

Theta Notation

We formally introduce theta notation, which forms the basis of mathematical analysis of algorithms.

Example

Let's consider the following three functions:

$$f(n) = n^2 + 2$$

$$f(n) = 2n^2 - 1$$

$$f(n) = 2n^2 + 4$$

The table below shows how these functions grow as the value of n grows from 0 and onwards. Note that we aren't considering negative values for n , but we'll explain why shortly.

$f(n)$	$n^2 + 2$	$2n^2 - 1$	$2n^2 + 4$
$f(0)$	2	-1	4
$f(1)$	3	1	6
$f(2)$	6	7	12
$f(3)$	11	17	22
$f(4)$	18	31	36
$f(5)$	27	49	54

$f(6)$	38	71	76
$f(7)$	51	97	102
$f(8)$	66	127	132
$f(9)$	83	161	166
$f(10)$	102	199	204

We can observe from the output that the function $f(n) = 2n^2 - 1$ grows faster than the function $f(n) = n^2 + 2$, but it grows slower than $f(n) = 2n^2 + 4$ once n becomes greater than or equal to 2. The astute reader would notice that the output of the function $f(n) = 2n^2$ is, in a sense, being **sandwiched** by the output of the other two functions when $n \geq 2$. All the yellow rows in the above table, show the values of the function $f(n) = 2n^2 - 1$ being sandwiched by the output of the other two functions.

Formal Definition

Whenever we can bound the output of a function $f(n)$ by the outputs of two other functions in the following form, we say that the function $f(n)$ **belongs to the set $\Theta(g(n))$** (pronounced as theta of $g(n)$)

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$$

where $n > n_0$ and c_1 and c_2 are positive

Let's apply the formal definition to the example functions we just discussed above.

$$n^2 + 2 < 2n^2 - 1 < 2n^2 + 4$$

$$1(n^2 + 2) < 2n^2 - 1 < 2(n^2 + 2)$$

where $n \geq 2$ and $c_1 = 1$ $c_2 = 2$ and $g(n) = n^2 + 2$

We can choose positive constants so that our function $2n^2 - 1$ becomes sandwiched by the other two functions as soon as n becomes greater than 2. Hence we can say that the function $2n^2 - 1$ is Θ of $g(n)$ where $g(n)$ equals $n^2 + 2$.

Explanation

There are several points to ponder about the above relationship.

- Note that to satisfy the formal definition, none of the three functions can produce negative values after n has crossed a certain threshold marked by n_0 . In our example, $f(n)$ produces a negative value when $n=1$. However, once $n \geq 2$ all subsequent values produced by all three functions are nonnegative. Consider the following function.

$$f(n) = (-1)^n n^2$$

The above function will not produce nonnegative values (i.e. 0 or positive numbers) for any threshold that n crosses, as it will oscillate between a negative and a positive number for odd and even values of n respectively. We require $g(n)$ to be *asymptotically nonnegative*, a mathematically-fancy term meaning that as n grows and passes a certain value n_0 , it should always produce nonnegative numbers as output. This restriction is partly because it makes it easier to reason about the complexity of algorithms, as input is either none (i.e. 0 or some positive number).

- Once our function $f(n)$ satisfies the inequality (i.e. we are able to sandwich it) we can claim that,

$$f(n) = \Theta g(n)$$

Note the use of equality instead of membership. Even though $\Theta g(n)$ is a set and $f(n)$ belongs to the set $\Theta g(n)$, we take the liberty to use the equality instead of membership when undertaking algorithmic analysis.

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- We say that $g(n)$ is a *asymptotically tight bound* for $f(n)$, which is just a mathematical way of saying that **the growth of $f(n)$, as n grows larger and larger, mirrors that of $g(n)$. Or $f(n)$ is equal to $g(n)$ within a constant factor when the values of n are past the threshold n_0 .**
- Note that in our example we used constants $c_1 = 1$ and $c_2 = 2$, but we could use any other set of constants that satisfy the inequality.

Quiz

Q

Can you determine the $\Theta(g(n))$ for the polynomial $5n^3 + n$?

☐ A) $\Theta(n)$

☐ B) $\Theta(n^2)$

☐ C) $\Theta(n^3)$

☐ D) $\Theta(n^4)$

CHECK ANSWERS

$\Theta(1)$

Let's say we have a function that always outputs the same constant value for any value of n .

$$f(n) = 7$$

The above function will always produce 7, no matter what the value of n is. Let's try to see if we can find a $\Theta(g(n))$ for our constant function.

$$c_1 n^0 < f(n) < c_2 n^0$$

$$c_1 n^0 < 7 < c_2 n^0$$

$$c_1 < f(n) < c_2$$

We can now choose $c_1 = 2$ or $c_2 = 8$ and claim that our function $f(n)$ belongs to the set $\Theta(n^0)$, which is equivalent to $\Theta(1)$ as we know $n^0 = 1$. This is considered slight abuse of the notation, as the expression $\Theta(1)$ doesn't indicate which variable is tending to infinity.

$\Theta(1)$ is used to denote either a constant or a constant function with respect to some variable.