Half a Heart

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1 Parametric Equations for the Heart Shape

We are given the parametric equations for the heart shape:

$$x(t) = 16\sin^3(t), \tag{1}$$
$$y(t) = 13\cos(t) - 5\cos(2t) - 2\cos(3t) - \cos(4t). \tag{2}$$

2 Finding the Derivatives

To determine the tangent lines, we need to compute $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

2.1 Computing $\frac{dx}{dt}$

Using the chain rule:

$$\frac{dx}{dt} = 16 \cdot 3\sin^2(t)\cos(t) \tag{3}$$
$$= 48\sin^2(t)\cos(t). \tag{4}$$

2.2 Computing $\frac{dy}{dt}$

Differentiating term by term:

$$\frac{dy}{dt} = -13\sin(t) + 10\sin(2t) + 6\sin(3t) + 4\sin(4t). \tag{5}$$

2.3 Slope of the Tangent Line

The slope of the tangent line is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \tag{6}$$
$$= \frac{-13\sin(t) + 10\sin(2t) + 6\sin(3t) + 4\sin(4t)}{48\sin^2(t)\cos(t)}. \tag{7}$$

3 Finding Vertical Tangent Lines

Vertical tangent lines occur where $\frac{dx}{dt} = 0$ but $\frac{dy}{dt} \neq 0$.

Setting $\frac{dx}{dt} = 0$:

$$48 \sin^2(t) \cos(t) = 0. \tag{8}$$

This is satisfied when:

$$\cos(t) = 0 \quad \text{or} \quad \sin(t) = 0. \tag{9}$$

Since $\sin(t) = 0$ would make $\frac{dx}{dt}$ nonzero, we solve $\cos(t) = 0$:

$$t = \frac{\pi}{2}, \frac{3\pi}{2}. \tag{10}$$

3.1 Finding Corresponding (x, y) Points

Plugging $t = \frac{\pi}{2}$ into the parametric equations:

$$x\left(\frac{\pi}{2}\right) = 16 \sin^3\left(\frac{\pi}{2}\right) = 16, \tag{11}$$

$$y\left(\frac{\pi}{2}\right) = 13 \cos\left(\frac{\pi}{2}\right) - 5 \cos\left(2 \times \frac{\pi}{2}\right) - 2 \cos\left(3 \times \frac{\pi}{2}\right) - \cos\left(4 \times \frac{\pi}{2}\right) \tag{12}$$

$$= 0 - 5(-1) - 2(0) - (-1) = 0 + 5 + 0 + 1 = 6. \tag{13}$$

Similarly, for $t = \frac{3\pi}{2}$:

$$x\left(\frac{3\pi}{2}\right) = 16 \sin^3\left(\frac{3\pi}{2}\right) = -16, \tag{14}$$

$$y\left(\frac{3\pi}{2}\right) = 13 \cos\left(\frac{3\pi}{2}\right) - 5 \cos\left(2 \times \frac{3\pi}{2}\right) - 2 \cos\left(3 \times \frac{3\pi}{2}\right) - \cos\left(4 \times \frac{3\pi}{2}\right) \tag{15}$$

$$= 0 - 5(-1) - 2(0) - (-1) = 0 + 5 + 0 + 1 = 6. \tag{16}$$

Thus, vertical tangents occur at:

$$(16, 6) \quad \text{and} \quad (-16, 6). \tag{17}$$