For neutral mesons, P and $\bar{P}$, the mixing matrix is

$$\mathbf{M} = \begin{pmatrix} M & M_{12} \\ M_{12}^* & M \end{pmatrix}, \tag{1}$$

and the decay matrix is

$$\mathbf{\Gamma} = \begin{pmatrix} \Gamma & \Gamma_{12} \\ \Gamma_{12}^* & \Gamma \end{pmatrix}. \tag{2}$$

The effective Hamiltonian is thus

$$\mathcal{H}_{\text{eff}} = \mathbf{M} - \frac{\mathbf{i}}{\mathbf{2}}\mathbf{\Gamma}. \tag{3}$$

To find the eigenvalues, one has

$$\begin{pmatrix} M - \frac{i}{2}\Gamma & M_{12} - \frac{i}{2}\Gamma_{12} \\ M_{12}^* - \frac{i}{2}\Gamma_{12}^* & M - \frac{i}{2}\Gamma \end{pmatrix} \begin{pmatrix} p \\ \pm q \end{pmatrix} = \lambda_{1,2} \begin{pmatrix} p \\ \pm q \end{pmatrix}, \tag{4}$$

with

$$\lambda_{1,2} = m_{1,2} - \frac{i}{2}\Gamma_{1,2}, \tag{5}$$

so

$$\mathrm{Re}(\lambda_{1,2}) = m_{1,2}, \tag{6}$$

$$-2\mathrm{Im}(\lambda_{1,2}) = \Gamma_{1,2}. \tag{7}$$

Then one has

$$\lambda_{1,2} p = \left(M - \frac{i}{2}\Gamma\right) p \pm \left(M_{12} - \frac{i}{2}\Gamma_{12}\right) q, \tag{8}$$

$$\pm\lambda_{1,2} q = \left(M_{12}^* - \frac{i}{2}\Gamma_{12}^*\right) p \pm \left(M - \frac{i}{2}\Gamma\right) q. \tag{9}$$

which gives

$$\lambda_{1,2} = \left(M - \frac{i}{2}\Gamma\right) \pm \left(M_{12} - \frac{i}{2}\Gamma_{12}\right) \frac{q}{p}, \tag{10}$$

$$= \left(M - \frac{i}{2}\Gamma\right) \pm \left(M_{12}^* - \frac{i}{2}\Gamma_{12}^*\right) \frac{p}{q}. \tag{11}$$

TODO: work out how to get to the eigenvalues.

Thus the eigenvalues are:

$$\lambda_{1,2} = M \mp |M_{12}| - \frac{i}{2}(\Gamma \mp |\Gamma_{12}|), \tag{12}$$

$$\equiv M \mp \frac{1}{2}\Delta m - \frac{i}{2}\left(\Gamma \mp \frac{1}{2}\Delta\Gamma\right), \tag{13}$$

$$\equiv \lambda_{L,H}. \tag{14}$$

The corresponding mass eigenstates are

$$|P_{L,H}\rangle = p|P\rangle \pm q|\bar{P}\rangle, \tag{15}$$

with time evolution

$$|P_{L,H}(t)\rangle = \exp(-i\lambda_{L,H}t)|P_{L,H}(0)\rangle, \tag{16}$$

$$= \exp(-i\lambda_{L,H}t)(p|P(0)\rangle \pm q|\bar{P}(0)\rangle. \tag{17}$$

Then,

$$|P(t)\rangle = \frac{1}{2p}(|P_H(t)\rangle + |P_L(t)\rangle) \tag{18}$$

$$= \frac{1}{2p}\big(\exp(-i\lambda_H t)(p|P(0)\rangle - q|\bar{P}(0)\rangle) \tag{19}$$

$$+ \exp(-i\lambda_L t)(p|P(0)\rangle + q|\bar{P}(0)\rangle)\big) \tag{20}$$

$$= \frac{1}{2}\Bigg[\left(\exp(-i\lambda_H t) + \exp(-i\lambda_L t)\right)|P(0)\rangle \tag{21}$$

$$- \frac{q}{p}\left(\exp(-i\lambda_H t) - \exp(-i\lambda_L t)\right)|\bar{P}(0)\rangle\Bigg]. \tag{22}$$

Similarly,

$$|\bar{P}(t)\rangle = \frac{1}{2q}(|P_L(t)\rangle - |P_H(t)\rangle) \tag{23}$$

$$= \frac{1}{2q}\big(\exp(-i\lambda_L t)(p|P(0)\rangle + q|\bar{P}(0)\rangle) \tag{24}$$

$$- \exp(-i\lambda_H t)(p|P(0)\rangle - q|\bar{P}(0)\rangle)\big) \tag{25}$$

$$= \frac{1}{2}\Bigg[\left(\exp(-i\lambda_H t) + \exp(-i\lambda_L t)\right)|\bar{P}\rangle \tag{26}$$

$$- \frac{p}{q}\left(\exp(-i\lambda_H t) - \exp(-i\lambda_L t)\right)|P\rangle\Bigg]. \tag{27}$$

Defining $P_{+1} \equiv P$ and $P_{-1} \equiv \bar{P}$, then

$$|P_k\rangle = \frac{1}{2}\Bigg[\left(\exp(-i\lambda_H t) + \exp(-i\lambda_L t)\right)|P_k(0)\rangle \tag{28}$$

$$+ \frac{q}{p}^{k}\left(\exp(-i\lambda_H t) - \exp(-i\lambda_L t)\right)|P_{-k}(0)\rangle\Bigg], \tag{29}$$

$$\equiv g_+(t)|P_k(0)\rangle + \left(\frac{q}{p}\right)^k g_-(t)|P_{-k}(0)\rangle, \tag{30}$$

using

$$g_\pm(t) \equiv \frac{1}{2}\left(\exp(-i\lambda_H t) \pm \exp(-i\lambda_L t)\right), \tag{31}$$

$$= \frac{1}{2}\left(\exp\left(-i\left(M + \frac{1}{2}\Delta m - \frac{i}{2}\left(\Gamma + \frac{1}{2}\Delta\Gamma\right)\right)t\right) \pm \exp\left(-i\left(M - \frac{1}{2}\Delta m - \frac{i}{2}\left(\Gamma - \frac{1}{2}\Delta\Gamma\right)\right)t\right)\right), \tag{32}$$

$$= \frac{1}{2}\exp\left(-iMt + \frac{1}{2}\Gamma t\right)\left(\exp\left(-\frac{i}{2}\Delta m t - \frac{1}{4}\Delta\Gamma t\right) \pm \exp\left(\frac{i}{2}\Delta m t + \frac{1}{4}\Delta\Gamma t\right)\right). \tag{33}$$

Then,

$$|A_{f,k}(t)|^2 = |\langle f|\mathcal{H}|P_k(t)\rangle|^2, \tag{34}$$

$$= \left(g_+^*(t)\langle f|\mathcal{H}|P_k(0)\rangle^* + \left(\frac{q}{p}\right)^{k*} g_-^*(t)\langle f|\mathcal{H}|P_{-k}(0)\rangle^*\right) \tag{35}$$

$$\left(g_+(t)\langle f|\mathcal{H}|P_k(0)\rangle + \left(\frac{q}{p}\right)^{k} g_-(t)\langle f|\mathcal{H}|P_{-k}(0)\rangle\right), \tag{36}$$

$$= |g_+(t)|^2\,|A_{f,k}(0)|^2 + \left(\frac{q}{p}\right)^{k} g_+^*(t)g_-(t)A_{f,k}^*(0)A_{f,-k}(0) \tag{37}$$

$$+ \left(\frac{q}{p}\right)^{k*} g_-^*(t)g_+(t)A_{f,-k}^*(0)A_{f,k}(0) + \left|\frac{q}{p}\right|^{2k} |g_-(t)|^2\,|A_{f,-k}(0)|^2, \tag{38}$$

$$= |g_+(t)|^2\,|A_{f,k}(0)|^2 + \left|\frac{q}{p}\right|^{2k} |g_-(t)|^2\,|A_{f,-k}(0)|^2 \tag{39}$$

$$+ 2\mathrm{Re}\left(\left(\frac{q}{p}\right)^{k} g_+^*(t)g_-(t)A_{f,k}^*(0)A_{f,-k}(0)\right). \tag{40}$$

Using,

$$|g_\pm(t)|^2 = \frac{1}{4}\exp(-\Gamma t)\left[\exp\left(\frac{i}{2}\Delta m t - \frac{1}{4}\Delta\Gamma t\right) \pm \exp\left(\frac{-i}{2}\Delta m t + \frac{1}{4}\Delta\Gamma t\right)\right] \tag{41}$$

$$\left[\exp\left(\frac{-i}{2}\Delta m t - \frac{1}{4}\Delta\Gamma t\right) \pm \exp\left(\frac{i}{2}\Delta m t + \frac{1}{4}\Delta\Gamma t\right)\right], \tag{42}$$

$$= \frac{1}{4}\exp(-\Gamma t)\left[\exp\left(-\frac{1}{2}\Delta\Gamma t\right) \pm \exp\left(i\Delta m t\right) \pm \exp(-i\Delta m t) + \exp\left(\frac{1}{2}\Delta\Gamma t\right)\right], \tag{43}$$

$$= \frac{1}{2}\exp(-\Gamma t)\left[\cosh\left(\frac{1}{2}\Delta\Gamma t\right) \pm \cos(\Delta m t)\right], \tag{44}$$

and

$$g_+^*(t)g_-(t) = \frac{1}{4}\exp(-\Gamma t)\left[\exp\left(\frac{i}{2}\Delta m t - \frac{1}{4}\Delta\Gamma t\right) + \exp\left(\frac{-i}{2}\Delta m t + \frac{1}{4}\Delta\Gamma t\right)\right] \tag{45}$$

$$\left[\exp\left(\frac{-i}{2}\Delta m t - \frac{1}{4}\Delta\Gamma t\right) - \exp\left(\frac{i}{2}\Delta m t + \frac{1}{4}\Delta\Gamma t\right)\right], \tag{46}$$

$$= \frac{1}{4}\exp(-\Gamma t)\left[\exp\left(-\frac{1}{2}\Delta\Gamma t\right) - \exp\left(i\Delta m t\right) + \exp(-i\Delta m t) - \exp\left(\frac{1}{2}\Delta\Gamma t\right)\right], \tag{47}$$

$$= \frac{1}{2}\exp(-\Gamma t)\left[-\sinh\left(\frac{1}{2}\Delta\Gamma t\right) + i\sin(\Delta m t)\right], \tag{48}$$

then

$$|A_{f,k}(t)|^2 = \frac{1}{2}\exp(-\Gamma t)\Bigg[\left(\cosh\left(\frac{1}{2}\Delta\Gamma t\right) + \cos(\Delta m t)\right)|A_{f,k}(0)|^2 \tag{49}$$

$$+ \left|\frac{q}{p}\right|^{2k}\left(\cosh\left(\frac{1}{2}\Delta\Gamma t\right) - \cos(\Delta m t)\right)|A_{f,-k}(0)|^2 \tag{50}$$

$$+ 2\mathrm{Re}\left(\left(\frac{q}{p}\right)^k\left(-\sinh\left(\frac{1}{2}\Delta\Gamma t\right) + i\sin(\Delta m t)\right)A_{f,k}^*(0)A_{f,-k}(0)\right)\Bigg], \tag{51}$$

$$= \frac{1}{2}\exp(-\Gamma t)\Bigg[\left(|A_{f,k}(0)|^2 + \left|\frac{q}{p}\right|^{2k}|A_{f,-k}(0)|^2\right)\cosh\left(\frac{1}{2}\Delta\Gamma t\right) \tag{52}$$

$$+ \left(|A_{f,k}(0)|^2 - \left|\frac{q}{p}\right|^{2k}|A_{f,-k}(0)|^2\right)\cos(\Delta m t) \tag{53}$$

$$- 2\mathrm{Re}\left(\left(\frac{q}{p}\right)^k A_{f,k}^*(0)A_{f,-k}(0)\right)\sinh\left(\frac{1}{2}\Delta\Gamma t\right) \tag{54}$$

$$- 2\mathrm{Im}\left(\left(\frac{q}{p}\right)^k A_{f,k}^*(0)A_{f,-k}(0)\right)\sin(\Delta m t)\Bigg] \tag{55}$$