

For neutral mesons, P and \bar{P} , the mixing matrix is

$$\mathbf{M} = \begin{pmatrix} M & M_{12} \\ M_{12}^* & M \end{pmatrix}, \quad (1)$$

and the decay matrix is

$$\mathbf{\Gamma} = \begin{pmatrix} \Gamma & \Gamma_{12} \\ \Gamma_{12}^* & \Gamma \end{pmatrix}. \quad (2)$$

The effective Hamiltonian is thus

$$\mathcal{H}_{\text{eff}} = \mathbf{M} - \frac{i}{2}\mathbf{\Gamma}. \quad (3)$$

To find the eigenvalues, one has

$$\begin{pmatrix} M - \frac{i}{2}\Gamma & M_{12} - \frac{i}{2}\Gamma_{12} \\ M_{12}^* - \frac{i}{2}\Gamma_{12}^* & M - \frac{i}{2}\Gamma \end{pmatrix} \begin{pmatrix} p \\ \pm q \end{pmatrix} = \lambda_{1,2} \begin{pmatrix} p \\ \pm q \end{pmatrix}, \quad (4)$$

with

$$\lambda_{1,2} = m_{1,2} - \frac{i}{2}\Gamma_{1,2}, \quad (5)$$

so

$$\text{Re}(\lambda_{1,2}) = m_{1,2}, \quad (6)$$

$$-2\text{Im}(\lambda_{1,2}) = \Gamma_{1,2}. \quad (7)$$

Then one has

$$\lambda_{1,2}p = \left(M - \frac{i}{2}\Gamma\right)p \pm \left(M_{12} - \frac{i}{2}\Gamma_{12}\right)q, \quad (8)$$

$$\pm\lambda_{1,2}q = \left(M_{12}^* - \frac{i}{2}\Gamma_{12}^*\right)p \pm \left(M - \frac{i}{2}\Gamma\right)q. \quad (9)$$

which gives

$$\lambda_{1,2} = \left(M - \frac{i}{2}\Gamma\right) \pm \left(M_{12} - \frac{i}{2}\Gamma_{12}\right)\frac{q}{p}, \quad (10)$$

$$= \left(M - \frac{i}{2}\Gamma\right) \pm \left(M_{12}^* - \frac{i}{2}\Gamma_{12}^*\right)\frac{p}{q}. \quad (11)$$

TODO: work out how to get to the eigenvalues.

Thus the eigenvalues are:

$$\lambda_{1,2} = M \pm |M_{12}| - \frac{i}{2}(\Gamma \pm |\Gamma_{12}|), \quad (12)$$

$$\equiv M \pm \frac{1}{2}\Delta m - \frac{i}{2}\left(\Gamma \pm \frac{1}{2}\Delta\Gamma\right), \quad (13)$$

$$\equiv \lambda_{H,L}. \quad (14)$$

The corresponding mass eigenstates are

$$|P_{H,L}\rangle = p|P\rangle \pm q|\bar{P}\rangle, \quad (15)$$

with time evolution

$$|P_{H,L}(t)\rangle = \exp(-i\lambda_{H,L}t)|P_{H,L}(0)\rangle, \quad (16)$$

$$= \exp(-i\lambda_{H,L}t)(p|P(0)\rangle \pm q|\bar{P}(0)\rangle). \quad (17)$$

Then,

$$|P(t)\rangle = \frac{1}{2p}(|P_H(t)\rangle + |P_L(t)\rangle) \quad (18)$$

$$= \frac{1}{2p}(\exp(-i\lambda_H t)(p|P(0)\rangle + q|\bar{P}(0)\rangle) \quad (19)$$

$$+ \exp(-i\lambda_L t)(p|P(0)\rangle - q|\bar{P}(0)\rangle)) \quad (20)$$

$$= \frac{1}{2} \left[(\exp(-i\lambda_H t) + \exp(-i\lambda_L t)) |P(0)\rangle \quad (21)$$

$$+ \frac{q}{p} (\exp(-i\lambda_H t) - \exp(-i\lambda_L t)) |\bar{P}(0)\rangle \right]. \quad (22)$$

Similarly,

$$|\bar{P}(t)\rangle = \frac{1}{2q}(|P_H(t)\rangle - |P_L(t)\rangle) \quad (23)$$

$$= \frac{1}{2q}(\exp(-i\lambda_H t)(p|P(0)\rangle + q|\bar{P}(0)\rangle) \quad (24)$$

$$- \exp(-i\lambda_L t)(p|P(0)\rangle - q|\bar{P}(0)\rangle)) \quad (25)$$

$$= \frac{1}{2} \left[(\exp(-i\lambda_H t) + \exp(-i\lambda_L t)) |\bar{P}(0)\rangle \quad (26)$$

$$+ \frac{p}{q} (\exp(-i\lambda_H t) - \exp(-i\lambda_L t)) |P(0)\rangle \right]. \quad (27)$$

Defining $P_{+1} \equiv P$ and $P_{-1} \equiv \bar{P}$, then

$$|P_k\rangle = \frac{1}{2} \left[(\exp(-i\lambda_H t) + \exp(-i\lambda_L t)) |P_k(0)\rangle \quad (28)$$

$$+ \frac{q^k}{p} (\exp(-i\lambda_H t) - \exp(-i\lambda_L t)) |P_{-k}(0)\rangle \right], \quad (29)$$

$$\equiv g_+(t)|P_k(0)\rangle + \left(\frac{q}{p}\right)^k g_-(t)|P_{-k}(0)\rangle, \quad (30)$$

using

$$g_{\pm}(t) \equiv \frac{1}{2} (\exp(-i\lambda_H t) \pm \exp(-i\lambda_L t)), \quad (31)$$

$$= \frac{1}{2} \left(\exp \left(-i \left(M + \frac{1}{2} \Delta m - \frac{i}{2} \left(\Gamma + \frac{1}{2} \Delta \Gamma \right) \right) t \right) \pm \exp \left(-i \left(M - \frac{1}{2} \Delta m - \frac{i}{2} \left(\Gamma - \frac{1}{2} \Delta \Gamma \right) \right) t \right) \right), \quad (32)$$

$$= \frac{1}{2} \exp \left(-iMt - \frac{1}{2}\Gamma t \right) \left(\exp \left(-\frac{i}{2}\Delta mt - \frac{1}{4}\Delta \Gamma t \right) \pm \exp \left(\frac{i}{2}\Delta mt + \frac{1}{4}\Delta \Gamma t \right) \right) \quad (33)$$

$$= \frac{1}{2} \exp \left(-i \left(M + \frac{1}{2} \Delta m \right) t - \frac{1}{2} \left(\Gamma + \frac{1}{2} \Delta \Gamma \right) t \right) \left(1 \pm \exp \left(i\Delta mt + \frac{1}{2} \Delta \Gamma t \right) \right), \quad (34)$$

$$= \exp \left(-iMt - \frac{1}{2}\Gamma t \right) \frac{\cosh}{\sinh} \left(-\frac{i}{2}\Delta mt - \frac{1}{4}\Delta \Gamma t \right), \quad (35)$$

$$\equiv \exp \left(-iMt - \frac{1}{2}\Gamma t \right) \frac{\cosh}{\sinh} \left(\frac{1}{2}z\Gamma t \right), \quad (36)$$

using

$$z \equiv - \left(\frac{\Delta \Gamma}{2\Gamma} + i \frac{\Delta m}{\Gamma} \right), \quad (37)$$

$$\equiv -(y + ix). \quad (38)$$

Then,

$$|A_{f,k}(t)|^2 = |\langle f | \mathcal{H} | P_k(t) \rangle|^2, \quad (39)$$

$$= \left(g_+^*(t) \langle f | \mathcal{H} | P_k(0) \rangle^* + \left(\frac{q}{p} \right)^{k*} g_-^*(t) \langle f | \mathcal{H} | P_{-k}(0) \rangle^* \right) \quad (40)$$

$$\left(g_+(t) \langle f | \mathcal{H} | P_k(0) \rangle + \left(\frac{q}{p} \right)^k g_-(t) \langle f | \mathcal{H} | P_{-k}(0) \rangle \right), \quad (41)$$

$$= |g_+(t)|^2 |A_{f,k}(0)|^2 + \left(\frac{q}{p} \right)^k g_+^*(t) g_-(t) A_{f,k}^*(0) A_{f,-k}(0) \quad (42)$$

$$+ \left(\frac{q}{p} \right)^{k*} g_-^*(t) g_+(t) A_{f,-k}^*(0) A_{f,k}(0) + \left| \frac{q}{p} \right|^{2k} |g_-(t)|^2 |A_{f,-k}(0)|^2, \quad (43)$$

$$= |g_+(t)|^2 |A_{f,k}(0)|^2 + \left| \frac{q}{p} \right|^{2k} |g_-(t)|^2 |A_{f,-k}(0)|^2 \quad (44)$$

$$+ 2\text{Re} \left(\left(\frac{q}{p} \right)^k g_+^*(t) g_-(t) A_{f,k}^*(0) A_{f,-k}(0) \right). \quad (45)$$

Using,

$$|g_{\pm}(t)|^2 = \frac{1}{4} \exp(-\Gamma t) \left[\exp\left(\frac{i}{2}\Delta mt - \frac{1}{4}\Delta\Gamma t\right) \pm \exp\left(\frac{-i}{2}\Delta mt + \frac{1}{4}\Delta\Gamma t\right) \right] \quad (46)$$

$$\left[\exp\left(\frac{-i}{2}\Delta mt - \frac{1}{4}\Delta\Gamma t\right) \pm \exp\left(\frac{i}{2}\Delta mt + \frac{1}{4}\Delta\Gamma t\right) \right], \quad (47)$$

$$= \frac{1}{4} \exp(-\Gamma t) \left[\exp\left(-\frac{1}{2}\Delta\Gamma t\right) \pm \exp(i\Delta mt) \pm \exp(-i\Delta mt) + \exp\left(\frac{1}{2}\Delta\Gamma t\right) \right], \quad (48)$$

$$= \frac{1}{2} \exp(-\Gamma t) \left[\cosh\left(\frac{1}{2}\Delta\Gamma t\right) \pm \cos(\Delta mt) \right], \quad (49)$$

and

$$g_+^*(t)g_-(t) = \frac{1}{4} \exp(-\Gamma t) \left[\exp\left(\frac{i}{2}\Delta mt - \frac{1}{4}\Delta\Gamma t\right) + \exp\left(\frac{-i}{2}\Delta mt + \frac{1}{4}\Delta\Gamma t\right) \right] \quad (50)$$

$$\left[\exp\left(\frac{-i}{2}\Delta mt - \frac{1}{4}\Delta\Gamma t\right) - \exp\left(\frac{i}{2}\Delta mt + \frac{1}{4}\Delta\Gamma t\right) \right], \quad (51)$$

$$= \frac{1}{4} \exp(-\Gamma t) \left[\exp\left(-\frac{1}{2}\Delta\Gamma t\right) - \exp(i\Delta mt) + \exp(-i\Delta mt) - \exp\left(\frac{1}{2}\Delta\Gamma t\right) \right], \quad (52)$$

$$= \frac{1}{2} \exp(-\Gamma t) \left[-\sinh\left(\frac{1}{2}\Delta\Gamma t\right) - i \sin(\Delta mt) \right], \quad (53)$$

then

$$|A_{f,k}(t)|^2 = \frac{1}{2} \exp(-\Gamma t) \left[\left(\cosh \left(\frac{1}{2} \Delta \Gamma t \right) + \cos(\Delta m t) \right) |A_{f,k}(0)|^2 \right. \quad (54)$$

$$+ \left| \frac{q}{p} \right|^{2k} \left(\cosh \left(\frac{1}{2} \Delta \Gamma t \right) - \cos(\Delta m t) \right) |A_{f,-k}(0)|^2 \quad (55)$$

$$+ 2 \operatorname{Re} \left(\left(\frac{q}{p} \right)^k \left(-\sinh \left(\frac{1}{2} \Delta \Gamma t \right) - i \sin(\Delta m t) \right) A_{f,k}^*(0) A_{f,-k}(0) \right) \Big], \quad (56)$$

$$= \frac{1}{2} \exp(-\Gamma t) \left[\left(|A_{f,k}(0)|^2 + \left| \frac{q}{p} \right|^{2k} |A_{f,-k}(0)|^2 \right) \cosh \left(\frac{1}{2} \Delta \Gamma t \right) \right. \quad (57)$$

$$+ \left(|A_{f,k}(0)|^2 - \left| \frac{q}{p} \right|^{2k} |A_{f,-k}(0)|^2 \right) \cos(\Delta m t) \quad (58)$$

$$- 2 \operatorname{Re} \left(\left(\frac{q}{p} \right)^k A_{f,k}^*(0) A_{f,-k}(0) \right) \sinh \left(\frac{1}{2} \Delta \Gamma t \right) \quad (59)$$

$$+ 2 \operatorname{Im} \left(\left(\frac{q}{p} \right)^k A_{f,k}^*(0) A_{f,-k}(0) \right) \sin(\Delta m t) \Big] \quad (60)$$