Filter Design

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1 Introduction

We are supposed to design the equivalent FIR and IIR filter realizations for the given filter number. We will be designing a bandpass filter.

2 Filter Specifications

The sampling rate for the filter has been specified as $F_s = 48$ kHz. If the un-normalized discrete-time (natural) frequency is F, the corresponding normalized digital filter (angular) frequency is given by $\omega = 2\pi\left(\frac{F}{F_s}\right)$.

2.1 The Digital Filter

1. *Tolerances:* The passband (δ_1) and stopband (δ_2) tolerances are given to be equal, so we let $\delta_1 = \delta_2 = \delta = 0.15$.

2. Passband: The passband is from {4 + 0.6(j)} kHz to {4 + 0.6(j+2)} kHz.
   where

$$j = (r - 11000) \mod \sigma \tag{1}$$

   where σ is sum of digits of roll number and r is roll number.

$$r = 11208 \tag{2}$$
$$\sigma = 12 \tag{3}$$
$$j = 4 \tag{4}$$

   substituting $j = 4$ gives the passband range for our bandpass filter as 6.4 kHz - 7.6 kHz. Hence, the un-normalized discrete time filter passband frequencies are $F_{p2} = 6.4$ kHz and $F_{p1} = 7.6$ kHz.

The corresponding normalized digital filter passband frequencies are

$$\begin{aligned}\omega_{p2} &= 2\pi\frac{F_{p2}}{F_s} &(5)\\ &= 2\pi\frac{6.4KHz}{48KHz} &(6)\\ &= 0.266\pi &(7)\\ \omega_{p1} &= 2\pi\frac{F_{p1}}{F_s} &(8)\\ &= 2\pi\frac{7.6KHz}{48KHz} &(9)\\ &= 0.3166\pi &(10)\end{aligned}$$

3. *Stopband:* The *transition band* for bandpass filters is $\Delta F = 0.3$ kHz on either side of the passband. Hence, the un-normalized *stopband* frequencies are $F_{s2} = 6.4-0.3 = 6.1$ kHz and $F_{s1} = 7.6+0.3 = 7.9$ kHz. The corresponding normalized frequencies are

$$\begin{aligned}\omega_{s2} &= 2\pi\frac{F_{s2}}{F_s} &(11)\\ &= 2\pi\frac{6.1KHz}{48KHz} &(12)\\ &= 0.254\pi &(13)\\ \omega_{s1} &= 2\pi\frac{F_{s1}}{F_s} &(14)\\ &= 2\pi\frac{7.9KHz}{48KHz} &(15)\\ &= 0.329\pi &(16)\end{aligned}$$

2.2 The Analog filter

In the bilinear transform, the analog filter frequency (Ω) is related to the corresponding digital filter frequency (ω) as $\Omega = \tan\frac{\omega}{2}$. Using this relation, we obtain the analog passband and stopband frequencies as:
$\Omega_{p2} = 0.4439$, $\Omega_{p1} = 0.5429$ and $\Omega_{s2} = 0.4217$, $\Omega_{s1} = 0.5685$.

3 The IIR Filter Design

Filter Type: We are supposed to design filters whose stopband is monotonic and passband equiripple. Hence, we use the *Chebyschev approximation* to design our bandpass IIR filter.

3.1 The Analog Filter

1. *Low Pass Filter Specifications:* If $H_{a,BP}(j\Omega)$ be the desired analog band pass filter, with the specifications provided in Section 2.2, and $H_{a,LP}(j\Omega_L)$ be the equivalent low pass filter, then

$$\Omega_L = \frac{\Omega^2 - \Omega_0^2}{B\Omega} \tag{17}$$

where $\Omega_0 = \sqrt{\Omega_{p1}\Omega_{p2}} = 0.4914$ and $B = \Omega_{p1} - \Omega_{p2} = 0.0971$. The low pass filter has the stopband edges at ,

$$\Omega_{Ls1} = \frac{\Omega_{s1}^2 - \Omega_0^2}{B\Omega_{s1}} = 1.476 \tag{18}$$

$$\Omega_{Ls2} = \frac{\Omega_{s2}^2 - \Omega_0^2}{B\Omega_{s2}} = -1.56 \tag{19}$$

We will choose the minimum of these two stopband edges

$$\Omega_{Ls} = \min(|\Omega_{Ls_1}|, |\Omega_{Ls_2}|) = 1.476 \tag{20}$$

2. The Low Pass Chebyschev Filter Paramters: The magnitude of frequency response of the low pass filter is given by

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \epsilon^2 c_N^2(\Omega_L/\Omega_{Lp})} \tag{21}$$

The passband edge of the low pass filter is chosen as $\Omega_{Lp} = 1$. Therfore ,

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \epsilon^2 c_N^2(\Omega_L)} \tag{22}$$

Here c_N denote the chebyshev polynomials for a particular order N of the filter.

$$c_N(x) = \cosh(N \cosh^{-1} x), x = \Omega_L \tag{23}$$

$$c_0(x) = 1 \tag{24}$$

$$c_1(x) = x \tag{25}$$

There exists a recurssive relation from which all the polynomials can be found out.

$$c_{N+2} = 2xc_{N+1} + c_N \tag{26}$$

Imposing the band restrictions on (21)

$$|H_{a,LP}(j\Omega_L)|^2 < \delta_2 \ \text{ for } \ \Omega_L = \Omega_{Ls} \tag{27}$$

$$1 - \delta_1 < |H_{a,LP}(j\Omega_L)|^2 < 1 \ \text{ for } \ \Omega_L = \Omega_{Lp} \tag{28}$$

$$\tag{29}$$

we obtain :

$$\frac{\sqrt{D_2}}{c_N(\Omega_{Ls})} \leq \epsilon \leq \sqrt{D_1},$$
$$N \geq \left\lceil \frac{\cosh^{-1}\sqrt{D_2/D_1}}{\cosh^{-1}\Omega_{Ls}} \right\rceil, \tag{30}$$

where $D_1 = \frac{1}{(1-\delta)^2} - 1$ and $D_2 = \frac{1}{\delta^2} - 1$ and $\lceil . \rceil$ is known as the ceiling operator .
$\delta = 0.15$. Thus we have $D_1 = 0.384$ and $D_2 = 43.44$.
Using (23), $c_4(\Omega_s) = 21.54$. Hence $N \geq 4$ and $0.304 \leq \epsilon \leq 0.6196$

Parameter	Value
D_1	0.384
D_2	43.44
N	4
$c_4(x)$	$8x^4 + 8x^2 + 1$

Table 1: Parameter Table

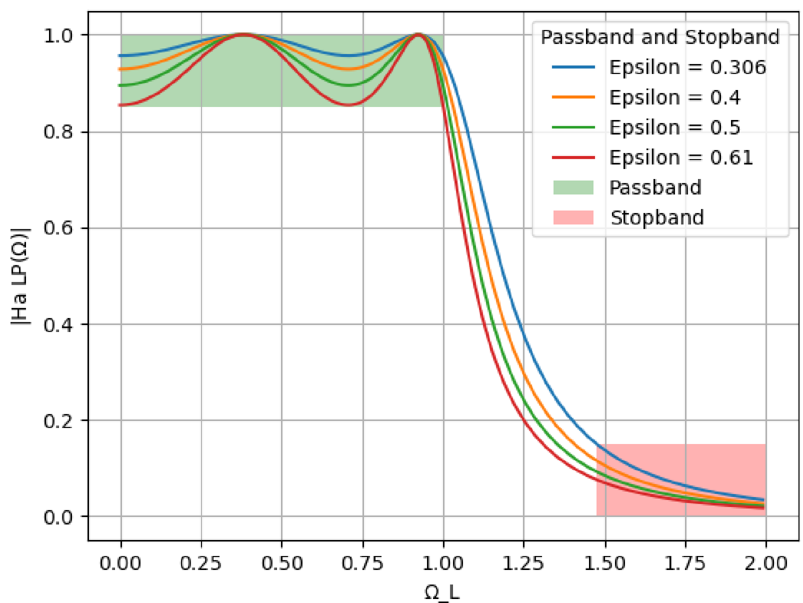

Figure 1: The Analog Low-Pass Frequency Response for $0.304 \leq \epsilon \leq 0.61$

In Fig. 1 we can observe the equiripple behaviour in passband and monotonic behaviour in stopband. As the value of ϵ increases the value of $|H_{a,LP}(j\Omega_L)|$ decreases. We choose $\epsilon = 0.4$ for our IIR filter design.

3. The Low Pass Chebyschev Filter: The next step in design is to find an expression for magnitude response in s domain.

   Using $s = j\Omega$ or in this case $s_L = j\Omega_L$ we obtain:

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \epsilon^2 c_N^2(\frac{s_L}{j})} \tag{31}$$

   To find poles equate the denominator to zero:

$$1 + \epsilon^2 c_N^2 \left(\frac{s_L}{j}\right) = 0 \text{ where } \quad c_N(x) = cos\left(Ncos^{-1}(x)\right) \tag{32}$$

   On solving (32) we obtain poles :

$$s_k = -\Omega_{Lp} \sin(A_k) \sinh(B_k) - j\Omega_{Lp} \cos(A_k) \cosh(B_k) \tag{33}$$

   where k is the index of the pole and

$$A_k = (2k+1) \frac{\pi}{2N} \tag{34}$$

$$B_k = \frac{1}{N} \sinh^{-1}\left(\frac{1}{\epsilon}\right) \tag{35}$$

   The poles obtained are formulated in the table below.

$Pole$	$Value$
s_1	0.1621 + j1.0033
s_2	0.3913 + j0.4156
s_3	0.1621 + j-1.0033
s_4	0.3913 + j-0.4156
s_5	-0.3913 + j-0.4156
s_6	-0.3913 + j0.4156
s_7	-0.1621 + j1.0033
s_8	-0.1621 + j-1.0033

Table 2: Values of s_k

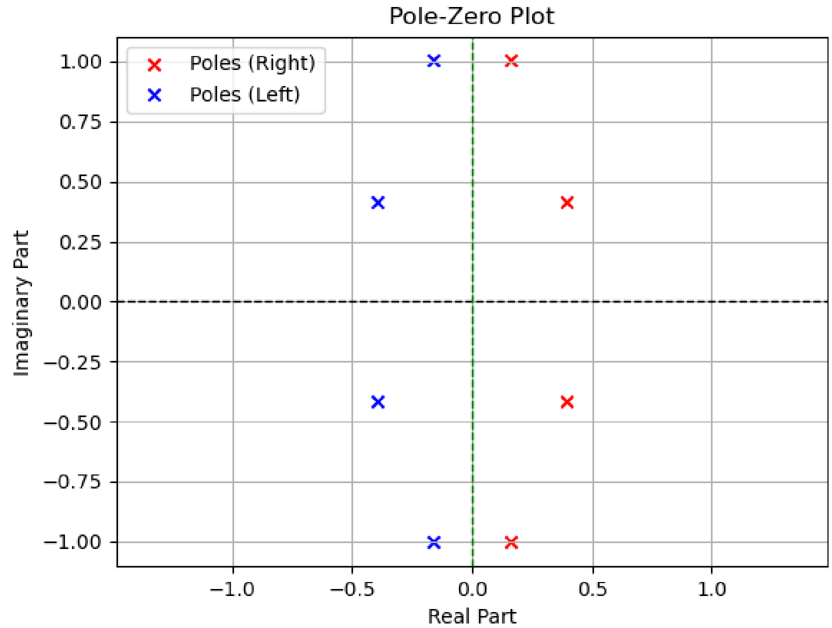

Figure 2: The Pole zero plot and all the poles lie on an ellipse. The left and right poles have been identified as shown.

The poles in the left half of the plane are considered in the design as we intend to design a stable system.
Therefore the magnitude response is written as :-

$$H_{a,LP}(s_L) = \frac{G_{LP}}{(s_L - s_5)(s_L - s_6)(s_L - s_7)(s_L - s_8)} \tag{36}$$

where G_{LP} is the gain of the Low pass filter. Refer to Table 2 for s_k values.

We know that from (21):-

$$\left|H_{a,LP}(s_L)\right| = \frac{1}{\sqrt{1+\epsilon^2}} \text{at } \Omega_L = 1 \implies s_L = j \tag{37}$$

Substituting respective values in (37) we get $G_{LP} = 0.4166$

$$H_{a,LP}(s_L) = \frac{0.3125}{(s_L - s_5)(s_L - s_6)(s_L - s_7)(s_L - s_8)} \tag{38}$$

$$= \frac{0.3125}{s_L^4 + 1.1068s_L^3 + 1.9325s_L^2 + 0.78_L + 0.3562} \tag{39}$$

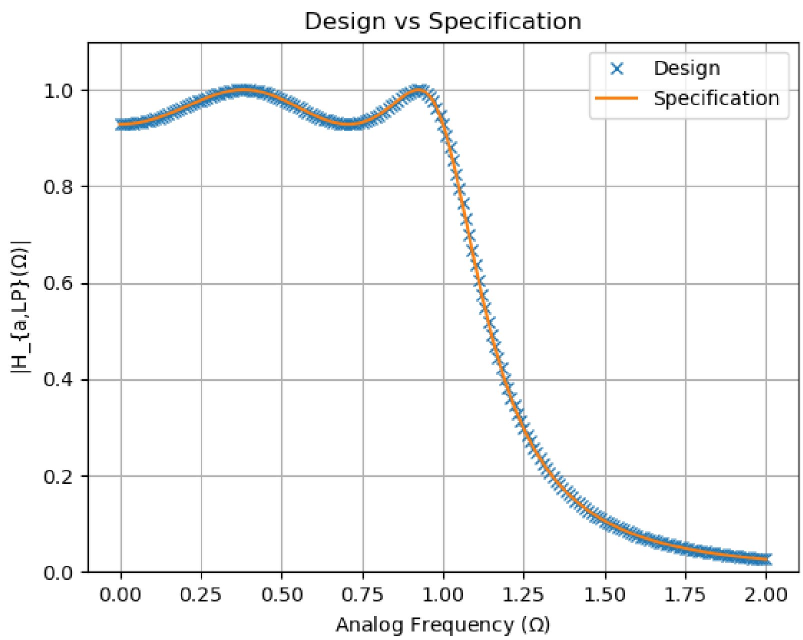

Figure 3: Design vs Specification plot

4. The Band Pass Chebyschev Filter: After verifying design with the required specifications the next step in design is to jump to required type of filter using frequency transformation.

$$s_L = \frac{s^2 + \Omega_0^2}{Bs} \tag{40}$$

$$H_{a,BP}(s) = G_{BP} H_{a,LP}(s_L)\Big|_{s_L = \frac{s^2+\Omega_0^2}{Bs}}, \tag{41}$$

As there is one to one correspondence between the filters so $\Omega = \Omega_{p1}$ should correspond to Ω_{Lp}

$$s = j\Omega_{p1} \tag{42}$$

$$s_L = \frac{(j\Omega_{p1})^2 + \Omega_0^2}{B(j\Omega_{p1})} \tag{43}$$

$$\left|H_{a,BP}(j\Omega_{p1})\right| = 1 \tag{44}$$

$$G_{BP}\left|H_{a,LP}(s_L)\right| = 1 \tag{45}$$

Substituting (43) in (45) we obtain Gain of required bass pass filter:

$$G_{BP} = 1.08 \tag{46}$$

Thus the response in s domain

$$H_{a,BP}(s) = \frac{9.8138 \times 10^{-5} s^4}{s^8 + 0.108s^7 + 0.982s^6 + 0.079s^5 + 0.358s^4 + 0.0192s^3 + 0.0574s^2 + 0.0015s + 0.0034} \tag{47}$$

The expressions in the s-domain and gain factors are computed by writing a Python code.
In Figure 3, we plot $|H_{a,BP}(j\Omega)|$ as a function of Ω for both positve as well as negative frequencies. We find that the passband and stopband frequencies in the figure match well with those obtained analytically through the bilinear transformation.

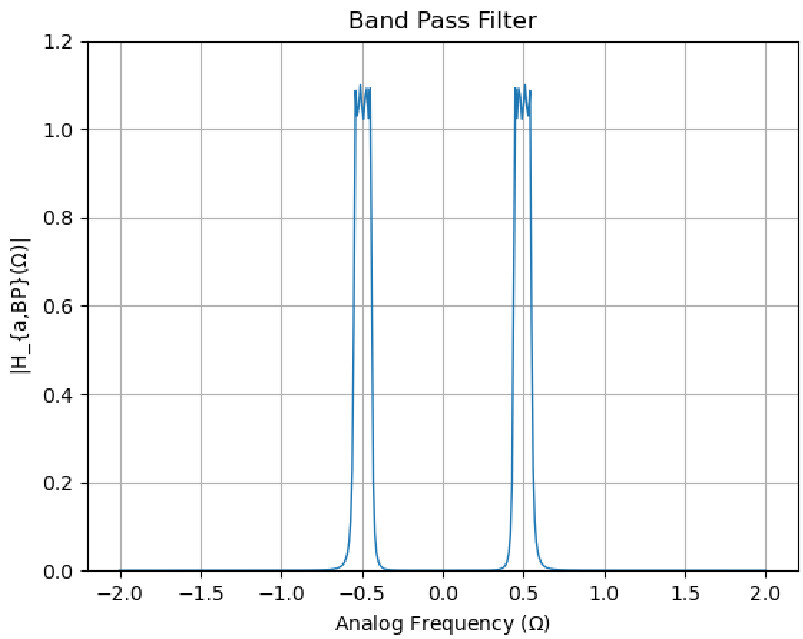

Figure 4: The Analog Bandpass Magnitude Response from (47).The filter design specifications are satisfied

3.2 The Digital Filter

From the bilinear transformation, we obtain the digital bandpass filter from the corresponding analog filter as

$$H_{d,BP}(z) = GH_{a,BP}(s)\big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \tag{48}$$

Substituting $s = \frac{1-z^{-1}}{1+z^{-1}}$ in (47) and calculating expression using a python code we get :

$$H_{d,BP}(z) = \frac{G\left(z^{-8} - 4z^{-6} + 6z^{-4} - 4z^{-2} + 1.0\right)}{2.61 - 12.43z^{-1} + 32.16z^{-2} - 53.24z^{-3} + 61.99z^{-4} - 50.98z^{-5} + 29.48z^{-6} - 10.91z^{-7} + 2.19z^{-8}} \tag{49}$$

where $G = 9.8138 \times 10^{-5}$

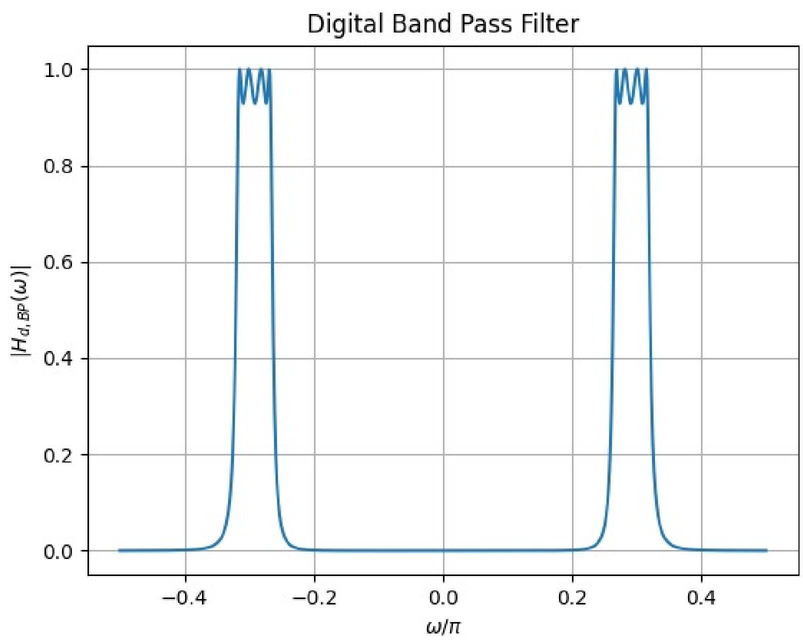

Figure 5: Digital Specifications are met. Passband and stopband frequencies are same

4 The FIR Filter

We design the FIR filter by first obtaining the (non-causal) lowpass equivalent using the Kaiser window and then converting it to a causal bandpass filter.

4.1 The Equivalent Lowpass Filter

The lowpass filter has a passband frequency ω_l and transition band $\Delta\omega = 2\pi\frac{\Delta F}{F_s} = 0.0125\pi$. The stopband tolerance is $\delta = 0.15$.The cutoff-frequency is given by :

$$\omega_l = \frac{\omega_{p1} - \omega_{p2}}{2} \tag{50}$$

$$= 0.025\pi \tag{51}$$

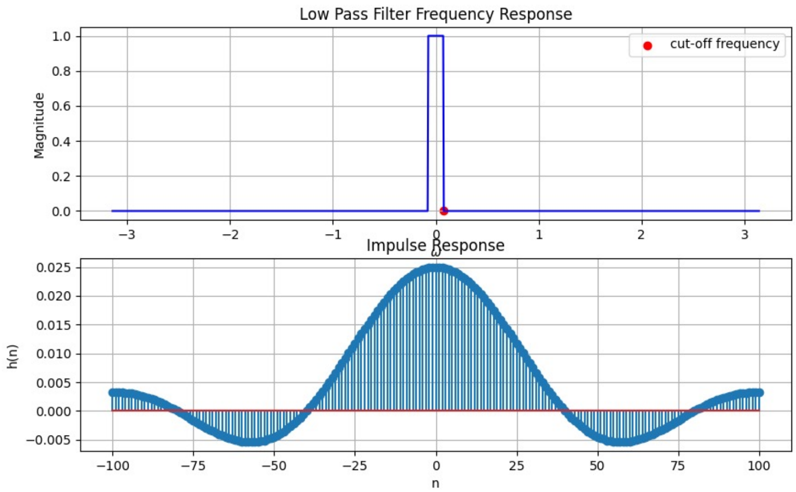

Figure 6: Frequency response and impulse response of an ideal Low Pass Filter

The impulse response of ideal Low Pass Filter is given by :

$$h(n) = \begin{cases} \frac{w_l}{\pi}, & \text{if } n = 0 \\ \frac{\sin(w_l n)}{n\pi}, & \text{if } n \neq 0 \end{cases} \tag{52}$$

From (52) we conclude that $h(n)$ for an ideal Low Pass Filter is not causal and can neither be made causal by introducing a finite delay. And $h(n)$ do not converge and hence the system is unstable.

4.2 The Kaiser Window

Therefore we move on windowing the impulse response.A window function is chosen and multiplied. The Kaiser window is defined as

$$w(n) = \begin{cases} \frac{I_0\left[\beta N\sqrt{1-\left(\frac{n}{N}\right)^2}\right]}{I_0(\beta N)}, & -N \leq n \leq N, \quad \beta > 0 \\ 0 & \text{otherwise,} \end{cases}$$

1. N is chosen according to

$$N \geq \frac{A-8}{4.57\Delta\omega}, \tag{53}$$

where $A = -20\log_{10}\delta$. Substituting the appropriate values from the design specifications, we obtain $A = 16.4782$ and $N \geq 48$.

2. β is chosen according to

$$\beta N = \begin{cases} 0.1102(A - 8.7) & A > 50 \\ 0.5849(A - 21)^{0.4} + 0.07886(A - 21) & 21 \leq A \leq 50 \\ 0 & A < 21 \end{cases} \tag{54}$$

The window function is defined as :

$$w\left(n\right) = \begin{cases} 1, & \text{for } -48 \leq n \leq 48 \\ 0, & \text{otherwise} \end{cases} \tag{55}$$

Therefore the desired impulse response is :

$$h_{lp} = h_n w_n \tag{56}$$

$$h\left(n\right) = \begin{cases} \frac{\sin(w_l n)}{n\pi}, & \text{for } -48 \leq n \leq 48 \\ 0 & \text{otherwise} \end{cases} \tag{57}$$

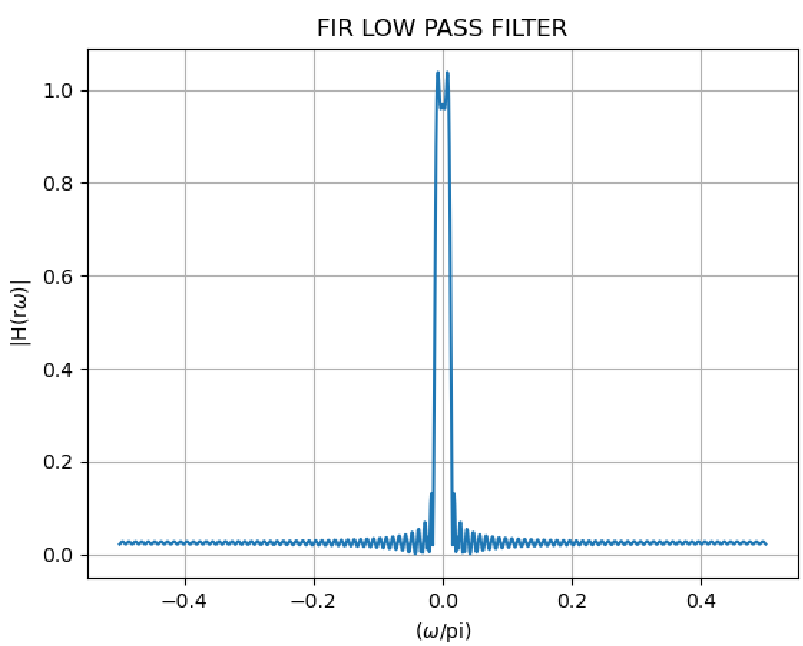

Figure 7: Magnitude Response of Low Pass Filter after using Kaiser Window

4.3 The Equivalent Band Pass Filter

A Band-Pass Filter (BPF) can be obtained by subtracting the magnitude response of a Low-Pass Filter (LPF) with cutoff frequency ω_{p1} from another LPF magnitude response with cutoff frequency ω_{p2}.

$$h_{BP}(n) = \begin{cases} \frac{\sin(w_{p2}n)}{n\pi} - \frac{\sin\left(\omega_{p1}n\right)}{n\pi}, & \text{for } n \neq 0 \\ \frac{\omega_{p2}-\omega_{p1}}{\pi} & \text{for } n = 0 \end{cases} \tag{58}$$

$$\frac{\sin(\omega_{p2}n)}{n\pi} - \frac{\sin\left(\omega_{p1}n\right)}{n\pi} = 2\cos\left(\frac{\omega_{p2}n + \omega_{p1}n}{2}\right)\sin\left(\frac{\omega_{p2}n - \omega_{p1}n}{2}\right) \tag{59}$$

$$= \frac{2\cos(0.292n\pi)\sin(0.025n\pi)}{n\pi} \tag{60}$$

Multipying by window function we get :

$$h_{BP}(n) = \begin{cases} \frac{2\cos(0.292n\pi)\sin(0.025n\pi)}{n\pi}, & \text{for } -48 \leq n \leq 48 \\ 0 & \text{otherwise} \end{cases} \tag{61}$$

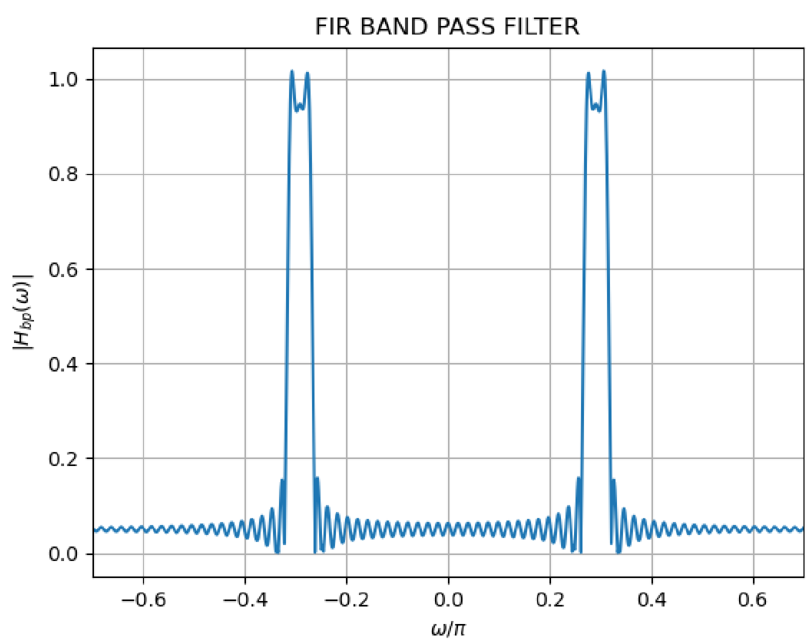

Figure 8: Magnitude Response of Band Pass Filter after using Kaiser Window