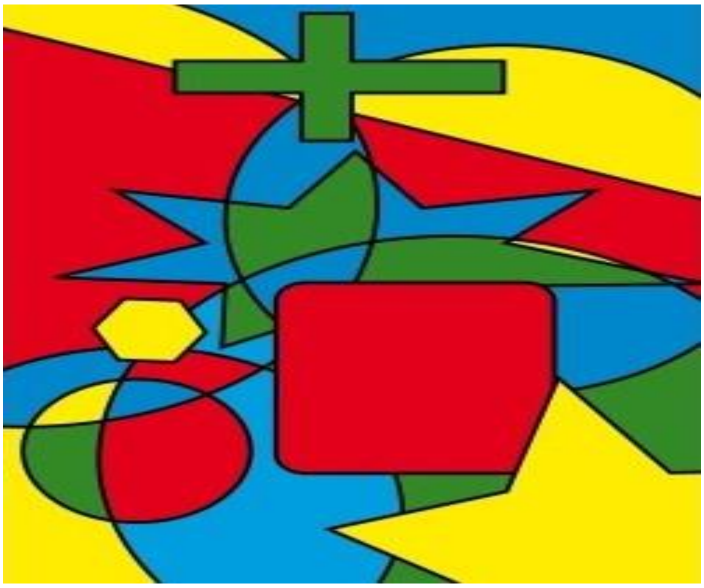
18CS36

Discrete Mathematical Structures

(For the 3rd Semester Computer Science and Engineering Students)



Module 3

Relations & Functions

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Content

MODULE -3

RELATIONS AND FUNCTIONS

☻**Syllabus:**

Relations and Functions: Cartesian Products and Relations, Functions – Plain and One-to One, Onto Functions. The Pigeon-hole Principle, Function Composition and Inverse Functions. **Relations:** Properties of Relations, Computer Recognition – Zero-One Matrices and Directed Graphs, Partial Orders – Hasse Diagrams, Equivalence Relations and Partitions.

☻**Cartesian Products:**

For set $A, B \subseteq U$, the Cartesian product of A and B is denoted by $A \times B$ and equals $\{(a, b) \mid a \in A, b \in B\}$

Example: Let $U = \{1,2,3, \dots .7\}, A = \{2,3,4\}, B = \{4,5\}$

Then (a). $A \times B = \{(2,4), (2,5), (3,4), (3,5), (4,4), (4,5)\}$

(b). $B^2 = B \times B = \{(4,4), (4,5), (5,4), (5,5)\}$

(c). $B^3 = B \times B \times B = \{(a, b, c) \mid a, b, c \in B\}$

☻**Relation:**

For sets $A, B \subseteq U$ any subset of $A \times B$ is Called a Relation From A to B and any subset of $A \times A$ is called a Binary relation on A.

Example:

Let A and B be finite sets with $|B| = 3$. If there are 4096 relations from A to B what is $|A|$?

Solution: If $|A| = m, |B| = n$ then there are 2^{mn} relations from A to B.

Given $n = 3, 2^{mn} = 4096 \therefore m = 4 = |A|$.

☻**Functions:**

Let **A** and **B** be two non-empty sets. Then a function **f** from **A** to **B** is a relation from A to B such that for each **a** in **A** there is a unique **b** in **B** such that $(\mathbf{a}, \mathbf{b}) \in \mathbf{f}$

Types of Functions:

(a). Floor function:

The function $f: R \to Z$, is given by

$f(x) = \lfloor x \rfloor =$ The greatest integer less than or equal to x.
$\lfloor 3.8 \rfloor = 3$
$\lfloor -3.8 \rfloor = -4$

(b). Ceiling Function:

The function $g: R \to Z$ is defined by $g(x) = \lceil x \rceil$

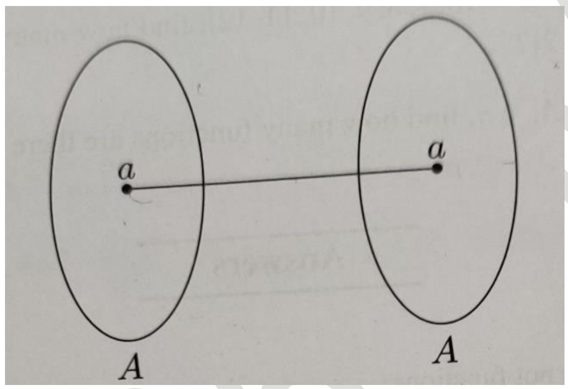
$\lceil 3 \rceil = 3, \lceil 3.01 \rceil = \lceil 3.7 \rceil = 4 = \lceil 4 \rceil$

$\lceil -3.01 \rceil = \lceil -3.7 \rceil = -3$

(c). Identity function:

A function $f: A \to A$ such that $f(a) = a$ for every $a \in A$ is called the identity function (or identity mapping) on A.

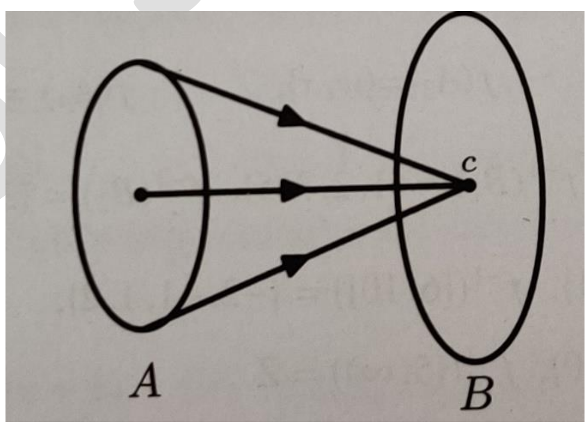
In other words, a function f on a set A is an identity function if the image of every element of A (under f) is itself.



(d). Constant function:

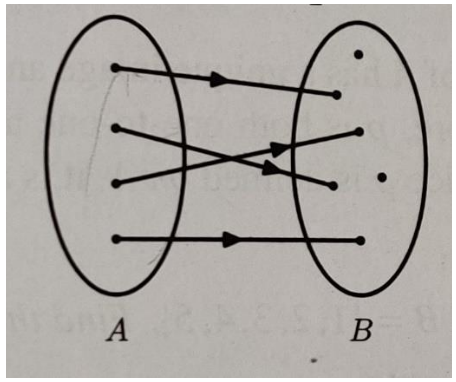
A function $f: A \to B$ such that $f(a) = c$ for every $a \in A$, where c is a fixed element of B, is called a Constant function.

In other words, a function f from A to B is a constant function if all elements of A have the same image (say c) in B.



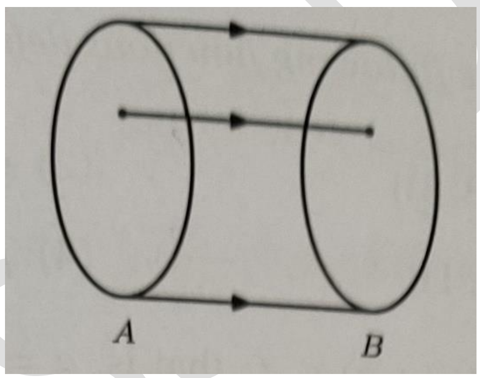
(e). Injective or one-to-one: A function $f: A \to B$ is called one-to-one, if each element of **B** appears at most once as the image of an element of **A**.

In other words, If different elements of A have different images in B under f; If whenever $f(a_1) = f(a_2)$ for $a_1, a_2 \in A$, then $a_1 = a_2$

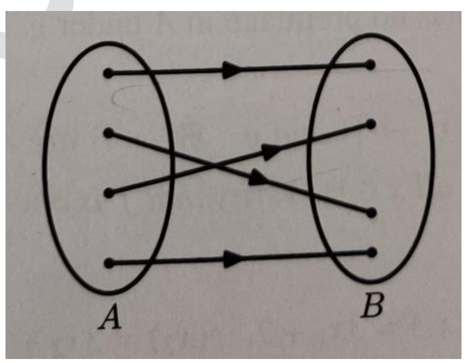


(f). Surjective or onto: A function $f: A \to B$ is called onto if for every element **b** of **B** there is an element **a** of **A** such that $f(a) = b$

In other words. f is an onto function from A to B if every element of B has a Preimage in A.



(g). Bijective or one-to-one correspondence: A function which is both one-to-one and onto is called Bijective.



Note: Number of one-to-one functions from A to B is

$P(n, m) = \frac{n!}{(n-m)!}$ Where $|A| = m, |B| = n$ & $m >= n$

Number of onto functions from A to B is

$$P(n,m) = \sum_{k=0}^{n} (-1)^k \binom{n}{n-k} (n-k)^m$$

Problems:

1. Let $A = \{1,2,3,4,5,6,7\}, B = \{w,x,y,z\}$. Find the number of Onto Functions from A to B.

Solution: Given $m = |A| = 7$ & $n = |B| = 4$

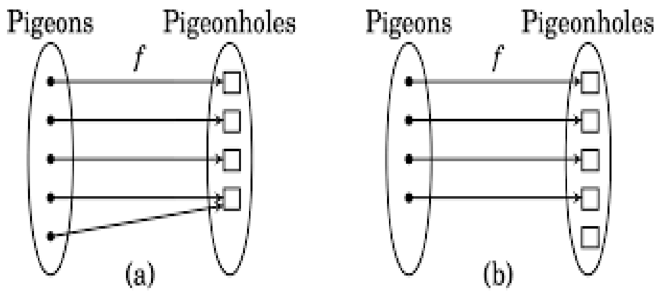
$$P(7,4) = \sum_{k=0}^{n} (-1)^k \binom{4}{4-k} (4-k)^7 = 8400$$

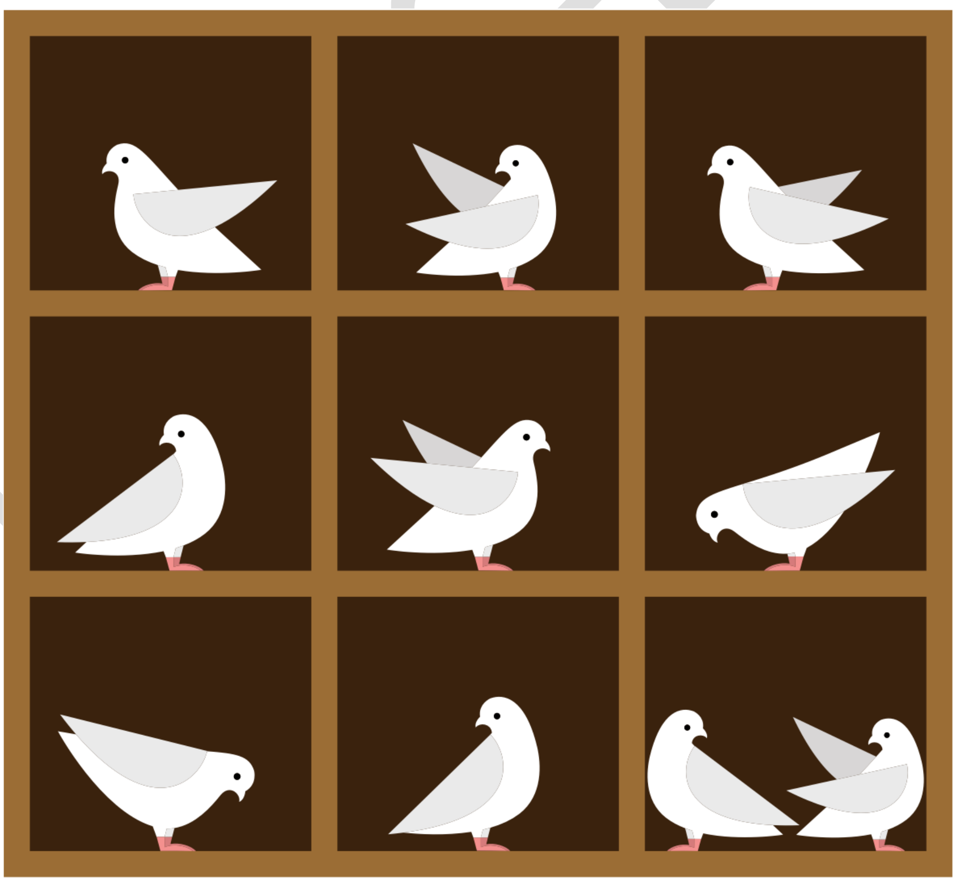
☻ **Pigeonhole Principle**:

If m pigeons occupy n pigeon holes and if $m > n$, then two or more pigeons occupy the same pigeonhole.

Generalization:

If m pigeons occupy n pigeonholes, then at least one pigeonhole must contain $(p+1)$ or more pigeons, where $p = \left\lfloor \frac{(m-1)}{n} \right\rfloor$



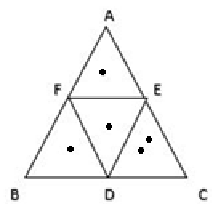


Problems:

1. ABC is an equilateral triangle whose sides are of length 1cm each. If we select 5 points inside the triangle, prove that at least 2 of these points are such that the distance between them is less than $\frac{1}{2}$ cm.

Solution:

Consider the triangle DEF formed by the mid points of the sides BC, CA and AB of the given triangle ABC. Then the triangle ABC is partition into 4 small equilateral triangles, each of which has sides equal to $\frac{1}{2}$ cm treating each of these four portions as a pigeonhole and 5 points chosen inside the triangle as pigeons, we find by using the pigeonhole principle that at least one portion must contain two or more points. Evidently the distance between such points is $< \frac{1}{2}$ cm.



2. A magnetic tape contains a collection of 5 lakh strings made up to four or fewer number of English Letters can all the strings is the collection be distinct?

Solution:

Each place is an n letter string can be filled in 26 ways. Therefore, the possible number of strings made up of n letters is 26^n consequently, the total number of different possible strings made up of four or fewer letter is $26^4 + 26^3 + 26^2 + 26 = 4{,}75{,}254$.

Therefore, if there are 5 lakh strings in the tape, then at least one string is repeated. Thus, all the strings in the collection cannot be distinct.

3. Shirts numbered consecutively from 1 to 20 are worn by 20 students of a class. When any 3 of these students are chosen to be a debating team from the class, the sum of their shirt numbers is used as a code number of the team. Show that if any 8 of the 20 are selected, then from these 8 we may form at least two different teams having the same code number.

Solution:

From the 8 of the 20 students selected the numbers of teams of 3 students that can be formed is ${}^8C_3$=56. According to the way in which the code number of a team is determined, we note that the smallest possible code number is $1 + 2 + 3 = 6$ and the largest possible code number is $18 + 19 + 20 = 57$. Thus, the code number vary from 6 to 57, and these are 52 in number. As such only 52 code number are available for 56 possible teams, consequently by the pigeonhole principle, at least two different teams will have the same code number.

☻ **Composition of functions:**

Consider three non-empty sets A, B, C and the functions $f: A \to B$ and $g: B \to C$. the composition of these two functions is defined as the function $gof: A \to C$ with $(gof)(a) = g\{f(a)\}$ for all $a\epsilon A$.

Problems:

1. Consider the function f and g defined by $f(x) = x^3$ and $g(x) = x^2 + 1$ $\forall$ $x\epsilon R$ find gof, fog, f^2 and g^2

Solution:

Here, both f and g are defined on R, therefore all of the functions $gof, fog, f^2 = fof$and $g^2 = gog$ are defined on R and we find

$(gof)(x) = g\{f(x)\} = g(x^3) = (x^3)^2 + 1 = x^6 + 1$

$(fog)(x) = f\{g(x)\} = f(x^2 + 1) = (x^2 + 1)^3$

$f^2(x) = (fof)(x) = f\{f(x)\} = f(x^3) = (x^3)^3 = x^9$

$g^2(x) = (gog)(x) = g\{g(x)\} = g(x^2 + 1) = (x^2 + 1)^2 + 1$

2. Let f and g be function from R to R defined by $f(x) = ax + b$ and $g(x) = 1 - x + x^2$ if $(gof)(x) = 9x^2 - 9x + 3$ determine a, b.

Solution: We have $(gof)(x) = 9x^2 - 9x + 3 = g\{f(x)\}$

$= g\{ax + b\}$

$= 1 - (ax + b) + (ax + b)^{\,2}$

$= a^2x^2 + (2ab - a)x + (1 - b + b^2)$

Comparing the corresponding coefficients

$$9 = a^2, 9 = a - 2ab, 3 = 1 - b + b^2.$$

$$a = \pm 3, \qquad b = -1,2$$

☻ **Invertible Functions:**

A function $f: A \to B$ is said to be invertible if there exists a function $g: B \to A$ such that $gof = I_A$ and $fog = I_B$ where I_A is the identity function on A and $I_B$is the identity function on B.

Problems:

1. Let $A = \{1,2,3,4\}$ and f and g be function From A to A given by $f = \{(1,4), (2,1), (3,2), (4,3)\}$ $g = \{(1,2), (2,3), (3,4), (4,1)\}$. Prove that f and g are inverse of each other.

Solution:

$$(gof)(1) = g\{f(1)\} = g(4) = 1 = I_A(1)$$
$$(gof)(2) = g\{f(2)\} = g(1) = 2 = I_A(2)$$
$$(gof)(3) = g\{f(3)\} = g(2) = 3 = I_A(3)$$
$$(gof)(4) = g\{f(4)\} = g(3) = 4 = I_A(4)$$
$$(fog)(1) = f\{g(1)\} = f(2) = 1 = I_B(1)$$
$$(fog)(2) = f\{g(2)\} = f(3) = 2 = I_B(2)$$
$$(fog)(3) = f\{g(3)\} = f(4) = 3 = I_B(3)$$
$$(fog)(4) = f\{g(4)\} = f(1) = 4 = I_B(4)$$

Thus, for all $x \epsilon A$, we have $(gof)(x) = I_A(x)$ and $(fog)(x) = I_B(x)$, therefore g is an inverse of f and f is an inverse of g.

2. Consider the function $f: R \to R$ defined by $f(x) = 2x + 5$. Let a function $g: R \to R$ be defined by $g(x) = \frac{1}{2(x-5)}$ Prove that g is an inverse of f.

Solution:

We check that for any $x \epsilon R$

$(gof)(x) = g[f(x)] = g(2x + 5)$

$= 1/2(2x + 5 - 5) = x = I_R(x)$

$(fog)(x) = f[g(x)] = f\{1/2(x - 5)\}$

$= 2\{1/2(x - 5)\} + 5 = x = I_R(x)$

☻ **Properties of Functions:**

Theorem 1: A function $f: A \to B$ is invertible if and only if one-to-one and onto.

Proof: Suppose that f is invertible then there exists a unique function $g: B \to A$ such that $gof = I_A$ and $fog = I_B$ Take any $a_1, a_2 \epsilon A$ then

$f(a_1) = f(a_2) \Rightarrow g\{f(a_1)\} = g\{f(a_2)\}$

$\Rightarrow (gof)(a_1) = (gof)(a2)$

$\Rightarrow I_A(a_1) = I_A(a_2)$

$\Rightarrow a_1 = a_2$

This prove f is one-to-one

Next, take any $b \epsilon B$. Then $g(b) \epsilon A$ and $b = I_B(b)$

$$= (fog)(b) = f\{g(b)\}.$$

Thus, b is the image of an element $g(b) \epsilon A$ under f. therefore, f is onto as well.

Conversely, suppose that f is one-to-one and onto then for each $b \epsilon B$ there is a unique $a \epsilon A$ such that $b = f(a)$ now consider the function $g: B \to A$ defined by $g(b) = a$ then

$(gof)(a) = g\{f(a)\} = g(b) = a = I_A(a)$ and $(fog)(b) = f\{g(b)\} = f(a) = b = I_B(b)$

These show that f is invertible with g as the inverse. This completes the proof of the theorem.

Theorem 2: If $f: A \to B$ and $g: B \to C$ are invertible functions, then

$gof: A \to C$ is an invertible function and $(gof)^{-1} = f^{-1}og^{-1}$.

Proof: Since $\boldsymbol{f}$ and $\boldsymbol{g}$ are invertible functions; they are both one-to-one and onto consequently gof is both one-to-one and onto therefore, gof is invertible. Now the inverse f^{-1} of f is a function from B to A and the inverse g^{-1} of g is a function from C to B.

Therefore, if $h = f^{-1}og^{-1}$ then h is a function from C to A.

We find that

$$(gof)oh = (gof)o(f^{-1}og^{-1}) = go(fof^{-1})og^{-1} = goI_B og^{-1}$$
$$= gog^{-1} = I_C$$

And

$$ho(gof) = (f^{-1}og^{-1})o(gof) = f^{-1}o(g^{-1}og)of = f^{-1}oI_B of$$
$$= f^{-1}o\,f = I_A$$

The above expression show that h is the inverse of gof,

i.e., $h = (gof)^{-1}$. Thus $(gof)^{-1} = h = f^{-1}og^{-1}$ this completes the proof of the theorem.

☻ Zero-one matrices and Directed graphs:

Power of R:

Given a set A and a relation R on A we define the powers of R recursively by

(a) $RI = R$ (b) for $n \in Z^+, R^{n+1} = RoR^n$

Example:

If $A = \{1,2,3,4\}$ and $R = \{(1,2)\ (1,3)\ (2,4)\ (3,2)\}$ then $R^2 = \{(1,4),(1,2),(3,4)\}, R^3 = \{(1,4)\}$ and for $n \geq 4, R^n = \phi$.

Zero Matrix:

An $m \times n$ Zero-matrix $E = (e_{ij})_{m\times n}$ is a rectangular array of number arranged is m rows and n columns, where each e_{ij}, for $1 \leq i \leq m$ and $1 \leq i \leq n$ denote the entry is the i^{th}row and j^{th} column of E, and each such entry is 0 or 1.

$\boldsymbol{n \times n}$ $\boldsymbol{(0,1)}$ matrix:

For $n \in Z^+$, $I_n = \left(\delta_{ij}\right)_{n\times \boldsymbol{n}}$ is the $n \times n$ (0,1)-matrix where

$$\delta_{ij} == \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

☻ Digraph of a relation:

Let V be a finite nonempty set. A directed graph G on V is made up of the elements of V, called the vertices or nodes of G, and a subset E, of $V \times V$ that contains the edges or arcs, of G. The set V is called the vertex set of G, the set E edge set. We then write $G = (V, E)$ to denote the graph.

If $a, b \in V$ and $(a, b) \in E$ then there is an edge from a to b vertex a is called the origin or source of the edge with b the terminus or terminating vertex and we say that b is adjacent from a and that a is adjacent to b. In addition, if $a \neq b$, then $(a, b) \neq (b, a)$. An edge of the form (a, a) is called a loop.

Problems:

1. Let $A = \{1,2,3,4\}$ and let R be the relation on A defined by xRy if and only if $y = 2x$.

   a) Write down R as asset of ordered pairs.

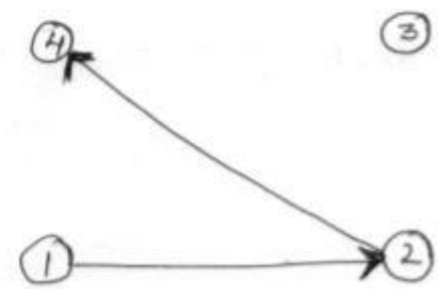
   b) Draw the digraph of R.

   c) Determine the in-degrees and out-degrees of the vertices in the digraph.

Solution:

a) We observe that for $x, y \in A, (x, y) \in R$ if and only if $y = 2x$. thus $R = \{(1,2), (2,4)\}$.

b) The digraph of R is as shown below



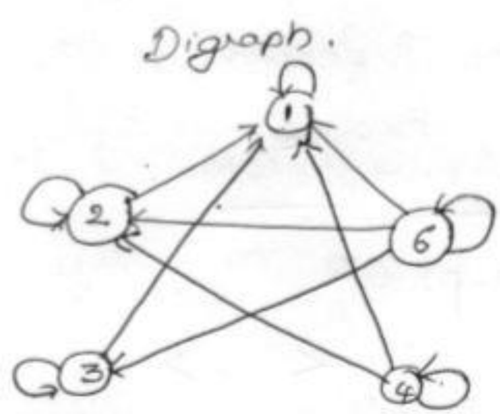
c) From the above digraph, we note that 3 is an isolated vertex and that for the vertex 1,2,4 the in-degrees and out-degrees are as shown in the table

Vertex	**1**	**2**	**4**
In-degree	0	1	1
Out-degree	1	1	0

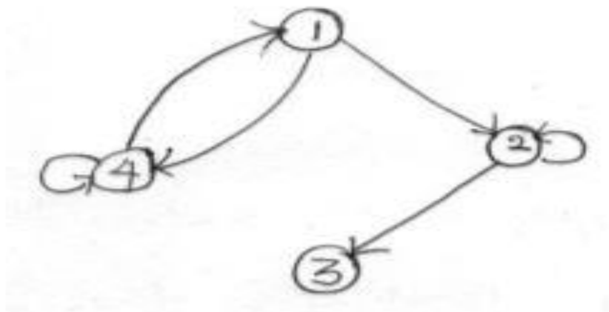
2. Let $A = \{1,2,3,4,6\}$ and R be a relation on A defined by aRb if and only if a is a multiple of b. Represent the relation R as a matrix and draw its digraph.

Solution: $R = \{(1,1),(2,1),(2,2),(3,1),(3,3),(4,1),(4,2),(4,4),(6,1),(6,2),(6,3),(6,6)\}$

$$M_R = \begin{bmatrix} 1&0&0&0&0\\ 1&1&0&0&0\\ 1&0&1&0&0\\ 1&1&0&1&0\\ 1&1&1&0&1 \end{bmatrix}$$



3. Find the relation represented by the digraph given below. Also write down its matrix.



Solution:

By examining the given digraph which has 4 vertices, we note that the relation R represented by it is defined on the set $A = \{1,2,3,4\}$ and is given by $R = \{(1,2), (1,4), (2,2), (2,3), (4,1), (4,4)\}$. The matrix of R is

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

☻ **Properties of Relations**:

1. **Reflexive relation**:

A relation R on a set A is said to be reflexive, if $(a, a) \in R$, for all $a \in A$.

Example: $\leq$

2. **Irreflexive relation**:

A relation is said to be irreflexive, if $(a, a) \notin R$ for any $a \in A$.

Example: $<, >$

3. **Symmetric Relation:**

A relation R on a set is said to be symmetric, If $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

A relation which is not symmetric is called an **Asymmetric relation**.

Example: If $A = \{1,2,3\}$ and $R_1 = \{(1,1), (1,2), (2,1)\}$, $R_2 = \{(1,2), (2,1), (1,3)\}$

R_1 is symmetric and R_2 is asymmetric.

4. **Antisymmetric relation:**

A relation R on a set A is said to be antisymmetric, if whenever $(a, b) \in R$ and $(b, a) \in R$ then $a = b$.

Example: is less than or equal to.

5. **Transitive Relation:**

A relation on a set A is said to be transitive if whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$ for all $a, b, c \in A$.

Examples:

1. Determine nature of the relations.

[1] $A = \{1,2,3\}$, $R_1 = \{(1,2), (2,1), (1,3), (3,1)\}$
  - Symmetric but not reflexive.

[2] $R_2 = \{(1,1), (2,2), (3,3), (2,3)\}$
  - Reflexive but not symmetric.

[3] $R_3 = \{(1,1), (2,2), (3,3)\}$
  - Reflexive and symmetric.

[4] $R_4 = \{(1,1), (2,2), (3,3), (2,3), (3,2)\}$
  - Both reflexive and symmetric.

[5] $R_5 = \{(1,1), (2,3), (3,3)\}$
  - Neither reflexive nor symmetric

2. If $A = \{1,2,3,4\}$, $R_1 = \{(1,1),(2,3),(3,4),(2,4)\}$ is transitive $R_2 = \{(1,3),(3,2)\}$ is not transitive.

☻ **Equivalence relation:**

A relation that is reflexive, symmetric and transitive.

Problems:

1. A relation R on a set $A = \{a,b,c\}$ is represented by the following matrix. $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ determine whether R is an Equivalence relation.

Solution: $R = \{(a,a),(a,c),(b,b),(c,c)\}$ we note that $(a,c) \in R$ but $(c,a) \notin R$

$\therefore$ R is not symmetric
$\therefore$R is not equivalence

2. For a fixed integer $n > 1$ prove that the relation congruent modulo n is an equivalence relation on the set of all integers Z.

Solution: For $a,b \in z$, we say that a is congruent to b modulo n if $a - b$ is a multiple of n or equivalently, $a - b = kn$ for some $k \in Z$.

Let us denote this relation by R so that aRb means $a \equiv b\ (mod\ n)$ we have to prove that R is an equivalence relation.

We note that for every $a \in Z$, $a - a = 0$ is a multiple of n ie, $a \equiv a(mod\ n), aRa$

R is reflexive. Next for all $a,b \in z$

$aRb \rightarrow a \equiv b\ mod\ n.$

$\rightarrow a - b$ is a multiple of n

$\rightarrow b - a$ is a multiple of n

$\rightarrow\ b \equiv a\ mod\ n$
$\rightarrow bRa$

R is symmetric.

Lastly, we note that for all $a,b,c\ \in Z$
aRb and $bRc \Rightarrow a \equiv b(mod\ n)$ and $b \equiv c(mod\ n)$
$=\ a - b$ and $b - c$ are multiples of n
$=\ (a - b) + (b - c)\ =\ (a - c)$ is a multiple of n
$=\ a \equiv c(\ mod\ n)\ =\ aRc$
R is transitive. This proves that R is equivalence relation.

☻ **Equivalence Class:**

Let R be an equivalence relation on a set A and $a \in A$. Then the set of all those elements x of A which are related to a by R is called the equivalence class of a with respect to R.

$$\bar{a} = [a] = R(a) = \{x \in A | (x, a) \in R\}$$

Example:

$R = \{(1,1), (1,3), (2,2), (3,1), (3,3)\}$ defined on the set $A = \{1,2,3\}$ we find elements x of A for which $(x, 1) \in R$ are $x = 1, x = 3$. Therefore $\{1,3\}$ is the equivalence class of 1

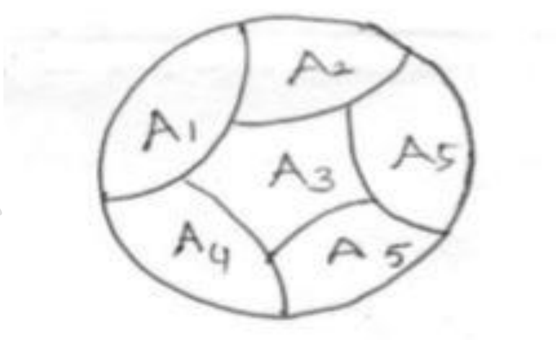
i.e., $[1] = \{1,3\}, [2] = [2], [3] = \{1,3\}$

☻ **Partition of a set:**

Let A be a non-empty set suppose that there exist non-empty subsets $A_1, A_2, A_3, \ldots\ldots A_K$ of A such that the following two conditions hold.

1) A is the union of $A_1, A_2, A_3, \ldots\ldots A_K$ that is $A = A_1 U A_2 U A_3, \ldots U A_K$
2) Any two of the subsets $A_1, A_2, A_3, \ldots\ldots A_K$ are disjoint i.e., $A_i \cap A_j = \phi$ for $i \neq j$ then the set $P = \{A_1, A_2, A_3, \ldots\ldots A_K\}$ is called a partition of A. also $A_1, A_2, A_3, \ldots\ldots A_K$ are called the blocks or cells of the partition.

A partition of a set A with 6 blocks is as shown below



$A = \{1,2,3,4,5,6,7,8\}$ and its following subsets $A_1 = \{1,3,5,7\}$, $A_2 = \{2,4\}$, $A_3 = \{6,8\}$

$P = \{A_1, A_2, A_3\}$ is a Partition of A with A_1 A_2 A_3 as blocks of the partition?

$A_4 = \{1,3,5\}$ then $P_1 = \{A_2, A_3, A_4\}$ in not a partition of the set A. Because although the subsets A_2, A_3 and A_4 are mutually disjoint A is not the union of these subsets. We find if $A_5 = \{5,6,8\}$ then $P_2 = \{A_1, A_2, A_5\}$ is also not a partition of A because A is the union of A_1, A_2, A_5. A_1, A_5 are not disjoint.

Problems:

1. For the set A and the relation R on A

$$A = \{1,2,3,4,5\}, R = \{(1,1), (2,2), (2,3), (3,2), (3,3), (4,4), (4,5), (5,4), (5,5)\}$$

Defined on A find the partition of A induced by R.

Solution:

By examining the given R_1 we find that $[1] = \{1\}$, $[2] = \{2,3\}$, $[3] = \{2,3\}$, $[4] = \{4,5\}$, $[5] = \{4,5\}$ of these equivalence classes only $[1]$, $[2]$ and $[4]$ are distinct these constitute the partition P of A determined by R then

$P = \{[1],\ [2],\ [4]\}$ is the partition induced by R

$A = [1]\ U\ [2]\ U\ [4] = \{1\}U\{2,3\}U\{4,5\}$

☻ **Partial orders:**

A relation R on a set A is said to be a partial ordering relation or a partial order on A if (i) R is reflexive (ii) R is antisymmetric and (iii) R is transitive on A.

Poset:

A set with a partial order R defined on it is called a partially ordered set or Poset.

Example: less than or equal to. On set of integers.

Total Order:

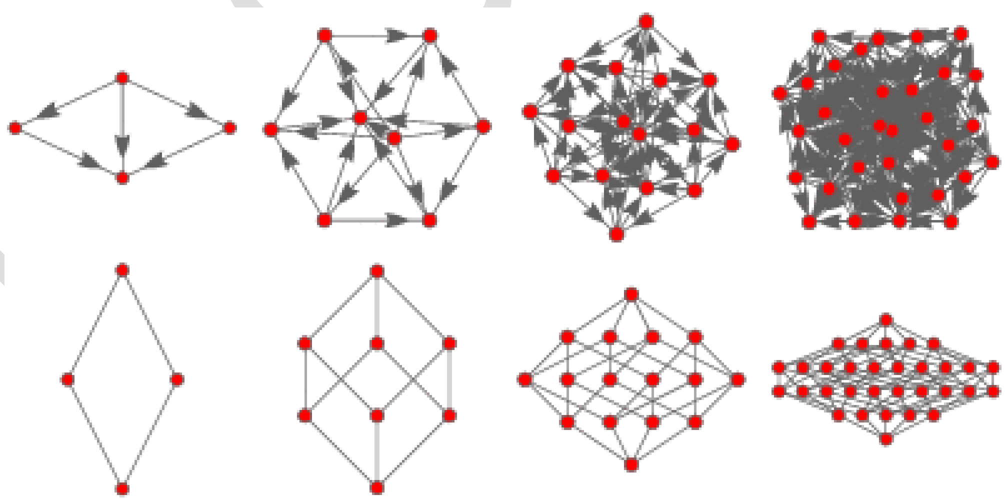
Let R be a partial order on a set A. Then R is called a total order on A. if for all $x, y \in A$ either xRy or yRx. In this case the poset (A, R) is called a totally ordered set.

Hasse Diagram:

A Hasse diagram is a graphical rendering of a partially ordered set displayed via the cover relation of the partially ordered set with an implied upward orientation. A point is drawn for each element of the poset, and line segments are drawn between these points according to the following two rules:

1. If $x < y$ in the poset, then the point corresponding to x appears lower in the drawing than the point corresponding to y.

2. The line segment between the points corresponding to any two elements x and y of the poset is included in the drawing iff x covers y or y covers x .



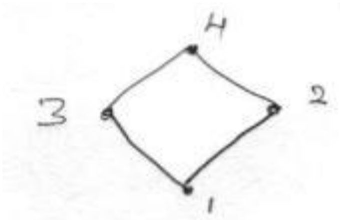
Problems:

1. Let $A = \{1,2,3,4\}$ and $R = \{(1,1), (1,2), (2,2), (2,4), (1,3), (3,3), (3,4), (1,4), (4,4)\}$. Verify that R is a partial order on A. also write down the Hasse diagram for R.

Solution:

We observe that the given relation R is reflexive and transitive. Further R does not contain ordered pairs of the form (a, b) and (b, a) with $b \neq a$. R is antisymmetric as such R is a partial order on A.

The Hasse diagram for R must exhibit the relationships between the elements of A as defined by R. if $(a, b) \in R$ there must be an upward edge from a to b.

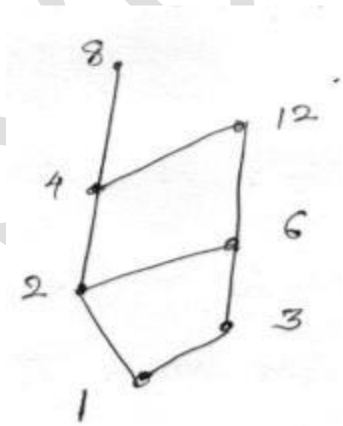


2. Let $A = \{1,2,3,4,6,8,12\}$ on A, define the partial ordering relation R by xRy if and only if x/y draw the Hasse diagram for R.

Solution:

$$R = \{(1,1),(1,2),(1,3),(1,4),(1,6),(1,8),(1,12),(2,2),(2,4),(2,6),(2,8),(2,12),$$
$$(3,3),(3,6),(3,12),(4,4),(4,8),(4,12),(6,6),(6,12),(8,8),(12,12)\}.$$

The Hasse diagram for this R is as shown below.



3. Draw the Hasse diagram representing the positive divisors of 36.

Solution:

The set of positive divisors of 36 is

$D_{36} = \{1,2,3,4,6,9,12,18,36\}$ The relation R of divisibility (that is aRb if and only if a divides b) is a partial order on this set. The Hasse diagram for this partial order is required here.

1 is related to all elements of D_{36}

2 is related to 2,4,6,12,18,36

3 is related to 3,6,9,12,18,36

4 is related to 4,12,36

6 is related to 6,12,18,36

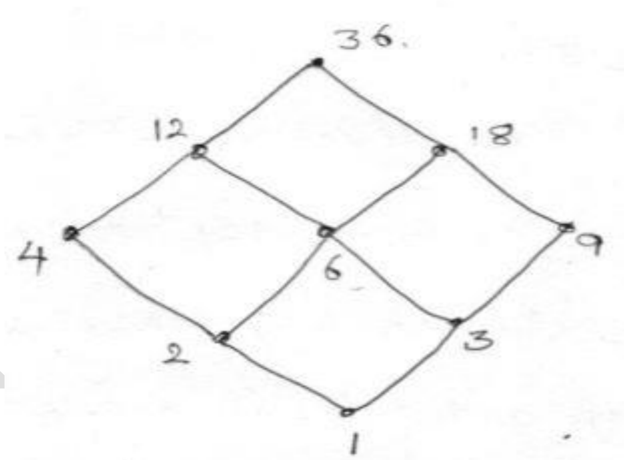
9 is related to 9,18,36

12 is related to 12 and 36

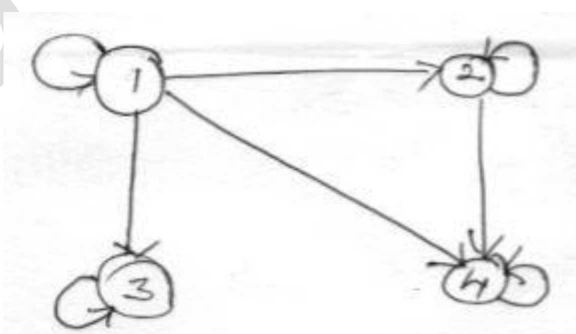
18 is related to 18 and 36

36 is related to 36.

The Hasse diagram for R must exhibit all of the above facts.

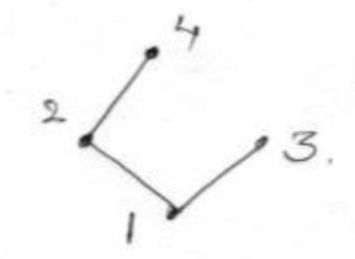


4. A partial order R on set $A = \{1,2,3,4\}$ is represented by the following diagraph. Draw the Hasse diagram for R.



Solution:
By observing the given diagraph, we note that

$$R = \{(1,1),(2,2),(3,3),(4,4),(1,2),(1,3),(1,4),(2,4)\}$$



☻ **External elements in Posets:**

Upper bond of a subset B of A: an element $a \in A$ is called an upper bound of a subset B of A if xRa for all $x \in B$.

Lower bound of a subset B of A: an element $a \in A$ is called lower bound of a subset B is A if aRx for all $x \in B$.

Supremum (LUB): An element $a \in A$ is called the LUB of a subset B of A if the following two conditions hold.

i) A is an upper bound of B.

ii) If a^I is an upper bound of B then aRa^I.

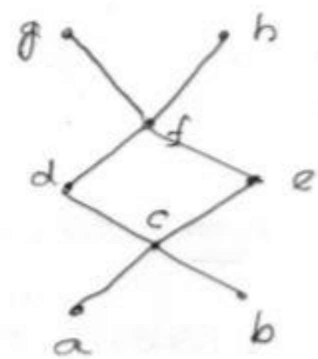
Infimum (GLB): An element $a \in A$ is called the GLB of a subset B of A if the following two conditions hold

i) A is a lower bound of B.

ii) If a^I is a lower bound of B then $a^I Ra$.

Problems:

1. Consider the Hasse diagram of a Poset (A, R) given below.



If $B = \{c, d, e\}$ find (if they exist).

i) All upper bounds of B

ii) All lower bounds of B

iii) The least upper bound of B

iv) The greatest lower bound of B

Solution:

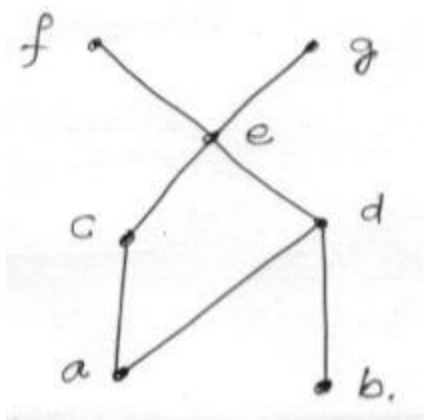
(i) All of c, d, e which are is B are related to f, g, h therefore f, g, h re upper bounds of B.

(ii) The elements a, b and c are related to all of c, d, e which are in B. therefore a, b and c are lower bounds of B.

(iii) The upper bound f of B is related to the other upper bounds g and h of B. Therefore, f is the LUB of B.

(iv) The lower bounds a and b of B are related to the lower bound c of B. therefore C is the GLB of B.

2. Consider the Poset whose Hasse diagram is shown below. Find LUB and GLB of $B = \{c, d, e\}$



By examining all upward paths from c, d, e is the given Hasse diagram. We find that LUB $(B) = e$. by examining all upward paths to c, d, e we find that $GLB(B) = a$.

☻ **Lattice:**

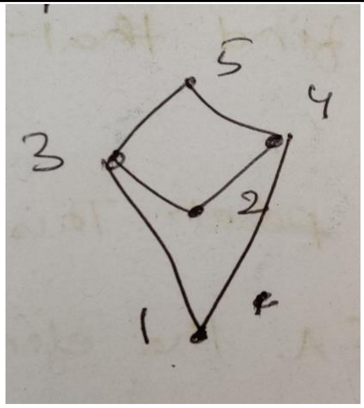
Let (A, R) be a Poset this Poset is called a lattice. For all $x, y \in A$ the elements $LUB\ \{x, y\}$ and $GLB\ \{x, y\}$ exist is A.

Example: Let (A, R) be Poset. The Poset is called a.

1). Consider the set N of all-natural numbers and let R be the partial order "less than or equal to" then for any $x, y \in N$, we note that $LUB\ \{x, y\} = Max\{x, y\}$ and $GLB\ \{x, y\} = min\{x, y\}$ and both of these belong to N. Therefore, the Poset $(N, \leq)$ is a lattice.

2). Consider the Poset $(Z^+, |)$ where Z^+ is set of all positive integer & | is the divisibility set. We can check that for any $a, b \in Z^+$, the least common multiple of a & b is the $LUB\ \{a, b\}$ & the GCD of a & b is $GLB\ \{a, b\}$. Since these belongs to Z^+ we infer that $(Z^+, |)$ is a lattice.

3). Consider the poset where Hasse Diagram is



By examining the Hasse diagram, we note that $GLB\ \{3, 4\}$ does not exist.

∴ The poset is not a Lattice