Calculus without Plotting Graphs

We said earlier; calculus is about understanding how things change in a mathematically precise way. Let's see if we can do that by applying this idea of ever smaller Δx to the mathematical expressions that define these things — things like our car speed curves. To recap, the speed is a function of the time that we know to be $s = t^2$. We want to know how the speed changes as a function of time. We've seen that is the slope of *s* when it is plotted against *t*.

This rate of change ∂s / ∂t is the height divided by the extent of our constructed lines but where the Δx gets infinitely small. What is the height? It is (t + Δx)^2 - (t - Δx)^2 as we saw before. This is just $s = t^2$ where *t* is a bit below and a bit above the point of interest. That amount of bit is Δx. What is the extent? As we saw before, it is simply the distance between (t + Δx) and (t - Δx) which is 2Δx. We're almost there,

$$\frac{\delta s}{\delta t} = \frac{height}{extent}$$

$$= \frac{(t + \Delta x)^2 - (t - \Delta x)^2}{2\Delta x}$$

Let's expand and simplify that expression,

$$\frac{\delta s}{\delta t} = \frac{t^2 + \Delta x^2 + 2t\Delta x - t^2 - \Delta x^2 + 2t\Delta x}{2\Delta x}$$

$$= \frac{4t\Delta x}{2\Delta x}$$

$$\frac{\delta s}{\delta t} = 2t$$

We've actually been very lucky here because the algebra simplified itself very neatly. So we've done it! The mathematically precise rate of change is $\partial s / \partial t = 2t$. That means for any time t; we know the rate of change of speed $\partial s / \partial t = 2t$. At $t = 3$ minutes we have $\partial s / \partial t = 2t = 6$. We actually confirmed that before using the approximate method. For $t = 6$ minutes, $\partial s / \partial t = 2t = 12$, which nicely matches what we found before too. What about $t = 100$ minutes? $\partial s / \partial t = 2t = 200$ mph per minute. That means after 100 minutes, the car is speeding up at a rate of 200 mph per minute.

Let's take a moment to ponder the magnitude and coolness of what we just did. We have a mathematical expression that allows us to precisely know the rate of change of the car speed at any time. And following our earlier discussion, we can see that changes in s do indeed depend on time. We were lucky that the algebra simplified nicely, but the simple $s = t^2$ didn't give us an opportunity to try reducing the Δx in an intentional way. So let's try another example where the speed of the car is only just a bit more complicated,

$$s = t^2 + 2t$$

$$\frac{\delta s}{\delta t} = \frac{\text{height}}{\text{extent}}$$

What is the height now? It is the difference between s calculated at $t+\Delta x$ and s calculated at $t-\Delta x$. That is, the height is $(t + \Delta x)^2 + 2(t + \Delta x) - (t - \Delta x)^2 - 2(t - \Delta x)$. What about the extent? It is simply the distance between $(t + \Delta x)$ and $(t - \Delta x)$ which is still $2\Delta x$.

$$\frac{\delta s}{\delta t} = \frac{(t + \Delta x)^2 + 2(t + \Delta x) - (t - \Delta x)^2 - 2(t - \Delta x)}{2\Delta x}$$

Let's expand and simplify that expression,

$$\frac{\delta s}{\delta t} = \frac{t^2 + \Delta x^2 + 2t\Delta x + 2t + 2\Delta x - t^2 - \Delta x^2 + 2t\Delta x - 2t + 2\Delta x}{2\Delta x}$$

$$= \frac{4t\Delta x + 4\Delta x}{2\Delta x}$$

$$\frac{\delta s}{\delta t} = 2t + 2$$

That's a great result! Sadly the algebra again simplified a little too easily. It wasn't wasted effort because there is a pattern emerging here which we'll come back to. Let's try another example, which isn't that much more complicated. Let's set the speed of the car to be the cube of the time.

$$s = t^3$$

$$\frac{\delta s}{\delta t} = \frac{height}{extent}$$

$$\frac{\delta s}{\delta t} = \frac{(t + \Delta x)^3 - (t - \Delta x)^3}{2\Delta x}$$

Let's expand and simplify that expression,

$$\frac{\delta s}{\delta t} = \frac{t^3 + 3t^2\Delta x + 3t\Delta x^2 + \Delta x^3 - t^3 + 3t^2\Delta x - 3t\Delta x^2 + \Delta x^3}{2\Delta x}$$

$$= \frac{6t^2\Delta x + 2\Delta x^3}{2\Delta x}$$

$$\frac{\delta s}{\delta t} = 3t^2 + \Delta x^2$$

Now, this is much more interesting! We have a result which contains a Δx, whereas before they were all canceled out. Well, remember that the gradient is only correct as the Δx gets smaller and smaller, infinitely small.

This is the cool bit! What happens to the Δx in the expression $\partial s / \partial t = 3t^2 + \Delta x^2$ as Δx gets smaller and smaller? It disappears! If that sounds surprising, think of a very small value for Δx. If you try, you can think of an even smaller one. And an even smaller one … and you could keep going forever, getting ever closer to zero. So let's just get straight to zero and avoid all that hassle. That gives is the mathematically precise answer we were looking for:

$$\frac{\delta s}{\delta t} = 3t^2$$

That's a fantastic result, and this time we used a powerful mathematical tool to do calculus, and it wasn't that hard at all.