CS492: Probabilistic Programming

Denotational Semantics of Probabilistic Programs

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KAIST

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      y (observe (normal x 1) 2)]
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2. Compiler optimisation.
3. Detection of ill-defined models.
4. Clear meaning of complex models.

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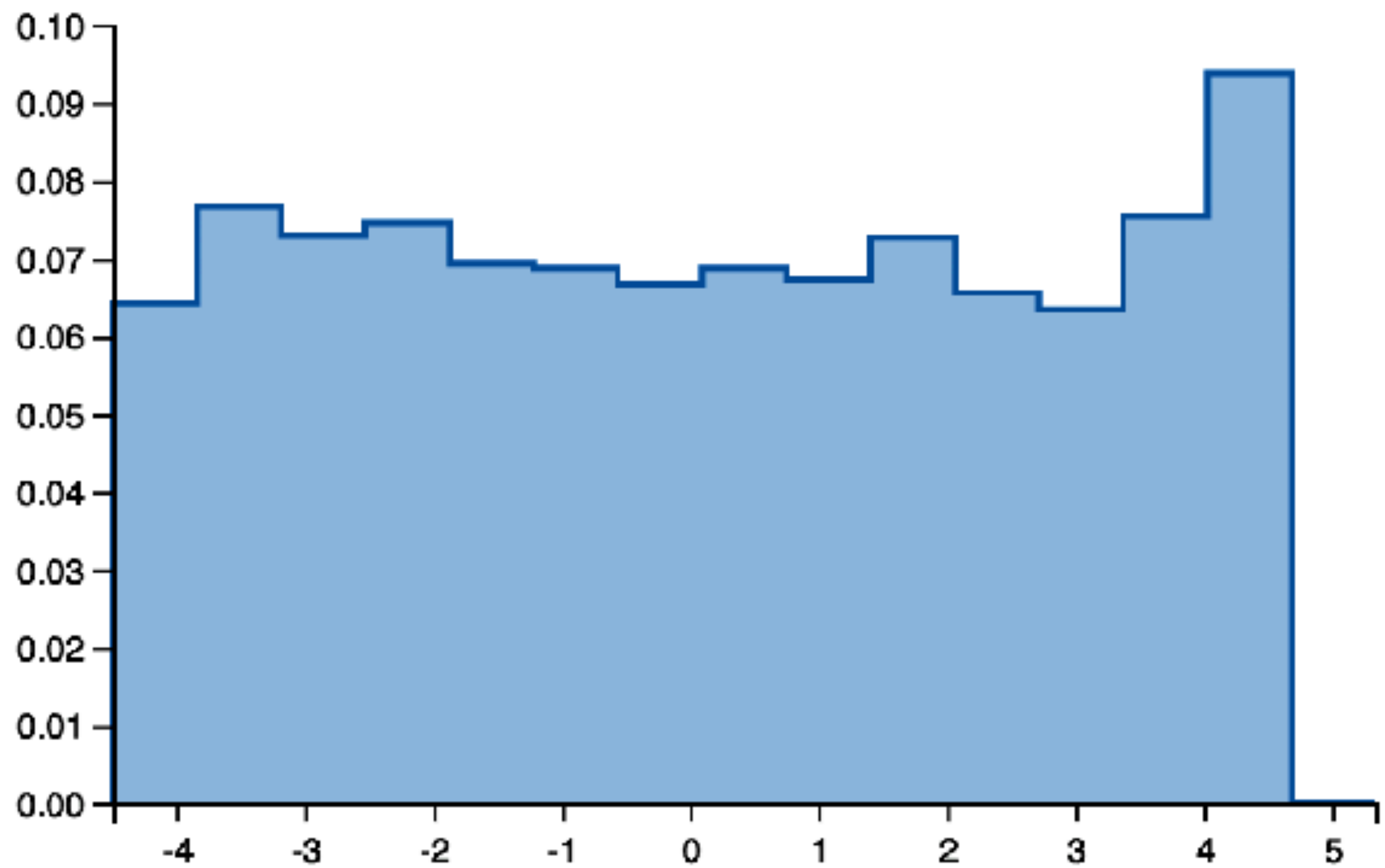
```
(def lazy-samples (doquery :lmh example1 []))
(def samples (map :result (take-nth 100 (take 500000 (drop 100000 lazy-samples)))))
(plot/histogram samples :normalize :probability)
```

```
#'uppsala-pp17/lazy-samples

#'uppsala-pp17/samples
```

5K samples



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```
(def lazy-samples (doquery :lmh example1 []))
(def samples (map :result (take-nth 100 (take 1000000 (drop 100000 lazy-samples)))))
(plot/histogram samples :normalize :probability)
```

```
#'uppsala-pp17/lazy-samples

#'uppsala-pp17/samples
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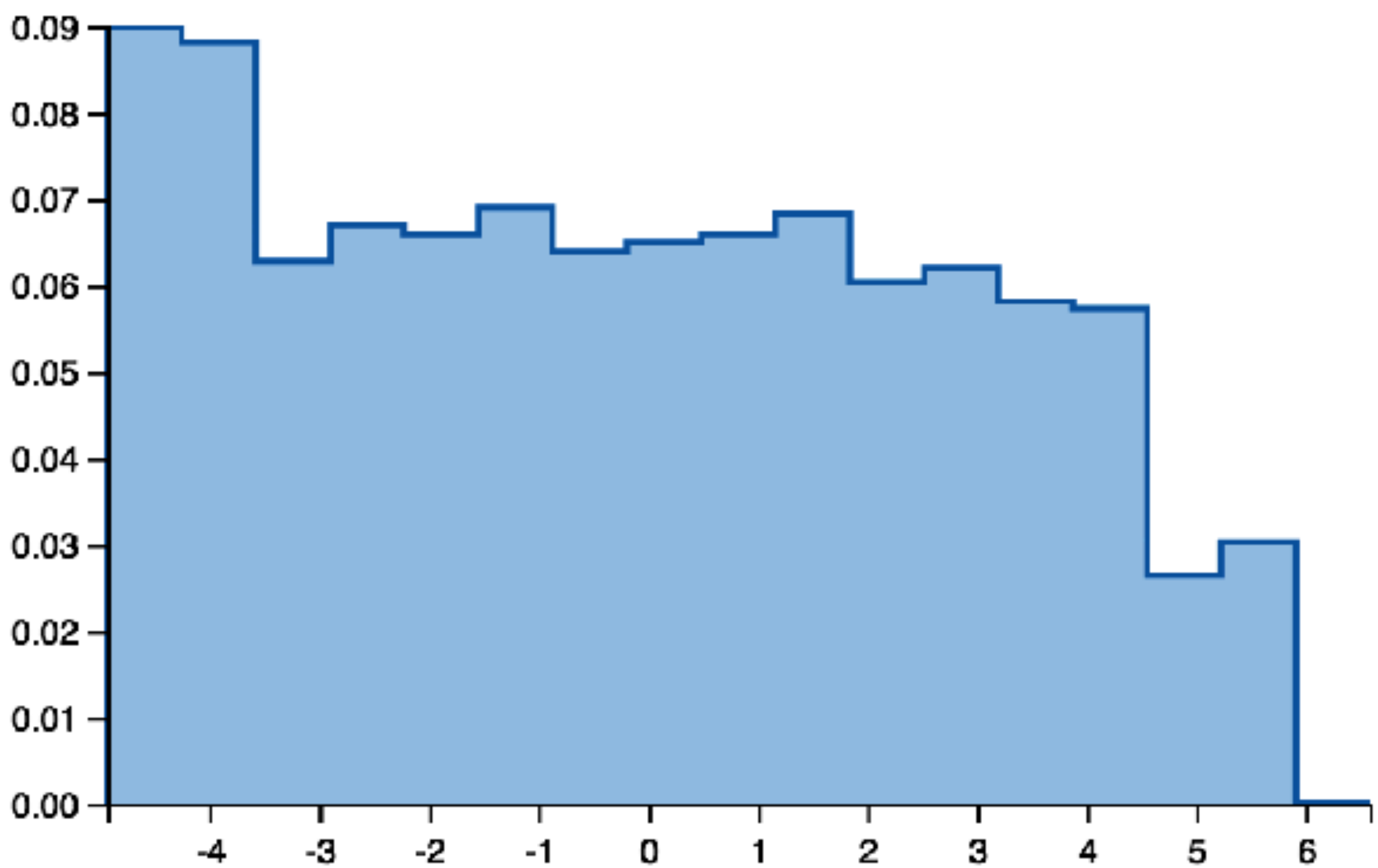
10K samples



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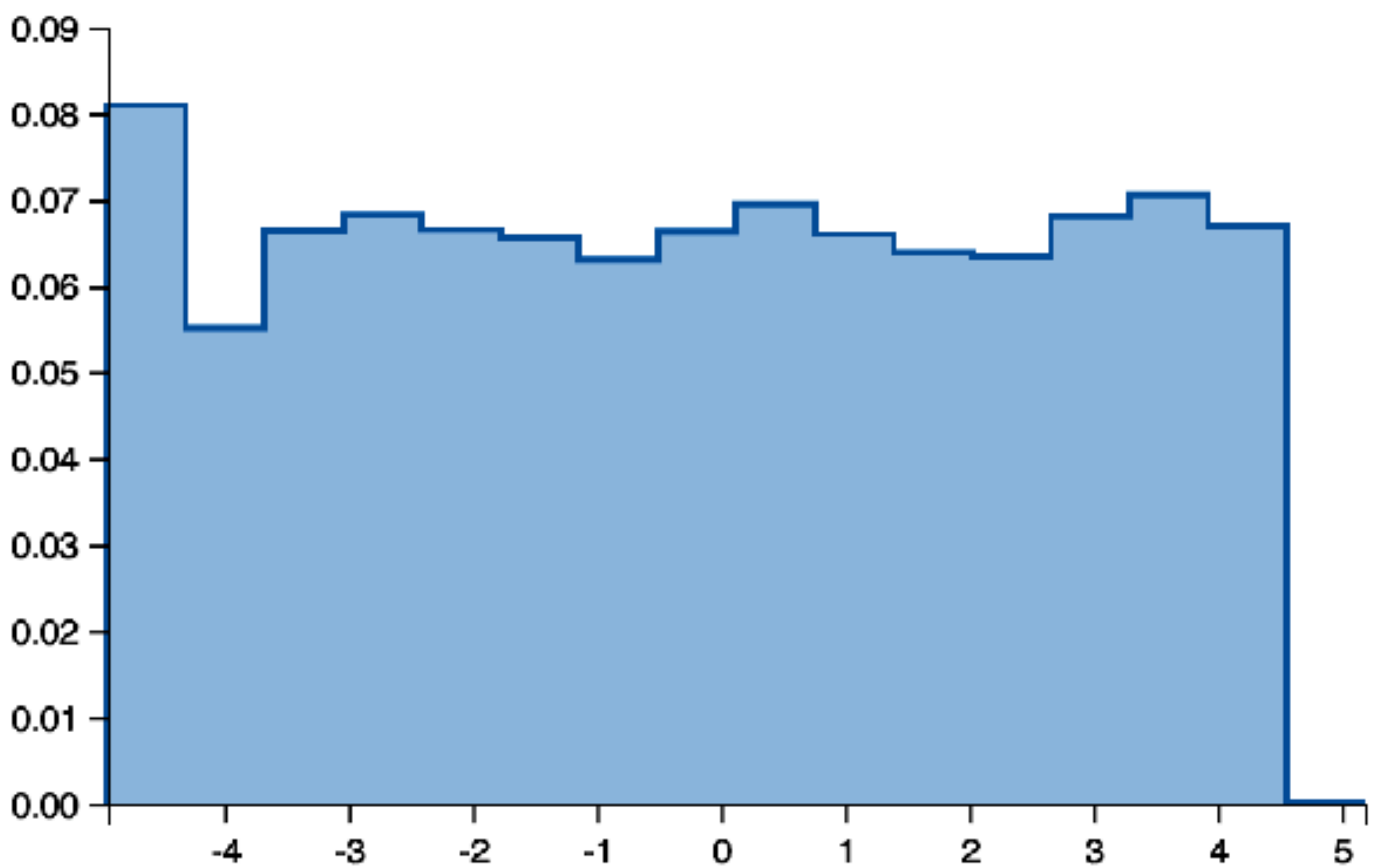
```
(def lazy-samples (doquery :lmh example1 []))
(def samples (map :result (take-nth 100 (take 1500000 (drop 100000 lazy-samples)))))
(plot/histogram samples :normalize :probability)
```

#'uppsala-pp17/lazy-samples

#'uppsala-pp17/samples

15K samples



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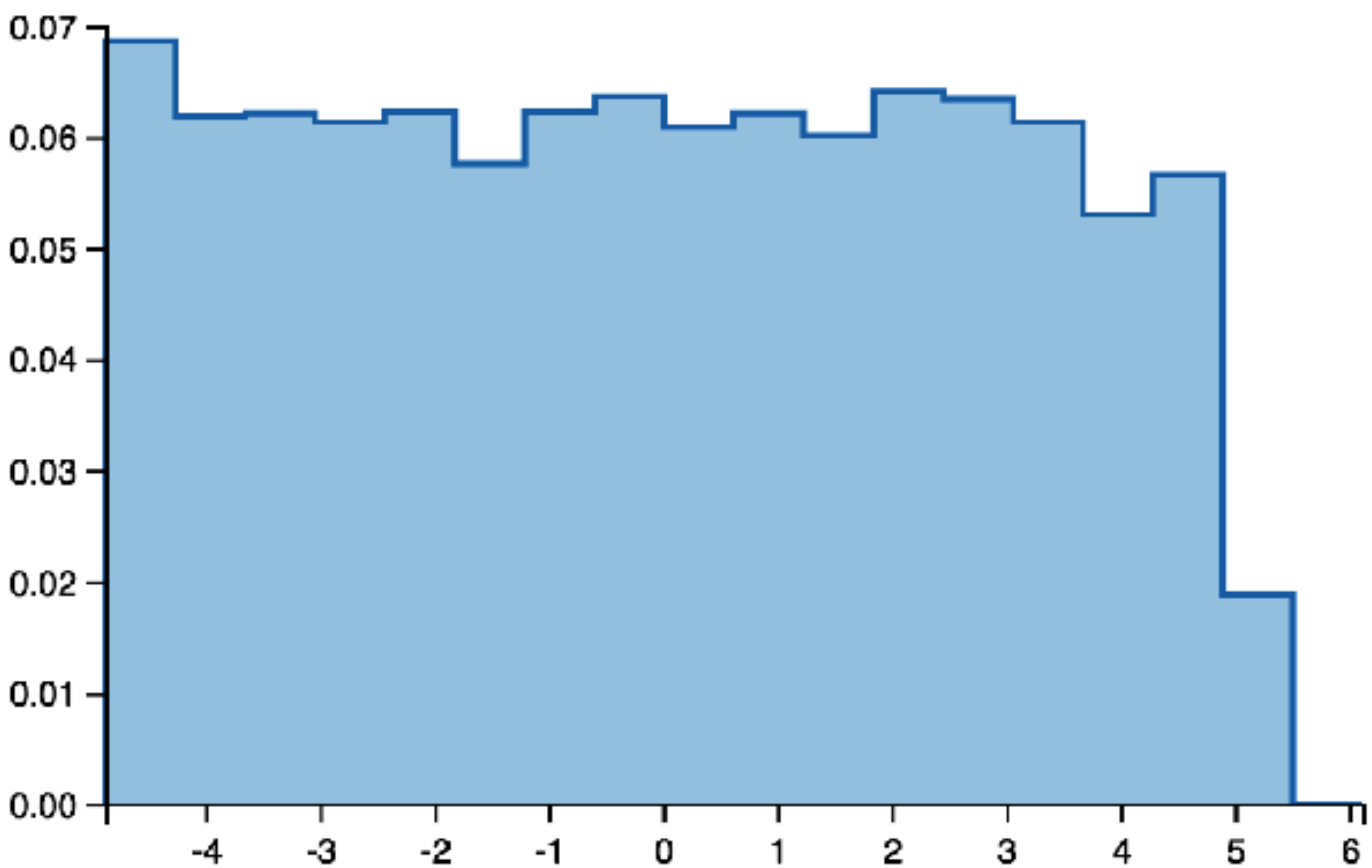
20K samples



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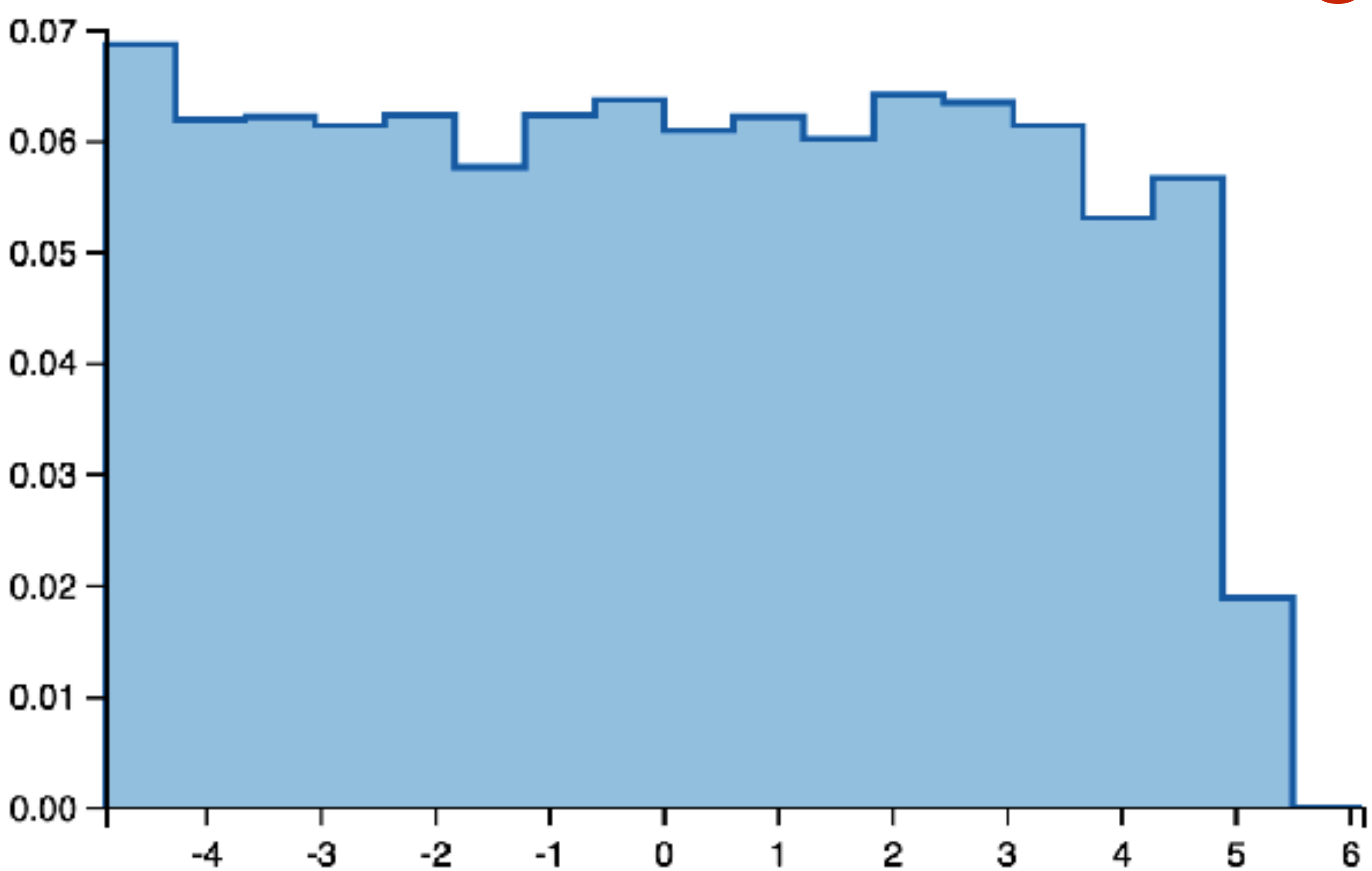

20K samples

Uniform[-6,6]?

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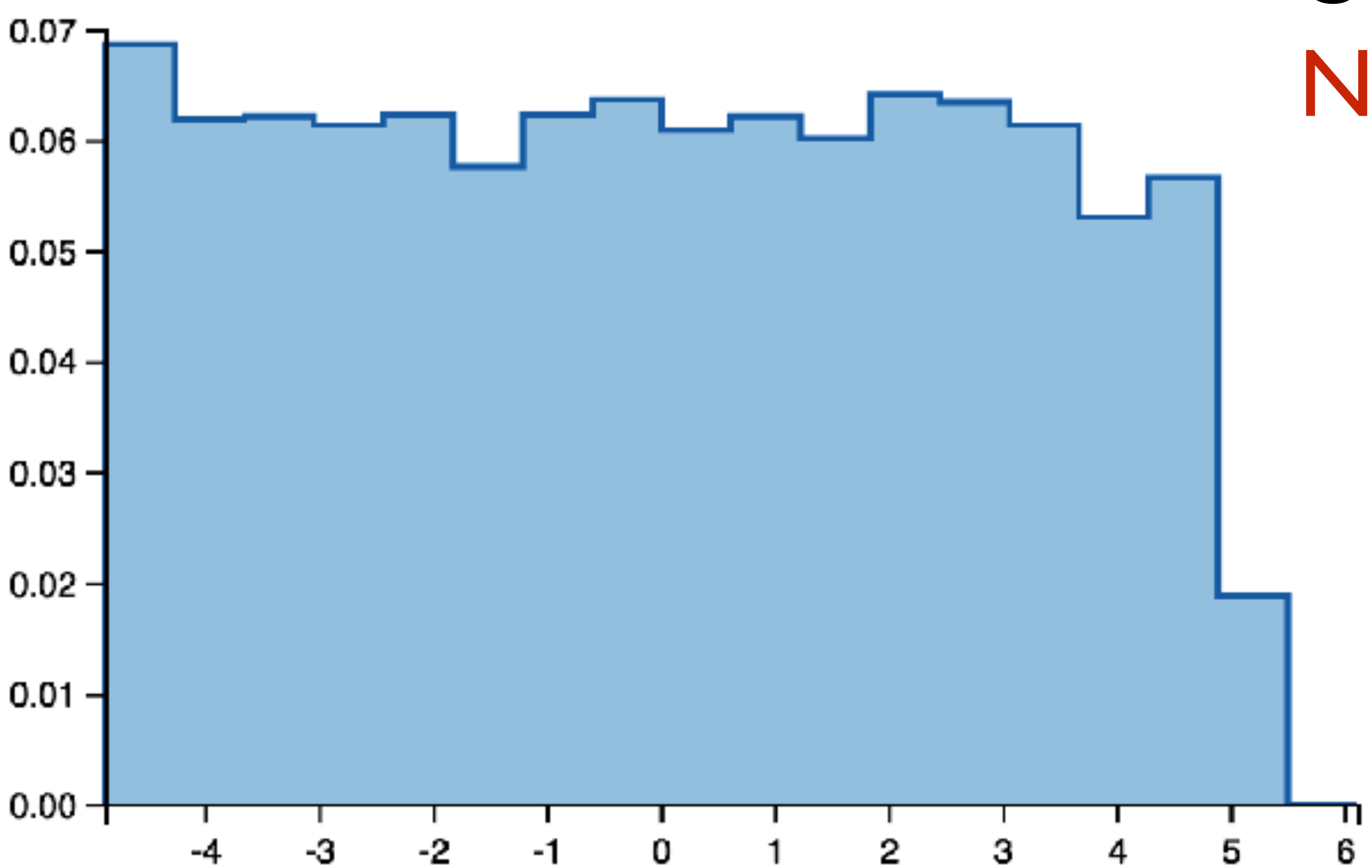

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No posterior.

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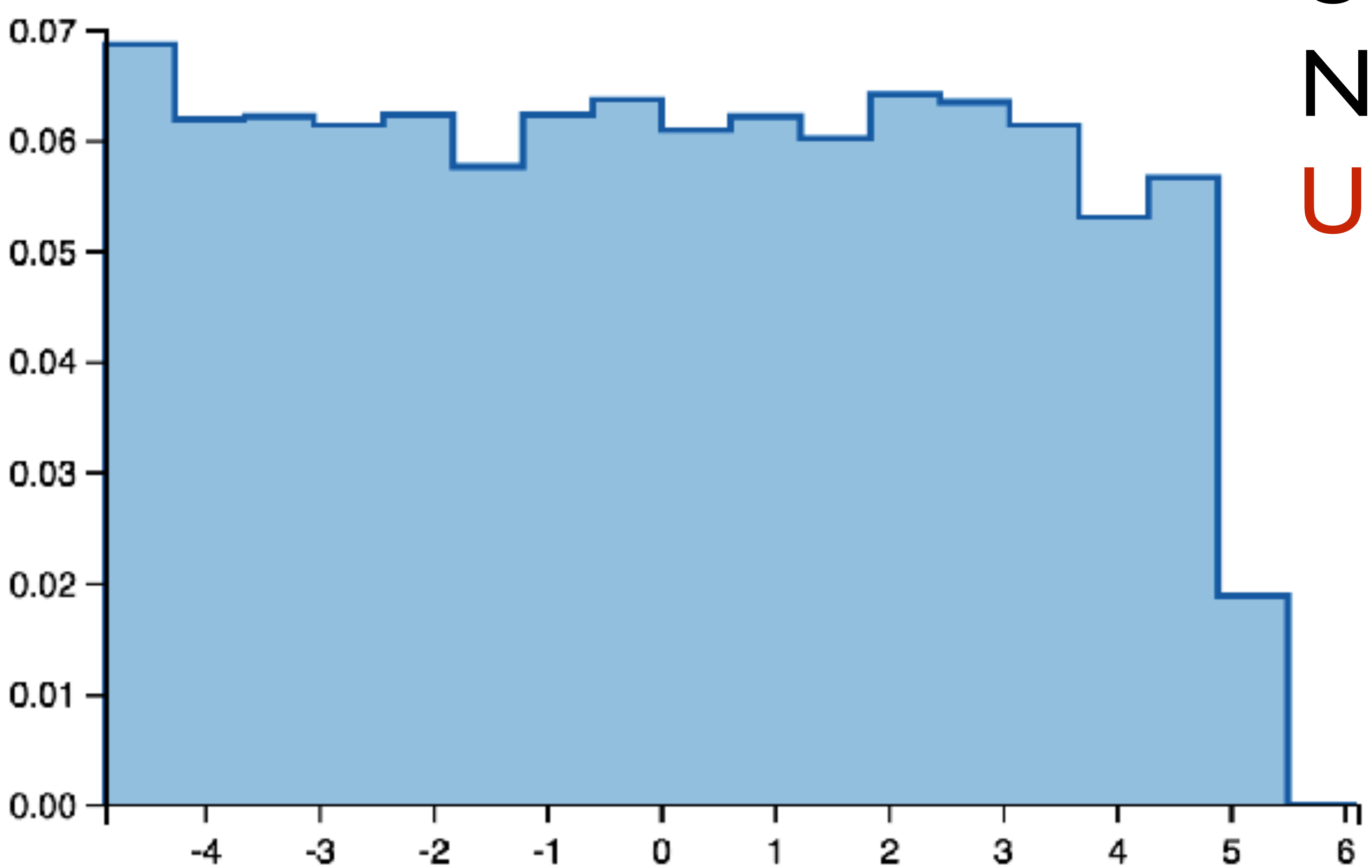

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Uniform[-∞,∞].

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p(x) = normal-pdf(x;0,1)

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$p(y=0) = \int p(x,y=0)dx = \int dx = \infty$

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1. Specification of inference algorithms.
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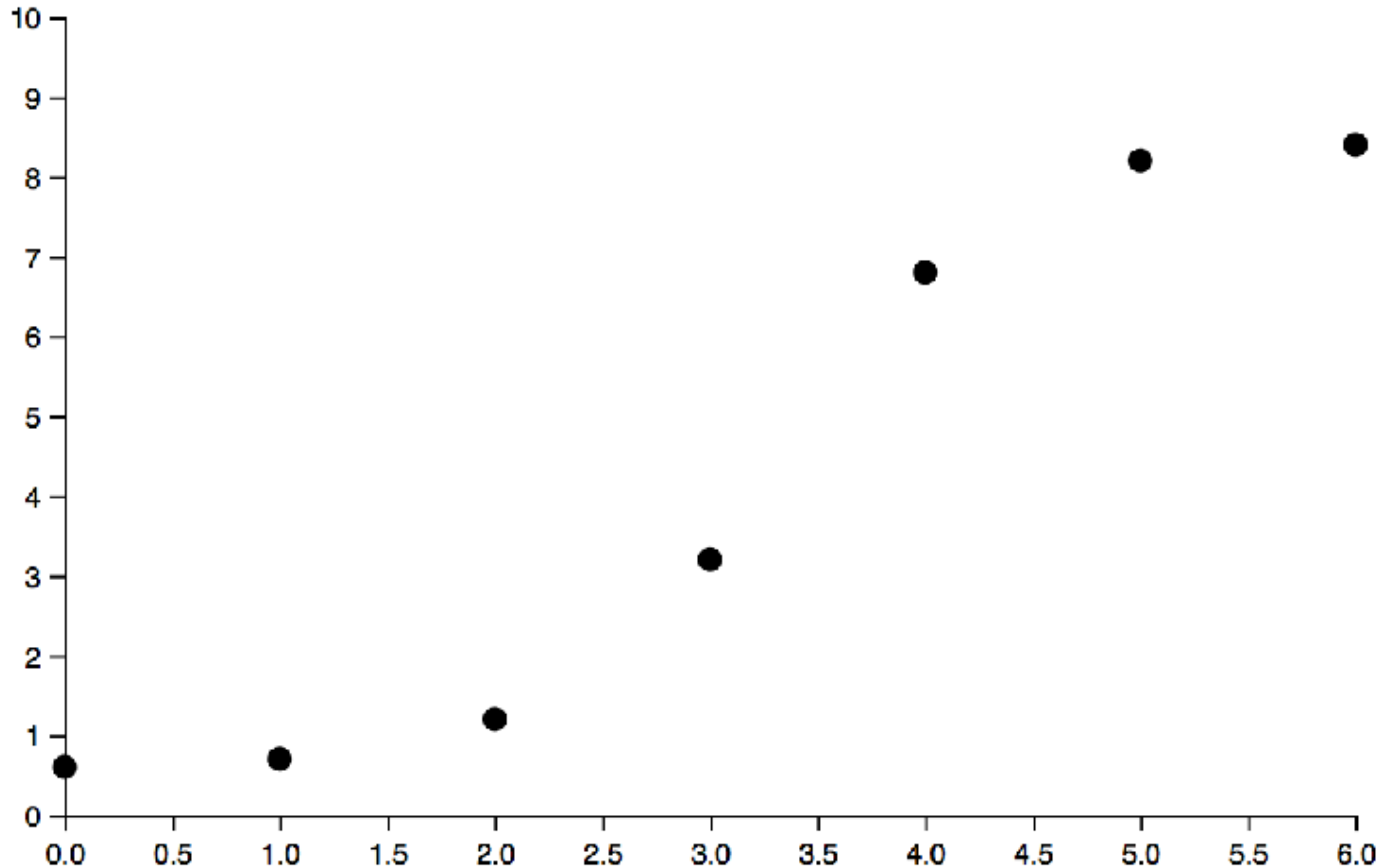
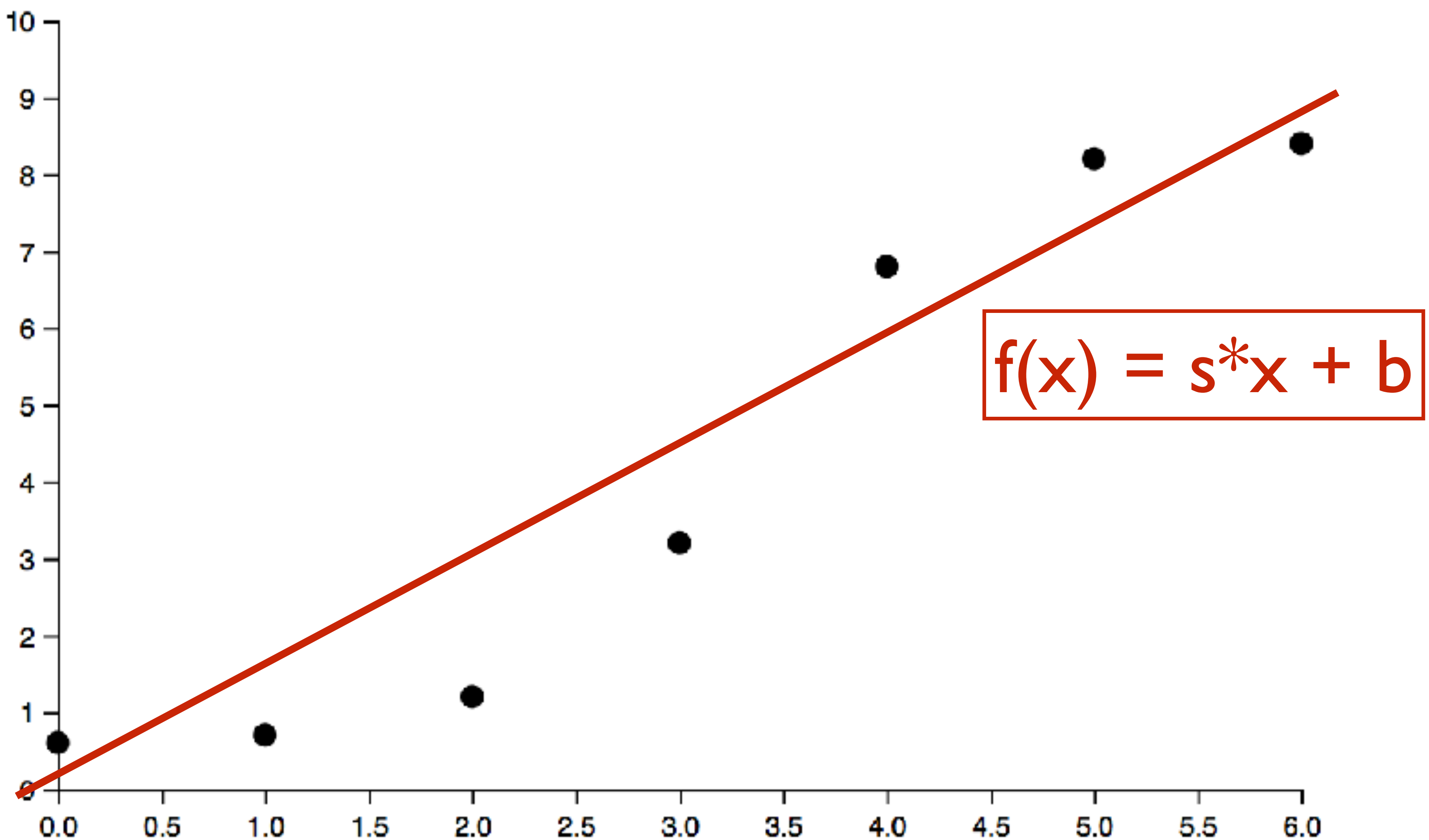
Line fitting



Line fitting



Anglican program

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(defquery lin-regression []
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(defquery lin-regression []
  (let [s (sample (normal 0 2))
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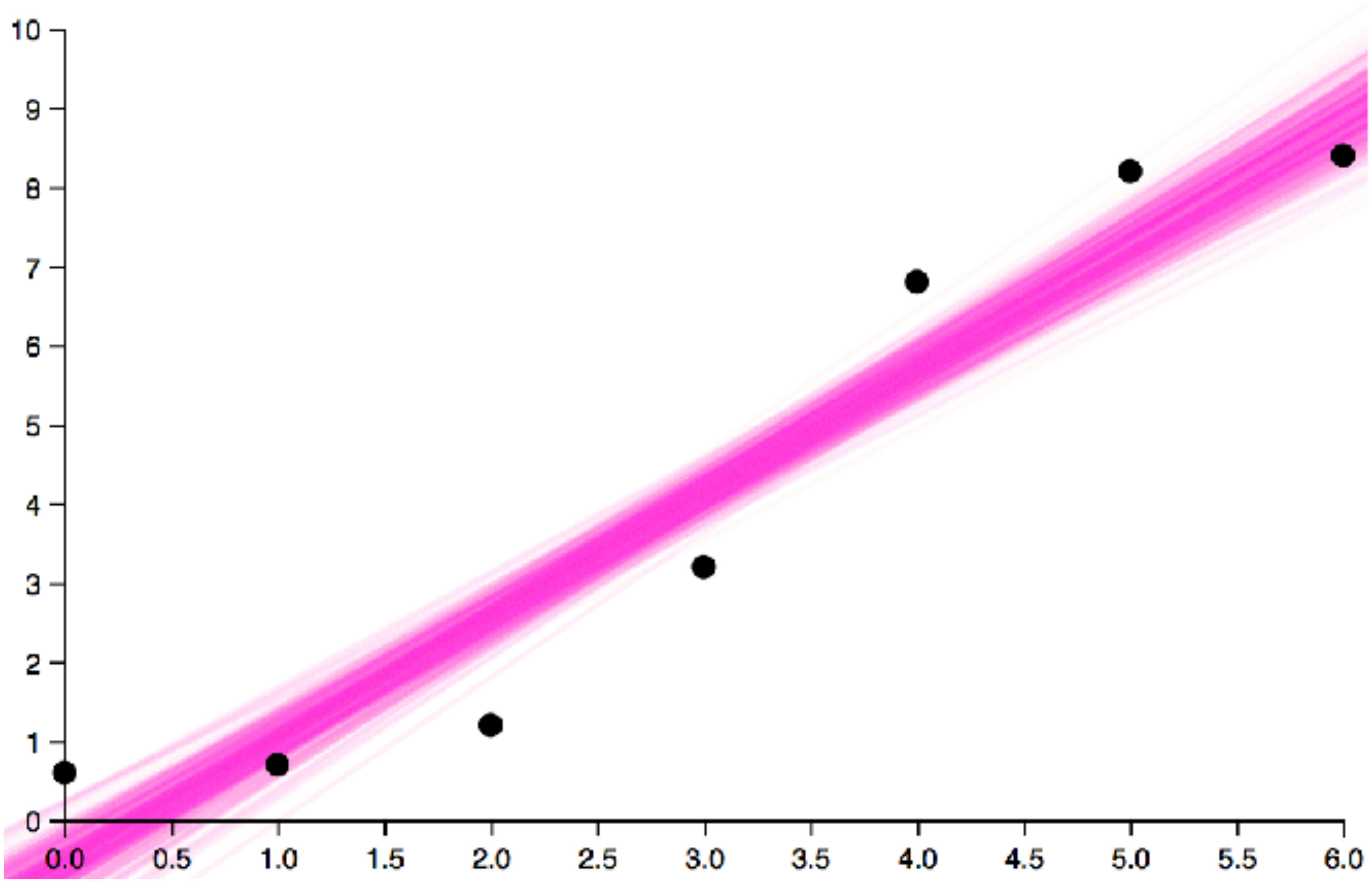
Samples from posterior



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[Q] Which posterior?

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[Q] Which posterior?

Inference algo. gives only approximation.

Should define distr. on functions. Not easy.

Foundational question

Measure theory provides a foundation for modern probability theory.

But it doesn't support higher-order fns well.

$$\mathrm{ev} : (\mathbb{R} \rightarrow_m \mathbb{R}) \times \mathbb{R} \rightarrow \mathbb{R}, \qquad \mathrm{ev}(f,x) = f(x).$$

[Aumann 61] ev is not measurable no matter which σ-algebra is used for $\mathbb{R} \rightarrow_m \mathbb{R}$.

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Denotational semantics: Compositional method. Answers a deep Q.

Learning outcome

- Can define a denotational semantics for a simple programming language.
- Can use measure theory to interpret prob. prog. with continuous distributions.
- Can use quasi-Borel space to interpret higher-order prob. prog.

References

1. A convenient category for higher-order probability theory. Heunen et al. LICS'17.
2. Commutative semantics for probabilistic programs. Staton. ESOP'17.
3. Reynolds's "Theories of Programming Languages".
4. Billingsley's "Probability and Measure".

Plan for the rest

1. Denotational semantics.
   PL with discrete random choices.

2. Baby measure theory.
   PL with cont. distribution.

3. Quasi-Borel space (QBS).
   PL with cont. distr. & higher-order (HO) fns.

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   PL with cont. distr., HO fns & conditioning.

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   linear regression example

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- Defines formal meanings of programs.
- Interprets type and expr. mathematically.
  - Type as space (e.g. set, measurable space).
  - Expr. as good function between spaces.
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First-order PL with discrete random choices

t ::= bool | rational | dist[bool] | dist[rational]

e ::= c | x | (p e ... e) | (let [x e] e) | (if e e e)

c ::= true | false | 0 | 1 | ...

p ::= sample | flip | poisson | and | + | ...

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No function type.

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Only primitive functions can be applied.

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[Q] Denotational semantics of this PL?

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[Q] Denotational semantics of this PL?

Interpret type as set and expr. as function.

Types mean sets

$[\![t]\!]$ is the meaning of t.

$$[\![\texttt{bool}]\!] = \ldots$$

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[Q] Fill in . . .

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$[\![\mathtt{t}]\!]$ is the meaning of $\mathtt{t}$.

$$[\![\mathtt{bool}]\!] = \mathbb{B} = \{\mathrm{tt}, \mathrm{ff}\}$$

$$[\![\mathtt{rational}]\!] = \mathbb{Q} = \{0, 1, -1/3, 1/7, \ldots\}$$

$$[\![\mathtt{dist[bool]}]\!] = \mathrm{DiscProb}([\![\mathtt{bool}]\!])$$

$$[\![\mathtt{dist[rational]}]\!] = \mathrm{DiscProb}([\![\mathtt{rational}]\!])$$

[Q] Fill in . . .

Typed expressions

$$\mathtt{x_1{:}t_1,\ x_2{:}t_2,\ \ldots,\ x_n{:}t_n \vdash e : t}$$

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- Γ is a finite map from variables to types.

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- Γ is a finite map from variables to types.
- Denotes a $\mathtt{t}$-typed expr. $\mathtt{e}$ under Γ.

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- Γ is a finite map from variables to types.
- Denotes a t-typed expr. e under Γ.

x:bool, y:bool ⊢ (if x y y): bool

x:bool, y:rational ⊢ (if x y y) : rational

x:rational ⊢ (sample (flip x)) : bool

Typed expressions mean fns to discrete prob. distr.

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$[\![\mathtt{x_1{:}t_1,\ ...,\ x_n{:}t_n} \vdash \mathtt{e : t}]\!]$

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$$[\![x_1{:}t_1, \ldots, x_n{:}t_n \vdash e : t]\!]$$
$$= \; g : [\![x_1{:}t_1, \ldots, x_n{:}t_n]\!] \to \mathrm{DiscProb}([\![t]\!])$$

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   $\{$map η from $\{x_1,\ldots,x_n\} \mid \eta(x_i) \in [\![t_i]\!]$ for all $i\}$

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[Q] Any problem with $\mathrm{DiscProb}([\![\mathtt{dist[bool]}]\!])$?

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2. $\mathrm{DiscProb}(A) = \{p{:}A \to [0,1] \mid \sum_a p(a) = 1\}$

   p(a)=0 except for countably many a's and

[Q] Define the interpretation recursively.

Compiler optimisation

Show the following equations:

$[\![\Gamma \vdash (\texttt{if true } e_1\ e_2) : t]\!] = [\![\Gamma \vdash e_1 : t]\!]$

$[\![\Gamma \vdash (\texttt{sample (flip (+ 0.1 0.2))}) : \texttt{bool}]\!]$
$= [\![\Gamma \vdash (\texttt{sample (flip 0.3)}) : \texttt{bool}]\!]$

Plan for the rest

1. Denotational semantics.
   PL with discrete random choices.

2. Baby measure theory.
   PL with cont. distribution.

3. Quasi-Borel space (QBS).
   PL with cont. distr. & higher-order (HO) fns.

4. [To be skipped] SFinKer monad on QBS.
   PL with cont. distr., HO fns & conditioning.

First-order PL with discrete random choices

t ::= bool | rational | dist[bool] | dist[rational]

e ::= c | x | (p e ... e) | (let [x e] e) | (if e e e)

c ::= true | false | 0 | 1 | ...

p ::= sample | flip | poisson | and | + | ...

First-order PL with discrete random choices

and continuous

t ::= bool | ~~rational~~ real | dist[bool] | dist[~~rational~~ real]

e ::= c | x | (p e … e) | (let [x e] e) | (if e e e)

c ::= true | false | 0 | 1 | …

p ::= sample | flip | poisson | and | + | …
| normal | uniform-continuous | …

How to define denotational semantics?

Types as spaces and expressions as functions.

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$$\mathtt{t} \ ::= \ \mathtt{bool} \mid \mathtt{real} \mid \mathtt{dist[bool]} \mid \mathtt{dist[real]}$$

[Try] Interpret $[\![\mathtt{t}]\!]$ as a set.

How to define denotational semantics?

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$$\mathtt{t} \ ::= \ \mathtt{bool} \mid \mathtt{real} \mid \mathtt{dist[bool]} \mid \mathtt{dist[real]}$$

[Try] Interpret $[\![\mathtt{t}]\!]$ as a set. Then, we get stuck since $[\![\mathtt{dist[real]}]\!]$ can't be DiscProb($[\![\mathtt{real}]\!]$).

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Types as spaces and expressions as functions.

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[Try] Interpret $[\![\texttt{t}]\!]$ as a set. Then, we get stuck since $[\![\texttt{dist[real]}]\!]$ can't be DiscProb($[\![\texttt{real}]\!]$).

[Sol] Use measure theory. $[\![\texttt{t}]\!]$ as a measurable space, and $[\![\Gamma \vdash \texttt{e} : \texttt{t}]\!]$ as a measurable function.

How to specify prob. p?

How to specify prob. p?

$X = \{0, 1, 2\}$.

Define $p : X \to [0,1]$. E.g., $p = [0.4, 0.4, 0.2]$.

Lifted $p : 2^X \to [0,1]$ by $p(A) = \sum_{x \in A} p(x)$.

How to specify prob. p?

$X = \mathbb{R}$.

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Uncountable sum.
Typically ∞.

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~~Define~~

Pick a good collection $\Sigma \subseteq 2^X$.

Define $p : \Sigma \to [0,1]$ with some care.

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σ-algebra

Pick a good collection $\Sigma \subseteq 2^X$.

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probability measure

Let $\Sigma \subseteq 2^X$.

Σ is a σ-algebra if it contains X, and is closed under countable union and set subtraction.

(X, Σ) is a measurable space if Σ is a σ-algebra.

Let $\Sigma \subseteq 2^X$.

Σ is a <u>σ-algebra</u> if it contains X, and is closed under countable union and set subtraction.

(X, Σ) is a <u>measurable space</u> if Σ is a σ-algebra.

$p : \Sigma \to [0,1]$ is a <u>probability measure</u> if $p(X)=1$ and $p(\biguplus_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} p(A_n)$ for all disjoint A_n's.

(X,Σ,p) is a <u>probability space</u> if …

[Q] What are not measurable spaces?

1. $(\mathbb{B}, 2^{\mathbb{B}})$.
2. $(\mathbb{B}\times\mathbb{B}, \{ A\times B \mid A\in 2^{\mathbb{B}} \text{ and } B\in 2^{\mathbb{B}} \})$.
3. $(\mathbb{R}, \{ A\subseteq\mathbb{R} \mid A \text{ or } (\mathbb{R}-A) \text{ finite or countable} \})$.
4. $(\mathbb{R}, \{ (r,s] \mid r<s \})$.

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[Q] Convert them to measurable spaces.

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[Q] Convert them to measurable spaces.

1. $(\mathbb{B}, 2^{\mathbb{B}})$.
2. $(\mathbb{B}\mathrm{x}\mathbb{B}, \sigma\{\, A\mathrm{x}B \mid A \in 2^{\mathbb{B}} \text{ and } B \in 2^{\mathbb{B}} \,\})$.
3. $(\mathbb{R}, \{\, A \subseteq \mathbb{R} \mid A \text{ or } (\mathbb{R}-A) \text{ finite or countable} \,\})$.
4. $(\mathbb{R}, \sigma\{\, (r,s] \mid r<s \,\})$.

Closure exists.
$\sigma(\Pi)$ smallest σ-algebra containing Π.

$(X, \Sigma), (Y, \Theta)$ - mBle spaces.

Product σ-algebra: $\Sigma \otimes \Theta = \sigma\{A x B \mid A \in \Sigma, B \in \Theta\}$.

Product space: $(X,\Sigma) x_m (Y, \Theta) = (X x Y, \Sigma \otimes \Theta)$.

Borel σ-algebra on $\mathbb{R}$: $\mathfrak{B} = \sigma\{(r,s] \mid r<s\}$.

Borel space: $(\mathbb{R}, \mathfrak{B})$.

(X, Σ), (Y, Θ) - mBle spaces.

$\Pr(\Sigma) = \ \ldots$

Probability space: $\Pr(X,\Sigma) = (\Pr(X), \Pr(\Sigma))$

[Q] What is $\Pr(\Sigma)$?

(X, Σ), (Y, Θ) - mBle spaces.

$\Pr(\Sigma) = \sigma\{ \{p \mid p(A)<r\} \mid A\in\Sigma, r\in\mathbb{R} \}$.

Probability space: $\Pr(X,\Sigma) = (\Pr(X), \Pr(\Sigma))$

[Q] What is $\Pr(\Sigma)$?

Types mean mBle spaces

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$$[\![\texttt{bool}]\!] = (\mathbb{B}, 2^{\mathbb{B}})$$

$$[\![\texttt{dist[bool]}]\!] = \mathrm{Pr}([\![\texttt{bool}]\!])$$

Types mean mBle spaces

$$[\![\texttt{bool}]\!] = (\mathbb{B}, 2^{\mathbb{B}})$$

$$[\![\texttt{real}]\!] = \ \ldots$$

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[Q] Fill in ...

Types mean mBle spaces

$$[\![\texttt{bool}]\!] = (\mathbb{B}, 2^{\mathbb{B}})$$

$$[\![\texttt{real}]\!] = (\mathbb{R}, \mathfrak{B})$$

$$[\![\texttt{dist[bool]}]\!] = \mathsf{Pr}([\![\texttt{bool}]\!])$$

$$[\![\texttt{dist[real]}]\!] = \mathsf{Pr}([\![\texttt{real}]\!])$$

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$$[\![\mathtt{x}_1 : \mathtt{t}_1, \ldots, \mathtt{x}_n : \mathtt{t}_n]\!] = (X, \Sigma)$$

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$$[\![\mathtt{x}_1\mathtt{:t}_1, \ldots, \mathtt{x}_n\mathtt{:t}_n]\!] = (X, \Sigma)$$

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$X = \ldots$

$\Sigma = \ldots$

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$\Sigma = \sigma\{ \{\eta \mid \eta(\mathtt{x}_i) \in A_i \text{ for all } i\} \mid A_i \in \Sigma_i \text{ for all } i\}$

[Q] Fill in …

(X, Σ), (Y, Θ) - mBle spaces.

$f{:}X \to Y$ is measurable (denoted $f{:}X \to_m Y$) if $f^{-1}(A) \in \Sigma$ for all $A \in \Theta$.

Exprs mean mBle fns

$[\![\Gamma \vdash \mathtt{e} : \mathtt{t}]\!]$ is a mBle fn from $[\![\Gamma]\!]$ to Pr$[\![\mathtt{t}]\!]$.

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$= \int_A \text{density-norm}(s \mid \eta(y), 1)\, ds.$

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$= \int_A$ density-norm(s | η(y), 1) ds.

Defined recursively. Complex but doable.

Plan for the rest

1. Denotational semantics.
   PL with discrete random choices.

2. Baby measure theory.
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   PL with cont. distr. & higher-order (HO) fns.

4. [To be skipped] SFinKer monad on QBS.
   PL with cont. distr., HO fns & conditioning.

Prob. PL with HO fns and continuous random choices

t ::= bool | real | dist[bool] | dist[real] | $(t_1,\ldots,t_n) \to t$

e ::= c | x | (fn [x ... x] e) | (e e ... e) | (if e e e)

c ::= true | false | 0 | 1 | 2 | and | + | ...
| sample | flip | normal | ...

Prob. PL with HO fns and continuous random choices

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Function type.

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Function type.
General fn decl. and app.

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Function type.
General fn decl. and app.
General constants.

Measure theory insufficient due to HO fns.

Use a new foundation of probability theory based on quasi-Borel spaces.

Interpret $[\![\mathtt{t}]\!]$ as a quasi-Borel space (QBS), and $[\![\Gamma \vdash \mathtt{e} : \mathtt{t}]\!]$ as a QBS morphism.

High-level idea:
Random variable first.

Random variable α in X

Random variable α in X

$$\alpha : \Omega \rightarrow X$$

- X - set of values.
- Ω - set of random seeds.
- Random seed generator.

Random variable α in X in measure theory

$$\alpha : \Omega \to X$$

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Random variable α in X
in measure theory

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1. $\Sigma \subseteq 2^{\Omega}, \Theta \subseteq 2^{X}$

Random variable α in X in measure theory

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1. $\Sigma \subseteq 2^{\Omega}$, $\Theta \subseteq 2^{X}$
2. $\mu : \Sigma \to [0,1]$

Random variable α in X
in measure theory

$\alpha : \Omega \to X$ is a random variable
if $\alpha^{-1}(A) \in \Sigma$ for all $A \in \Theta$

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Random variable α in X in quasi-Borel spaces

$$\alpha : \Omega \to X$$

- X - set of values.
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Random variable α in X in quasi-Borel spaces

$$\alpha : \mathbb{R} \to X$$

- X - set of values.
- $\mathbb{R}$ - set of random seeds.
- Random seed generator.

1. $\mathbb{R}$ as random source
2. Borel subsets $\mathfrak{B} \subseteq 2^{\mathbb{R}}$

Random variable α in X
in quasi-Borel spaces

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3. $M \subseteq [\mathbb{R} \to X]$

Random variable α in X in quasi-Borel spaces

$\alpha : \mathbb{R} \to X$ is a random variable if $\alpha \in M$

- X - set of values.
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1. $\mathbb{R}$ as random source
2. Borel subsets $\mathfrak{B} \subseteq 2^{\mathbb{R}}$
3. $M \subseteq [\mathbb{R} \to X]$

- Measure theory:
  - Measurable space $(X, \Theta \subseteq 2^X)$.
  - Random variable is an induced concept.
- QBS:
  - Quasi-Borel space $(X, M \subseteq [\mathbb{R} \rightarrow X])$.
  - M is the set of random variables.

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  - Measurable space $(X, \Theta \subseteq 2^X)$.
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Quasi-Borel space - set with random variables.

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$$(X, M \subseteq [\mathbb{R} \to X])$$

such that M has enough random variables.

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3. If $\mathbb{R} = \uplus_{i \in \mathbb{N}} R_i$ with $R_i \in \mathfrak{B}$ and $\alpha_1, \alpha_2, \ldots \in M$, then $(\alpha_i \text{ when } R_i)_{i \in \mathbb{N}} \in M$.

   Here $(\alpha_i \text{ when } R_i)_{i \in \mathbb{N}}(r) = \alpha_i(r)$ when $r \in R_i$.

[Q] Pick a non-QBS.

1. $(\mathbb{R},\ \{\alpha{:}\mathbb{R}\to\mathbb{R} \mid \alpha \text{ is a constant function}\})$.
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Standard way of converting a mBle space to a QBS.

(QBS) morphism

(X,M), (Y,N) - QBSes.

$f : X \to Y$ is a morphism if $(f \circ \alpha) \in N$ for all $\alpha \in M$.

Maps random variables to random variables.

We will write $f : X \to_q Y$.

[Th] QBSes support higher-order functions well.
(The category of QBSes is cartesian closed.)

[Q] What are the sets of random variables?

1. Product: $(X,M) \times_q (Y,N) = (Z,O)$.

- $Z = X \times Y$, $\pi_1(x,y) = x$, $\pi_2(x,y) = y$.
- $O = ???$

2. Fn space: $[(X,M) \to_q (Y,N)] = (Z,O)$

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[Q] What are the sets of random elements?

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2. Fn space: $[(X,M) \rightarrow_q (Y,N)] = (Z,O)$
   - $Z = \{ f \mid f : X \rightarrow_q Y \}$, $ev(f,x) = f(x)$
   - $O = \{ g : \mathbb{R} \rightarrow Z \mid r \mapsto g(\gamma(r))(\alpha(r)) \in N$ for all $\gamma : \mathbb{R} \rightarrow_m \mathbb{R}$ and $\alpha \in M\}$.

Why works?

[NO] ev : $(\mathbb{R} \to_m \mathbb{R}) \times_m \mathbb{R} \to_m \mathbb{R}$

vs

[YES] ev : $(\mathbb{R} \to_q \mathbb{R}) \times_q \mathbb{R} \to_q \mathbb{R}$

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Because the QBS product is more permissive.

Types mean QBSes

$$[\![\mathtt{bool}]\!] = \mathsf{MStoQBS}(\mathbb{B}, 2^{\mathbb{B}})$$

$$[\![\mathtt{dist[bool]}]\!] = \mathsf{Pr_q}([\![\mathtt{bool}]\!])$$

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$M = \{r \longmapsto (\mathtt{x}_i \longmapsto \alpha_i(r)) \mid \alpha_i \in M_i \text{ for all } i\}$

[Q] Fill in …

Exprs mean QBS morphisms

$[\![\Gamma \vdash \mathtt{e} : \mathtt{t}]\!]$ is a QBS morphism from $[\![\Gamma]\!]$ to $\mathrm{Pr_q}[\![\mathtt{t}]\!]$.

A probability measure on a QBS (X,M) is a pair (α,μ) of $\alpha\in M$ and a prob. measure μ on $(\mathbb{R},\mathfrak{B})$.

A <u>probability measure</u> on a QBS (X,M) is a pair (α,μ) of $\alpha \in$M and a prob. measure μ on ($\mathbb{R}$,$\mathfrak{B}$).

random seed generator

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E.g.

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μ' = uniform(0,2]/2, $\alpha'(r)$ = if ($r < 1$) true false

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Equate two QBS prob. measures:

$$(\alpha,\mu) \sim (\beta,\nu)$$

if $\mu \circ \alpha^{-1} = \nu \circ \beta^{-1}$.

$[\alpha,\mu]$ - equivalence class.

QBS of prob. measures

$$\mathrm{Pr}_q(X,M) = (Y,N)$$

$Y = \{ [\alpha,\mu] \mid (\alpha,\mu)$ is a prob. meas. on $(X,M) \}$.

$N = \{ \lambda r.[\alpha,k(r)] \mid \alpha \in M$ and $k : \mathbb{R} \times \mathfrak{B} \to [0,1]$ is a probability kernel $\}$.

k(r,-) is a probability measure
and k(-,A) is measurable for all r,A

Completing the definitions

$$[\![t \rightarrow t']\!] = [[\![t]\!] \rightarrow_q Pr_q([\![t']\!])]$$

$[\![\Gamma \vdash e : t]\!]$ is a morphism $[\![\Gamma]\!] \rightarrow_q Pr_q[\![t]\!]$

We couldn't cover:

1. SFinKer Monad on QBSes and semantics of conditioning.

If you want to know about them, look at:

1. A convenient category for higher-order probability theory. Heunen et al. LICS'17.
2. Commutative semantics for probabilistic programs. Staton. ESOP'17.