Shortest Reconfiguration of Matchings

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Presenters: Niranjan and Mano Prakash

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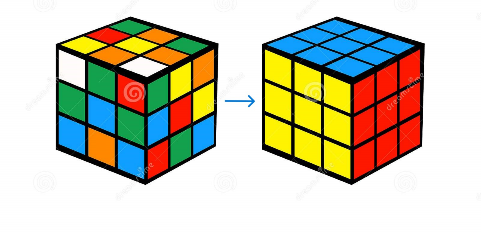
Reconfiguration Problem: In General

Reconfiguration Problem

Goal: Ask for the existence of a **step-by-step** transformation between two given configurations.

Example: Rubik's Cube Configurations.



- In this paper, we consider matchings in graphs as configurations.

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Two matchings M and M' of a graph G are **adjacent** if one can be obtained from the other by relocating a single token.

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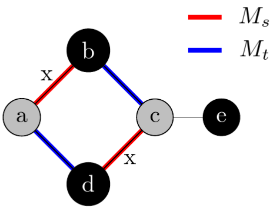
Reconfiguration sequence

$M_0, M_1, ...M_l$ of matchings of G is a **reconfiguration sequence** of length l from M to M', if $M_0 = M, M_l = M'$, and the matchings M_{i-1} and M_i are adjacent for each $i \in \{1, 2, ...l\}$.

An Example of Reconfiguration Problem: In Matchings



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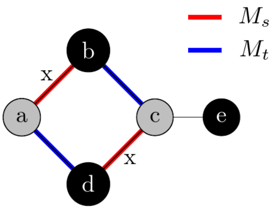

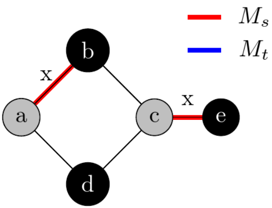

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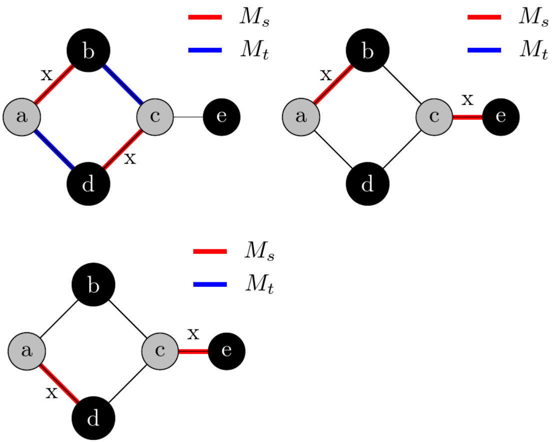

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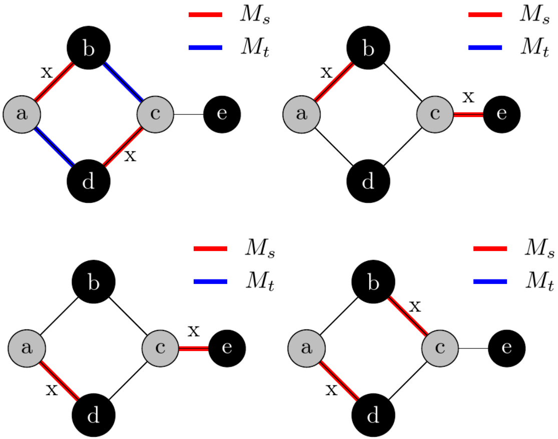

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Input Graph G and two matchings M_s, M_t of G.

Question Is there a reconfiguration sequence from M_s to M_t ?

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- The reachability variant is solvable in polynomial time.
- This paper deals with the **shortest variant**.
- This is an NP-hard problem.

Main Result of the paper

Theorem (Shorter Version)

Matching Distance in bipartite graphs is **fixed parameter tractable (FPT)**.

- Before looking into the complete version, we shall look into what fixed parameter tractability is.

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Fixed Parameter Tractability

A problem is *fixed-parameter tractable* if there is an $f(k)n^c$ algorithm for some constant c and a parameter k.

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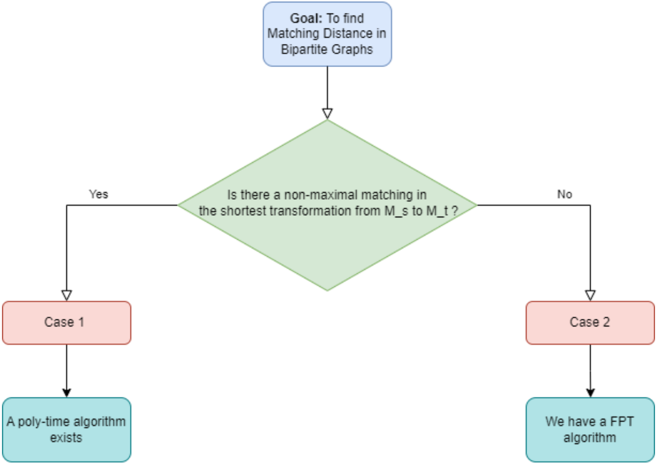
Theorem

Matching Distance in bipartite graphs can be solved in $2^d n^{\mathcal{O}(1)}$, where d is the size of the symmetric difference of the two given matchings.

Overview of the algorithm



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- Directed Steiner Tree Problem is known to be FPT.
- This will give us an FPT algorithm for matching distance in bipartite graphs.

The Directed Steiner Tree Problem

Directed Steiner Tree

Input Directed Graph $D = (V, A)$, integral arc weights c_a for each $a \in A$, root vertex $r \in V$, and terminals $T \subseteq V$

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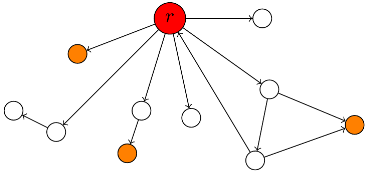

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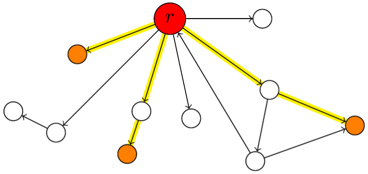

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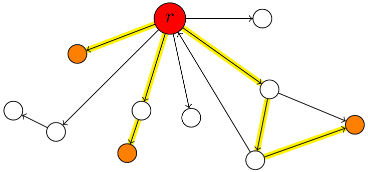

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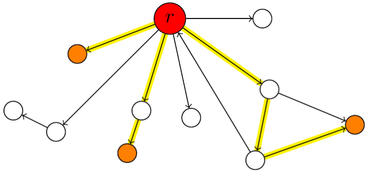

Directed Steiner Tree has an algorithm which runs in $2^{|T|} n^{\mathcal{O}(1)} W$ where W is the maximum arc weight.

Reducing Matching Distance to DST

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Goal We will convert I to an instance $I' := (D, c, r, T)$ of DIRECTED STEINER TREE such that given a Steiner Tree F for I', we can construct in polynomial time a transformation from M_s to M_t of cost at most $c(F)$.

Reducing Matching Distance to DST

Let X_s be the set of M_s-exposed vertices in G. We construct the digraph $D = (U', A)$ as follows.

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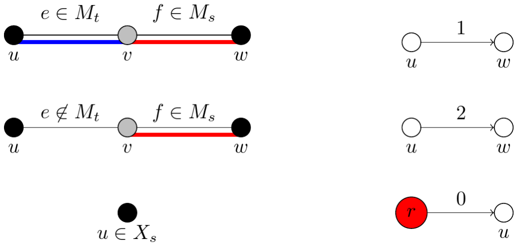

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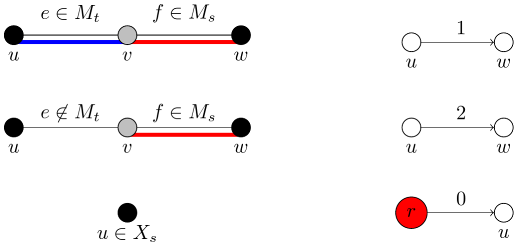
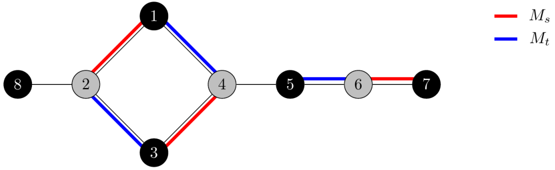


The terminals of the DIRECTED STEINER TREE instance are

$$T := [\bigcup_{Z \in \mathcal{C}} V(Z) \bigcup_{Z \in \mathcal{P}} (V(Z) \setminus X_s)] \cap U'$$

where $\mathcal{P}$ and $\mathcal{C}$ are the paths and cycles respectively which form connected components of $M_s \oplus M_t$.

An example



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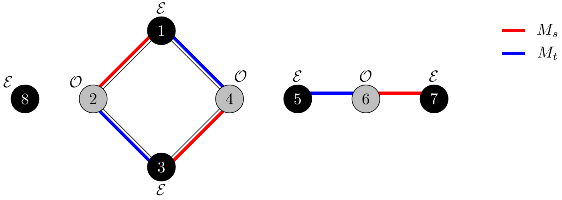

$\mathcal{E}$
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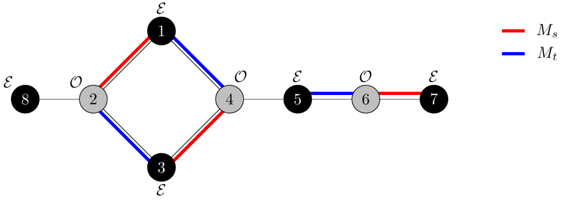
$\mathcal{E}$
8
$\mathcal{O}$
2
$\mathcal{O}$
4
$\mathcal{E}$
5
$\mathcal{O}$
6
$\mathcal{E}$
7
3
$\mathcal{E}$
M_s
M_t

An example



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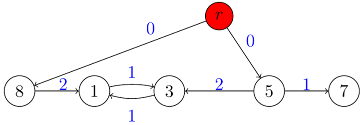

r
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0
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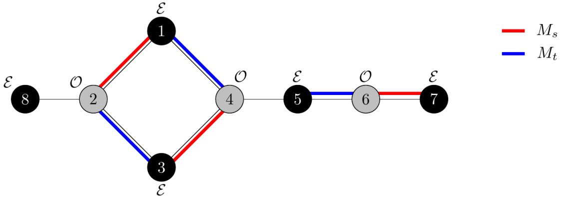
2
1
1
3
1
2
5
1
7

An example



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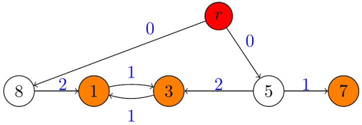

r
0
0
8
2
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7
1

We don't need to look at all Steiner trees!

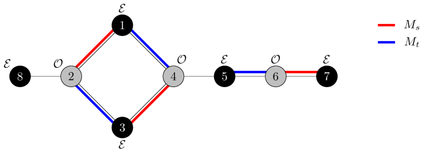
Proposition

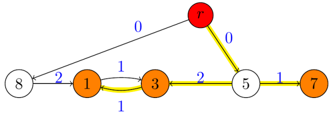
Given a Steiner Tree F for I', we can construct a Steiner Tree F' with $c(F') \leq c(F)$ which satisfies the following properties:

(i) For each $P \in \mathcal{P}$, F' contains all arcs of P.

(ii) For each $C \in \mathcal{C}$, F' misses exactly one arc of C.

(iii) For each $P \in \mathcal{P}$, r is joined to the M_s-exposed vertex of P.





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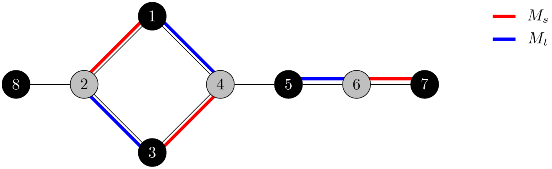
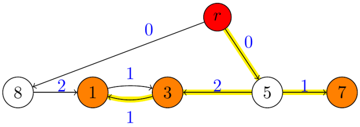
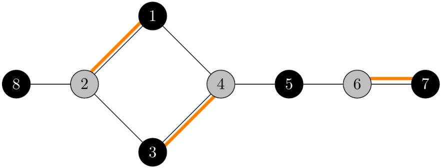
- Perform DFS traversal of F giving preference to the largest weight arcs.
- Each arc of nonzero weight corresponds to a token.
- When traversing down an arc of weight 1, move its token to target destination. No move when backtracking.
- When traversing down an arc of weight 2, we move its token away from the target destination. When backtracking, move it to the target position.

An example







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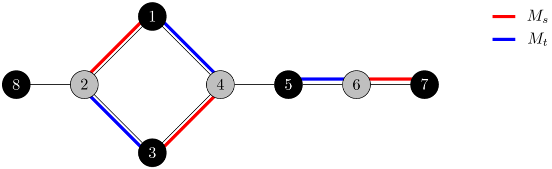

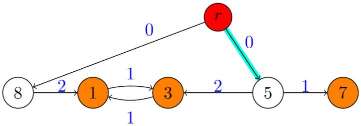

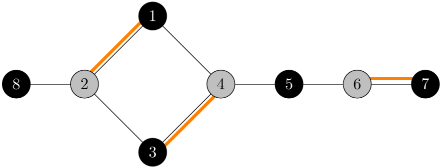

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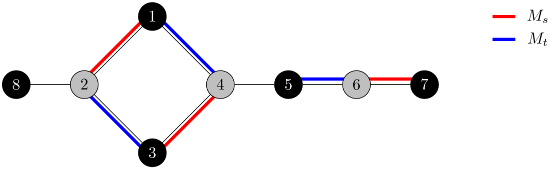

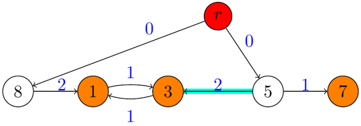

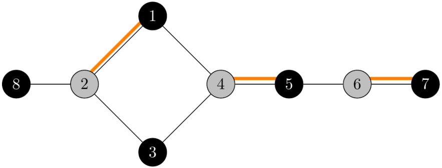

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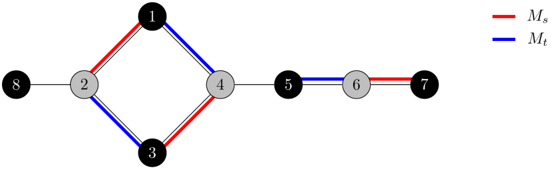

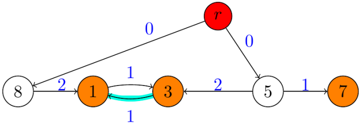

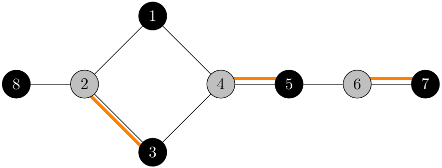

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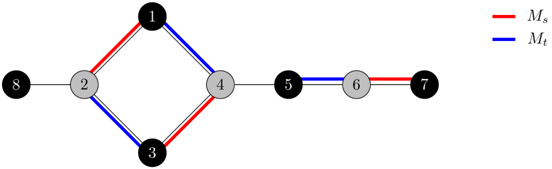

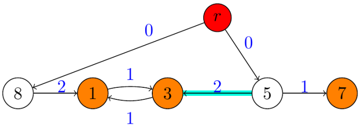

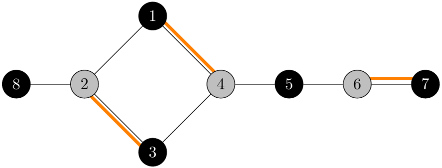

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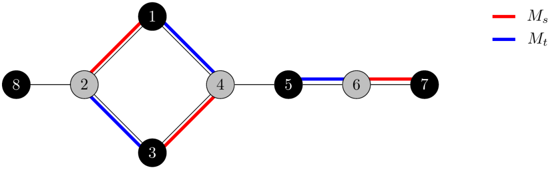

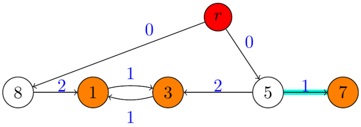

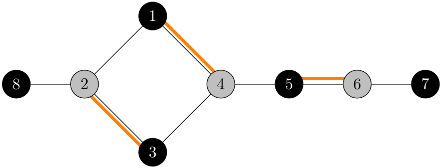

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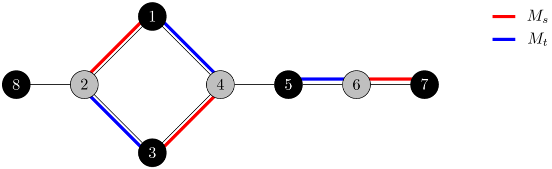

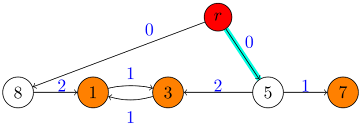

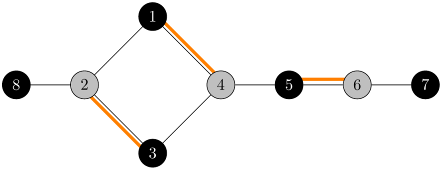

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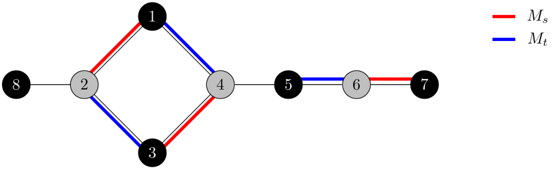

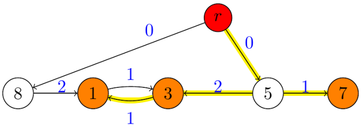

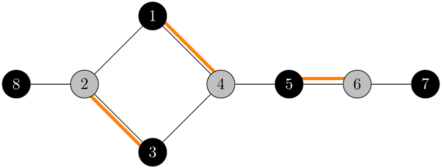

A necessary and sufficient condition

Lemma

There is a transformation from M_s to M_t $\iff$ for each cycle $C \in \mathcal{C}$, there is an M_s-alternating path starting from an M_s-exposed vertex and ending at a vertex in C.

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Note: This is true for Case 2 in general, not just for maximum matchings.

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Note: This is true for Case 2 in general, not just for maximum matchings. For each cycle $C \in \mathcal{C}$, we have to slide the token along this M_s-alternating path and then keep moving the token to its target position.

The algorithm for Case 2

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- For the second sub-instance, delete the M_s-exposed vertices of W.
- Solve each sub-instance using the algorithm for maximum matchings and combine the optimal solutions of the two sub-instances.

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Hence it can be solved in time $2^{\frac{d}{2}} n^{\mathcal{O}(1)}$.
For the general case 2, we branch over at most $2^{\frac{d}{4}}$ choices. Since each of these can be solved in $2^{\frac{d}{2}} n^{\mathcal{O}(1)}$, we can solve case 2 in time $2^{\frac{d}{4}} \times 2^{\frac{d}{2}} n^{\mathcal{O}(1)}$.
$\square$

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- If one of M_s or M_t is non-maximal, the shortest transformation has length $\frac{d}{2}$ if $M_s \oplus M_t$ contains no cycles, or $\frac{d}{2} + 1$ otherwise.
- If M_s and M_t are maximal, we can transform M_s into a non-maximal matching by sliding tokens along an M_s-augmenting path.

Thank You !