtain

$$B_q(\rho, \sigma(s)) \geq s + 2C \ln(d(\eta, \xi)).$$

From the assumption that there are no atoms, there exists a set $E := \{\xi \in \partial X : d(\eta, \xi) \leq \epsilon\}$ with measure $\mu(E) > 1 - \delta$. Note that $B_q(\rho, \sigma(s)) \geq s + 2\ln(\epsilon)$ if ρ

belongs to a boundary point in E and $B_q/(\rho, \sigma(s) \geq -s$ if ρ belongs to a boundary point outside of E. We now compute:

$$\begin{aligned}\mathcal{B}(\sigma(s)) &\geq \big(s + 2\ln(\epsilon)\big)\,\mu(E) - s(1-\mu(E))\\ &\geq s\mu(E) + 2\mu(E)\ln(\epsilon) - s + s\mu(E)\\ &= 2\mu(E)(s+\ln(\epsilon)) - s \to \infty\end{aligned}$$

as $2\mu(E) > 1$. Therefore, $b_q(\rho, \sigma(s)) \to \infty$ for $s \to \infty$, which means that no ray starting at q is fully contained in A_c. Thus, by convexity of A_c, the set must be bounded. Thus, the fact that we have $\mathcal{B}(y) \to \infty$ for $y \to \partial Y$ implies that the function $\mathcal{B}$ must have a minimum.

We claim that the function $\mathcal{B}$, which can be thought of an average of the convex function $B_{q'}$, is strictly convex. If it were strictly convex, the sub-level sets - which are bounded - do indeed have a minimum, thus ending our proof. For a proof of the strict convexity of $\mathcal{B}$ we refer to Lemma 4.2. $\square$

Lemma 4.2. The function $\mathcal{B}$ is strictly convex.

Proof. Let $c : \mathbb{R} \to Y$ be a unit speed geodesic with $c(0) =: y$ and let $\theta \in \partial Y$ be an element of the border with a ray τ in its class. The angle ψ is defined as the angle between the derivatives τ' and c'. We write $\sphericalangle_y(\tau'(0), c'(0)) =: \psi$. First off, we will show that for all $s \in \mathbb{R}$, we have

$$B_y(\tau, c(s)) \geq \log(\cosh(s) - \cos(\psi)\sinh(s)) \tag{4.1}$$

by comparison with the hyperbolic upper half-plane $U^+ \subset \mathbb{C}$. We define the geodesic $\bar{c} : \mathbb{R} \to \mathbb{C}$ by $s \mapsto \bar{c}(s) := \tanh(a-s) + \cosh(a-s)^{-1}$, where we pick a so that $\bar{c}(0) = e^{i\psi} =: \bar{y}$. Let $\bar{\tau}$ be the upward vertical ray starting at $\bar{y}$. Then, the Busemann function satisfies

$$B_{\bar{y}}(\bar{\tau}, u + iv) = -\log(v\cosh(a))$$

for all points $u + iv \in U^+$. For the angle, we then have $\sphericalangle_{\bar{q}}(\bar{\tau}'(0), \bar{c}'(0)) = \psi$, as well as $d(c(s)\tau(t)) \geq d(\bar{c}(s), \bar{\tau}(t))$ so by Hinge comparison, [Lan21], definition 5.7, we have

$$B_y(\tau, c(s)) \geq B_{\bar{y}}(\bar{\tau}, \bar{c}(s))$$

for all $s \in \mathbb{R}$. Furthermore,

$$B_{\bar{y}}(\bar{\tau}, \bar{c}(s)) = \log\left(\frac{\cosh(a-s)}{\cosh(a)}\right) = \log\big(\cosh(s) - \tanh(a)\sinh(s)\big)\,,$$

as we have $\tanh(a) = \cosh(\psi)$. This then gives rise to Equation (4.1). For $s > 0$ we compute

$$\begin{aligned} B_y(\tau, c(s)) + B_y(\tau, c(-s)) &\geq \log(\cosh^2(s) - \cos^2(\psi)\sinh^2(s)) \\ &= \log\left(1 + \sin^2(\psi)\sinh^2(s)\right). \end{aligned}$$

For the set $A := X \cup \partial X \setminus \{c_\infty, c_{-\infty}\}$ we note that $\mu(A) > 0$. Additionally, for all $\theta \in A$, the angle $\psi_\theta \in (0, \pi)$. By integration, we conclude that $B_y(\tau, c(s)) + B_y(\tau, c(-s)) - 2B_y(\tau, y) > 0$ and thus $\mathcal{B}$ is strictly convex.

$$\int_{\partial Y} B_y(\tau, c(s))d\mu(\tau) + \int_{\partial Y} B_y(\tau, c(-s))d\mu(\tau) > 2\int_{\partial Y} B_y(\tau, q)d\mu(\tau)$$
$$\frac{1}{2}\left(\mathcal{B}(c(s)) + \mathcal{B}(c(-s))\right) > \mathcal{B}(y).$$

□

Using the barycenter construction, we can assign a point $\mathrm{bar}(\mu)$ in the space Y to any atom free measure μ on the boundary ∂Y through the following assignment:

$$\mu \longmapsto \arg\min_{y \in Y} \mathcal{B}(y) =: \mathrm{bar}(\mu) \in Y.$$

This assignment will be crucial when we define the natural map in the next chapter. As a final remark, assume we have an isometry $\gamma : Y \to Y$. Then we have a homeomorphic extension to its boundary, also denoted by γ. Let us study the integral under the pushforward measure of the isometry where we denote the boundary element of $\gamma(\tau)$ by τ_γ.

$$\int_{\partial Y} B_q(\tau, y)d(\gamma_*\mu)(\tau) = \int_{\partial Y} B_q(\tau_\gamma, y)d\mu(\tau).$$

By using the fact that we have $B_q(\tau_\gamma, y) = B_{\gamma^{-1}(q)}(\tau, \gamma^{-1}(y))$, we have

$$\mathrm{bar}(\gamma_*\mu) = \gamma \mathrm{bar}(\mu).$$

5 The natural map

5.1 Consequences of the statement

Let N be as described in Theorem 1.1 with universal cover Y. That is, compact, connected, locally symmetric, n-dimensional and with negative curvature. We can normalize the metric on N, so that Y has maximal sectional curvature sec ≤ -1. By theorem 5.12 of [Lan21], Y is as $CAT(-1)$ space and thus by theorem III.H 1.2 in [BH99] it is also a Gromov hyperbolic space. The fundamental group of M acts on X through deck transformations and is a subgroup of the group of isometries of X. The same applies for N and Y. Assume we have continuous maps $f : M \to N$ and $h : N \to M$ that are homotopically equivalent. This means that both of the concatenations of the maps are homotopic to their respective identity maps.

Lemma 5.1. [Hat02] For a path $h : [0,1] \to Y$, let β_h be the group isomorphism $\beta_h : \pi_1(Y, h(0)) \to \pi_1(Y, h(1))$. Then, if $f : [0,1] \times X \to Y$ is a homotopy and for a fixed basepoint x_0, we define the path h as $t \mapsto f_t(x_0) \in Y$. We then have

$$(f_0)_* = (\beta_h) \circ (f_1)_*.$$

Proof. First, let γ be a loop in X at x_0. An overscript defines the inverse loop. Define the following map:

$$h_{[0,t]} * (f_t \circ \gamma) * \bar{h}_{[0,t]}.$$

It describes a path that goes first from x_0 to $h(t)$, then follows the loop $f_t \circ \gamma$ and at last returns back to x_0. This gives a homotopy of paths, $f_{0*}([\gamma]) = \beta_h f_{1*}([\gamma])$ that establishes the statement. □

Lemma 5.2. [Hat02], proposition 1.18. Let M and N be compact Riemannian manifolds and let $f : M \to N$ be a homotopy equivalence. Then, the map between fundamental groups $f_* : \pi_1(M) \to \pi_1(N)$ is an isomorphism.

Proof. Let $f : X \to Y$ and $g : Y \to X$ be homotopy inverse maps. Observe:

$$\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, f(x_0)) \xrightarrow{g_*} \pi_1(X, g(f(x_0))) \xrightarrow{f_*} \pi_1(Y, f(g(f(x_0)))).$$

Here, the composition of the first two maps is an isomorphism, the same for the latter two. We thus conclude that f_* is an isomorphism as claimed. □

By Lemma 5.2, the fundamental groups of M and N are isomorphic as abstract groups, say through the map $\tilde{f}_* : \pi_1(M) \to \pi_1(N)$. We denote the lift of $\pi_1(M)$ to the group of isometries on X by Γ. We think of it as a subgroup $\Gamma < \mathrm{Isom}(X)$. We then have for the lifted map $\tilde{f}(\gamma(x)) = \tilde{f}_*(\gamma)\tilde{f}(x)$ for $x \in X$ and $\gamma \in \Gamma$. The first link to the theory established in the last chapters is the following:

Lemma 5.3. [BP92], lemma C.1.4. The map $\tilde{f}$ is a quasi-isometry between X and Y.

Proof. Recall that we assumed both M and N to be compact. Due to the fact that f is continuous, we can assume it to be homotopic to a differentiable map, which we will call f again. The same applies for the map h.

By compactness, the norm of the derivative maps df and dg is universally bounded by a constant c_1. The same constant also bounds the lifted maps $d\tilde{f}$ and $d\tilde{h}$. This gives us the following inequalities:

$$d(\tilde{f}(x_1), \tilde{f}(x_2)) \leq c_1 d(x_1, x_2)$$
$$d(\tilde{g}(x_1), \tilde{g}(x_2)) \leq c_1 d(x_1, x_2).$$

Up to a choice of basepoint for our deck transformations, we have

$$\tilde{g} \circ \tilde{f} \circ \gamma = \tilde{g} \circ \tilde{f}_*(\gamma) \circ \tilde{f} = (\tilde{g}_* \circ \tilde{f}_*)(\gamma) \circ \tilde{g} \circ \tilde{f} = \gamma \circ \tilde{g} \circ \tilde{f}.$$

Again, by compactness, there exists a constant $b > 0$ so that $d(y, (\tilde{g} \circ \tilde{f})y) \leq b$. Then, we have the bound $d((\tilde{g} \circ \tilde{f})y_1, (\tilde{g} \circ \tilde{f})y_2) \geq d(y_1, y_2) - 2b$, giving us the final inequality

$$d(\tilde{f}(x_1), \tilde{f}(x_2)) \geq \frac{1}{c_1} d\left((\tilde{g} \circ \tilde{f})x_1, (\tilde{g} \circ \tilde{f})x_2\right) \geq \frac{1}{c_1} d(x_1, x_2) - \frac{2b}{c_1}.$$

We have now shown that $\tilde{f}$ is a quasi-isometry. □

5.2 Towards the natural map

Lemma 2.26 states that any given quasi-isometry $f : X \to Y$ on Gromov hyperbolic spaces can be extended to a homeomorphism on the boundary $f : X \cup \partial X \to Y \cup \partial Y$. We apply this to the map $\tilde{f} : X \to Y$ from Lemma 5.3 and, consistent with the notation, denote the extended map in the same way:

$$\tilde{f} : X \cup \partial X \to Y \cup \partial Y.$$

Recall that this in particular holds true for isometries, meaning that an isometry $\gamma : X \to X$ extends to a homeomorphism on the boundary. This then allows for the following statement.

Lemma 5.4. The homeomorphism $\tilde{f}$ still commutes with the isometries given by the deck transformations, that is

$$\tilde{f} \circ \gamma = \tilde{f}_*(\gamma) \circ \tilde{f}.$$

Note that the map $\tilde{f}_*$ takes elements of $\mathrm{Isom}(X)$ and maps them to $\mathrm{Isom}(Y)$.

Proof. We aim to show that $\tilde{f}(\gamma(x)) = \tilde{f}_*(\gamma)(\tilde{f}(x))$ for all $x \in X \cup \partial X$.

If $x \in X$, then the statement has been established just above Lemma 5.3.

Suppose the ray $\rho : [0, \infty] \to X$ belongs to the boundary point η and starts at p. Then the quasi-geodesic $\gamma \circ \rho$ lies finitely close to a ray σ. Again, the quasi-geodesic $\tilde{f} \circ \sigma$ lies finitely close to a geodesic ψ that starts at $\tilde{f}(\gamma(p))$ and belongs to the boundary point $\tilde{f}(\gamma(\eta)) \in \partial Y$.

Similarly, the quasi-geodesic $\tilde{f} \circ \rho$ lies finitely close to a ray τ. Then, $\tilde{f}_*(\gamma) \circ \tau$ lies finitely close to a ray π that starts at $\tilde{f}_*(\gamma) \circ \tilde{f}(p)$ and belongs to the boundary point $\tilde{f}_*(\gamma) \circ \tilde{f}(\eta)$. We know that for $p \in X$, $\tilde{f}_*(\gamma) \circ \tilde{f}(p) = \tilde{f}(\gamma(p))$.

Hence, we have multiple rays that are finitely close to each other. In particular, ψ and π have the same starting point and are equivalent. Thus, they have the same boundary point, proving the statement.

□

In particular, that means that the extended, lifted map $\tilde{f} : X \to Y$ is stable under deck transformations. In Chapter 3, we defined a measure on the boundary which was quasi-invariant with respect to a group action. Recall that in Chapter 3, we fixed an orbit Θ that contained a point $x \in X$. If we chose another point x' in another orbit, the Poincaré series and the measures behave differently. Thus, we have a set of measures given by $\{\mu_x, x \in X/\Gamma\}$ which are all quasi-conformal measures. Let $\tilde{f} : X \cup \partial X \to Y \cup \partial Y$ be a homeomorphism. Then $\tilde{f}_*(\mu_x)$ defines a measure on the boundary ∂Y for each $x \in X/\Gamma$.

Definition 5.5. We define the **natural map on the universal cover** as follows:

$$\begin{aligned} \tilde{F} : X &\to Y \\ x &\longmapsto \mathrm{bar}(\tilde{f}_*(\mu_x)). \end{aligned}$$

As the map is constant on orbits of the deck transformations, we can pick a representative of each orbit, which amounts to the set M. We define the **natural map** as:

$$F : M \to N.$$

Observe that the map F also (or rather still) induces the isomorphism $\tilde{f}_*$ between the fundamental group and hence is homotopic to f.

6 Computations

6.1 Computing the natural map

Keeping in mind the definition of the natural map, we wish to study it more in depth now. For that sake, recall its definition given by $\tilde{F} : X \to Y$, $x \longmapsto \mathrm{bar}(f_*\mu_x)$. We will omit the tilde on the F in this section for neatness' sake. By the definition of the barycenter, the map F is the argument minimum of the term

$$\begin{aligned} &\int_{\partial Y} B_q(\xi, y) d(f_*\mu_x)(\xi) \\ = &\int_{\partial X} B_q(f(\alpha), y) d\mu_x(\alpha). \end{aligned}$$

Because it is the argument minimum, the derivative of this term is zero. We can write

$$\begin{aligned} d\left(\int_{\partial X} B_q(f(\alpha), F(x)) d\mu_x(\alpha)\right) &= 0 \\ \int_{\partial X} dB_q(f(\alpha), y) d\mu_x(\alpha) &= 0. \end{aligned}$$

Now, recall from quasi-conformality that we have the following estimates for the Radon-Nikodym derivatives

$$C^{-1} e^{-hB_p(\alpha, \gamma^{-1}x)} \leq \frac{d\mu_x}{d\mu_p} \leq C^{-1} e^{-hB_p(\alpha, \gamma^{-1}x)},$$

where we have $\gamma^{-1}x = p$. If we plug this equation into

$$\int_{\partial X} dB_q(f(\alpha), F(x)) \frac{d\mu_x}{d\mu_p} d\mu_p(\alpha) = 0,$$

we obtain

$$\int_{\partial X} dB_q(f(\alpha), F(x)) e^{-hB_p(\alpha, \gamma^{-1}x)} d\mu_x(\alpha) = 0.$$

Differentiating this statement by using the chain rule, we obtain a relationship between the derivative of F and Busemann functions on X.

$$\begin{aligned} D\left(\int_{\partial X} dB_q(f(\alpha), F(x)) e^{-hB_p(\alpha, \gamma^{-1}x)} d\mu_x(\alpha)\right) &= \\ \int_{\partial X} DdB_q(f(\alpha), F(x))(D_x F(u), v) e^{-hB_p(\alpha, \gamma^{-1}x)} d\mu_p(\alpha) & \\ -h \int_{\partial X} DdB_q(f(\alpha), F(x))(v) dB_p(\alpha, \gamma^{-1}x)(u) e^{-hB_p(\alpha, \gamma^{-1}x)} d\mu_x &= 0. \end{aligned}$$

Hence, we have $(u,v) \in TX_x \times TY_{F(x)}$. We now revert back to the measure μ_x

$$\begin{aligned} &\int_{\partial X} DdB_q(f(\alpha), F(x))(D_x F(u), v) d\mu_x(\alpha) \qquad (6.1)\\ &= h \int_{\partial X} DdB_q(f(\alpha)), F(x))(v) dB_p(\alpha, \gamma^{-1}x)(u) d\mu_x(\alpha). \end{aligned}$$

6.2 Symmetric spaces

This subsection gives an insight into symmetric spaces, of which a more in-depth insight is given in [Mer14].

Lemma 6.1. *Let N be an n-dimensional symmetric space with universal cover Y and dimension of the division algebra of N given by $k \in \{1,2,4,8\}$. Then we have $h(N) = n + k - 2$ for the entropy as defined in Definition 3.2.*

Proof. In order to compute the entropy, we want to compute $\mathrm{vol}(B(y,r))$. We denote the $(n-1)$-dimensional volume of the unit sphere in $\mathbb{R}^n$ by ω_{n-1}. Properties of rank one symmetric spaces allow us to make some statements on Y.

The space Y is one of the following: the real hyperbolic space H^m, or the complex hyperbolic space $\mathbb{C}H^m$ with $n = 2m$ or $\mathbb{H}H^m$ for $n = 4m$ or lastly the hyperbolic Cayley plane $\mathbb{O}H^2$ where $n = 16$. In all cases but the last the dimension n of the locally symmetric spaces is thus given by $n = km$ where $k \in \{1,2,4,8\}$.

In a locally symmetric space there exist $(1,1)$-tensor fields $J_1, ..., J_{k-1} : \Gamma(Y) \to \Gamma(Y)$ satisfying $J_i^2 = -id$ and $\langle J_i(V), J_i(W)\rangle = \langle V, W\rangle$ as well as being pairwise orthogonal. Next, the Jacobian of exp is given by

$$J \exp_y(tv) = \frac{\big(sn_{-1}(t)\big)^{n-1-(k-1)} \big(sn_{-4}(t)\big)^{k-1}}{t^{n-1}} = \frac{\sinh^{n-1}(t)\cosh^{k-1}(t)}{t^{n-1}}.$$

We refer to [Lan21], example 3.8, for the computation. Thus we have

$$\mathrm{vol}\big(B(y,r)\big) = \omega_{n-1} \int_0^r \sinh^{n-1}(t)\cosh^{k-1}(t) dt$$

and the entropy of any locally symmetric space N with universal cover is $h(N) = n + k - 2$. $\square$

6.3 First estimates

In order to understand the Jacobian JF we will make use of the gradient of the Busemann function B_p through two maps.

Definition 6.2. We define the maps $K_x : TY_y \to TY_y$ and $H_x : TY_y \to TY_y$ as follows:

- $\langle K_x(v), v\rangle = \int_{\partial Y} DdB_q(\theta, y)(v, v)d(f_*\mu_x)(\theta)$
- $\langle H_x(v), v\rangle = \int_{\partial Y} \Big(d\left(B_q(\theta, y)\right)(v)\Big)^2 d(f_*\mu_x)(\theta)$

Lemma 6.3. With Definition 6.2 and notation from before we have

$$|\langle K_x \circ DF_x(u), v\rangle| \leq h\langle H_x(v), v\rangle^{1/2} \left|\int_{\partial X} dB_p(\alpha, x)(u)^2 d\mu_x(\alpha)\right|^{1/2}$$

for every pair $(u, v) \in TX_x \times TY_{F(x)}$.

Proof. We see that

$$\langle K_x \circ DF_x(u), v\rangle = \int_{\partial Y} DdB_q(\theta, y))(DF_x(u), v)d(f_*\mu_x)(\theta),$$

where we suppose that $\theta = f(\alpha)$ and $y = F(x)$. This is equal to the right-hand side of (6.1), so we have

$$\begin{aligned}
&= h\int_{\partial X} DdB_q(\theta, y)(v)B_p(\alpha, x)d\mu_x(\alpha)\\
&\leq \sqrt{h^2\int_{\partial X}\left(DdB_q(\theta, y)(u)\right)^2 d\mu_x(\alpha)\int_{\partial X}\left(dB_p(\alpha, x)(u)\right)^2 d\mu_x(\alpha)}\\
&= h\sqrt{\int_{\partial X}\left(DdB_q(\theta, y)(u)\right)^2 d\mu_x(\alpha)}\sqrt{\int_{\partial X}\left(dB_p(\alpha, x)(u)\right)^2 d\mu_x(\alpha)}\\
&= h\langle H_x(v), v\rangle^{1/2}\left|\int_{\partial X}\left(dB_p(\alpha, x)(u)\right)^2 d\mu_x(\alpha)\right|^{1/2}
\end{aligned}$$

proving the statement.

∎

Lemma 6.4. We have

$$JF(x) \leq \frac{h(M)^n}{n^{n/2}}\frac{\det(H_x)^{1/2}}{\det(K_x)}$$

and additionally, H_x has trace 1.

Proof. Suppose the map dF_x is invertible. Otherwise, we have $JF(x) = 0$. Let $(v_i)_{i=0\ldots n-1}$ be an orthonormal basis of TY_y that diagonalizes H_x. We set $u_i' := (K_x \circ dF_x)^{-1}(v_i)$ and denote the orthonormal basis arising from u_i' by applying the Gram-Schmidt method simply by u_i. The matrix with respect to the bases v_i and u_i of

$K_x \circ dF_x$, now denoted by $(\lambda_{i,j})_{i,j}$, is triangular. This is the reason why we can write u_j as $u_j = \sum_{i=0}^{j} \lambda_{ij} u'_i$. Therefore, $JF(x)$ can be computed through the absolute value of the determinant of the matrix dF_x with respect to the bases v_i and u_i. By applying Lemma 6.3, we have

$$\begin{aligned}
\det(K_x)JF(x) &= |\det\left((\lambda_{i,j})_{i,j}\right)| \\
&= \prod_{i=1}^{n} |\langle K_x \circ dF_x(u_i), v_i\rangle| \\
&\le h^n \det(H_x)^{1/2} \prod_{i=1}^{n} \left(\int_{\partial X} dB_p(\theta, x)(u_i)^2 d\nu_x(\theta)\right)^{1/2} \\
&:= h^n \det(H_x)^{1/2} \prod_{i=1}^{n} \mathcal{B}_i^{1/2}.
\end{aligned}$$

We now want to study the terms $\mathcal{B}_i$. Observe that we have the well-known inequality linking the geometric and arithmetic means $\prod \mathcal{B}_i \le \left(\frac{1}{n}\sum \mathcal{B}_i\right)^n$ as well as

$$\sum_{i=1}^{n} \langle \operatorname{grad} B_p(\xi, x), u_i\rangle^2 = |\operatorname{grad} B_p(\xi, x)|^2 = 1,$$

so we have $\sum_i \mathcal{B}_i = 1$. We now compute as before:

$$\begin{aligned}
\det(K_x)JF(x) &\le h^n \det(H_x)^{1/2} \prod_i (\mathcal{B}_i)^{1/2} \\
&\le h^n \det(H_x)^{1/2} \left(\frac{1}{n}\sum_i \mathcal{B}_i\right)^{n/2} \\
&\le h^n \det(H_x)^{1/2} \frac{\left(\sum_i \mathcal{B}_i\right)^{n/2}}{n^{n/2}}.
\end{aligned}$$

After dividing by $\det(K_x)$ and plugging in $\sum_i \mathcal{B}_i = 1$, we get the claimed statement. □

Lemma 6.5. Let $B : Y \to \mathbb{R}$ be any Busemann function on a symmetric space Y of rank one as before and let $v \in TY_y$. Then, we have the following formula:

$$\operatorname{Hess}(B)_y(v, v) = |v|^2 - dB_y(v)^2 + \sum_{i=1}^{k-1} dB_y(J_i(v))^2.$$

Proof. Let $\{v_0, ..., v_{n-1}\}$ be an orthonormal basis of TY_y obtained by normalizing the vectors and set $v_0 = \operatorname{grad} B(y)$ and $v_i = J_i(v_0)$. Let $c : \mathbb{R} \to Y$ be a geodesic starting at y with derivative $c'(0) = v_0$. Let V_i be the parallel vector field along c with $V_i(0) = v_i$ and let us define the Jacobi fields W_i as

$$W_i(t) = e^{\kappa_i t} V_i(t),$$

where $\kappa_i = 2$ for $i \in \{1, ..., k-1\}$ and 1 for $i \in \{k, ..., n-1\}$.

For any geodesic curve $\sigma : (-\epsilon, \epsilon) \to Y$ with $\sigma'(0) = v_i$ there exist complete integral curves γ_s of the gradient $\operatorname{grad} B$ through $\sigma(s) = \gamma_s(0)$ that form a geodesic variation with variation vector field W_i along c. From that we infer

$$D_{v_i} \operatorname{grad} B = \frac{D}{ds}\Big|_{s=0} (\operatorname{grad} B \circ \sigma) = \frac{D}{dt}\Big|_{t=0} W_i = \kappa_i v_i.$$

Hence, by the definition of the Hessian, we have

$$\operatorname{Hess}(B)_y(v_i, v_i) = \langle D_{v_i}(\operatorname{grad} B), v_i \rangle = \kappa_i$$
$$\operatorname{Hess}(B)_y(v_0, v_0) = 0.$$

We can now check the identity for all basis vectors and then the general case follows handily. $\square$

6.4 Algebraic lemmas

Lemma 6.6. Let H be a symmetric positive definite $n \times n$ matrix with trace equal to 1 and $n \geq 3$. Let I denote the n-dimensional identity matrix. Then, for every positive real number $r \leq n-2$ we have

$$\frac{\det(H)}{\det\left(\frac{1}{n-1}(I - H)\right)^{r+1}} \leq n^{nr}$$

and equality holds if and only if $H = \frac{1}{n} I$.

Proof. We define the set

$$\Lambda := \{(\lambda_1, ..., \lambda_n) \in (0,1)^n : \sum_i \lambda_i = 1\}$$

and note that any $\lambda \in \Lambda$ is a possible sorted array of eigenvalues of H. Next, we study the component wise function ψ defined by

$$\psi(\lambda) := \ln(\lambda) - (r-1)\ln(1-\lambda).$$

Suppose there is a local maximum of ψ at the point $\mu \in \Lambda$. Then, the gradient at μ would be equal to a perpendicular vector $(c, ..., c)$, meaning that $\psi'(\mu_i) = c$ for all $i = 1...n$. For every distinct pair of canonical basis vectors e_i and e_j we note

$$0 \geq \frac{d^s}{ds^2}\Big|_{s=0} \psi(\mu + s(e_i - e_j)) = \frac{d}{ds}(\psi'(\mu_i + s) - \psi'(\mu_j - s)$$
$$0 \geq \psi''(\mu_i) + \psi''(\mu_j).$$

Next, we observe that there exists a $t \in (0,1)$ so that $\psi'' < 0$ on the interval $(0,t)$ and $\psi'' \geq 0$ on the interval $(t,1)$. Therefore there exist exactly one a and b in their respective intervals so that $\psi'(a) = c = \psi'(b)$. If $a < b$, which is always the case unless they are equal, then $\psi''(b) > 0$ and at most one eigenvalue is equal to b. Hence, in any case there is an index l so that $\mu_i = a$ for all $i \neq l$ and $\mu_l \in \{a, b\}$. So we have $\psi'(\mu_l) = \psi'(a) = \frac{1+\mu r}{\mu_l}(1-\mu_l)$. This implies that

$$(\mu_l - a)(1 - \mu_l - a - a\mu_l r) = 0,$$

where we see that due to $(1-\mu_l) = (n-1)a$ and $n-2 \geq r$, the second term in this equation is strictly greater than zero. This implies $\mu_l = a$, too. Moreover, in the equality case, we have $\lambda_i = \frac{1}{n}$ for all is.

Now, we analyze the values near the boundary. Let us fix an index $j \in \{1, ..., n\}$. By the geometric-arithmetic mean inequality, we have

$$\frac{\prod_i \lambda_i}{\prod_i \left(\frac{1}{n-1}(1-\lambda_i)\right)^{r-1}} \leq (n-1)^{nr+1} \frac{\lambda_j (1-\lambda_j)^{n-r-2}}{\prod_{i \neq j} (1-\lambda_i)^{r-1}}.$$

If all λ_i with $i \neq j$ stay bounded away from 1 and $\lambda_j \to 0$, the term on the right tends to zero.

If we have $\lambda_j \geq 1-\epsilon$, then $\lambda_i \leq \epsilon$ for all $i \neq j$. This gives us

$$(n-1)^{nr+1}(1-\epsilon)^{(r+1)(1-n)}\epsilon^{n-r-2}$$

as an upper bound, which tends to zero if ϵ tends to 0 for $r < n-2$ and tends to $(n-1)^{nr+1}$ for $r = n-2$. By the following computation we can check that $(n-1)^{nr+1} < n^{nr}$:

$$\left(\frac{n}{n-1}\right)^{nr} = \left(1 + \frac{1}{n-1}\right)^{n(n-2)} \geq 1 + \frac{n(n-2)}{n-1} > n-1.$$

This then establishes the statement. $\square$

Lemma 6.7. Suppose A and B are two symmetric positive definite $n \times n$ matrices. Then we have

$$\det\big((1-t)A + tB\big) \geq \det(A)^{t-1}\det(B)^t$$

for all $t \in [0,1]$. That is to say that the determinant is log-concave on the set of symmetric positive matrices.

Proof. [BCG95] Let A and B as in the statement. We pick an orthonormal basis a_i, so that A can be written as the product of the transposes of the basis matrix: $A = P^T P$.

Then, as B is diagonalizable, we also have $B = P^T DP$ where D is a diagonal matrix with entries d_{ii}. This results in

$$\det(tA + (1-t)B) = det(P)^2 \prod_{i=1}^{n} (t + (1-t)d_{ii}).$$

This function is concave on $\mathbb{R}$ for any set of positive numbers d_{ii} and thus, the statement is proven. □

6.5 Final preparations

For the sake of recapitulation, let us state the whole set up. Let M and N be two closed Riemannian manifolds and suppose N is locally symmetric with negative sectional curvature. Let the universal covers of M and N be denoted X and Y respectively. Next, suppose there is a continuous map of degree one, which we call f. This then readily translates to an isomorphism between subgroups of the isomorphism groups of X and Y. We denote this through $\psi : \Gamma \to \Gamma'$. Given that Y is negatively curved, in particular Gromov hyperbolic, X is as well and we can define the boundary of both X and Y. For a measure on ∂Y we established the barycenter method that assigned a point $y \in Y$ to such a measure. Using this and the map arising from the isomorphism of isometries, we were able to define the natural map F as the barycenter of the f-pushforward of the Patterson-Sullivan measure on X. Recall that the natural map F remained *insensitive* with respect the group action of Γ and Γ', which will allow us to make statements on the original manifolds M and N. We then proceeded on defining and studying the Jacobian JF, so that we are in the process of working towards a way of showing a bound on the Jacobian. Now that the formalism is in place, we can prove the following lemma.

Lemma 6.8. Suppose the whole setup and notation as before and let $x \in X$. Then we have $JF(x) \leq 1$ and if $JF(x) = 1$ is satisfied for some $x \in X$, and subsequently we have $JF(x) = 1$ for all $x \in X$.

Proof. Let us set again $F(x) =: y$. By integrating the identity of the Hessian of the Busemann function obtained in Lemma 6.4 against the measure $f_*\nu_x$ we obtain

$$K_x = 1 - H_x \sum_{i=1}^{k-1} J_i H_x J_i.$$

We now set $L_x = \frac{1}{n-1}$ and, because the space N is locally symmetric, we have $h(N) =$

$n+k-2$. We write $\frac{1}{h}K_x$ as the following convex combination:

$$\frac{1}{h}K_x = \frac{n-1}{h}L_x + \frac{1}{h}\sum_{i=1}^{k-1}(-J_iH_xJ_i)\,.$$

In the case $k=1$, this means $\frac{1}{h}K_x = L_x$. Recall that we assume that $JF(x) > 0$. From Lemma 6.6 we know that both H_x and L_x have positive eigenvalues and trace 1. We conclude that

$$\det\left(\frac{1}{h}K_x\right) \geq \det(L_x)^{(n-1)/h}\det(H_x)^{k-1/h}.$$

Inserting this term of $\det(K_x)$ into Lemma 6.4 yields

$$JF(x) \leq \frac{\det(H_x)^{1/2-(k-1)/h}}{n^{n/2}\det(L_x)^{(n-1)/h}} \leq 1,$$

where the last inequality is due to Lemma 6.6 for $r = \frac{h}{n-k}$, independently of $x \in X$. The exponents of $\det(H_x)$ and $\det(L_x)$ can be written as follows:

$$\begin{aligned}\frac{1}{2} - \frac{k-1}{h} &= \frac{1}{2r},\\ \frac{n-1}{h} &= \frac{r+1}{2r}.\end{aligned}$$

Suppose now that $JF(x)=1$ Then, by Lemma 6.6, we have $H_x = \frac{1}{n}I$. Hence,

$$K_x = I - H_x + \sum_{i=1}^{k-1}(-J_iH_xJ_i) = \frac{n+k-2}{n}I = \frac{h}{n}I.$$

Thus, the inequality in Lemma 6.3 reduces to

$$|dF_x(u)|^2 \leq n\int_{\partial X} d(B_p(\xi,x))(u)^2 d\nu_x(\xi).$$

Let now u_i be an orthonormal basis of TX_x. Let $G := \big(\langle dF_x(u_i), dF_x(u_j)\rangle\big)$ be the Gram matrix. Then we have $\mathrm{trace}(G) \leq n$ due to $|\operatorname{grad}(B_p(\xi,x))| = 1$ and $\nu_x(\partial X) = 1$. In the end we thus have

$$1 = JF(x)^2 = \det(G) \leq \left(\frac{1}{n}\mathrm{trace}(G)\right)^n \leq 1.$$

The equality requires G to be a matrix of the form $(\delta_{ij})_{i,j}$ and thus dF_x is an isometry.

$\square$

7 The proof of the theorem

We have now collected all the tools required to prove the statement. We will state the statement again, and then prove it.

Theorem 7.1. First off, let $n \geq 3$. Next, let N be a compact, connected, locally symmetric, n-dimensional, Riemannian manifold with negative curvature, and let M be any compact, connected, n-dimensional Riemannian manifold. Assume that there exists a homotopy equivalence between M and N which we denote $f : M \to N$. We then have

(1) $h^n(M)\mathrm{vol}(M) \geq h^n(N)\mathrm{vol}(N)$.

(2) If $h(M) = h(N)$ and $\mathrm{vol}(M) = \mathrm{vol}(N)$, then the manifolds are isometric.

Proof. First off, we note that the quantity $h(\cdot)^n\,\mathrm{vol}(\cdot)$ is scale invariant. Next, we normalize the metrics on M and N so that we have $h(N) = n+k-2$ and $h(M) = h(N)$. In Section 5.1 we established that the universal cover Y is Gromov hyperbolic and Lemma 5.3 showed us that there is a quasi-isometry between X and Y. Theorem 2.11 allowed us to recover Gromov hyperbolicity of the universal cover X of M. Section 2.4 introduced us to the notions of the boundary and in Theorem 3.15 we were able to show that there is a set of measures on the boundary of a Gromov hyperbolic space that is quasi-conformal. We called these measures the Patterson-Sullivan measures. Recall that the Radon-Nikodym derivatives of the measures behave well enough: conformal when changing the basepoint and quasi-conformal with respect to the pushforward of an isometry. We can thus assign to each point on X a measure in this family.

Recall the study of the so-called barycenters in Section 4. The barycenter is the inverse notion of the Patterson-Sullivan method, so to speak, as we can use it to assign a point in the underlying space to a probability measure on the boundary.

A further consequence of the statement was that the map f behaved well with respect to the fundamental group and universal coverings. We leveraged the assumptions in the statement in order to compute a homeomorphism on the boundary of the universal covers that was equivariant with respect to elements of the fundamental group, which become isometries when lifted to the universal cover.

With our new knowledge on the Patterson-Sullivan measure, equivariance of the map f and lastly the barycenter, we were able to define the so-called natural map $\tilde{F}$ by taking the barycenter of the Patterson-Sullivan measure on X pushed forward through f.

$$\tilde{F} : X \to Y$$

In Section 6 we studied the map $\tilde{F}$ with the goal of showing that its Jacobian is less than 1 as well as showing that if the Jacobian is 1 at some point, it is 1 everywhere. All computations in that section were made pointwise, so that the equivalent statements hold on the map $F : M \to N$.

The main step is then Lemma 6.8. By pulling down the map $F : X \to Y$ to a surjective map $F : M \to N$, we get the following computation

$$\operatorname{vol}(N) \leq \int_N \#F^{-1}\{q\} d \operatorname{vol}(q) = \int_M JF(p) d \operatorname{vol}(p) \leq \operatorname{vol}(M).$$

In the case that $JF(x) = 1$ everywhere, we have equalities and $\#F^{-1}\{q\} = 1$ everywhere.

We have thus proven the statement. $\square$

Appendices

Appendix A

The equivalent definitions of Gromov hyperbolic spaces

This appendix relies on chapter 1 of [CDP90] and aims to establish some properties and equivalent definitions of Gromov hyperbolic spaces.

Definition A.1. Let X be a geodesic metric space. A **geodesic triangle** is the union of three geodesic segments which we denote by $[x, y]$, $[y, z]$ and $[z, x]$ in what is to follow. We normally call a geodesic triangle $\Delta := [x, y, z]$. Lastly, we call these segments the **edges** and the points x, y, z the **vertices** of the geodesic triangle.

Definition A.2. The **tripod** of a geodesic triangle is the metric space obtained by the following procedure: Suppose we have three segments of length equal to the three Gromov products obtained when plugging the vertices in each entry of the Gromov product. (For example $(x_1, x_2)_{x_3}$ and $(x_2, x_3)_{x_1}$ and $(x_3, x_1)_{x_2}$.) The metric space obtained by connecting the segments isometrically at one point in the shape of a Y is called the tripod of the geodesic triangle. We denote it by T_Δ. There exists a map $f : \Delta \to T_\Delta$, so that $f|_{[x_i, x_j]}$ is an isometry for each pair of vertices $\{x_i, x_j\}$ with $i \neq j \in \{1, 2, 3\}$.

Definition A.3. We call a geodesic triangle Δ **δ-thin** if for each $y_1, y_2 \in \Delta$ satisfying $f(y_1) = f(y_2)$ under the map f from Definition A.2, we have $d(y_1, y_2) \leq \delta$.

Proposition A.4. Let X be a metric space. We observe:

1. If X is a δ-hyperbolic metric space, then all geodesic triangles are 4δ-thin.

2. If all geodesic triangles are δ-thin, then X is a δ-hyperbolic space.

Proof. Let us define a geodesic triangle by $\Delta = [x_0, x_1, x_2]$. Next, let $x \in [x_0, x_1]$ and $y \in [x_1, x_2]$ so that $f(x) = f(y)$. Picking x_0 as base point, δ-hyperbolicity implies

$$(x, y)_{x_0} \geq \min\{(x, x_1)_{x_0}, (x_1, x_2)_{x_0}, (x_2, y)_{x_0}\} - 2\delta.$$

Thus, we have

$$(x, x_1)_{x_0} = (y, x_2)_{x_0} = d(x, x_0) = d(y, x_0) \leq (x_1, x_2)_{x_0}.$$

This then implies

$$(x, y)_{x_0} = \frac{1}{2}(d(x, x_0) - d(x, y)) \geq d(x, x_0) - 2\delta,$$

which implies

$$d(x, y) \leq 4\delta,$$

meaning that geodesic triangles are 4δ-thin.

For the second statement, let again $x_0 \in X$ be a base point and let $x, y, z \in X$. Consider the geodesic triangles given by $[x_0, x, y]$, $[x_0, x, z]$ and $[x_0, y, z]$. Pick points x', y' as well as z' so that

$$d(x_0, x') = d(x_0, y') = d(x_0, z') = \min\{(x, z)_{x_0}, (y, z)_{x_0}\}.$$

As the triangles $[x_0, x, z]$ and $[x_0, y, z]$ are δ-slim, we have $d(x', z') < \delta$ and $d(y', z') < \delta$. This then implies $d(x', y') < 2\delta$. By the triangle inequality, we obtain

$$\begin{aligned} d(x, y) &\leq d(x, x') + d(x', y') + d(y', y) \\ &\leq d(x_0, x) + d(x_0, y) - 2\min\{(x, z)_{x_0}, (y, z)_{x_0}\} + d(x', y'). \end{aligned}$$

Applying $d(x', y') < 2\delta$, we have

$$(x, y)_{x_0} \geq \min\{(x, z)_{x_0}, (y, z)_{x_0}\} - \delta,$$

which establishes the statement.

□

Definition A.5. We define the preimage of the informal *midpoint* of the tripod as the **internal points** c_1, c_2, c_3 of $\Delta = [x_1, x_2, x_3]$. We label each of the three internal points by the index that does not appear in the segment on which it lies, that is to say that c_2, for example, lies on $[x_1, x_3]$.

Definition A.6. The **insize** of a geodesic triangle Δ is defined as

$$\max_{i,j=1,2,3} d(c_i, c_j) =: \text{insize}(\Delta).$$

Proposition A.7. Let X be a metric space. We observe:

1. The space X is δ-hyperbolic implies that $\text{insize}(\Delta) \leq 4\delta$.

2. If we have $\text{insize}(\Delta) \leq \delta$, then the space X is δ-hyperbolic.

Proof. For the first statement, we know that all geodesic triangles are 4δ-thin by Proposition A.4.

For the second statement, assume that $\text{insize}(\Delta) \leq \delta$. We show that Δ is δ-thin, which will then establish the statement, again by Proposition A.4. Let $[x_0, x_1, y_1]$ be a geodesic triangle. Let $x \in [x_0, x_1]$ and $y \in [x_0, y_1]$, so that $f_\Delta(x) = f_\Delta(y)$. This then implies that $d(x_0, x) = d(x_0, y) \leq (x_1, y_1)_{x_0}$. For any $t \in [0, 1]$ we define the points $x_t \in [x_0, x_1]$ and $y_t \in [x_0, y_1]$, so that $d(x_0, x_t) = t$ and $d(x_0, y_t) = t$, where we parametrized the speed of the geodesic segments accordingly. By the mean value theorem, there exists an $\alpha \in [0, 1]$, so that

$$(x_\alpha, y_\alpha)_{x_0} = d(x_0, x) = d(x_0, y).$$

Then, the geodesic triangle $\Delta' = [x_0, x_\alpha, y_\alpha]$, also satisfying $\text{insize}(\Delta') < \delta$, yields $d(x, y) \leq \delta$. $\square$

Definition A.8. We define the quantity **minsize**(Δ) of a geodesic triangle $\Delta = [x_1, x_2, x_3]$ as

$$\text{minsize}(\Delta) := \min_{i,j,k} \left\{ \max_{y_i} d(y_i, x_i) \right\}$$

for all $y_i \in [x_j, x_k]$ for any $i \neq j \neq k \in \{1, 2, 3\}$. Due to compactness, the maximum is always achievable.

Lemma A.9. In a δ-hyperbolic space X, let Δ be a geodesic triangle. We have the following inequality:

$$\text{minsize}(\Delta) \leq \text{insize}(\Delta) \leq 4 \text{ minsize}(\Delta).$$

Proof. The left inequality follows by definition. The right one does not. Let $\Delta = [x_1, x_2, x_3]$ be a geodesic triangle and let us define $y_i \in [x_j, x_k]$ and the internal points

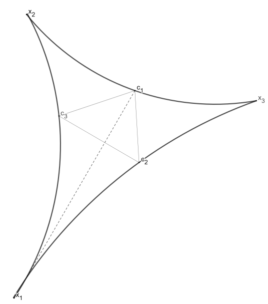

Figure A.1: The insize is bounded by the minsize.

c_1, c_2, c_3. We set $d_{ij} = d(x_i, y_j)$ and $d'_{i,j} = d(x_i, c_i)$ for $i \neq j$. We then have the following relations: $d_{ik} + d_{jk} = d'_{ik} + d'_{jk} = d(x_i, x_j)$ for all $i \neq j$ and $d'_{ij} = d'_{ik}$. Hence,

$$d_{12} - d'_{12} = \frac{1}{2}\left((d_{12} - d_{13}) + (d_{21} - d_{23}) + (d_{31} - d_{32})\right),$$

as well as the same statements obtained by permuting the indices.

By the triangle inequality, we have $|d_{ij} - d_{ik}| \leq d(y_j, y_k)$. This then implies $d(y_j, c_j) = |d_{ij} - d'_{ij}| \leq \frac{3}{2}\mathrm{diam}(\{y_1, y_2, y_3\})$. We now have

$$d(c_i, c_j) \leq d(c_i, y_i) + d(y_i, y_j) + d(y_j, c_j) \leq 4\ \mathrm{diam}(\{y_1, y_2, y_3\}),$$

which establishes the proof.

□

Proposition A.10. Let X be a metric space. We observe:

1. The metric space X is δ-hyperbolic. This implies that all geodesic triangles have $\mathrm{minsize}(\Delta) \leq 4\delta$.

2. If all geodesic triangles in a metric space have $\mathrm{minsize}(\Delta) \leq \delta$, then the metric space is δ-hyperbolic.

Proof. This statement follows from the two inequalities in Lemma A.9. □

Definition A.11. A geodesic triangle $\Delta = [x, y, z]$ is said to be δ**-slim** if any edge is contained in the δ-neighbourhood of the other two.

Lemma A.12. In a metric space X, all δ-slim triangles have $\mathrm{minsize}(\Delta) \leq 2\delta$.

Proof. Let $[x, y, z]$ be a geodesic triangle so that $[x, z] \subseteq B([x, y] \cup [y, z], \delta)$. We define the function $f : [x, z] \to \mathbb{R}$ by $f(m) = d(m, [x, y]) - d(m, [y, z])$. Immediately, we see that $f(x) \leq 0$ and $f(z) \geq 0$. Therefore, there must exist an m so that $f(m) = 0$. This implies that $d(m, [x, z] = d(m, [y, z]) \leq \delta$. Thus, there exist p and q in $[x, z]$ and $[y, z]$ respectively, so that $d(p, m) \leq \delta$ and $d(q, m) \leq \delta$. Thus, the diameter of the circle going through $\{p, q, m\}$ is less than or equal to 2δ, which proves the statement. $\square$

Proposition A.13. Let X be a metric space. We observe:

1. If X is a δ-hyperbolic space, then all geodesic triangles are δ-slim.
2. If all geodesic triangles are δ-slim, then X is a 8δ-hyperbolic space.

Proof. For the first statement, use Proposition A.4 to show that the entirety of the edge of a δ-thin geodesic triangle is contained in a δ-neighbourhood of the other two sides.

For the second statement, use Proposition A.10 as well as Lemma A.12 in that order. $\square$

Definition A.14. We define $\mathbb{H}^n$ to be the, up to isometries, unique, complete and simply connected Riemannian manifold of dimension n and constant sectional curvature -1.

Lemma A.15. The space $\mathbb{H}^2$ is δ-slim with $\delta = \log 3$.

Proof. Let $[x, y, z]$ be a geodesic triangle in $\mathbb{H}^2$. Increase the size of the triangle in a regular way, for example by moving up 25% the continuation of the clockwise-adjacent segment for each corner. Then, the diameter of the incircle does not decrease at any step. By going to a limit with this process, we obtain the biggest possible triangle, which turns out to be unique up to isometries. In the Poincaré upper half-plane realization, this would coincide with a geodesic triangle given by $[-1, 1, \infty]$. Let us study how far point i lies from the other two edges. The Euclidean distance is $|(i + 1) - i| = 1$. The hyperbolic distance can be computed by using the Poincaré metric. As we do not know the exact closest point to i on the vertical geodesic, due to symmetry, we compute it as half the distance between i and $i + 2$, notably

$$d(u, v) := 2 \tanh^{-1} \left(\frac{|u - v|}{|u - \bar{v}|} \right).$$

Consequently, we have

$$d(i, i + 2) = 2 \tanh^{-1} \left(\frac{|i + 2 - i|}{|i + 2 + i|} \right) = 2 \tanh^{-1}(1/\sqrt{2}) = 2 \ln(1 + \sqrt{2}).$$

This means that the bottom edge of the triangle is at most $\ln(1+\sqrt{2})$ far away from the two other edges. Thus, we can conclude that in the space H^2, all triangles are $\ln(1+\sqrt{2})$-slim. $\square$

Corollary A.16. The space $\mathbb{H}^2$ is δ-hyperbolic with $\delta = \log(1+\sqrt{2})$. The same holds for $\mathbb{H}^n$.

Theorem A.17. Every simply connected, complete Riemannian manifold with sectional curvature bounded from above by $-a^2 < 0$ is hyperbolic with $\delta = a^2 \ln(1-\sqrt{2})$.

In order to prove this statement we will need to apply the Alexandrov-Toponogov comparison theorem as seen in [Lan21], chapter 5. For this, let $S(a)$ be the simply connected hypersurface of constant curvature $-a^2$. For a geodesic triangle $\Delta = [x_1, x_2, x_3]$ in X, we define its counterpart $\Delta(a) = [m_1, m_2, m_3]$, a geodesic triangle in $S(a)$ together with a map f_Δ, so that $d(x_i, x_j) = d(m_i, m_j)$ and $f|_{[x_i,x_j]}$ is an isometry for all pairs $i \neq j \in \{1, 2, 3\}$. The Alexandrov-Toponogov comparison theorem then yields $d(y, z) \leq d(f(y), f(z))$ for all $y, z \in \Delta$.

We now proceed to the proof of the Theorem A.17:

Proof. Distances between points in $S(a)$ can be computed by multiplying distances by a^{-1} in $\mathbb{H}^n$, as it is just a scaling of the original space. Geodesic triangles in $S(a)$ are thus $\delta = (a^{-1}\ln(1+\sqrt{2})3)$-slim by Corollary A.16. By applying the map f and the Alexandrov-Toponogov comparison theorem, the statement then follows. $\square$

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