A Curved Line

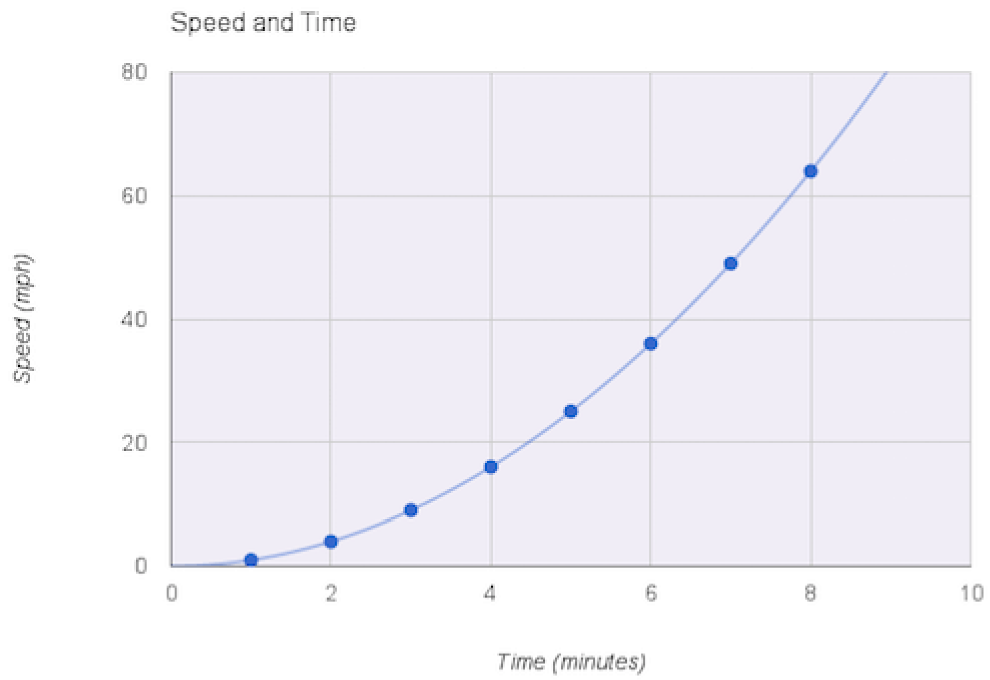
Imagine I started the car from stationary and hit the accelerator hard and kept it pressed hard. Clearly, the starting speed is zero because we're not moving initially. Imagine we're pressing the accelerator so hard that the car doesn't increase its speed at a constant rate. Instead, it increases its speed in a faster way. That means, it is not adding 10 miles per hour every minute. Instead, it is adding speed at a rate that itself goes up the longer we keep that accelerator pressed. For this example, let's imagine the speed is measured every minute as shown in this table.

Time (mins)	Speed (mph)
0	0
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64

If you look closely, you might have spotted that I've chosen to have the speed as the square of the time in minutes. That is, the speed at time 2 is $2^2 = 4$, and at time three is $3^2 = 9$, and time 4 is $4^2 = 16$, and so on. The expression for this is easy to write too.

$$s = t^2$$

Yes, I know this is a very contrived example of car speed, but it will illustrate really well how we might do calculus. Let's visualise this so we can get a feel for how the speed changes with time.



You can see the speed zooming upwards at an ever faster rate. The graph isn't a straight line now. You can imagine that the speed would explode to very high numbers quite quickly. At 20 minutes the speed would be 400 miles per hour. And at 100 minutes the speed would be $10,000$ miles per hour! The interesting question is — what's the rate of change for the speed with respect to time? That is, how does the speed change with time?

This isn't the same as asking, what is the actual speed at a particular time. We already know that because we have the expression $s = t^2$. What we are asking is this — at any point in time, what is the *rate of change* of speed? What does this even mean in this example where the graph is curved?

If we think back to our previous two examples, we saw that the rate of change was the slope of the graph of speed against time. When the car was cruising at a constant 30, the speed wasn't changing, so it's rate of change was zero. When the car was getting faster steadily, it's rate of change was 10 miles per hour every minute. And that 10 mph every minute was true at any point in time. At time 2 minutes the rate of change was 10 mph/min. And it was true at 4 minutes and would be true at 100 minutes too. Can we apply this same thinking to this curved graph? Yes, we can — but let's go extra slowly here.