classmate

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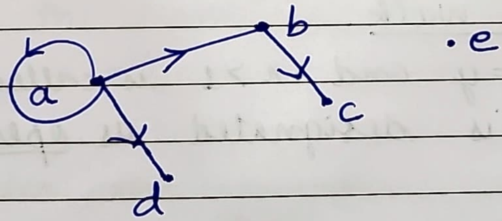
UNIT- 4 : Graphs

Graph

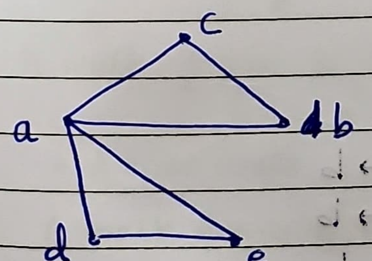
Let V be a finite ~~nom~~ non empty set and let ~~E~~ $E \subseteq V \times V$, then the pair (V, E) is called graph G where V is a set of vertices and
E is a set of edges
symbolically we write $G = (V, E)$

There are two different types of graphs : Directed and Undirected graph.
Directed graph will have the directions associated with the edge indicating the source and terminating vertex.



The given graph is symbolically represented as $G = (V, E)$ where
$V = \{a, b, c, d, e\}$
$E = \{(a,a), (a,b), (b,c), (a,d)\}$

In the above graph, (a,a) is called the self-loop and the vertex 'e' is called isolated vertex.



The given undirected graph is symbolically represented as
$G = (V, E)$ where
$V = \{a, b, c, d, e\}$
$E = \{\{a,b\}, \{a,c\}, \{b,c\}, \{a,e\}, \{a,d\}, \{d,e\}\}$

Here $\{a,b\}$ represents (a,b) and (b,a).

Terminologies used in graph

Defⁿ Walk in a graph

Let x and y be the vertices in an undirected graph $G = (V, E)$. ~~and x~~ An x-y walk in G is a finite alternating sequence of vertices and edges from starting at the vertex x and ending at the vertex y.

An x-y walk can be denoted as

$x = x_0\ e_1\ x_1\ e_2\ x_2 \ldots x_{n-1}\ e_n\ x_n = y$

The walk involves n edges where

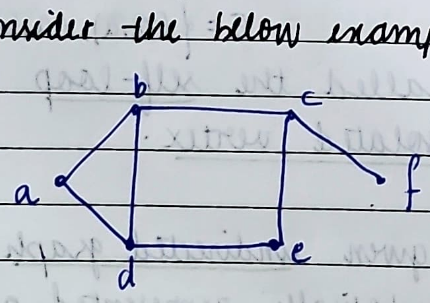
$e_i = \{x_{i-1}, x_i\},\ 1 \le i < n$

The length of the walk is the number of edges constituting the walk.

note: i) If the number of edges/ ~~path~~ length of walk is zero, then the walk is called trivial walk.

ii) Any x-y walk where $x = y$ and $n > 1$ is called closed walk. otherwise the walk is designated as open walk.

Eg: Consider the below example:



i) Determine a walk b/w a-b of length 6.

① a → b → c → e → d → a → b
② a → d → b → c → e → d → b
③ a → d → e → c → f → c → b

ii) Determine the walk from b-f of length = 2 and 5

Length 2: b → c → f

Length 5: b → a → d → e → c → f

iii) Determine the walk from f - a

f → c → b → a

iv) Determine the closed walk from b-b of length 3 and 5.

Length 3 : b → a → d → b

Length 5 : b → c → e → d → a → b

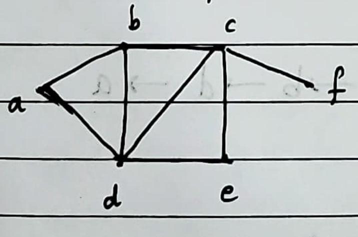
Defn: ~~Trial~~ Trail and Circuit

If no edge in the x-y walk is repeated, then the walk is called as x-y trail. A closed x-x trail is called as circuit.

Defn: Path and Cycle

If no vertex of the x-y walk occurs more than once, then the walk x-y is called as x-y path. When x=y, the term cycle is used to describe the closed path x-x.

Eg: Consider the example:



i) A walk from b to f

b → c → d → e → c → f

* A open walk * Trail - yes * Path - no [c is repeated]

ii) A walk from f to a

f → c → d → a

* open walk * Trail - yes * Path - yes

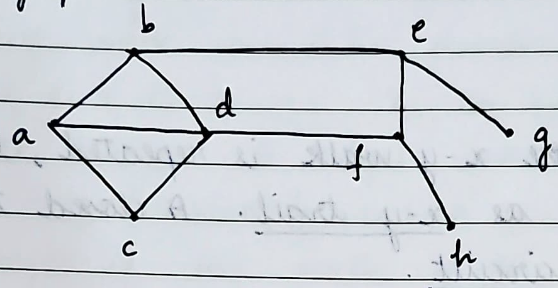
iii) A walk from a to a

a → b → d → c → e → d → a

* closed walk * circuit - yes * cycle - no [d is repeated]

iv) A walk from a to a
* closed walk * circuit - yes * cycle - yes

① Determine which of the following sequence are walk, closed walk, trail, circuit, path and cycle. Consider the below graph:



(i) a → b → e → f → d → a → c → d → b
Walk, Trail [not a path since d is repeated]

(ii) d → f → d
closed walk, cycle

(iii) a → b → e → f → d → c → a
closed walk, circuit, cycle

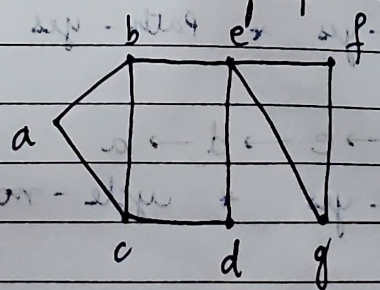
(iv) a → b → d → f → e → b → d → c
walk,

(v) a → c → d → f → e → b → d → a
Closed walk, Circuit

(vi) h
Trivial walk, Trail, path

(vii) b → e → f → g
None

② For the below graph determine



(i) A cycle from b to b

$b \to a \to c \to b$

$b \to e \to d \to c \to b$

$b \to e \to b$

(ii) A closed walk from b to b that is not a circuit

$b \to e \to g \to e \to b$ $\quad$ * $\{e, g\}$

(iii) A path from b to d

$b \to c \to d$ $\quad$ $b \to a \to c \to d$

$b \to e \to d$

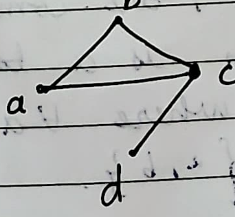
(iv) A walk from b to d that is not a trail

$b \to e \to b \to c \to d$

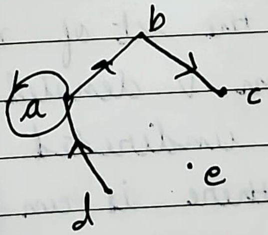
Properties of graph

[1] Connected graph

A graph $G = (V, E)$ we call G connected, if there is a path b/w any two distinct vertices of G.



(G1)



(G2)

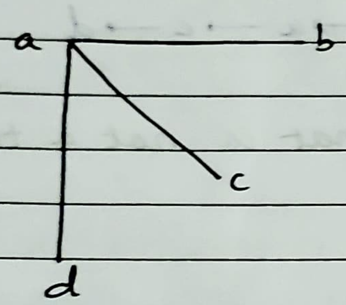
* G1 is connected
* G2 is not connected, path from a to d, a to e ... doesn't exist

Since e is isolated there is no path from e to other vertices.

2 Complement of a graph

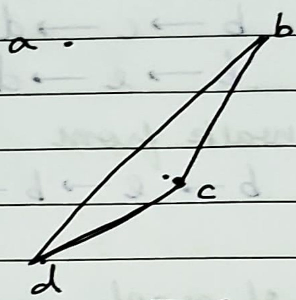
Let G be a loop free undirected graph in vertices, the complement of G denoted by $\overline{G}$ is the subgraph of K_n consisting of all the vertices in G and all edges that are not in G.

Eg:



G

K_4



$\overline{G}$ (Complement of G)

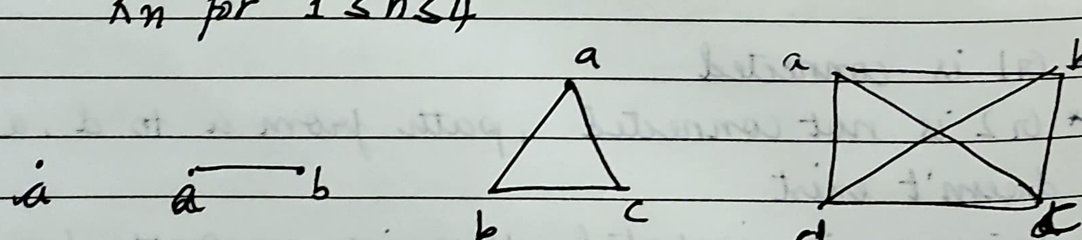
K_4

3 Complete graph

Let V be the set of n vertices, the complete graph on V denoted by K_n, is a ~~loplace~~ loop free undirected graph where $\forall a, b \in V$, $a \neq b$ there is an edge $\{a, b\}$.

Eg:

K_n for $1 \leq n \leq 4$



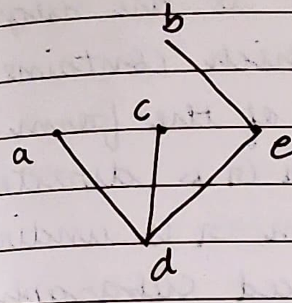
K_1 K_2 K_3 K_4

2/2024 <u>Sub graphs</u>

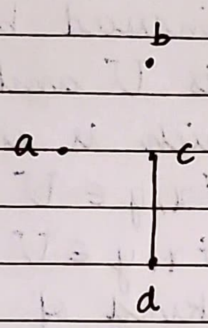
If $G = (V, E)$ is a graph (directed / undirected) then ~~G1 = V1, E1~~ $G1 = (V_1, E_1)$ is called sub graph of G if ~~$\phi \neq V_1 \neq$~~ $\phi \neq V_1 \subseteq V$ and $E_1 \subseteq E$ where each edge in E_1 is incident with the vertices in V_1.

Eg:

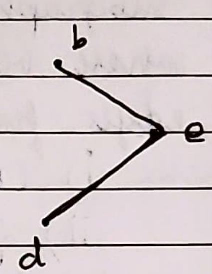
(1)



(G)



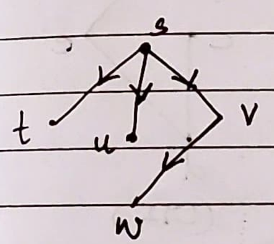
(G1)



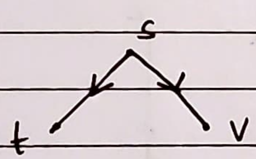
(G2)

In the above example, G1 & G2 are subgraphs of G.

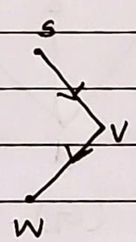
(2)



(G)



(G1)



(G2)

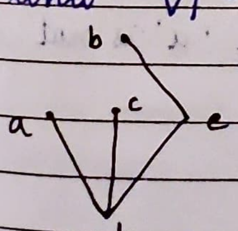
G1 & G2 are subgraphs of G.

<u>Special Types of Sub graphs</u>

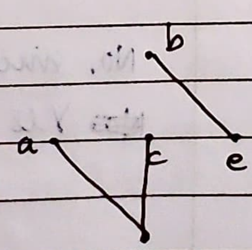
1 Spanning Subgraph

Given a graph $G = (V, E)$, G1 is called spanning subgraph of G iff $G_1 = (V_1, E_1)$ is a subgraph of G and $V_1 = V$.

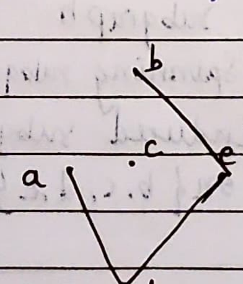
Eg:



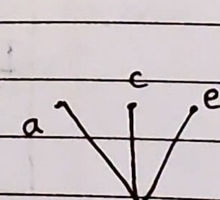
(G)



(G1)



(G2)



(G3)

G1 & G2 are spanning subgraphs of G whereas G3 is a subgraph but not a spanning subgraph the vertex b is not included.

[2] Induced subgraph

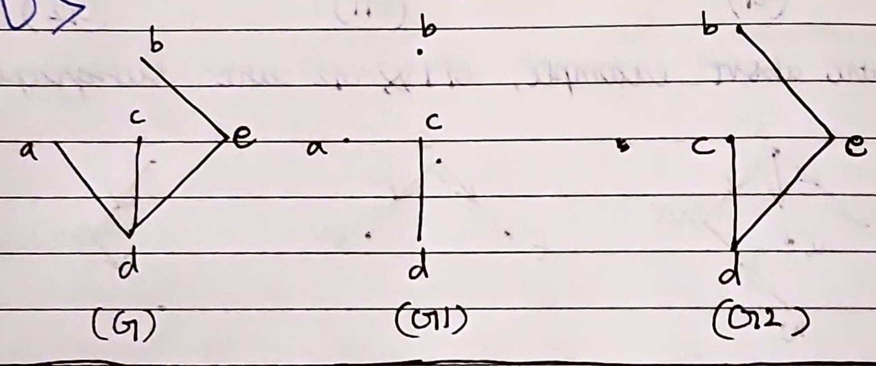
Let $G = (V, E)$ be a graph, if $\phi \neq U \subseteq V$, the subgraph of G induced by U is the subgraph whose vertex set is U and which contains all the edges from G which is either of the form

i) (x, y) $\forall x, y \in U$ [When G is directed]

ii) $\{x, y\}$ $\forall x, y \in U$ [When G is undirected]

We denote such kind of induced subgraph by $\langle U \rangle$

Eg:

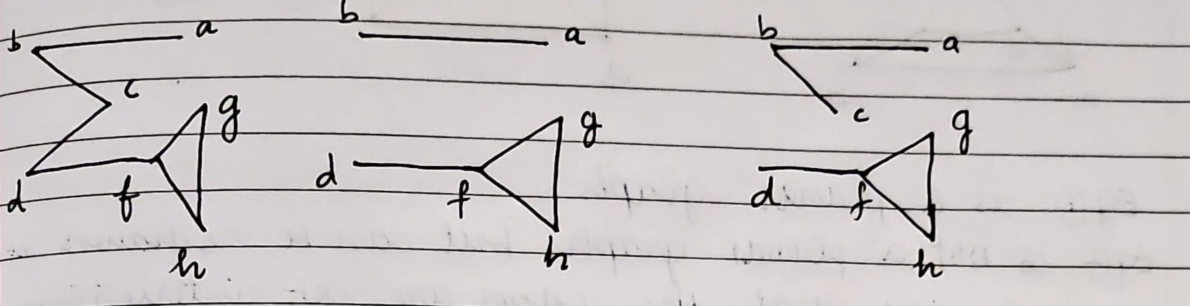


G1	Yes / No
Subgraph	Yes
spanning subgraph	No since vertex 'e' is not included
Induced subgraph on $\{a, b, c, d\}$	No since edge $\{a, d\}$ is not included

G2	Yes/No
Subgraph	Yes
Spanning subgraph	No, since vertex 'a' is not included
Induced subgraph on $\{b, c, d, e\}$	Yes

~~Oth~~ Other special types of subgraphs

They can be emerged when a particular vertex/edge is deleted from given graph.



G $\quad$ $G_1 = G - c$, where $V_1 = V - c$ $\quad$ $G_2 = G - \{e\}$ where $e = \{c, d\}$

note: Removal of vertex - induced subgraph
Removal of edge - spanning subgraph

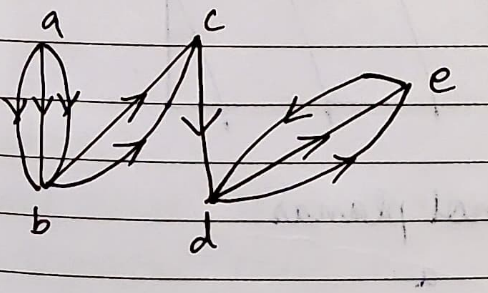
Multigraph

For a graph $G = (V, E)$ is referred to as a multigraph if there are two or more edges in E of the form

(i) (x, y) when G is directed

(ii) $\{x, y\}$ when G is undirected

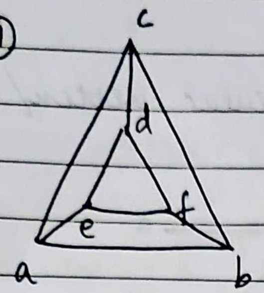
Eg:

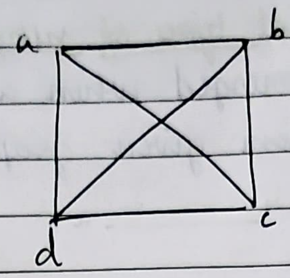


3 edges from a to b ∴ The edge (a,b) has multiplicity of 3 and that of $(b,c) = 2$, $(d,e) = 2$, $(e,d) = 1$

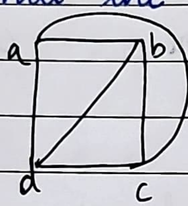
Planar graphs

graph $G = (V, E)$ is said to be planar iff G can be ~~grawn~~ drawn in the plane with its edges intersecting only at the vertices of G.

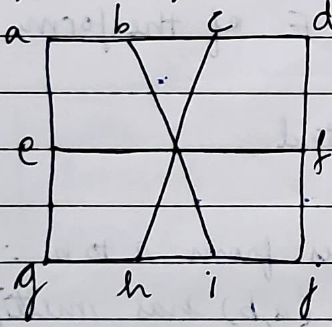
① 

② 

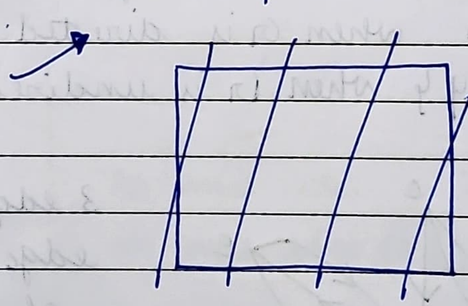
Eg① is a planar graph
Eg② is not a planar graph but can be redrawn in such a way that the edges are not intersecting at not a vertice.

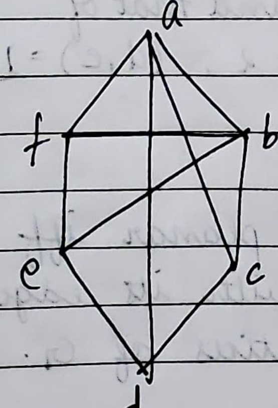


① Determine below which of the graphs are planar. If it is not a planar graph, redraw it such that no edges are intersecting in order to make it as a planar graph.

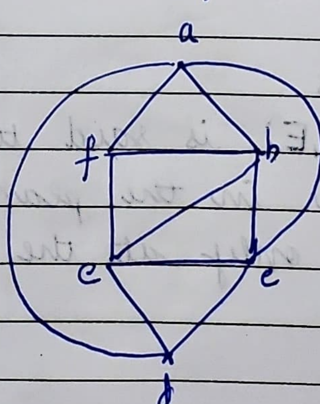
i) 

It is not planar.

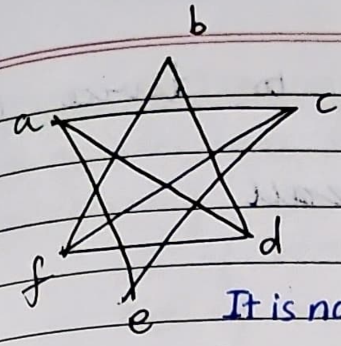


ii) 

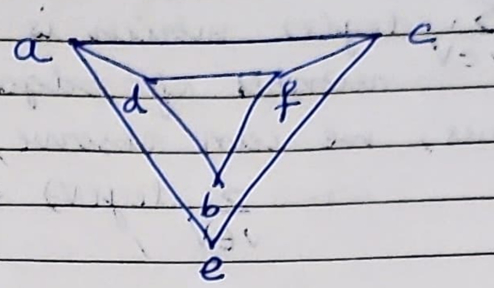
It is not planar



iii)



It is not planar



Graph Isomorphism

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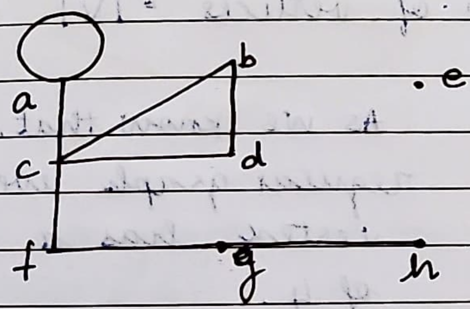
* **Degree of a Vertex:**

Let G be an undirected graph or multigraph, for each vertex v of G, the degree of v ~~is~~ written as deg (v) is the number of edges in G that are incident with v.

Here a loop at vertex V is considered as two incident edges for v.

Eg:



deg (a) = 3 (2+1)

deg (b) = 2

deg (c) = 4

deg (d) = 2

deg (e) = 0

deg (f) = 2

deg (g) = 2

deg (h) = 1

Theorem:

Prove that if G = (V, E) is an undirected graph or multigraph then

$$\sum_{v \in V} \deg(v) = 2|E|$$

Let us consider each edge {a, b} in graph G, we find that edge contributes a count of 1 to each deg (a) and deg (b).

And also consequently a ~~count~~ count of 2 to

$\sum_{v \in V} deg(v)$ which is equivalent to twice the number of edges i.e. $2|E|$.

Thus, we can observe that ~~for all~~

$$\sum_{v \in V} deg(v) = 2|E|$$

* **Regular graph:**

A graph G where each vertex has the same degree is called a regular graph.

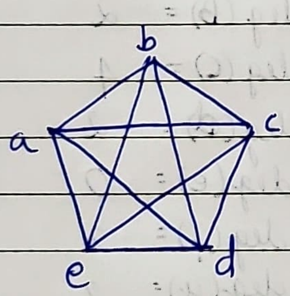
If all the vertices have $\forall v \ deg(v) = k$, then we call it has k-regular graph.

① Draw a 4-regular graph with 10 edges.

WKT $\sum_{v \in V} deg(v) = 2|E|$

$$4(x) = 2 \times 10$$

$\boxed{x = 5}$ → no. of vertices $= |V|$



As we know that, in 4-regular graph every vertex has a ~~degt~~ degree of 4.

② Draw a 4-regular graph with 15 edges.

According to theorem,

$$\sum_{v \in V} deg(v) = 2|E|$$

$$4|V| = 2 \times 15$$

$$|V| = 30/4 = 7.5.$$

∴ The number of vertices i.e. $|V|$ cannot be a fraction. Hence a 4-regular graph with 15 edges is not possible.

Graph ~~Ex~~ Isomorphism

Two graphs are said to be isomorphic that satisfies the following:

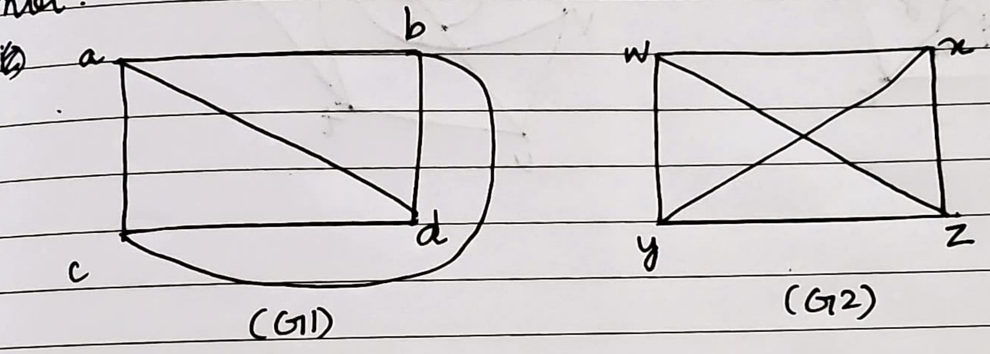
i) No. of vertices must be same.

ii) No. of ~~the~~ edges must be same.

iii) Degree sequence of vertices must be same.

iv) There must be one to one correspondance b/w the vertices and the adjacency of the edges ~~of~~ must also be preserved.

① Verify whether the below graphs are isomorphic or not.



i) Both the graph (G1) and (G2) have same number of vertices = 4

ii) Both the graph have same number of edges = 6.

iii) For graph (G1), degree sequence is {3, 3, 3, 3}.
For graph (G2), degree sequence is {3, 3, 3, 3}.

iv) One to one correspondance b/w the vertices of (G1) & (G2)

mapping — $f(a) = w$ $f(b) = x$ $f(c) = y$ $f(d) = z$

Adjacency of vertices should be verified i.e. check preserveness is maintained.

We consider the mapping and check if edges are present.

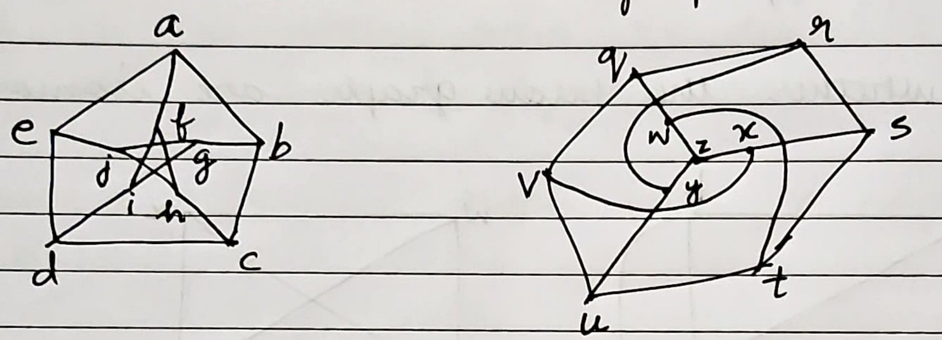
$\{a, b\} \leftrightarrow \{f(a), f(b)\} = \{w, x\}$

$\{a, c\} \leftrightarrow \{f(a), f(c)\} = \{w, y\}$

$\{a,d\} \leftrightarrow \{f(a), f(d)\} = \{w,z\}$
$\{b,d\} \leftrightarrow \{f(b), f(d)\} = \{x,z\}$
$\{c,d\} \leftrightarrow \{f(c), f(d)\} = \{y,z\}$
$\{b,c\} \leftrightarrow \{f(b), f(c)\} = \{x,y\}$

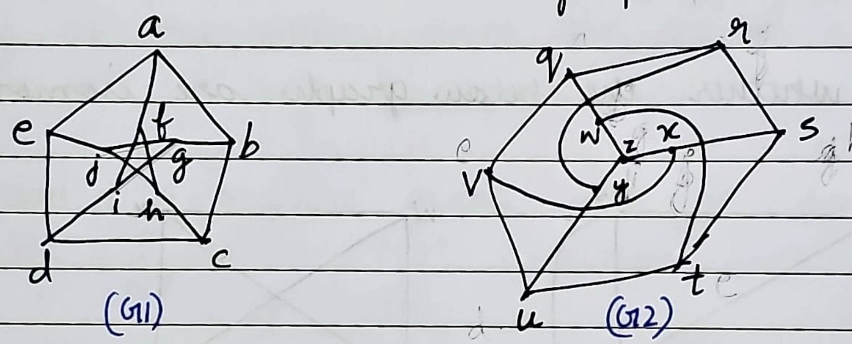
The edges preservancy is maintained $\forall \cdot E$.
Since all the 4 conditions are satisfied, the two graphs are isomorphic.

(2) Check whether the below graphs are isomorphic.



$\{a,d\} \leftrightarrow \{f(a), f(d)\} = \{w, z\}$
$\{b,d\} \leftrightarrow \{f(b), f(d)\} = \{x, z\}$
$\{c,d\} \leftrightarrow \{f(c), f(d)\} = \{y, z\}$
$\{b,c\} \leftrightarrow \{f(b), f(c)\} = \{x, y\}$
The edges preservancy is maintained $\forall$ E.
Since all the 4 conditions are satisfied, the two graphs are isomorphic.

② Check whether the below graphs are isomorphic.



i) Both (G1) & (G2) have same number of vertices = 10
ii) Both the graphs have same number of edges = 15
iii) For graph (G1), degree sequence is
$\{3, 3, 3, 3, 3, 3, 3, 3, 3, 3\}$
For graph (G2), degree sequence is
$\{3, 3, 3, 3, 3, 3, 3, 3, 3, 3\}$
iv) One to one correspondance b/w the vertices of (G1) & (G2)

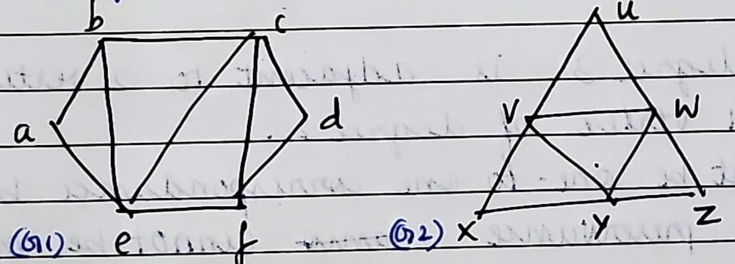
$f(a) = q$ $f(e) = w$ $f(i) = u$
$f(b) = r$ $f(f) = v$ $f(j) = z$
$f(c) = s$ $f(g) = y$
$f(d) = t$ $f(h) = x$

Adjacency of vertices should be verified i.e. check if preservency is maintained.
We consider mapping and check if edges are present
$\{a,b\} \leftrightarrow \{f(a), f(b)\} = \{q, r\}$

$\{b,c\} \leftrightarrow \{f(b), f(c)\} = \{r,s\}$
$\{c,d\} \leftrightarrow \{f(c), f(d)\} = \{s,t\}$
$\{d,e\} \leftrightarrow \{f(d), f(e)\} = \{t,w\}$
$\{a,e\} \leftrightarrow \{f(a), f(e)\} = \{q,w\}$
$\{a,f\} \leftrightarrow \{f(a), f(f)\} = \{q,v\}$
$\{b,g\} \leftrightarrow \{f(b), f(g)\} = \{r,y\}$
$\{c,h\} \leftrightarrow \{f(c), f(h)\} = \{s,x\}$
$\{d,i\} \leftrightarrow \{f(d), f(i)\} = \{t,u\}$
$\{e,j\} \leftrightarrow \{f(e), f(j)\} = \{w,z\}$
$\{f,h\} \leftrightarrow \{f(f), f(h)\} = \{v,x\}$
$\{f,i\} \leftrightarrow \{f(f), f(i)\} = \{v,u\}$
$\{g,j\} \leftrightarrow \{f(g), f(j)\} = \{y,z\}$
$\{g,i\} \leftrightarrow \{f(g), f(i)\} = \{y,u\}$
$\{h,j\} \leftrightarrow \{f(h), f(j)\} = \{x,z\}$

The edges preservancy is maintained $\forall E$.
Since all the 4 conditions are satisfied, the two graphs are <u>isomorphic</u>.

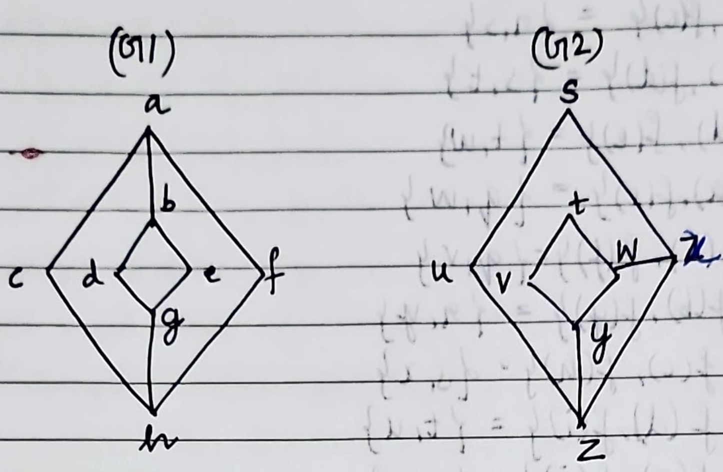
③



Same no. of vertices = 6
Same no. of edges = 9
But,

They are not isomorphic.
degree sequence of $G1 = \{2,3,4,2,4,3\}$
degree sequence of $G2 = \{2,4,4,2,4,2\}$
The degree sequence is not same. We can observe that in G1 there are two vertices of degree 4 whereas in G2 we have 3 vertices of degree 4.
Hence the two graphs are not isomorphic.

(4)



G1 and G2 have same no. of vertices = 8

G1 and G2 have same no. of edges = 10

Degree seq. of G1 = {3, 3, 2, 2, 2, 2, 3, 3}

Degree seq of G2 = {2, 2, 2, 2, 3, 3, 3, 3}

They have same ~~no. of~~ degree sequence.

In the above two graphs, we observe that in graph G1, the vertex with degree 3 is adjacent to 2 vertices of degree 2 and 1 ~~vertice~~ vertex of degree 3.

Whereas in G2,

the vertices of degree 3 is adjacent to 2 vertices of degree 3 and 1 vertex of degree 2.

Hence there cannot be one-to one correspondance b/w the vertices and edge preservance ~~canno~~ cannot be maintained. Hence the two graphs are not isomorphic.

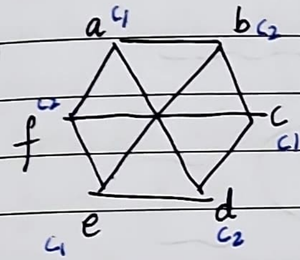
3/2/24 # Graph Coloring

Chromatic Number

A graph G is said to be k colourable if we ~~cam~~ can properly colour the graph with k colours.
The minimum number of colours required to properly colour a graph G is called as chromatic ~~graph~~ number of a graph denoted by $\chi(G)$.

① Find the chromatic number of the following:

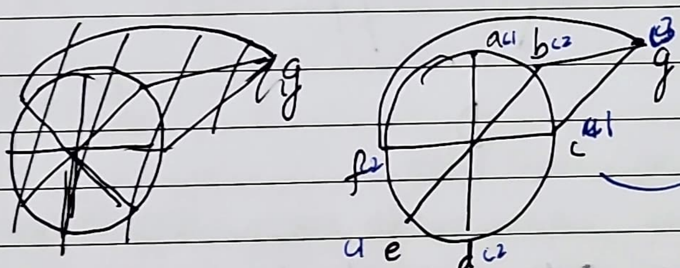
a)



a-1, b-2, c-1, d-2, e-1, f-2

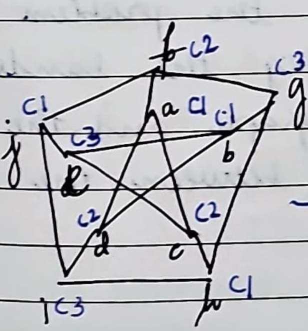
$\chi(G) = 2$

b)



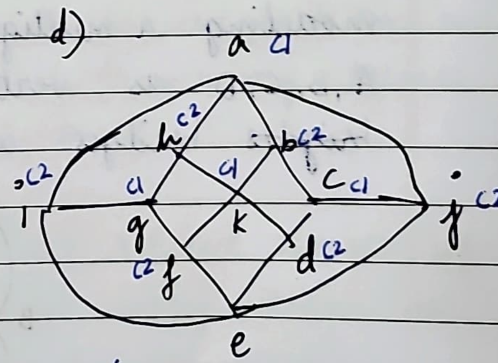
$\chi(G) = 3$

c)



$\chi(G) = 3$

d)

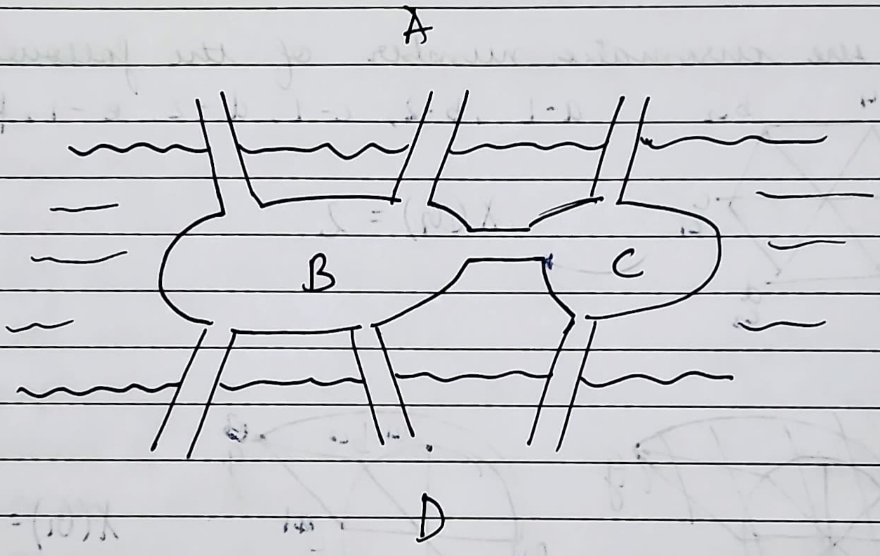


$\chi(G) = 2$

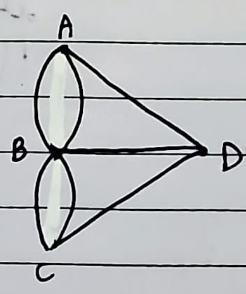
(3-4 marks)

Seven Bridges of Konigsberg Problem

The problem of seven bridges of Konigsberg is problem of traversing all the seven bridges across the River Pregal which is having two islands such a way that traversing from the four lands A, B, C, D without traversing any bridge twice and returning back to the same place from where the traversal began.



The Mathematician Euler formulated the problem by modeling a multigraph considering the lands A, B, C, D as vertices of the graph and the seven ~~brigdes~~ bridges as the edges between the lands.



classmate
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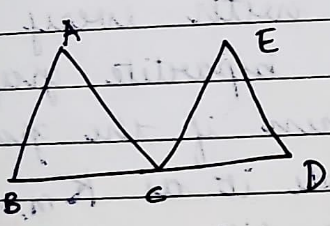
Page

9/2/24 # Euler circuit, euler trail

Euler circuit:
Let $G = (V, E)$ be an undirected graph or multigraph with no isolated vertices. Then G is said to have a euler circuit iff there is a circuit in G that traverses every edge of the graph exactly once.

Euler trail:
For $G = (V, E)$, the graph is said to have euler trail if there is a open trail from the vertex a to b in G such that it traverses each edge in G exactly once.

eg: Consider the given graph. Find the euler's circuit from A-A, and euler trail from A-D.



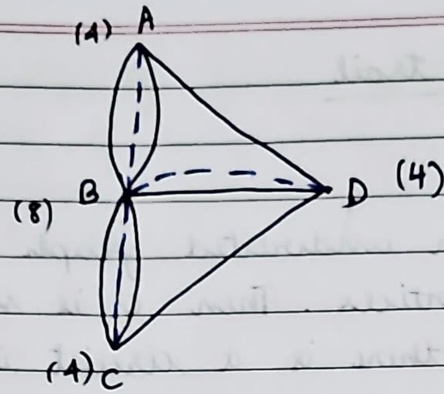
Euler circuit for A-A

$A \to B \to C \to D \to E \to C \to A$

Euler trail for A-D :

~~Abo~~ Observing the above graph, we ~~cannot~~ can note that there is a trail from A-D consisting of path $A \to B \to C \to E \to D$ but the edges $\{C, D\}$ and $\{A, C\}$ were not visited while traversing. Hence there is no euler trail from A-D.

note: Euler observed that the euler's circuit exists iff the graph G is connected and every vertex in the graph has even degree.
In the given seven bridge problem, we can observe that $deg(A) = \cancel{4}\ 3$, $deg(B) = \cancel{3}\ 5$, $deg(C) = \cancel{4}\ 3$, $deg(D) = 3$
None of the vertices has even degree. Hence we can conclude that there is no euler circuit



Euler circuit is possible in the ~~modied~~ modified graph.

~~Biparting~~

Bipartite Graph

A graph $G = (V, E)$ is called bipartite graph if $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \phi$ and every edge of G ~~if~~ is of the form $\{a, b\}$ with $a \in V_1$, $b \in V_2$

Complete Bipartite graph:

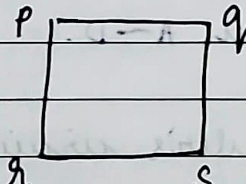
If every vertex of V_1 is joined with every vertex of V_2 then we have complete bipartite graph.

If $|V_1| = m$ and $|V_2| = n$, then if the graph is a bipartite graph, we denote it as $K_{m,n}$

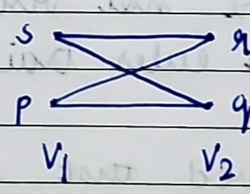
~~The~~ If the graph is complete bipartite graph then $K_{m,n}$ will have $m \times n$ edges.

① Verify whether the following graph is a bipartite graph

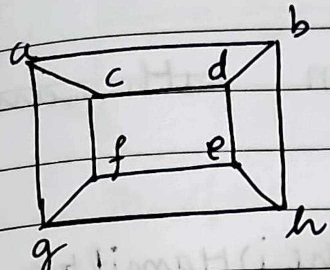
a)



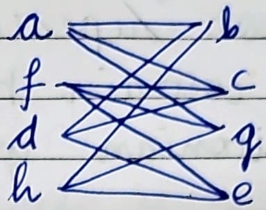
let



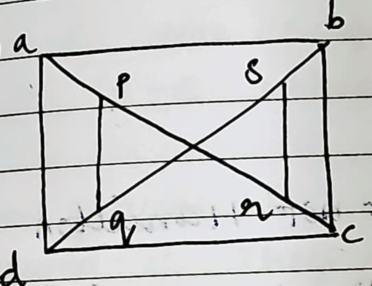
We can observe that the given graph is bipartite graph ~~and~~ denoted by $K_{2,2}$ where $V_1 = \{p, s\}$ & $V_2 = \{q, r\}$. We also observe that it is complete bipartite graph.

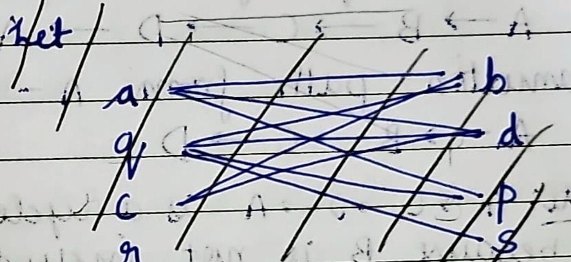
b) 

Let



We can observe that given graph is a bipartite graph denoted by $K_{4,4}$ where $V_1 = \{a, d, h, f\}$ $V_2 = \{b, c, e, g\}$
It is not a complete bipartite graph because a is not joined with e. Similarly f not with b, d not with g, h not with c.

c) 



~~In th~~ The above graph is not bipartite graph since we can split the vertices into two ~~disp~~ disjoint sets that are not related to itself.
For any division of vertices, two vertices of having edge will be in same set.

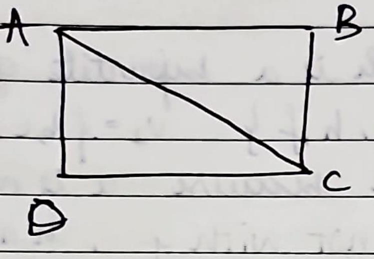
2/24 Hamilton~~ara~~ Path & cycle

Let $G = (V, E)$ is a connected graph then if there is a cycle is G then such a cycle is called Hamilton cycle. If there exists a Hamilton cycle in a graph, the cycle consists of n vertices and n edges.

A path in a connected graph which contains every vertex of the graph is called Hamilton path.

note: i) Both Hamilton cycle and path ensures that the vertices are not repeated.
ii) Graphs that contain Hamilton ~~path and~~ cycle is called Hamitonian graphs.

eg: ① Find the i) Hamilton cycle from A-A.
i) Hamilton path from A-D



Hamilton cycle from A-A

$A \rightarrow B \rightarrow C \rightarrow D \rightarrow A$

Hamilton path from A - D

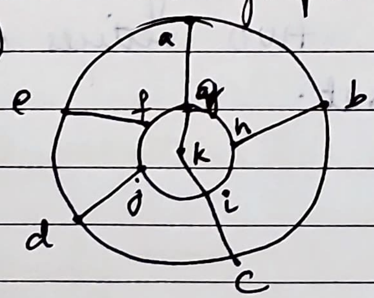
$A \rightarrow B \rightarrow C \rightarrow D$

note: $A \rightarrow C \rightarrow D \rightarrow A$ is a cycle but not Hamilton cycle because B is not included.

$A \rightarrow C \rightarrow D$ is not a Hamilton path since B not included

② Consider the below graph and show the graph is Hamiltonian graph.

ⓐ



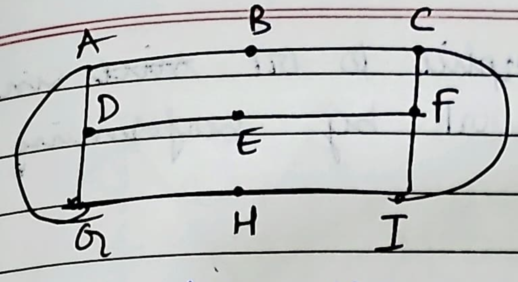
Hamilton cycle from a-a

~~$a \rightarrow g \rightarrow k \rightarrow i \rightarrow j \rightarrow f \rightarrow g \rightarrow h \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow a$~~

~~$a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow e$~~

$a \rightarrow b \rightarrow h \rightarrow g \rightarrow k \rightarrow i \rightarrow \cancel{j}\, c \rightarrow d \rightarrow j \rightarrow f \rightarrow e \rightarrow a$.

We can observe there is a Hamilton cycle for a-a, it is a Hamiltonian ~~cycle~~ graph.

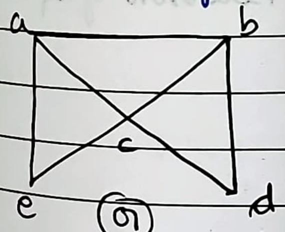
(b) 

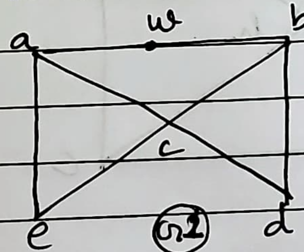
In the given graph, we can observe that there exists Hamilton path but there is no Hamilton cycle. Therefore the graph is not ~~an~~ a Hamiltonian graph.

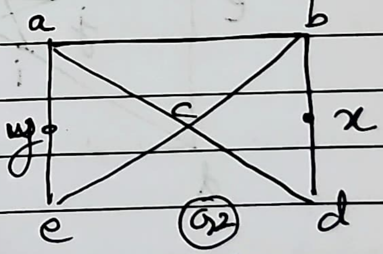
Homeomorphic graphs

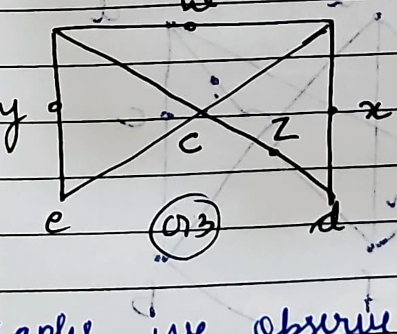
Let $G = (V, E)$ be a loop free undirected graph where $|E| \neq \Phi$ and elementary subdivision of G results when an edge $e = \{u, w\}$ is removed from G and the edges $\{u, v\}$ and $\{v, w\}$ are added to $G - e$ where $v \notin V$.

Two graphs $G1$ and $G2$ are called Homeomorphic graphs if they can obtained from the same loop free undirected graph G by a sequence of elementary sub divisions.









In the above graphs, we observe that

i) G1 and G2 are homeomorphic to G. [G1 - one operation, G2 - two operation]

ii) Also G3 is homeomorphic to G, G1, G2.

iii) Is G2 homeomorphic to G1?

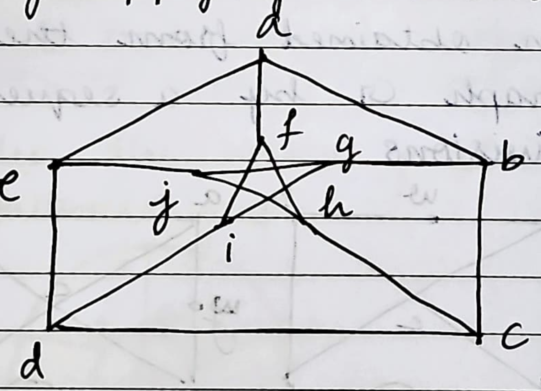
G_2 is not homeomorphic to G_1 ~~still~~ since we can't obtain G_2 from G_1 by performing elementary sub divisions.

Kurotowski's Theorem

A graph is non-planar iff it contains a subgraph that is homeomorphic to either K_5 (regular) or $K_{3,3}$ (bipartite).

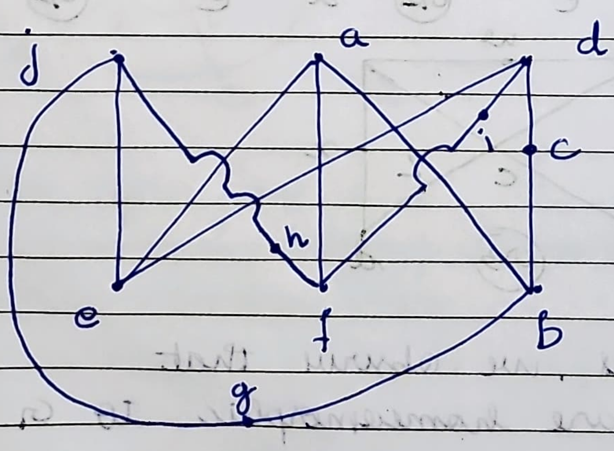
Check whether the following is planar ~~of~~ or non planar by finding a subgraph that is homeomorphic to either K_5 or $K_{3,3}$.

(OR) Show that the Petersen graph is a non planar graph by applying Kurotowskis theorem.

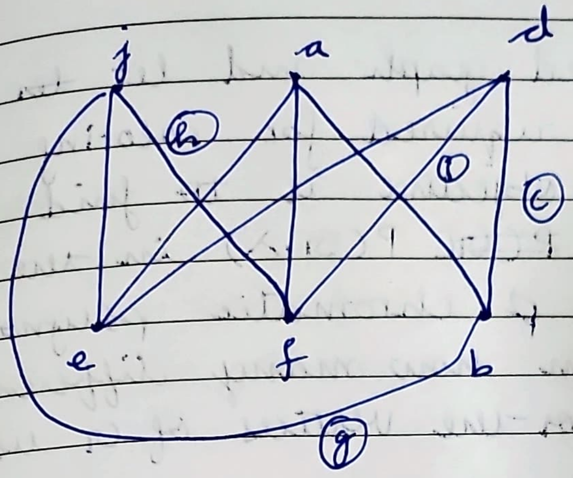


→ Petersen graph.

⇒



this subgraph should have all vertices but edges are missing $\{c,b\}$ & $\{i,g\}$

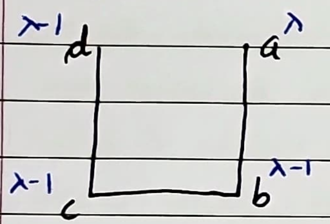


homeomorphic to $K_{3,3}$

Chromatic polynomial.

Let G be an undirected graph and let ~~ter~~ λ be the no. of colours required for coloring the vertices of G. The objective is to find a polynomial func^n ~~P(G)~~ $P(G,\lambda)$ in the variable λ which is called ~~a~~ chromatic polynomial of G which tells us in how many different ways we can properly color the vertices of G using atleast λ colors.

eg: Consider the example,



Here G represents the graph and λ indicates the colors. We can have

$$P(G,\lambda) = \lambda(\lambda-1)^3 \quad \text{[By rule of product]}$$

We can note that in the above example, $P(G,\lambda) = 0$ for $\lambda = 1$. This indicates that there are no ways of coloring the graph using one color. Similarly we can observe that $P(G, 2) = 2$ which indicates that there are two ways of coloring the graph when $\lambda = 2$. Through this, we can infer that the chromatic no. of graph $\chi = 2$.

Possible ways to color graph when $(\lambda = 4) = 4(4-1)^3 = \underline{108}$

note: If $G = (V,E)$ with ~~cardin~~ $|V| = n$ and $E = \phi$ then $P(G,\lambda) = \lambda^n$.

note: If $G = K_n$ then at least n colors are needed denoted as $P(G,\lambda) = \lambda^{(n)}$ where $P(G,\lambda) = \lambda(\lambda-1)\ldots(\lambda-n+1)$
(complete graph)

continued after 2 pages...

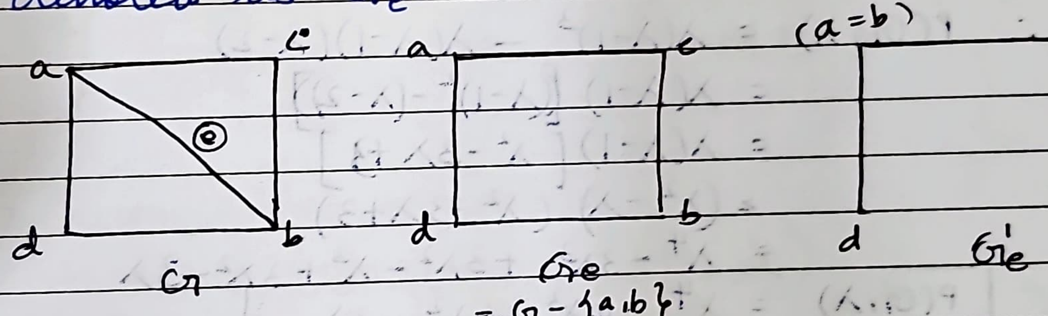
Decomposition Theorem for Chromatic Polynomial

If $G=(V,E)$ is a connected graph and ~~EtG~~ $e \in E$ then

$$P(G_e,\lambda) = P(G,\lambda) - P(G'_e,\lambda)$$

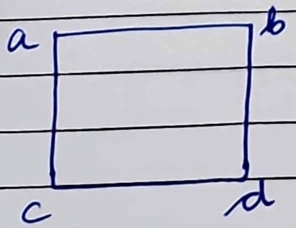
If $G=(V,E)$ is an undirected graph & $e=\{a,b\}$, then G_e denotes subgraph of G obtained by deleting edge e from G w/o removing the vertices a and b.
From G, we get a second subgraph of G by ~~sole~~ coalescing (merging) the vertices a and b denoted as G'_e.

eg:



The merging of two vertices a and b leads to merging of two pairs of edges i.e. $\{a,d\}$, ~~and~~ $\{d,b\}$ and $\{b,c\}$, $\{c,b\}$

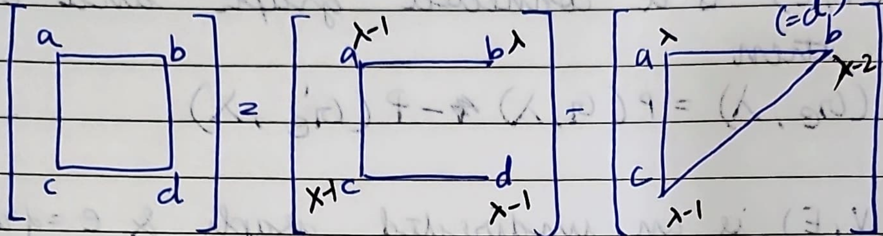
eg: ~~For the graph below~~ using decomposition theorem, find the chromatic polynomial and chromatic number for a graph that contains a cycle of length 4.



Consider the graph as G and $e=\{b,d\}$

By decomposition theorem we have

$$P(G,\lambda) = P(G_e,\lambda) - P(G'_e,\lambda)$$



$P(G,\lambda)$ $\quad$ $P(G_e,\lambda)$ $\quad$ $P(G'_e,\lambda)$

From the above, we can observe that

$$P(G_e,\lambda) = \lambda(\lambda-1)^3 \quad \cancel{\lambda(\lambda-1)(\lambda-2)}$$

$$P(G'_e,\lambda) = \lambda(\lambda-1)(\lambda-2)$$

$$\therefore\ P(G,\lambda) = \lambda(\lambda-1)^3 - \lambda(\lambda-1)(\lambda-2)$$

$$= \lambda(\lambda-1)\left[(\lambda-1)^2 - (\lambda-2)\right]$$

$$= \lambda(\lambda-1)\left[\lambda^2 - 3\lambda + 3\right]$$

$$= (\lambda^2-\lambda)(\lambda^2-3\lambda+3)$$

$$= \lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda^3 + 3\lambda^2 - 3\lambda$$

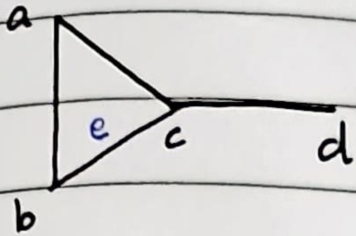
$$\boxed{P(G,\lambda) = \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda}$$

To find chromatic number, we ~~sub~~ substitute $\lambda = 1$ in $P(G,\lambda)$. Hence we get $P(G,1) = 0$ indicating no ways of coloring using 1 color.
Similarly $P(G,2) = 2 > 0$ ~~incal~~ indicating that it is possible to color the graph using 2 colors in 2 ways.
Hence $\chi(G) = 2$.

Find chromatic polynomial and number for:

i) A graph with cycle of length 5.

ii)



[don't take {c,d} since isolated vertex]

iii)