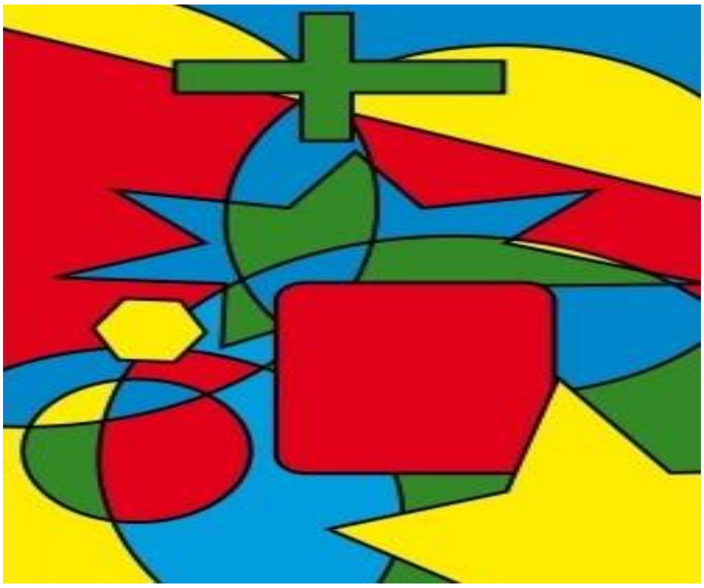
18CS36

Discrete Mathematical Structures

(For the 3rd Semester Computer Science and Engineering Students)



Module 2

Properties of Integers & Principles of Counting

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Content

MODULE 2

PROPERTIES OF INTEGERS & PRINCIPLES OF COUNTING

☻ **Syllabus:**

Properties of the Integers: The Well Ordering Principle – Mathematical Induction.

Fundamental Principles of Counting: The Rules of Sum and Product, Permutations, Combinations – The Binomial Theorem, Combinations with Repetition.

☻ **Mathematical Induction:**

Mathematical induction is a mathematical proof technique. It is essentially used to prove that a statement $P(n)$ holds for every natural number $n = 0, 1, 2, 3, \dots \dots$ i.e., the overall statement is a sequence of infinitely many cases $P(0), P(1), P(3), \dots \dots \dots$

Well ordering principle:

Every non empty subset of Z^+contains a smallest element. (we often express this by saying that Z^+ is well ordered).

Finite induction principle (principle of Mathematical induction):

Let $S(n)$ denote an open mathematical statement that involves one or more occurrences of the variable n. Which represents a positive integer

(a) If $S(1)$ is true; and

(b) If whenever $S(k)$is true (for some particular but arbitrarily chosen $k \in Z^+$), then $S\ (k+1)$is true, then $S(n)$ is true for all $n \in Z^+$.

Proof:

Let $S(n)$ be such an open statement satisfying conditions (a) and (b) and let $F = \{t \in Z^+ / \ S(t)$ is false$\}$. We wish to prove that $F = \emptyset$ so to obtain a contradiction we assume that $F \neq \emptyset$. Then by the well-ordering Principle, F has a least element. Since $S(1)$ is true. It follows that $S \neq 1$. so $s > 10$, and consequently $s - 1 \in Z^+$. With $s - 1 \notin F$, $S(s-1)$ we have true. So, by condition (b) it follows that $S\big((s-1)+1\big) = S(s)$ is true, contradicting $s \in F$. This contradiction arose from the assumption that $F \neq \emptyset$. Consequently $F = \emptyset$.

Problems:

1. Prove by mathematical induction that, for all positive integers $n \geq 1$.

$$1 + 2 + 3 + \cdots \cdots \cdots + n = \frac{1}{2}n(n+1)$$

Solution:

Here, we have to prove the statement

$$S(n) = 1 + 2 + 3 + \cdots \cdots \cdots + n = \frac{1}{2}n(n+1) \text{ for all integers } n \geq 1.$$

Basic step: We note that $S(1)$ is the statement

$$1 = \frac{1}{2} \cdot 1 \cdot (1+1)$$

Which is clearly true. thus, the statement $S(n)$is verified for $n = 1$.

Induction step: We assume that the statement $S(n)$ is true for $n = k$ where k is an integer ≥ 1; that is, we assume that the following statement is true:

$$S(k) = 1 + 2 + 3 + \cdots\cdots\cdots + k = \frac{1}{2} \cdot k(k+1)$$

Using this we find that (by adding $(k+1)$ to both side)

$$S(k) = 1 + 2 + 3 + \cdots\cdots\cdots + k + (k+1) = \frac{1}{2} \cdot k(k+1) + (k+1)$$

$$= (k+1)\left\{\frac{1}{2}k + 1\right\}$$

$$= \frac{1}{2}(k+1)(k+2)$$

This is precisely the statement $S(k+1)$. Thus, on the basis of the assumption that $S(n)$ is true for $n = k \geq 1$, the truth ness of $S(n)$ for $n = k + 1$ is established.

2. Prove that, for each $n \in Z^+$ $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

OR

Prove that, for each $n \in Z^+$, $1^2 + 2^2 + 3^2 + \cdots\cdots\cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Solution:

Let S(n) denote the given statement.

Basic step: We note that is $S(1)$ is the statement

$1^2 = \frac{1}{6} \cdot 1 \cdot (1+1) \cdot (2+1)$ which is clearly true.

Induction Step: We assume that the statement $S(n)$ is true for $n = k$ where k is an integer ≥ 1; that is, we assume that the following statement is true.

$$S(k) = 1^2 + 2^2 + 3^2 + \cdots\cdots\cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

Adding $(k+1)^2$ to both sides of this, we obtain

$$S(k) = 1^2 + 2^2 + 3^2 + \cdots\cdots\cdots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= (k+1)\left\{\frac{k\,(2k+1)}{6} + (k+1)\right\}$$

$$= \frac{1}{6}(k+1)\{k(2k+1)+6(k+1)\}$$

$$= \frac{1}{6}(k+1)\{2k^2+k+6k+6\}$$

$$= \frac{1}{6}(k+1)\{2k^2+7k+6\}$$

$$= \frac{1}{6}(k+1)(k+2)(2k+3)$$

This is precisely the statement $S(k+1)$. Thus, on the basis of the assumption that $S(n)$ is true for $n = k \geq 1$, the truth ness of $S(n)$ for $n = k+1$ is established.

3. By mathematical induction, Prove That $(n!) \geq 2^{n-1}$ for all integers$n \geq 1$.

Solution:

Basic step: For $n = 1$, $S(n)$ reads $(1!) \geq 2^{1-1}$ which is obviously true. Thus $S(n)$ is verified for $n = 1$.

Induction step: We assume that $S(n)$ is true for $n = k$, where k is an integer ≥ 1; that is, we assume that

$(k!) \geq 2^{k-1}$, or $2^{k-1} \leq k!$ is true

$$2^k = 2 \cdot 2^{k-1} \leq 2 \cdot k!$$

$\leq (k+1) \cdot k!$, because $2 < (k+1)$ for $k \geq 1$

$$= (k+1)!$$

$$(k+1)! \geq 2^k$$

This is precisely the statement $S(n)$ for $n = k+1$. Thus, on the assumption that $S(n)$is true for $n = k \geq 1$, We have proved that $S(n)$ is true for $n = k+1$.

Hence, by mathematical induction, it follows that the statement $S(n)$ is true for all integers $n \geq 1$.

4. Prove that every positive integer $n \geq 24$ can be written as a sum of 5's and/or 7's.

Solution:

Basic step: We note that $24 = (7+7)+(5+5)$

This shows $S(24)$ is true.

Induction step: We assume that $S(n)$ is true for $n = k$ where $k \geq 24$. Then

$$k = (7+7+\cdots\cdots)+(5+5+\cdots\cdots)$$

Suppose this representation of k has r number of 7's and s number of 5's. Since $k \geq 24$ we should have $r \geq 2$ and $s \geq 2$.

Using this representation of k, we find that

$$k+1=\left\{\underbrace{(7+7+\cdots\cdots)}_{r}+\underbrace{(5+5+\cdots\cdots)}_{s}\right\}+1$$

$$=\left\{\underbrace{(7+7+\cdots\cdots)}_{r-2}+(7+7)+\underbrace{(5+5+\cdots\cdots)}_{s}\right\}+1$$

$$=\left\{\underbrace{(7+7+\cdots\cdots)}_{r-2}+\underbrace{(5+5+\cdots\cdots)}_{s+3}\right\}$$

This shows that $k+1$is sum of 7's and 5's. Thus, $S(k+1)$ is true.

5. Prove by mathematical induction that, for all positive integers $n \geq 1$.

$$1\cdot 2+2\cdot 3+3\cdot 4+\cdots\cdots\cdots+n(n+1)=\frac{1}{3}n(n+1)(n+2)$$

Solution:

Here, we have to prove the statement

$S(n)=1\cdot 2+2\cdot 3+3\cdot 4+\cdots\cdots\cdots+n(n+1)=\frac{1}{3}n(n+1)(n+2)$ for all integers $n \geq 1$.

Basic step: We note that $S(1)$ is the statement

$$1\cdot 2=\frac{1}{3}\cdot 1\cdot(1+1)\cdot(2+1)$$

Which is clearly true. thus, the statement $S(n)$is verified for $n=1$.

Induction step: We assume that the statement $S(n)$ is true for $n=k$ where k is an integer ≥ 1; that is, we assume that the following statement is true:

$$S(k)=1\cdot 2+2\cdot 3+3\cdot 4+\cdots\cdots\cdots+k(k+1)=\frac{1}{3}k(k+1)(k+2)$$

Using this we find that (by adding $(k+1)(k+2)$ to both side)

$$S(k)=1\cdot 2+2\cdot 3+3\cdot 4+\cdots\cdots\cdots+k(k+1)+(k+1)(k+2)$$

$$=\frac{1}{3}k(k+1)(k+2)+(k+1)(k+2)$$

$$=(k+1)(k+2)\left\{\frac{1}{3}k+1\right\}$$

$$=\frac{1}{3}(k+1)(k+2)(k+3)$$

This is precisely the statement $S(k+1)$. Thus, on the basis of the assumption that $S(n)$ is true for $n=k\geq 1$, the truth ness of $S(n)$ for $n=k+1$ is established.

6. Prove, by mathematical induction that $1^2+3^2+5^2+\cdots\cdots\cdots+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}$ for all integers $n \geq 1$.

Solution:

Let S(n) denote the given statement.

Basic step: We note that is $S(1)$ is the statement

$1^2 = \frac{1}{3} \cdot 1 \cdot (2-1) \cdot (2+1)$ which is clearly true.

Induction Step: We assume that the statement $S(n)$ is true for $n = k$ where k is an integer ≥ 1; that is, we assume that the following statement is true.

$$S(k) = 1^2 + 3^2 + 5^2 + \cdots\cdots\cdots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}.$$

Adding $(2k+1)^2$ to both sides of this, we obtain

$$S(k) = 1^2 + 3^2 + 5^2 + \cdots\cdots\cdots + (2k-1)^2 + (2k+1)^2 = \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$$

$$= (2k+1)\left\{\frac{k\,(2k-1)}{3} + (2k+1)\right\}$$

$$= \frac{1}{3}((2k+1)\{k(2k-1) + 3(2k+1)\}$$

$$= \frac{1}{3}((2k+1)\{2k^2 - k + 6k + 3\}$$

$$= \frac{1}{3}((2k+1)\{2k^2 + 5k + 3\}$$

$$= \frac{1}{3}(2k+1)(k+2)(2k+3)$$

This is precisely the statement $S(k+1)$. Thus, on the basis of the assumption that $S(n)$ is true for $n = k \geq 1$, the truth ness of $S(n)$ for $n = k+1$ is established.

7. Prove by mathematical induction that, for all positive integers $n \geq 1$.

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots\cdots\cdots + n(n+2) = \frac{1}{6}n(n+1)(2n+7)$$

Solution:

Here, we have to prove the statement

$S(n) = 1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots\cdots\cdots + n(n+2) = \frac{1}{6}n(n+1)(2n+7)$ for all integers $n \geq 1$.

Basic step: We note that $S(1)$ is the statement

$1 \cdot 3 = \frac{1}{6} \cdot 1 \cdot (1+1) \cdot (2+7)$

Which is clearly true. thus, the statement $S(n)$is verified for $n = 1$.

Induction step: We assume that the statement $S(n)$ is true for $n = k$ where k is an integer ≥ 1; that is, we assume that the following statement is true:

$$S(k) = 1\cdot 3 + 2\cdot 4 + 3\cdot 5 + \cdots\cdots\cdots + k(k+2) = \frac{1}{6}k(k+1)(2k+7)$$

Using this we find that (by adding $(k+1)(k+3)$ to both side)

$$S(k) = 1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots\cdots\cdots + k(k+2) + (k+1)(k+3) = \frac{1}{6}k(k+1)(2k+7) + (k+1)(k+3)$$

$$= (k+1)\left\{\frac{1}{6}k(2k+7) + (k+3)\right\}$$

$$= (k+1)\{2k^2 + 7k + 6k + 18\}$$

$$= (k+1)\{2k^2 + 13k + 18\}$$

$$= \frac{1}{6}(k+1)(k+2)(2k+9)$$

This is precisely the statement $S(k+1)$. Thus, on the basis of the assumption that $S(n)$ is true for $n = k \geq 1$, the truth ness of $S(n)$ for $n = k+1$ is established.

8. Prove that every positive integer greater than or equal to 14 can be written as a sum of 3's and/or 8's.

Solution:

Basic step: We note that $14 = (3+3) + 8$

This shows $S(14)$ is true.

Induction step: We assume that $S(n)$ is true for $n = k$ where $k \geq 14$. Then

$$k = (3 + 3 + \cdots\cdots) + (8 + \cdots\cdots)$$

Suppose this representation of k has r number of 3's and s number of 8's. Since $k \geq 14$ we should have $r \geq 2$ and $s \geq 2$.

Using this representation of k, we find that

$$k + 1 = \left\{\underbrace{(3 + 3 + \cdots\cdots)}_{r} + \underbrace{(8 + \cdots\cdots)}_{s}\right\} + 1$$

$$= \left\{\underbrace{(3 + 3 + \cdots\cdots)}_{r} + \underbrace{(8 + \cdots\cdots)}_{s-1} + 8\right\} + 1$$

$$= \left\{\underbrace{(3 + 3 + \cdots\cdots)}_{r+3} + \underbrace{(8 + \cdots\cdots)}_{s-1}\right\}$$

This shows that $k+1$is sum of 3's and 8's. Thus, $S(k+1)$ is true.

9. Prove by mathematical induction for any integer $n \geq 1$

$$\frac{1}{2\cdot 5} + \frac{1}{5\cdot 8} + \cdots\cdots\cdots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4}$$

Solution:

Here, we have to prove the statement

$$S(n) = \frac{1}{2\cdot 5} + \frac{1}{5\cdot 8} + \cdots\cdots\cdots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4} \text{ for all integers } n \geq 1.$$

Basic step: We note that $S(1)$ is the statement

$$\frac{1}{2\cdot 5} = \frac{1}{6\cdot 1+4}$$

Which is clearly true. thus, the statement $S(n)$is verified for $n = 1$.

Induction step: We assume that the statement $S(n)$ is true for $n = k$ where k is an integer ≥ 1; that is, we assume that the following statement is true:

$$S(k) = \frac{1}{2\cdot 5} + \frac{1}{5\cdot 8} + \cdots\cdots\cdots + \frac{1}{(3k-1)(3k+2)} = \frac{k}{6k+4}$$

Using this we find that (by adding $\frac{1}{(3k+2)(3k+5)}$ to both side)

$$S(k) = \frac{1}{2\cdot 5} + \frac{1}{5\cdot 8} + \cdots\cdots\cdots + \frac{1}{(3k-1)(3k+2)} + \frac{1}{(3k+2)(3k+5)} = \frac{k}{6k+4} + \frac{1}{(3k+2)(3k+5)}$$

$$= \frac{k(3k+2)(3k+5)+(6k+4)}{(6k+4)(3k+2)(3k+5)}$$

$$= \frac{9k^3+21k^2+16k+4}{(6k+4)(3k+2)(3k+5)}$$

$$= \frac{(k-1)(3k+2)^2}{(6k+4)(3k+2)(3k+5)}$$

$$= \frac{(k+1)(3k+2)}{(6k+4)(3k+5)}$$

This is precisely the statement $S(k+1)$. Thus, on the basis of the assumption that $S(n)$ is true for $n = k \geq 1$, the truth ness of $S(n)$ for $n = k+1$ is established.

10. Prove by mathematical induction that, for every positive integer n, 5 divides $n^5 - n$

Solution:

Let $S(n)$ be the given statement.

Basic step: We note that $S(1)$ is the statement

5 divides $1^5 - 1$

Since $1^5 - 1 = 0$, this statement is true

Induction step: We assume that the statement $S(n)$ is true for $n = k$ where k is an integer ≥ 1; that is, we assume that the following statement is true:

5 divides $k^5 - k$,

This means that $k^5 - k$ is a multiple of 5; that is $k^5 - k = 5m$, for some positive integer m.

Consequently, we find that

$$\begin{aligned}(k+1)^5 - (k+1) &= (k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1) - (k+1) \\ &= (k^5 - k) + 5(k^4 + 2k^3 + 2k^2 + k) \\ &= 5m + 5(k^4 + 2k^3 + 2k^2 + k) \\ &= 5(m + k^4 + 2k^3 + 2k^2 + k)\end{aligned}$$

This shows that $(k+1)^5 - (k+1)$ is a multiple of 5; that is, 5 divides $(k+1)^5 - (k+1)$.

This is precisely the statement $S(n)$ for $n = k + 1$. Thus, on the assumption that $S(n)$is true for $n = k \geq 1$, We have proved that $S(n)$ is true for $n = k + 1$.

☻ **Recursive Definition**:

For describing a sequence, the two methods are commonly used.

(i) Explicit method (ii) Recursive method

In explicit method, the general term of the sequence is explicitly indicated

In recursive method, first few terms of the sequence must be indicated explicitly and in the second part the rule which will enable us to obtain new term if the sequence from the terms already known must be indicated.

Problems:

1. Find an explicit definition of the sequence defined recursively by

$a_1 = 7, a_n = 2a_{n-1} + 1$ for $n \geq 2$.

Solution: By repeated use of the given recursive definition we find that

$$a_n = 2a_{n-1} + 1 = 2\{2a_{n-2} + 1\} + 1$$

$$= 2\{2(2a_{n-3} + 1) + 1\} + 1 = 2^3 a_{n-3} + 2^2 + 2 + 1$$

$$\dots\dots\dots$$

$$\dots\dots\dots$$

$$= 2^{n-1}a_{n-(n-1)} + 2^{n-2} + 2^{n-3} + \cdots\cdots + 2^2 + 2 + 1$$

$$= 2^{n-1}a_1 + (1 + 2 + 2^2 + 2^3 + \cdots\cdots + 2^{n-3} + 2^{n-2})$$

Using $a_1 = 7$ and the standard result

$1 + a + a^2 + a^3 + \cdots\cdots + a^{n-1} = \frac{a^n - 1}{a - 1}$ for $a > 1$

This becomes $a_n = 7 \cdot 2^{n-1} + \frac{2^{n-1} - 1}{2 - 1} = 8 \cdot 2^{n-1} - 1$

2. Obtain the recursive definition for the sequence $\{a_n\}$ is each of the following cases.

$(i).\ a_n = 5n$ $(ii).\ a_n = 6^n$ $(iii).\ a_n = 3n + 7$

$(iv).\ a_n = n(n + 2)$ $(v).\ a_n = n^2$ $(vi).\ a_n = 2 - (-1)^n$

Solution:

(i). Here $a_1 = 5, a_2 = 10, a_3 = 15, a_4 = 20, \dots\dots\dots$

We can rewrite these as $a_1 = 5$ and $a_n = a_{n-1} + 5$ for $n \geq 2$.

This is the Recursive definition of the given sequence.

(ii). Here $a_1 = 6, a_2 = 6^2, a_3 = 6^3, a_4 = 6^4, \dots\dots\dots$

We can rewrite these as $a_1 = 6$ and $a_{n+1} = 6a_n$ for $n \geq 1$.

This is the Recursive definition of the given sequence.

(iii). Here $a_1 = 10, a_2 = 13, a_3 = 16, a_4 = 19, \dots\dots\dots$

We can rewrite these as $a_1 = 10$ and $a_n = a_{n-1} + 3$ for $n \geq 2$.

This is the Recursive definition of the given sequence.

(iv). Here $a_1 = 3, a_2 = 8, a_3 = 15, a_4 = 24, \dots\dots\dots$

We observe that $a_2 - a_1 = 5 = 2 \cdot 1 + 3, a_3 - a_2 = 7 = 2 \cdot 2 + 3, a_4 - a_3 = 9 = 2 \cdot 3 + 3$

We can rewrite these as $a_{n+1} - a_n = 2n + 3$ then $a_{n+1} = a_n + 2n + 3$ for $n \geq 1$.

Hence $a_1 = 3$ and $a_{n+1} = a_n + 2n + 3$ for $n \geq 1$.

This is the Recursive definition of the given sequence.

(v). Here $a_1 = 1, a_2 = 4, a_3 = 9, a_4 = 16, \dots\dots\dots$

We observe that $a_2 - a_1 = 3 = 2 \cdot 1 + 1, a_3 - a_2 = 5 = 2 \cdot 2 + 1, a_4 - a_3 = 7 = 2 \cdot 3 + 1$

We can rewrite these as $a_{n+1} - a_n = 2n + 1$ then $a_{n+1} = a_n + 2n + 1$ for $n \geq 1$.

Hence $a_1 = 1$ and $a_{n+1} = a_n + 2n + 1$ for $n \geq 1$.

This is the Recursive definition of the given sequence.

(vi). Here $a_1 = 3, a_2 = 1, a_3 = 3, a_4 = 1, \dots\dots\dots$

We observe that $a_2 - a_1 = -2 = 2\,(-1), a_3 - a_2 = 2 = 2(1), a_4 - a_3 = -2 = 2(-1)$

We can rewrite these as $a_{n+1} - a_n = 2(-1)^n$ then $a_{n+1} = a_n + 2(-1)^n$

Hence $a_1 = 3$ and $a_{n+1} = a_n + 2(-1)^n$ for $n \geq 1$.

This is the Recursive definition of the given sequence.

3. The Fibonacci numbers are defined recursively by $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ Evaluate F_2 to F_{10}

Solution:

Given $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$

$F_2 = F_1 + F_0 = 1 + 0 = 1$

$F_3 = F_2 + F_1 = 1 + 1 = 2$

$F_4 = F_3 + F_2 = 2 + 1 = 3$

$F_5 = F_4 + F_3 = 3 + 2 = 5$

$F_6 = F_5 + F_4 = 5 + 3 = 8$

$F_7 = F_6 + F_5 = 8 + 5 = 13$

$F_8 = F_7 + F_6 = 13 + 8 = 21$

$F_9 = F_8 + F_7 = 21 + 13 = 34$

$F_{10} = F_9 + F_8 = 34 + 21 = 55$

Note: The Sequence formed by the Fibonacci numbers is called the Fibonacci sequence.

4. The Lucas numbers are defined recursively by $L_0 = 2, L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$ Evaluate L_2 to L_{10}

Solution:

Given $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$

$L_2 = L_1 + L_0 = 1 + 2 = 3$

$L_3 = L_2 + L_1 = 3 + 1 = 4$

$L_4 = L_3 + L_2 = 4 + 3 = 7$

$L_5 = L_4 + L_3 = 7 + 4 = 11$

$L_6 = L_5 + L_4 = 11 + 7 = 18$

$L_7 = L_6 + L_5 = 18 + 11 = 29$

$L_8 = L_7 + L_6 = 29 + 18 = 47$

$L_9 = L_8 + L_7 = 47 + 29 = 76$

$L_{10} = L_9 + L_8 = 76 + 47 = 123$

Note: The Sequence formed by the Lucas numbers is called the Lucas sequence.

5. For the Fibonacci sequence $F_0, F_1, F_2, \ldots \ldots \ldots \ldots$ Prove that $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

Solution:

For $n = 0$ and $n = 1$, the required results read (respectively)

$$F_0 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^0 - \left(\frac{1-\sqrt{5}}{2}\right)^0\right] = \frac{1}{\sqrt{5}}[1 - 1] = 0$$

$$F_1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right] = \frac{1}{\sqrt{5}}[\sqrt{5}] = 1$$

Which is true.

Thus, the required result is true for $n = 0$ and $n = 1$. We assume that the result is true for $n = 0, 1, 2, \ldots . k$, where $k \geq 1$. Then, we find that

$$F_{k+1} = F_k + F_{k-1}$$

$$F_{k+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right]$$ using the assumption made

$$F_{k+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left\{\frac{1+\sqrt{5}}{2} + 1\right\} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left\{\frac{1-\sqrt{5}}{2} + 1\right\}\right]$$

$$F_{k+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left\{\frac{3+\sqrt{5}}{2}\right\} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left\{\frac{3-\sqrt{5}}{2}\right\}\right]$$

$$F_{k+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left\{\frac{6+2\sqrt{5}}{4}\right\} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left\{\frac{6-2\sqrt{5}}{4}\right\}\right]$$

$$F_{k+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]$$

This shows that the required result is true for $n = k + 1$. Hence by mathematical induction, the result is true for all non – negative integers n.

☻ **The Rules of Sum and Product:**

Rule of sum:

Suppose two tasks T_1 and T_2 are to be performed. if the task T_1 can be performed in m different ways and the task T_2 can be performed in n different ways and if these two tasks cannot be performed simultaneously, then one of the two tasks (T_1 or T_2) can be performed in $m + n$ ways.

Example: Suppose T_1 is the task of selecting a prime no. < 10 and T_2 is the task of selecting an even number < 10. then T_1 can be performed in 4 ways and T_2 can be performed in 4 ways. But since 2 is both a prime and an even number < 10 the task T1 or T_2 can be performed in 4 + 4 – 1 = 7 ways.

Rule of product:

Suppose two tasks are to be performed one after the other. If T_1 can be performed in n_1 different ways, and for each of these ways T_2 can be performed in $n_2$different ways. then both of the tasks can be performed in $n_1 * n_2$ different ways.

Example: Suppose a person has 8 shirts and 5 ties. Then He has 8 * 4 = 40 different ways of choosing a shirt and a tie.

Problems:

1. Cars of a particular manufacturer come in 4 models, 12 colours, 3 engine sizes and 2 transmission types (a) how many distinct cars can be manufactured? (b) of these how many have the same colour?

Solution:

(a) By the product rule, it follows that the number of distinct cars that can be manufactured is 4*12*3*2 =288

(b) for any chosen colour, the number of distinct cars that can be manufactured is 4*3*2=24

2. A bit is either 0 or 1. A byte is a sequence of 8 bits. Find (i) the number of bytes. (ii) the number of bytes that begin with 11 and end with 11. (iii) The number of bytes that begin with 11 and do not end with 11. (iv) the number of bytes that begin with 11 or end with 11.

Solution:

(i) Since each byte contains 8 bits and each bit is 0 or 1, the number of bytes is 2^8 = 256

(ii) In a byte beginning and ending with 11, there occur 4 open positions. These can be filled un 2^4 = 16 ways. Therefore, there are 16 bytes which begin and end with 11.

(iii)These occur 6 open positions in a byte beginning with 11. these positions can be filled is 2^6 = 64 ways. thus, there are 64 bytes that begin with 11. since there are 16 bytes that begin and end with 11, the number of bytes that begin with 11 but do not end with 11 is 64-16 = 48.

(iv) As in (iii) the numbers of bytes that end with 11 is 64. Also, the number of bytes that begin and end with 11 is 16. Therefore, the number of bytes that begin or end with 11 is 64 + 64 = 16 = 112.

3. Find the number of 3 digit even numbers with no repeated digits.

Solution:

Here we consider number of the form $x\,y\,z$, where each of x, y, z represents a digit under the given restrictions. Since $x\,y\,z$ has to be even, z has to be 0, 2, 4, 6 or 8. If z is 0, then x has 9 choices and $y\,z$ has 2, 4, 6, 8 (4 choices) then x has 8 choices (Note that x cannot be 0). Therefore, z and x can be chosen in $1 \times 9 + 4 \times 8 = 41$ ways. For each of these ways, y can be chosen in 8 ways.

Hence, the desired number is 41 * 8 = 328.

4. Find the number of proper divisors of 441000.

Solution:

We note that $441000 = 2^3 \times 3^2 \times 5^3 \times 7^2$. Therefore, every divisor of $n = 441000$ must be of the form $d = 2^p \times 3^q \times 5^r \times 7^s$ where $0 \le p \le 3, 0 \le q \le 2, 0 \le r \le 3, 0 \le s \le 2$.

Thus, for a divisor d, p can be chosen in 4 ways, q in 3 ways, r in 4 ways and s in 3 ways. Accordingly, the number of possible d's is $4 \times 3 \times 4 \times 3 = 144$. Of these, two divisors (namely 1 and n) are not proper divisors. Therefore, the number of proper divisors of the given number is $144 - 2 = 142$.

5. How many among the first 100,000 positive integers contain exactly one 3, one 4 and one 5 in their decimal representations?

Solution:

The number 100000 does not contain 3 or 4 or 5. Therefore, we have to consider all possible positive integers with 5 places that meet the given conditions. In a 5-place integer the digit 3 can be in any one of the 5 places. Subsequently, the digit 4 can be in any one of the 4 remaining places. Then the digit 5 can be in any one of the 3 remaining places. There are 2 places left and either of these may be filled by 5 digits (digits from 0 to 9 other tan 3, 4, 5). Thus, there are $5 \times 4 \times 3 \times 7 \times 7 = 2940$ integers of the required type.

☻ **Permutations:**

Suppose that we are given n distinct objects and wish to arrange r of these objects in a line. Since there are n ways of choosing the first object, and after this done $n-1$ ways of choosing the second object…. And finally, $n-r+1$ ways of choosing r^{th} object, it follows by the product rule of counting (stated in the preceding section) that the number of different arrangements, or permutations (as they are commonly called) is $n(n-1)(n-2)\cdots\cdots\cdots(n-r+1)$. We denote this number by $P(n,r)$ and is referred to as the number of permutations of size r of n objects.

$$\boldsymbol{P(n,r)=\frac{n!}{(n-r)!}}$$

Generalization

Suppose it is required to find the number of permutations that can be formed from a collection of n objects of which n_1 are of one type , n_2 are of a second type ,……… n_k are of k^{th} type, with $n_1+n_2+\cdots\cdots+n_k=n$. Then, the number of permutations of the objects is

$$\boldsymbol{\frac{n!}{n_1!\ n_2!\cdots\cdots n_k!}}$$

Problems:

1. Four different mathematics books, five different computer science books and two different control theory books are to be arranged in a shelf. How many different arrangements are possible if (a) The books in each particular subject must be together? (b) Only mathematics books must be together?

Solution:

(a) The mathematics books can be arranged among themselves in 4! Ways, the computer science books in 5! Ways the control theory books in 2! Ways, and the three groups in 3! Ways. Therefore, the number of possible arrangements is 4! * 5! * 2! * 3! = 34560.

(b) Consider the 4 mathematics boos as one single book. Then we have 8 books which can be arranged in 8! Ways. In all of these ways the mathematics books are together. But the mathematics books can be arranged among themselves in 4! Ways. Hence, the number of arrangements is 8! * 4! = 967680

2. Find the number of permutations of the letters of the word MASSASAUGA. In how many of these, all four 'A's are together? How many of them begin with S?

Solution:

The given word has 10 letters of which 4 are A, 3 are S and 1 each are M, U and G. Therefore, the required number of permutations is

$$\frac{10!}{4!*3!*1!*1!*1!}=25200$$

It is a permutation all A's are to be together, we treat all of A's as one single letter. Then the letters to be permuted read (AAAA), S, S, S, M, U, G (which are 7 in number) and the number of permutations is

$$\frac{7!}{1!*3!*1!*1!*1!} = 840$$

For permutations beginning with S, there occur nine open positions to fill, where two are S, four are A, and one each of M, U, G. The number of such permutations is

$$\frac{9!}{2!*4!*1!*1!*1!} = 7560$$

3. (a) How many arrangements are there for all letters in the word SOCIOLOGICAL?

(b) In how many of these arrangements (i) A and G are adjacent? (ii) all the vowels are adjacent?

Solution:

(a) The given word has 12 letters of which three are O, two each are C, I, L and one each are S, A, G. Therefore, the number of arrangements of these letters is

$$\frac{12!}{3!*2!*2!*2!*1!*1!*1!} = 25200$$

(b)

(i) If, in an arrangement, A and G are to be adjacent, we treat A and G together as a single letter, say X so that we have three numbers of O's, two each of C, L, I and one each of S and X, totalling 11 letters. These can be arranged in $\frac{11!}{3!*2!*2!*2!*1!}$ Ways

Further the letters A and G can be arranged among themselves in two ways.

Therefore, the total number of arrangements in this case is

$$\frac{11!}{3!*2!*2!*2!*1!} \times 2 = 1663200$$

(ii) If, in an arrangement, all the vowels are to be adjacent, we treat all the vowels present in the given word (A, O, I) as a single letter, say Y, so that we have two each of C and L and one each of S, G & Y totalling to 7 letters. These can be arranged in $\frac{7!}{2!*2!*1!*1!*1!}$ ways

Further, since the given words contains 3 O's, two I's and one A, the letters A, O, I (clubbed as Y) can be arranged among themselves is $\frac{6!}{3!*2!*1!}$ Ways.

Therefore, the total number of arrangements in this case is $\frac{7!}{2!*2!*1!*1!*1!} \times \frac{6!}{3!*2!*1!} = 75600$

4. How many Positive integers n can we form using the digits 3, 4, 4, 5, 5, 6, 7 if we want n to exceed 5,000,000?

Solution:

Here n must be of the form $n = x_1x_2x_3x_4x_5x_6x_7$

Where $x_1, x_2, \dots\dots, x_7$ are the given digits with $x_1 = 5, 6\ or\ 7$. Suppose we take $x_1 = 5$. Then where $x_2x_3x_4x_5x_6x_7$ is an arrangement of the remaining 6 digits which contains two 4's and one each of 3, 5, 6, 7. The number of such arrangements is

$$\frac{6!}{1!\,2!\,1!\,1!\,1!} = 360$$

Similarly, we take $x_1 = 6$. Then where $x_2x_3x_4x_5x_6x_7$ is an arrangement of the remaining 6 digits which contains two each of 4 & 5 and one each of 3 & 7. The number of such arrangements is

$$\frac{6!}{1!\,2!\,2!\,1!} = 180$$

Similarly, we take $x_1 = 7$. Then where $x_2x_3x_4x_5x_6x_7$ is an arrangement of the remaining 6 digits which contains two each of 4 & 5 and one each of 3 & 6. The number of such arrangements is

$$\frac{6!}{1!\,2!\,2!\,1!} = 180$$

Accordingly, by the Sum Rule, the number of n's of the desired type is $360 + 180 + 180 = 720$.

5. How many numbers greater than 1,000,000 can be formed by using the digits 1, 2, 2, 2, 4, 4, 0?

Solution:

Here n must be of the form $n = x_1x_2x_3x_4x_5x_6x_7$

Where $x_1, x_2, \dots\dots, x_7$ are the given digits with $x_1 = 1, 2\ or\ 4$. Suppose we take $x_1 = 1$. Then where $x_2x_3x_4x_5x_6x_7$ is an arrangement of the remaining 6 digits which contains three 2's and two 4's. The number of such arrangements is

$$\frac{6!}{3!\,2!} = 60$$

Similarly, we take $x_1 = 2$. Then where $x_2x_3x_4x_5x_6x_7$ is an arrangement of the remaining 6 digits which contains two 2's and two 4's. The number of such arrangements is

$$\frac{6!}{2!\,2!} = 180$$

Similarly, we take $x_1 = 4$. Then where $x_2x_3x_4x_5x_6x_7$ is an arrangement of the remaining 6 digits which contains three 2's and one 4. The number of such arrangements is

$$\frac{6!}{3!\,1!} = 120$$

Accordingly, by the Sum Rule, the number of n's of the desired type is $60 + 180 + 120 = 360$.

☻ **Combinations:**

Suppose we are interested in selecting (choosing) a set of r objects from a set of $n \geq r$ objects without regard to order. The set of r objects being selected is traditionally called a Combination of r objects (or briefly r-combination).

The total number of combinations of r-different objects that can be selected from n different objects can be obtained by proceeding in the following way. Suppose this number is equal to C, say; that is, suppose there is a total of C number of combinations of r different objects chosen from n different objects. Take any one of these combinations. The r objects in this combination can be arranged in $r!$ Different ways. Since there are C combinations, the total number of permutations is $C \cdot r!$. But this is equal to $P(n,r)$. Thus,

$$\boldsymbol{C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!\,r!} \text{ for } 0 \leq r \leq n}$$

Problems:

1. A certain question paper contains two parts A and B each containing 4 questions. How many different ways a student can answer 5 questions by selecting at least 2 questions from each part?

Solution: The different ways a student can select his 5 questions are.

(i) 3 questions from part A and 2 questions from part B. this can be done in $C(4,3) * C(4,2) = 24$ ways.

(ii) 2 questions from part A and 3 questions from part B. this can be done in $C(4,2) * C(4,3) = 24$ ways.

Therefore, the total number of ways a student can answer 5 questions under given restrictions is 24 + 24 = 48.

2. Prove the following identities.

$$C(n,r-1) + C(n,r) = C(n+1,r)$$

$$C(m,2) + C(n,2) = C(m+n,2) - mn$$

Proof:

(i). $C(n,r-1) + C(n,r) = \frac{n!}{(r-1)!\,(n-r+1)!} + \frac{n!}{r!\,(n-r)!}$

$$= \frac{n!}{(r-1)!\,(n-r)!} \left\{\frac{1}{n-r+1} + \frac{1}{r}\right\}$$

$$= \frac{n!}{(r-1)!\,(n-r)!} \cdot \frac{n+1}{r\,(n-r+1)}$$

$$= \frac{(n+1)!}{r!\,(n-r+1)!}$$

$$= C(n+1,r)$$

(ii). $C(m,2) + C(n,2) = \frac{m!}{(m-2)! \cdot 2} + \frac{n!}{(n-2)! \cdot 2}$

$$= \frac{1}{2}\{m(m-1) + n(n-1)\}$$

$$= \frac{1}{2}\{m^2 + n^2 - m - n\}$$

$$= \frac{1}{2}(m+n)(m+n-1) - mn$$

$$= \frac{(m+n)!}{2\,(m+n-2)!} - mn$$

$$= C(m+n, 2) - mn$$

3. A woman has 11 close relatives and she wishes to invite 5 of them to dinner. In how many ways can she invite them in the following situations:

(i). There is no restriction on the choice.

(ii). Two particular persons will not attend separately.

(iii). Two particular persons will not attend together.

Solution:

(i). Since there is no restriction on the choice of invitees, five out of 11 can be invited in

$$C(11,5) = \frac{11!}{6!\,5!} = 462 \text{ ways}$$

(ii). Since two particular persons will not attend separately, they should both be invited or not invited.

Suppose if both of them are invited, then three are more invitees are to be selected from the remaining 9 relatives. This can be done in

$$C(9,3) = \frac{9!}{6!\,3!} = 84 \text{ ways}$$

Suppose if both of them are not invited, then five invitees are to be selected from the remaining 9 relatives. This can be done in

$$C(9,5) = \frac{9!}{5!\,4!} = 126 \text{ ways}$$

Therefore, the total number of ways in which the invitees can be selected in this case is $84 + 126 = 210$.

(iii). Since two particular persons (Say P_1 & P_2) will not attend together, only one of them can be invited or none of them can be invited. The number of ways of choosing the invitees with P_1 invited is

$$C(9,4) = \frac{9!}{5!\,4!} = 126 \text{ ways}$$

Similarly, the number of ways of choosing the invitees with P_2 invited is 126 ways

If both P_1 & P_2 are not invited, then the number of ways of inviting the invitees is

$$C(9,5) = \frac{9!}{5!\,4!} = 126 \text{ ways}$$

Therefore, the total number of ways in which the invitees can be selected in this case is

$$126 + 126 + 126 = 378.$$

4. Find the number of arrangements of all the letters in TALLAHASSEE. How many of these arrangements have no adjacent A's?

Solution:

The number of letters in the given word is 11 of which 3 are A's, 2 each are L's, S's, E's and 1 each are T and H. Therefore, the number of arrangements (permutations) of the letters in the given word is

$$\frac{11!}{3!\,2!\,2!\,2!\,1!\,1!} = 831600$$

If we disregard the A's, the remaining 8 letters can be arranged in

$$\frac{8!}{2!\,2!\,2!\,1!\,1!} = 5040$$

In each of these arrangements, there are 9 possible locations for the three A's. These locations can be chosen in $C(9,3)$ ways. Therefore, the number of arrangements having no adjacent A's is

$$5040 \times C(9,3) = 5040 \times \frac{9!}{3!\,6!} = 5040 \times 84 = 423360$$

5. A committee of 12 is to be selected from 10 men and 10 women. In how many ways can the selection be carried out if
(a) there are no restrictions?
(b) there must be six men and six women?
(c) there must be an even number of women?
(d) there must be more women than men?
(e) there must be at least eight men?

Solution:

(a). If there is no restriction than it is a simple selection of 12 out of 20.

$$C(20,12) = \frac{20!}{12!\,8!} = 125970$$

(b). For 6 men out of 10 and 6 women out of 10. These are two different stages of selection that's why product rule is used

$$C(10,6) \times C(10,6) = \frac{10!}{6!\,4!} \times \frac{10!}{6!\,4!} = 44100$$

(c). 2, 4, 6, 8 or 10 can be the number of women in committee and corresponding to that men will be 10, 8, 6, 4 and 2.

$C(10,2) \times C(10,10) + C(10,4) \times C(10,8) + C(10,6) \times C(10,6) + C(10,8) \times C(10,4) + C(10,10) \times C(10,2) = 63090$

(d). Number of women can be 7, 8, 9 or 10 and number of men will be 5, 4, 3, 2 respectively.

$C(10,7) \times C(10,5) + C(10,8) \times C(10,4) + C(10,9) \times C(10,3) + C(10,10) \times C(10,2) = 40935$

(e). Number of men can be 8, 9 or 10 in this case and respectively number of women can be 4, 3 and 2.

$C(10,8) \times C(10,4) + C(10,9) \times C(10,3) + C(10,10) \times C(10,2) = 10695$

☻ **Binomial and Multinomial Theorems:**

Binomial Theorem:

On the basic properties of $C(n,r) = \binom{n}{r}$ is that it is the coefficient of $x^r y^{n-r}$ and $x^{n-r}y^r$ in the expansion of the expression $(x+y)^n$, where x and y are real numbers. In other words,

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} = \sum_{r=0}^{n} \binom{n}{r} x^{r-n} y^r$$

This result is known as the binomial theorem for a positive integral index.

Multinomial Theorem:

For positive integers n and k the coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \ldots\ldots. x_k^{n_k}$ in the expansion of $(x_1 + x_2 + x_3 + \cdots\cdots\cdots + x_k)^n$ is $\frac{n!}{n_1!\, n_2!\, n_3! \ldots\ldots n_k!}$

Problems:

1. Find the coefficient of

(i) $x^9 y^3$ in the expansion of $(2x - 3y)^{12}$

(ii) x^{12} in the expansion of $x^3(1-2x)^{10}$

(iii) x^0 in the expansion of $\left(3x^2 - \frac{2}{x}\right)^{15}$

Solution:

By the Binomial theorem, we have $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} = \sum_{r=0}^{n} \binom{n}{r} x^{r-n} y^r$

(i). $(2x - 3y)^{12} = \sum_{r=0}^{12} \binom{12}{r} (2x)^r (-3y)^{12-r}$

$$= \sum_{r=0}^{12} \binom{12}{r} 2^r (-3)^{12-r} x^r y^{12-r}$$

In the expansion, the coefficient of $x^9 y^3$ (which corresponds to $r = 9$) is

$$\binom{12}{9} 2^9 (-3)^{12-9} = -2^9 \times 3^3 \times \frac{12!}{9! \cdot 3!}$$

$$= -2^9 \times 3^3 \times \frac{12 \times 11 \times 10}{6}$$

$$= -(2^{10} \times 3^3 \times 11 \times 10)$$

(ii). $x^3(1-2x)^{10} = \sum_{r=0}^{10} \binom{10}{r} (-2x)^r 1^{10-r}$

$$x^3(1-2x)^{10} = \sum_{r=0}^{10} \binom{10}{r} (-2)^r x^{r+3}$$

In the expansion, the coefficient of x^{12} (which corresponds to $r = 9$) is

$\binom{10}{9}(-2)^9 = -(10 \times 2^9) = -5120$

(iii). $\left(3x^2 - \frac{2}{x}\right)^{15} = \sum_{r=0}^{15} \binom{15}{r} (3x^2)^r \left(-\frac{2}{x}\right)^{15-r}$

$= \sum_{r=0}^{15} \binom{15}{r} 3^r (-2)^{15-r} x^{3r-15}$

In the expansion, the coefficient of $x^9 y^3$ (which corresponds to $r = 5$) is

$\binom{15}{5} 3^5 (-2)^{15-5} = (-2)^{10} \times 3^5 \times \frac{15!}{5! \cdot 10!}$

$= 2^{10} \times 3^5 \times 3003$

2. Determine the coefficient of

(i) xyz^2 in the expansion of $(2x - y - z)^4$

(ii) $x^{11}y^4$ in the expansion of $(2x^3 - 3xy^2 + z^2)^6$

(iii) $x^2y^2z^3$ in the expansion of $(3x - 2y - 4z)^7$

(iv) $a^2b^3c^2d^5$ in the expansion of $(a + 2b - 3c + 2d + 5)^{16}$

(v) $w^3x^2yz^2$ in the expansion of $(2w - x + 3y - 2z)^8$

Solution:

By the multinomial theorem, we have $(x_1 + x_2 + x_3 + \cdots\cdots\cdots + x_k)^n$ is $\frac{n!}{n_1!\, n_2!\, n_3! \ldots\ldots n_k!}$

(i). The general term is the expansion of $(2x - y - z)^4$ is $\binom{4}{n_1,\ n_2,\ n_3} (2x)^{n_1}(-y)^{n_2}(-z)^{n_3}$

For $n_1 = 1, n_2 = 1, n_3 = 2$ this becomes

$\binom{4}{1,\ 1,\ 2} (2x)^1(-y)^1(-z)^2 = \binom{4}{1,\ 1,\ 2} (2)(-1)(-1)^2 xyz^2$

This shows that the required coefficient is $\binom{4}{1,\ 1,\ 2} (2)(-1)(-1)^2 = \frac{4!}{1!\,1!\,2!} \times (-2) = -12$

(ii). The general term is the expansion of $(2x^3 - 3xy^2 + z^2)^6$ is $\binom{6}{n_1,\ n_2,\ n_3} (2x^3)^{n_1}(-3xy^2)^{n_2}(z^2)^{n_3}$

For $n_3 = 0, n_2 = 2, n_1 = 3$ this becomes

$$\binom{6}{3,\ 2,\ 0} (2x^3)^3(-3xy^2)^2(z^2)^0 = \binom{6}{3,\ 2,\ 0} (2)^3(-3)^2(1)^0 x^{11}y^4$$

This shows that the required coefficient is $\binom{6}{3,\ 2,\ 0} (2)^3(3)^2 = \frac{6!}{3!\,2!} \times 72 = 4320$

(iii). The general term is the expansion of $(3x - 2y - 4z)^7$ is $\binom{7}{n_1,\ n_2,\ n_3} (3x)^{n_1}(-2y)^{n_2}(-4z)^{n_3}$

For $n_1 = 2, n_2 = 2, n_3 = 3$ this becomes

$$\binom{7}{2,\ 2,\ 3}(3x)^2(-2y)^2(-4z)^3 = \binom{7}{2,\ 2,\ 3}(3)^2(-2)^2(-4)^3 x^2y^2z^3$$

This shows that the required coefficient is

$$\binom{7}{2,\ 2,\ 3}(3)^2(-2)^2(-4)^3 = \frac{7!}{2!\,2!\,3!} \times 9 \times 4 \times (-64) = -483840$$

(iv). The general term is the expansion of $(a + 2b - 3c + 2d + 5)^{16}$ is

$$\binom{16}{n_1,\ n_2,\ n_3,\ n_4,\ n_5}(a)^{n_1}(2b)^{n_2}(-3c)^{n_3}(2d)^{n_4}(5)^{n_5}$$

For $n_1 = 2, n_2 = 3, n_3 = 2, n_4 = 5, n_5 = 16 - (2 + 3 + 2 + 5) = 4$, this becomes

$$\binom{16}{2,\ 3,\ 2,\ 5,\ 4}(a)^2(2b)^3(-3c)^2(2d)^5(5)^4 = \binom{16}{2,\ 3,\ 2,\ 5,\ 4}(2)^3(-3)^2(2)^5(5)^4 a^2b^3c^2d^5$$

This shows that the required coefficient is

$$\binom{16}{2,\ 3,\ 2,\ 5,\ 4}(2)^3(-3)^2(2)^5(5)^4 = \frac{16!}{2!\,3!\,2!\,5!\,4!} \times 2^8 \times 3^2 \times 5^4 = \frac{16!}{(4!)^2} \times 2^5 \times 3 \times 5^3$$

(v). The general term is the expansion of $(2w - x + 3y - 2z)^8$ is

$$\binom{8}{n_1,\ n_2,\ n_3,\ n_4}(2w)^{n_1}(-x)^{n_2}(3y)^{n_3}(-2z)^{n_4}$$

For $n_1 = 3, n_2 = 2, n_3 = 1, n_4 = 2$ this becomes

$$\binom{8}{3,\ 2,\ 1,\ 2}(2w)^3(-x)^2(3y)^1(-2z)^2 = \binom{8}{3,\ 2,\ 1,\ 2}(2)^3(-1)^2(3)^1(-2)^2)\ w^3x^2yz^2$$

This shows that the required coefficient is

$$\binom{8}{3,\ 2,\ 1,\ 2}(2)^3(-1)^2(3)^1(-2)^2) = \frac{8!}{3!\,2!\,1!\,2!} \times 2^3 \times 3 \times 2^2 = 161280$$

☻ **Combinations with repetitions**:

Suppose we wish to select, with repetition, a combination of r objects from a set of n distinct objects. The number of such selections is given by $C(n+r-1,r) \equiv \frac{(n+r-1)!}{r!\ (n-1)!} \equiv C(r+n-1,n-1)$.

In other words, $C(n+r-1,r) \equiv C(r+n-1,n-1)$ represents the number of combinations of m distinct objects, taken r at a time, with repetition allowed.

The following are other interpretations of this number:

$C(n+r-1,r) \equiv C(r+n-1,n-1)$ represents the number of ways in which r identical objects can be distributed among n distinct containers.

$C(n+r-1,r) \equiv C(r+n-1,n-1)$ represents the number of nonnegative integer solutions of the equation.

Problems:

1. In how many ways we can distribute 10 identical marbles among 6 distinct containers?

Solution:

The selection consists in choosing with repetitions $r = 10$ marbles for $n = 6$ distinct containers

The required number is $C(6+10-1,10) = C(15,10) = \frac{15!}{10!\ 5!} = 3003$

2. Find the number of non-negative integer solutions of the inequality $x_1 + x_2 + x_3 + \cdots + x_6 < 10$

Solution:

We have to find the number of nonnegative integer solutions of the equation

$$x_1 + x_2 + x_3 + \cdots + x_6 = 9 - x_7$$

where $9 - x_7 \leq 9$ so that x_7 is non negative integer. Thus, the required number in the number of nonnegative solutions of the equation.

x1 + x2 + x3 + ……+ x7 = 9

This number is $C(7+9-1,9) = C(15,9) = \frac{15!}{9!\ 6!} = 5005$

3. In How many ways can we distribute 7 apples and 6 oranges among 4 children so that each child gets at least 1 apple?

Solution:

Suppose we first give 1 apple to each child. This exhausts 4 apples. The remaining 3 apples can be distributed among 4 children in $C(4+3-1,3) = C(6,3)$ ways. Also, 6 oranges can be distributed among the 4 children in $C(4+6-1,6) = C(9,6)$ ways. Therefore, by the product rule, the number ways of distributing the given fruits under the given condition is

$C(6,3) \times C(9,6) = \frac{6!}{3!\ 3!} \times \frac{9!}{6!\ 3!} = 20 \times 84 = 1680$

4. A message is made up of 12 different symbols and it is to be transmitted through a communication channel. In addition to the 12 symbols, the transmitter will also send a total of 45 blank spaces between the symbols, with at least three spaces between each pair of consecutive symbols. In how many ways can the transmitter send such a message?

Solution:

The 12 symbols can be arranged in 12! Ways. For each of these arrangements, there are 11 positions between the 12 symbols. Since there must be at least three spaces between successive symbols, 33 of the 45 spaces will be used up. The remaining 12 spaces are to be accommodated in 11 positions. This can be done in $C(11+12-1,12) = C(22,12)$ ways. Consequently, by the product rule, the required number is

$$12! \times C(22,12) = 12! \times \frac{22!}{12! \times 10!} = 3.097445 \times 10^{14}$$

5. In how many ways can one distribute eight identical balls into four distinct containers so that (i) no container is left empty? (ii) the fourth container gets an odd number of balls?

Solution:

(i). First, we distribute one ball in to each container. Then we distribute the remaining 4 balls into 4 containers. The number of ways of doing this is the required number. This number is

$$C(4+4-1,4) = C(7,4) = \frac{7!}{4! \times 3!} = 35$$

(ii). If the fourth container has o get an odd number of balls, we have to put 1 or 3 or 5 or 7 balls into it.

Suppose we put 1 ball into the fourth container and the remaining 7 balls can be put into the remaining three containers in

$C(3+7-1,7) = C(9,7)$ ways

Similarly, we put 3 balls into the fourth container and the remaining 5 balls can be put into the remaining three containers in

$C(3+5-1,5) = C(7,5)$ ways

Similarly, we put 5 balls into the fourth container and the remaining 3 balls can be put into the remaining three containers in

$C(3+3-1,3) = C(5,3)$ ways

Similarly, we put 7 balls into the fourth container and the remaining 1 ball can be put into the remaining three containers in

$C(3+1-1,1) = C(3,1)$ ways

Thus, the total number of ways of distributing the given balls so that the fourth container gets an odd number of balls is

$C(9,7) + C(7,5) + C(5,3) + C(3,1) = \frac{9!}{7!\times 2!} + \frac{7!}{5!\times 2!} + \frac{5!}{3!\times 2!} + \frac{3!}{1!\times 2!} = 36 + 21 + 10 + 3 = 70$

6. Find the number of integer solutions of $x_1 + x_2 + x_3 + x_4 = 32$ where $x_i \geq 0, 1 \leq i \leq 4$.

Solution:

Given $x_1 + x_2 + x_3 + x_4 = 32$, where $x_i \geq 0, 1 \leq i \leq 4$.

The required number is $C(4 + 32 - 1, 32) = C(35, 32) = \frac{35!}{32!\times 3!} = 6545$

7. Find the number of positive integer solutions of the equation $x_1 + x_2 + x_3 = 17$

Solution:

Given $x_1 + x_2 + x_3 = 17$, we require $x_i \geq 1, 1 \leq i \leq 3$.

Let us set $y_1 = x_1 - 1,\ y_2 = x_2 - 1,\ y_3 = x_3 - 1,$ then y_1, y_2, y_3 are all nonnegative integers.

Then the given equation is reads $(y_1 + 1) + (y_2 + 1) + (y_3 + 1) = 17$ or $y_1 + y_2 + y_3 = 14$

The required number is $C(3 + 14 - 1, 14) = C(16, 14) = \frac{16!}{14!\times 2!} = 120$

8. Find the number of positive integer solutions of the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 30$ where $x_1 \geq 2, x_2 \geq 3, x_3 \geq 4, x_4 \geq 2, x_5 \geq 0$

Solution:

Given $x_1 + x_2 + x_3 + x_4 + x_5 = 30$

Let us set $y_1 = x_1 - 2,\ y_2 = x_2 - 3,\ y_3 = x_3 - 4, y_4 = x_4 - 2,\ y_5 = x_5$ then y_1, y_2, y_3, y_4, y_5 are all nonnegative integers.

Then the given equation is reads

$(y_1 + 2) + (y_2 + 3) + (y_3 + 4) + (y_4 + 2) + (y_5 + 0) = 30$ or $y_1 + y_2 + y_3 + y_4 + y_5 = 19$

The required number is $C(5 + 19 - 1, 19) = C(23, 19) = \frac{23!}{19!\times 4!} = 8855$