Probability Theory & Random Processes

Unit i

Probability basic concePts

Probability theory had its origin in the analysis of certain games of chance that were popular in the seventeenth century. It has since found applications in many branches of Science and Engineering and this extensive application makes it an important branch of study. Probability theory, as a matter of fact, is a study of random or unpredictable experiments and is helpful in investigating the important features of these random experiments.

Objectives

- Comprehend the different ways of assigning probability
- Understand and apply marginal, union, joint, and conditional probabilities
- Solve problems using the laws of probability including the laws of addition, multiplication and conditional probability
- Revise probabilities using Bayes' rule

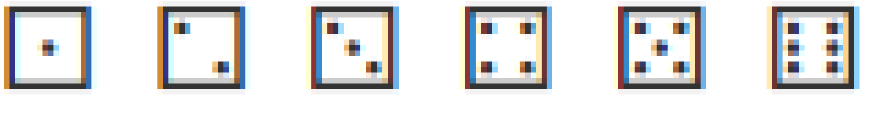
Experiments:

- Deterministic Experiment
- Random Experiment

An experiment whose outcome or result can be predicted with certainty is called a deterministic experiment. For example, if the potential difference E between the two ends of a conductor and the resistance R are known, the current I flowing in the conductor is uniquely determined by Ohm's law, $I = \frac{E}{R}$.

Although all possible outcomes of an experiment may be known in advance, the outcome of a particular performance of the experiment cannot be predicted owing to a number of unknown causes. Such an experiment is called a random experiment.

Whenever a fair 6-faced cubic dice is rolled, it is known that any of the 6 possible outcomes will occur, but it cannot be predicted what exactly the outcome will be, when the dice is rolled at a point of time.



Although the number of telephone calls received in a board in a 5-min. interval is a non-negative integer, we cannot predict exactly the number of calls received in the next 5-min. In such situations we talk of the chance or the probability of occurrence of a particular outcome, which is taken as a quantitative measure of the likelihood of the occurrence of the outcome.

Example of Random Experiments

- Toss of coin
- Rolling a dice
- Number of A’s a student can obtain in a year
- Customer arriving at a business outlet
- Component failure in system
- Transaction requests arriving at a sever
- Messages received

Probability Terminology

- Experiment
- Event
- Elementary Events
- Sample Space
- Unions and Intersections
- Mutually Exclusive Events
- Independent Events
- Collectively Exhaustive Events
- Complementary Events

Experiment, Trial, Elementary Event, Event

- Experiment: a process that produces outcomes
  - More than one possible outcome
  - Only one outcome per trial
- Trial: one repetition of the process
- Elementary Event: cannot be decomposed or broken down into other events
- Event: an outcome of an experiment
  - may be an elementary event, or
  - may be an aggregate of elementary events
  - usually represented by an uppercase letter, e.g., A, E1

Example

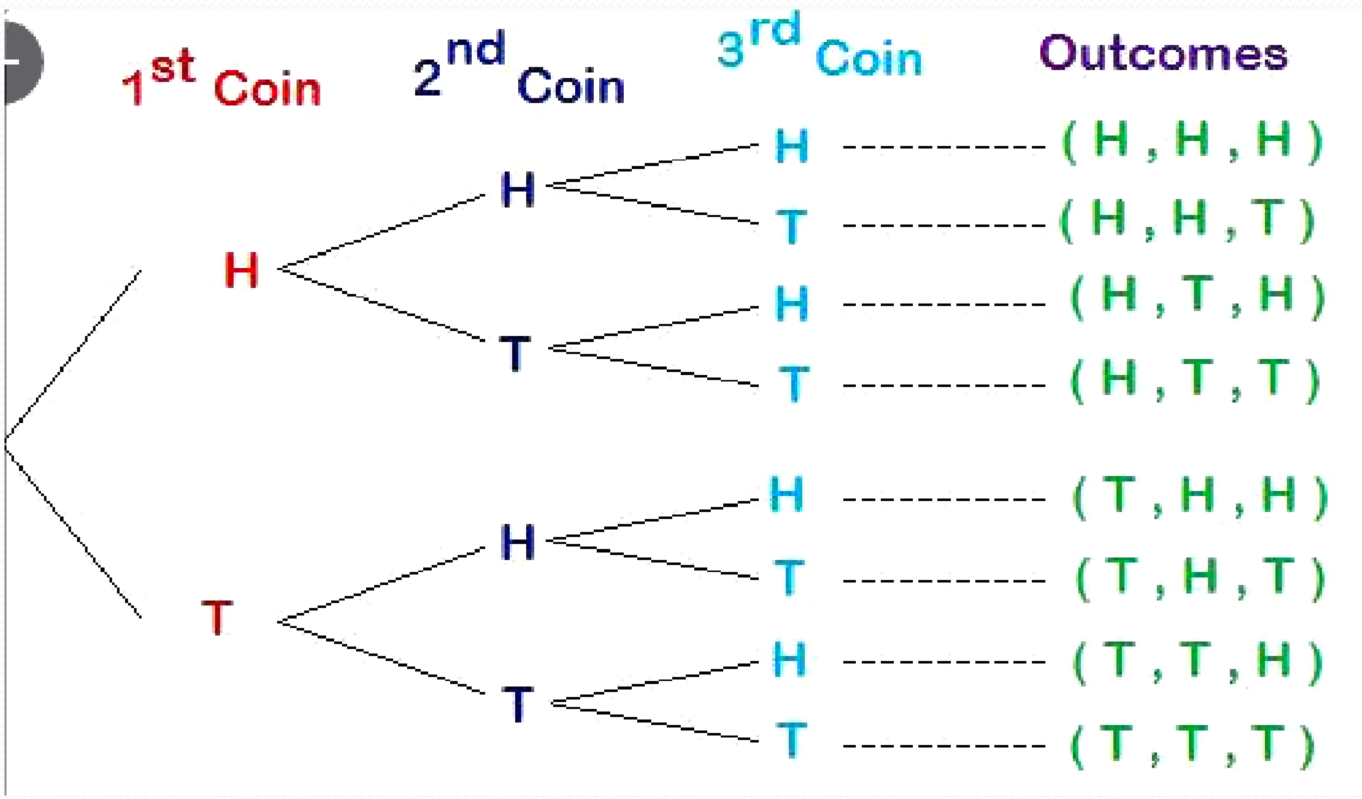
- If a die is tossed any one number from 1 to 6 appears. So the sample space of this experiment is

  $$S = \{1, 2, 3, 4, 5, 6\}$$

- The event “outcome is an even number” i set of S and is

  $$E = \{2, 4, 6\}$$

Sample Space of tossing three coins



Example

- Coin tossing experiment
- Outcomes are Head (H) and Tail (T)
- Space as result of three successive tosses is

  S= {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}
- The event "two heads and one tail" is

  E= {HHT, HTH, THH}
- Event "at least one head" is

  E= {HHH, HHT, HTH, HTT, THH, THT, TTH}

Example

- Rolling a dice twice or rolling two dice together
- Sample space
  S = { (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6),
  (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6),
  (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) }

$$S = \left\{ \begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{matrix} \right\}$$

- Event "sum of two numbers is 9"
  E = {(3,6), (4,5), (5,4), (6,3)}

Example

- Life span
  - Sample space

    $S = \{x \mid 0 \leq x < \infty\}$
  - Event : "Not more than 10 hours" is defined as

    $E = \{x \mid 0 \leq x < 10 \text{ (hours)}\}$

An Example Experiment

- Experiment: randomly select, without replacement, two families from the residents of Tiny Town
- Elementary Event: the sample includes families A and C
- Event: each family in the sample has children in the household
- Event: the sample families own a total of four automobiles

Tiny Town Population		
Family	Children in Household	Number of Automobiles
A	Yes	3
B	Yes	2
C	No	1
D	Yes	2

Sample Space

- The set of all elementary events for an experiment
- Methods for describing a sample space
  - roster or listing
  - tree diagram
  - set builder notation
  - Venn diagram

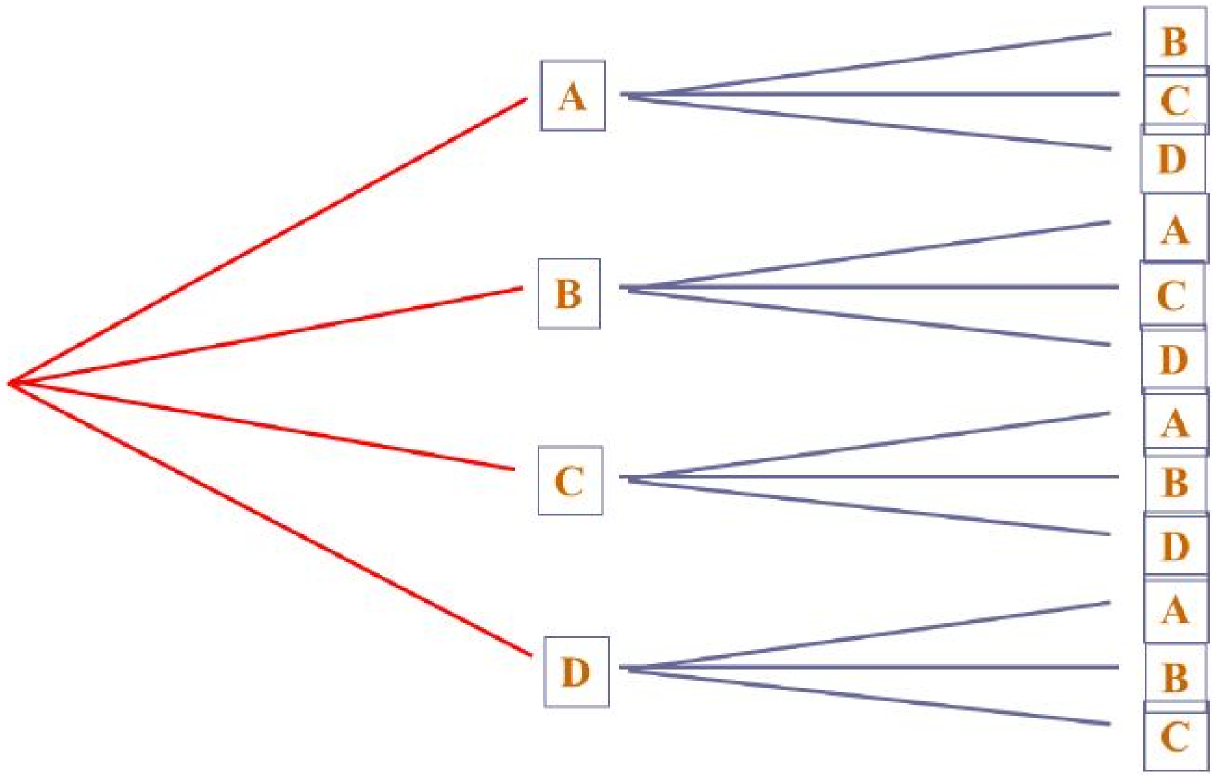
Sample Space: Roster Example

- Experiment: randomly select, without replacement, two families from the residents of Tiny Town
- Each ordered pair in the sample space is an elementary event, for example -- (D,C)

Family	Children in Household	Number of Automobiles
A	Yes	3
B	Yes	2
C	No	1
D	Yes	2

Listing of Sample Space
(A,B), (A,C), (A,D), (B,A), (B,C), (B,D), (C,A), (C,B), (C,D), (D,A), (D,B), (D,C)

Sample Space: Tree Diagram for Random Sample of Two Families



Sample Space: Set Notation for Random Sample of Two Families

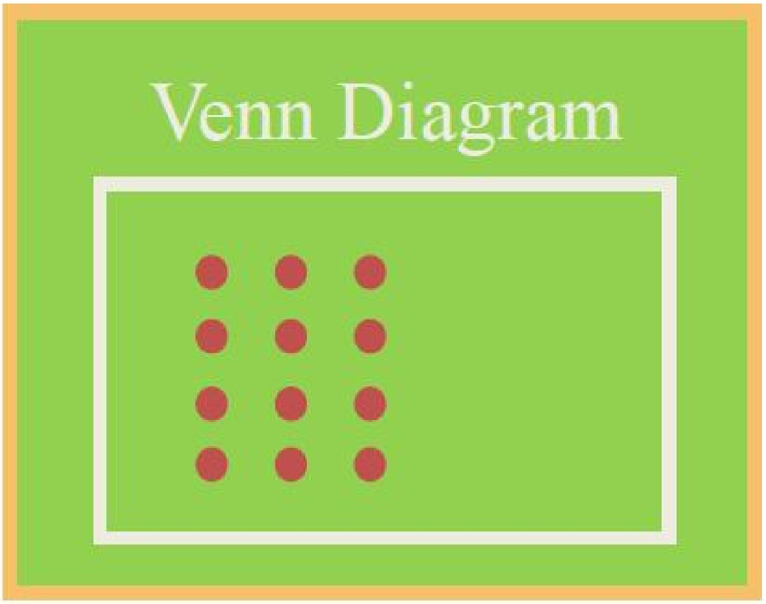
- S = {(x,y) | x is the family selected on the first draw, and y is the family selected on the second draw}
- Concise description of large sample spaces

Sample Space

- Useful for discussion of general principles and concepts

Listing of Sample Space

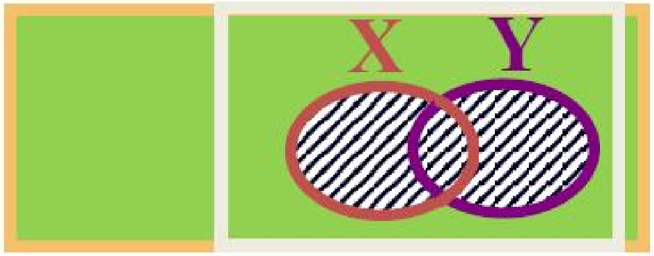
(A,B), (A,C), (A,D),
(B,A), (B,C), (B,D),
(C,A), (C,B), (C,D),
(D,A), (D,B), (D,C)



Union of Sets

- The union of two sets contains an instance of each element of the two sets.

$$X = \{1,4,7,9\}$$
$$Y = \{2,3,4,5,6\}$$
$$X \cup Y = \{1,2,3,4,5,6,7,9\}$$

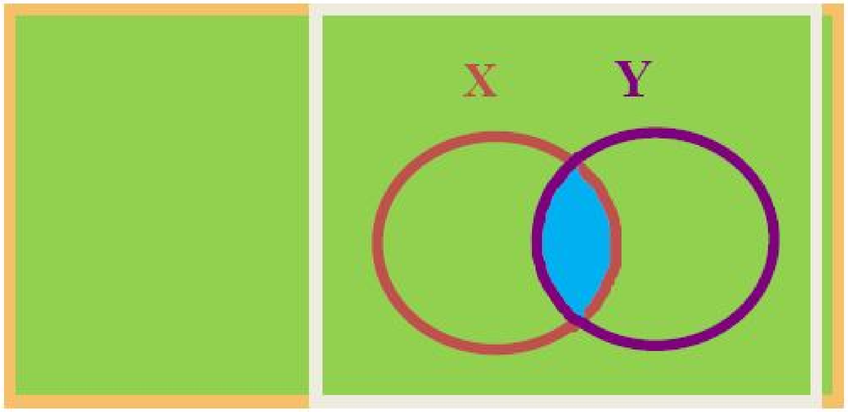


$$C = \{IBM, DEC, Apple\}$$
$$F = \{Apple, Grape, Lime\}$$
$$C \cup F = \{IBM, DEC, Apple, Grape, Lime\}$$

Intersection of Sets

- The intersection of two sets contains only those element common to the

$$X = \{1,4,7,9\}$$
$$Y = \{2,3,4,5,6\}$$
$$X \cap Y = \{4\}$$



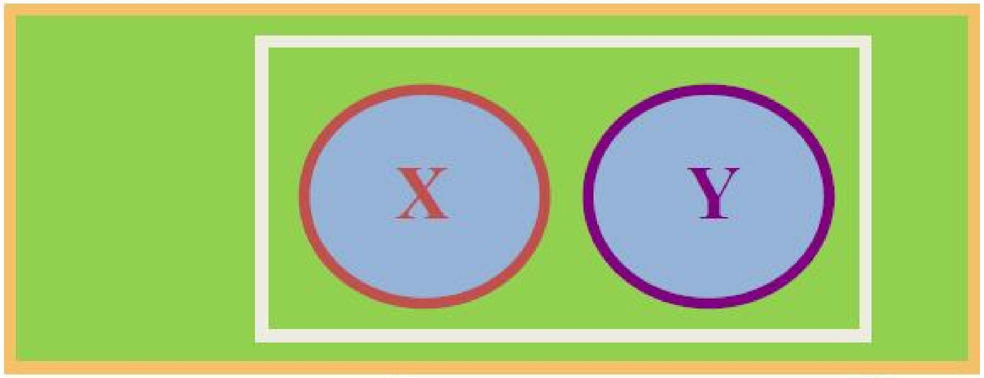
$$C = \{IBM, DEC, Apple\}$$
$$F = \{Apple, Grape, Lime\}$$
$$C \cap F = \{Apple\}$$

Mutually Exclusive Events

- Events with no common outcomes
- Occurrence of one event precludes the occurrence of the other event

$$C = \{IBM, DEC, Apple\}$$
$$F = \{Grape, Lime\}$$
$$C \cap F = \{\ \}$$

$$X = \{1,7,9\}$$
$$Y = \{2,3,4,5,6\}$$
$$X \cap Y = \{\ \}$$



$$P(X \cap Y) = 0$$

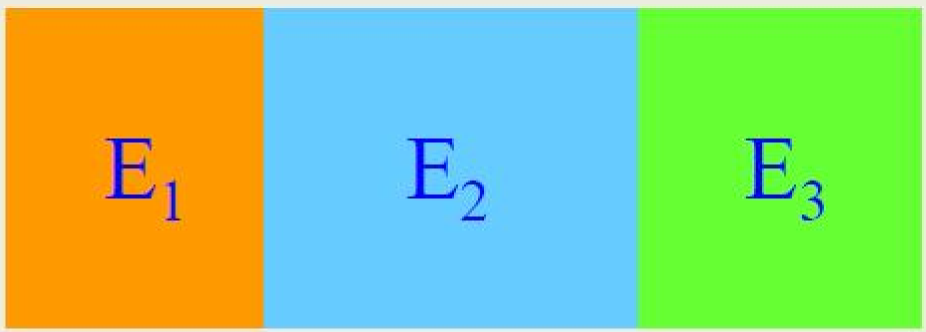
Independent Events

- Occurrence of one event does not affect the occurrence or nonoccurrence of the other event
- The conditional probability of X given Y is equal to the marginal probability of X.
- The conditional probability of Y given X is equal to the marginal probability of Y.

$$\boxed{P(X|Y) = P(X) \textit{ and } P(Y|X) = P(Y)}$$

Collectively Exhaustive Events

- Contains all elementary events for an experiment

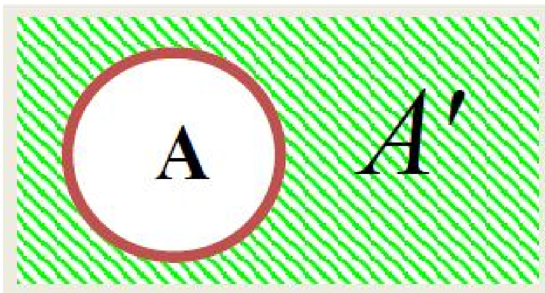


Sample Space with three
collectively exhaustive events

Complementary Events

- All elementary events not in the event 'A' are in its complementary event.

Sample Space



$$P(Sample\ Space) = 1$$

$$P(A') = 1 - P(A)$$

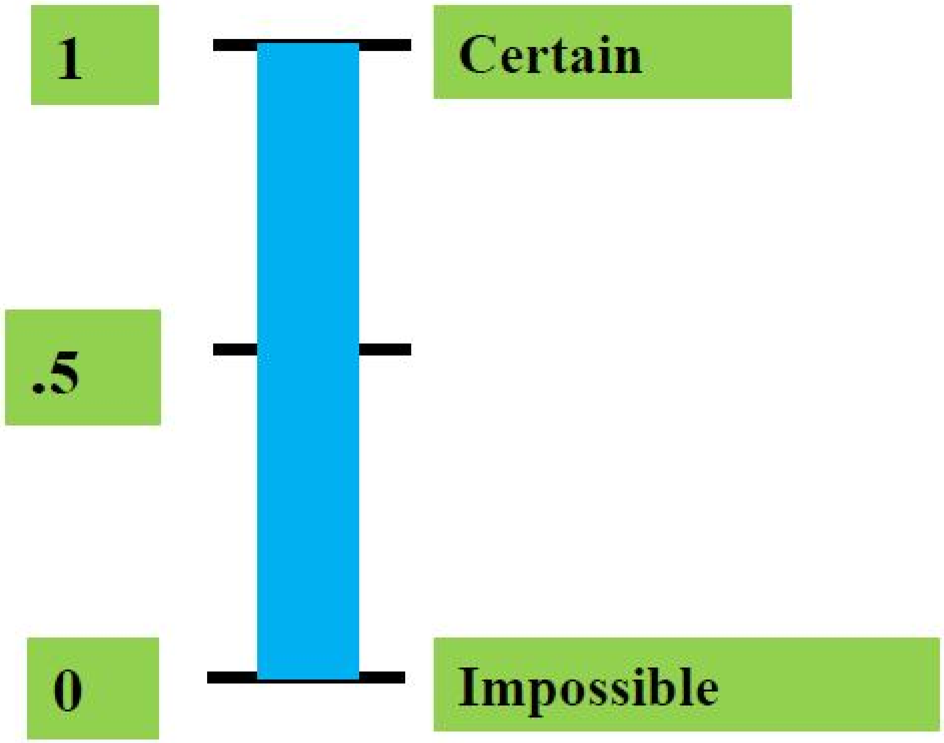
Probability

- Probability is the numerical measure of the likelihood that an event will occur.

- The probability of any event must be between 0 and 1, inclusively
  - $0 \leq P(A) \leq 1$ for any event A.

- The sum of the probabilities of all mutually exclusive and collectively exhaustive events is 1.
  - $P(A) + P(B) + P(C) = 1$
  - A, B, and C are mutually exclusive and collectively exhaustive

Range of Probability



Axiomatic Definition of Probability

- If a random experiment has a sample space then for each event A of S the probability of occurring of A is a number P(A) such that following hold
  - **Axiom 1:** $0 \leq P(A) \leq 1$
  - **Axiom 2:** $P(S) = 1$
  - **Axiom 3:** For any set of n mutually exclusive events A_1, A_2, A_3, ... , A_n of the same sample space $P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + P(A_3) + \ldots + P(A_n)$