Geometry of Linear equations

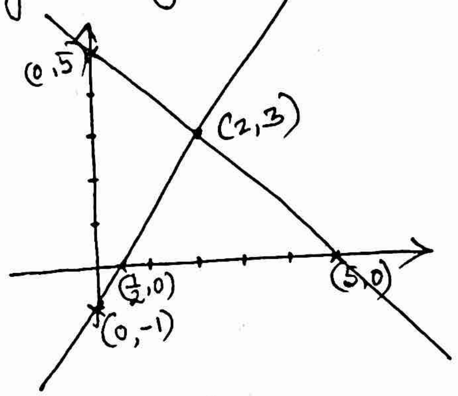
Consider the system of two equations in two unknowns

$2x - y = 1$; $x + y = 5$.

We can look at this system by rows or by columns.

1. The first approach concentrates on the separate equations (the rows).

The equations can be represented by straight lines in the x-y plane. The point of intersection $x=2$, $y=3$, gives the solution to the system.

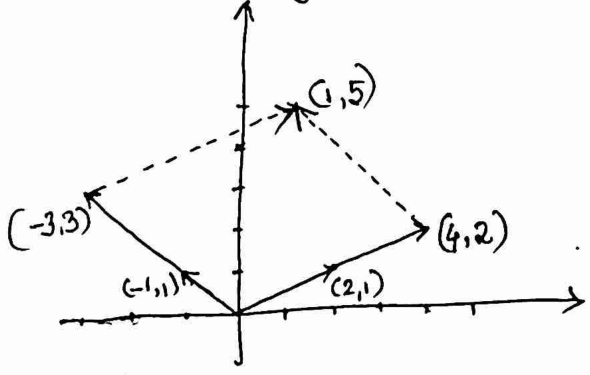


Row picture

2. The second approach looks at the columns of the linear system. Column form: $x\begin{bmatrix}2\\1\end{bmatrix} + y\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}1\\5\end{bmatrix}$

The problem is to find the combination of the column vectors on the left side that produces the vector on the right side.

2 times column 1 $\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}$ added to 3 times column 2 $\left\{\begin{bmatrix}-1\\1\end{bmatrix}\right\}$ algebraically produces the vector on the right side $\left\{\begin{bmatrix}1\\5\end{bmatrix}\right\}$. Geometrically it produces a parallelogram.

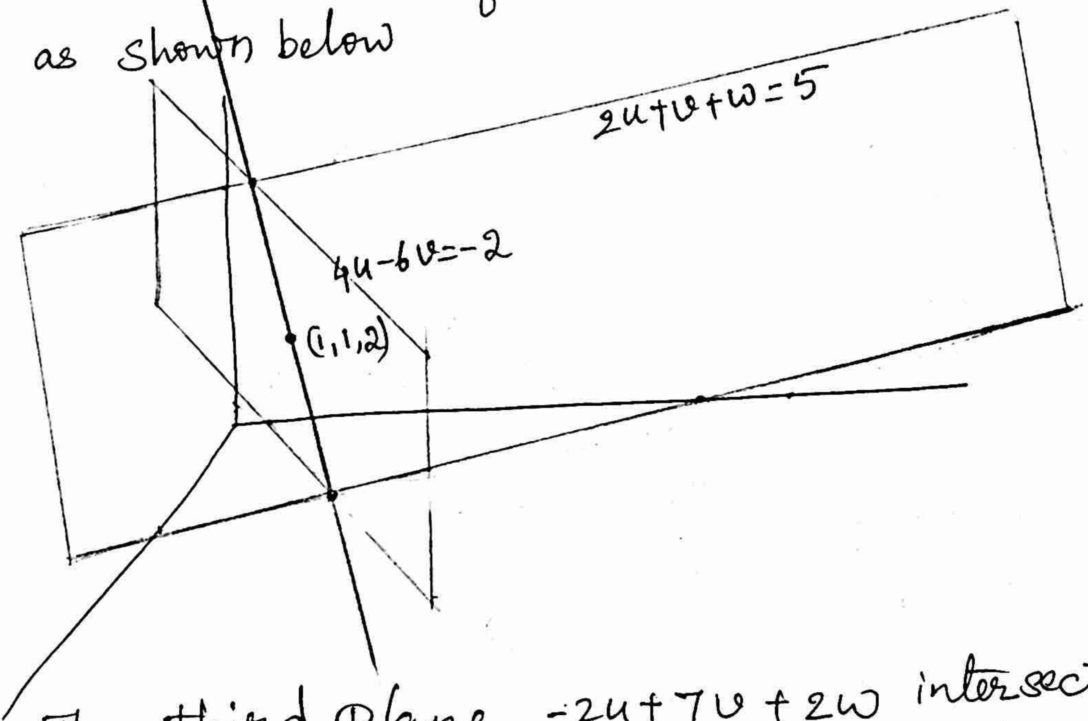


Column picture

Consider the system of three equations in three unknowns

$$2u + v + w = 5$$
$$4u - 6v \quad = -2$$
$$-2u + 7v + 2w = 9$$

Each equation describes a plane in three dimensions.
The first plane $2u+v+w=5$ contains the points $(\frac{5}{2},0,0)$, $(0,5,0)$, $(0,0,5)$
The second plane $4u-6v=-2$ is the vertical plane, as w can take any value, contains the points $(-\frac{1}{2},0,w)$, $(0,\frac{1}{3},w)$ OR $(-\frac{1}{2},0,0)$, $(0,\frac{1}{3},0)$
The intersection of these two planes will be a line, as shown below

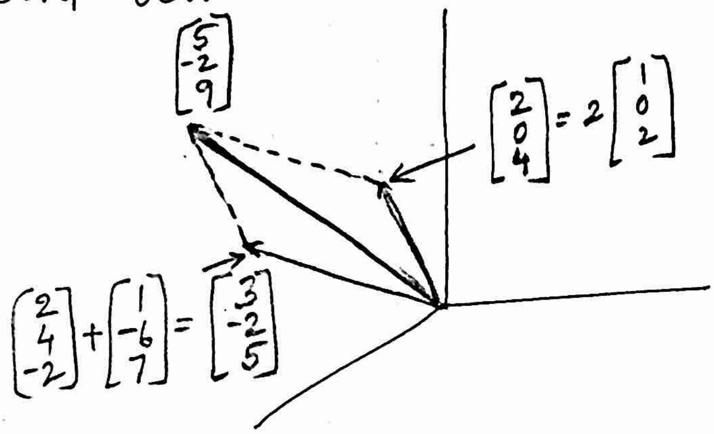


The third plane $-2u+7v+2w$ intersects this line at a point at $u=1$, $v=1$, $w=2$, which is the solution to the system.

The system in column vectors can be written as

$$u\begin{bmatrix}2\\4\\-2\end{bmatrix} + v\begin{bmatrix}1\\-6\\7\end{bmatrix} + w\begin{bmatrix}1\\0\\2\end{bmatrix} = \begin{bmatrix}5\\-2\\9\end{bmatrix}$$

The linear combination $1\begin{bmatrix}2\\4\\-2\end{bmatrix} + 1\begin{bmatrix}1\\-6\\7\end{bmatrix} + 2\begin{bmatrix}1\\0\\2\end{bmatrix} = \begin{bmatrix}5\\-2\\9\end{bmatrix}$ is graphically represent below.



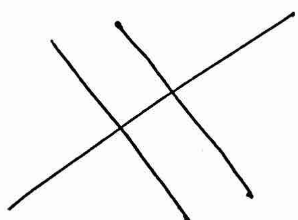
The Singular Case

In two dimensions ~~for the singular case~~

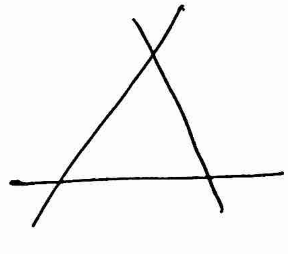
~~(a) lines do not intersect - no solution~~

(b) parallel lines is the only possibility for breakdown.

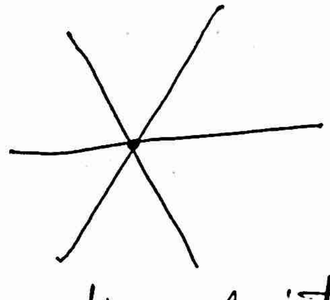
In three dimension



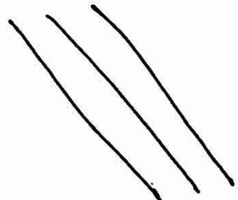
two parallel planes (no solution)



no intersection (no solution)



line of intersection

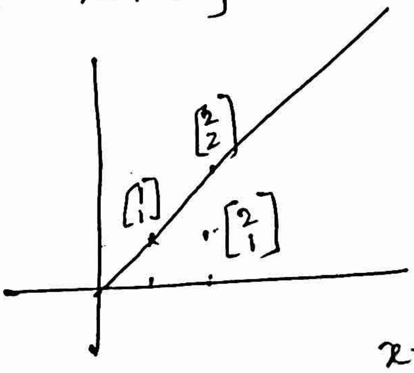


all planes parallel (no. solution)

In two dimension

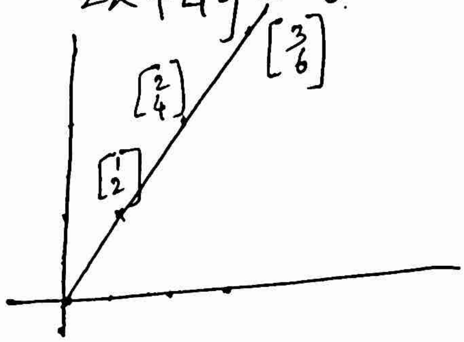
No solution

$x + 2y = 2$
$x + 2y = 1$



Infinite solution

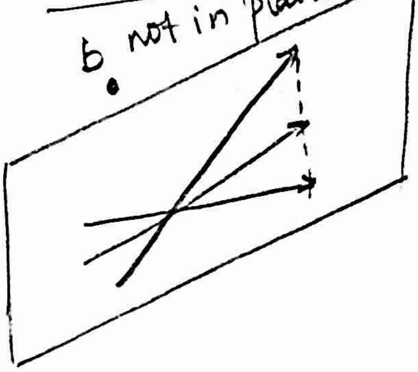
$x + 2y = 3$
$2x + 4y = 6$



$$\left.\begin{aligned} x + 2y &= 2 \\ x - y &= -4 \end{aligned}\right\} \text{unique solution.}$$

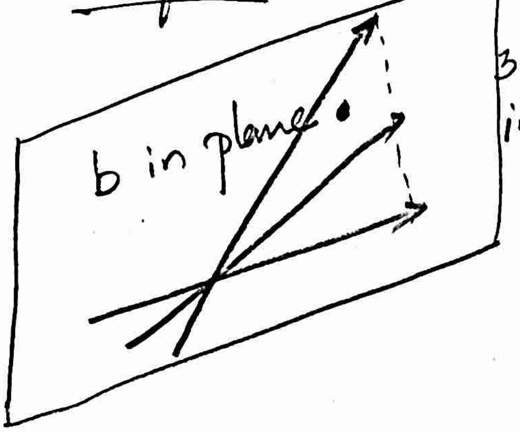
In three dimension

No solution



3 columns in a plane.

Infinite solution



3 columns in a plane

$$\left.\begin{aligned} u + v + w &= 2 \\ 2u \quad + 3w &= 5 \\ 3u + v + 4w &= 6 \end{aligned}\right\} \text{is not possible}$$

$$u\begin{bmatrix}1\\2\\3\end{bmatrix} + v\begin{bmatrix}1\\0\\1\end{bmatrix} + w\begin{bmatrix}1\\3\\4\end{bmatrix} = \begin{bmatrix}2\\5\\6\end{bmatrix}$$

$$u\begin{bmatrix}1\\2\\3\end{bmatrix} + v\begin{bmatrix}1\\0\\1\end{bmatrix} + w\begin{bmatrix}1\\3\\4\end{bmatrix} = \begin{bmatrix}2\\5\\7\end{bmatrix} \text{ possible.}$$

Vector Spaces

Let F be a field, V be a non empty set.

For every ordered pair $\alpha, \beta \in V$, let there be defined uniquely a sum $\alpha+\beta$ and for every $\alpha \in V$, and $c \in F$ a scalar product $c.\alpha$ in V.

The set V is called a vector space over the field F, if the following axioms are satisfied, for every $\alpha, \beta, \gamma \in V$ and for every $c, c' \in F$.

(i) $\alpha + \beta \in V$. Closed under addition.

(ii) $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$. Associative w.r.t addition.

(iii) Identity element w.r.t addition exists.
ie, $\exists\ e \in V$ s.t $\alpha+e = e+\alpha = \alpha$

(iv) Inverse element w.r.t addition exists.
ie, $\exists\ \alpha^{-1} \in V$ s.t $\alpha+\alpha^{-1} = e = \alpha^{-1}+\alpha$

(v) $\alpha+\beta = \beta+\alpha$. Commutative w.r.t addition.

(vi) $c.(\alpha+\beta) = c.\alpha + c.\beta$

(vii) $(c+c').\alpha = c.\alpha + c'.\alpha$

(viii) $(c.c').\alpha = c.(c'.\alpha)$

(ix) $1.\alpha = \alpha$, $\forall\ \alpha \in V$, where 1 is the unit element of F.

Examples:

Let F be a field and n be a positive integer.

Let $V_n(F)$ be the set of all ordered n tupples of the elements of the field F.

ie, $V_n(F) = \{(x_1, x_2, \ldots, x_n) / x_i \in F\}$

Define addition and scalar multiplication as below:

(a) $\alpha + \beta = (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n)$

$= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$

(b) $c.\alpha = c.(x_1, x_2, \ldots, x_n)$

$= (c.x_1, c.x_2, \ldots, c.x_n)$, $\forall\, c \in F$.

(i) $\alpha + \beta = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \in V_n(F)$

(ii) $(\alpha + \beta) + \gamma = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) + (z_1, z_2 \ldots, z_n)$

$= ((x_1 + y_1) + z_1, (x_2 + y_2) + z_2, \ldots, (x_n + y_n) + z_n)$

$= (x_1 + (y_1 + z_1), x_2 + (y_2 + z_2), \ldots, x_n + (y_n + z_n))$

$= (x_1, x_2, \ldots, x_n) + (y_1 + z_1, y_2 + z_2, \ldots y_n + z_n)$

$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

(iii) $\overbrace{(x_1, x_2, \ldots, x_n) + (0, 0, 0 \ldots, 0)}^{\alpha + 0} = (x_1, x_2 \ldots, x_n)$

$= \underbrace{(0, 0 \ldots, 0) + (x_1, x_2 \ldots, x_n)}_{0 + \alpha}$

$\therefore (0, 0, \ldots, 0)^{=0}$ is the additive identity.

(iv) $\alpha + (-\alpha) =$
$= (x_1, x_2 \dots x_n) + (-x_1, -x_2, \dots -x_n)$

$= (0, 0, \dots, 0)$

$= 0$

$= (-x_1, -x_2 \dots -x_n) + (x_1, x_2, \dots x_n)$

$= -\alpha + \alpha$

$\therefore -\alpha = (-x_1, -x_2 \dots -x_n)$ is the additive inverse of $\alpha = (x_1, x_2, \dots, x_n)$

(v) $\alpha + \beta = (x_1, x_2 \dots, x_n) + (y_1, y_2 \dots y_n)$

$= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$

$= (y_1 + x_1, y_2 + x_2, \dots, y_n + x_n)$

$= (y_1, y_2, \dots, y_n) + (x_1, x_2, \dots, x_n)$

$= \beta + \alpha$

(vi) $c.(\alpha + \beta) = c.(x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$

$= (c.x_1 + c.y_1, c.x_2 + c.y_2, \dots, c.x_n + c.y_n)$

$= (cx_1, cx_2, \dots, cx_n) + (cy_1, cy_2 \dots, cy_n)$

$= c.(x_1, x_2, \dots x_n) + c.(y_1, y_2 \dots y_n)$

$= c.\alpha + c.\beta.$

(vii) $(c + c')\alpha = (c + c')(x_1, x_2 \dots, x_n)$

$= ((c + c')x_1, (c + c')x_2 \dots, (c + c')x_n)$

$= (cx_1 + c'x_1, cx_2 + c'x_2 \dots, cx_n + c'x_n)$

$= (cx_1, cx_2 \dots, cx_n) + (c'x_1, c'x_2 \dots, c'x_n)$

$= c(x_1, x_2, \dots x_n) + c'(x_1, x_2 \dots x_n)$

$= c\alpha + c'\alpha$

(viii) $(c.c').\alpha = (c.c')(x_1, x_2 \dots, x_n)$

$= ((c.c')x_1, (c.c')\, x_2, \dots (cc')\, x_n)$

$= (c.(c'.x_1), c.(c'.x_2), \dots, c.(c'.x_n))$

$= c(c'.x_1, c'.x_2, \dots c'.x_n)$

$= c.(c'.(x_1, x_2 \dots, x_n))$

$= c.(c'.\alpha)$

(ix) $1.\alpha = 1.(x_1, x_2 \dots x_n)$

$= (1.x_1, 1.x_2 \dots, 1.x_n)$

$= (x_1, x_2 \dots, x_n)$

$= \alpha$

Thus $V_n(F)$ is a vector space over the field F.

Note :-

(i) With $F = \mathbb{R}$, $V_1(\mathbb{R})$, $V_2(\mathbb{R})$, $V_3(\mathbb{R})$ are all vector spaces. They are also denoted as $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^3$, ~~which~~ respectively. The elements of $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^3$ are real numbers, plane vectors and space vectors respectively.

(ii) If $F = \mathbb{R}$, $V_n(\mathbb{R})$ is denoted as $\mathbb{R}^n$.

If $F = \mathbb{C}$, $V_n(\mathbb{C})$ is denoted as $\mathbb{C}^n$.

2. Show that $V = \{a + b\sqrt{2} \mid a, b \in Q\}$, where Q is the set of all rationals, field Q, is a vector space under usual addition and scalar multiplication.

Let $\alpha = a_1 + b_1\sqrt{2}$, $\beta = a_2 + b_2\sqrt{2}$, $\gamma = a_3 + b_3\sqrt{2} \in V$

$c, c' \in Q$.

(i) $\alpha + \beta = (a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2}) \in V$.

(ii) $(\alpha + \beta) + \gamma = ((a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2})) + (a_3 + b_3\sqrt{2})$

$= \alpha + (\beta + \gamma)$

(iii) 0 is the additive identity,

as $0 + \alpha = \alpha = \alpha + 0$.

(iv) $-\alpha = -a_1 - b_1\sqrt{2}$ is the additive inverse of $\alpha = a_1 + b_1\sqrt{2}$, as $\alpha + (-\alpha) = 0 = (-\alpha) + \alpha$.

(v) $\alpha + \beta = (a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2}) = \beta + \alpha$

(vi) $c.(\alpha + \beta) = c.((a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2})) = c.\alpha + c.\beta$

(vii) $(c + c')\alpha = (c + c')(a_1 + b_1\sqrt{2}) = c.\alpha + c'.\alpha$

(viii) $(c.c')\alpha = (c.c')(a_1 + b_1\sqrt{2}) = c.(c'.\alpha)$

(ix) $1.\alpha = 1.(a_1 + b_1\sqrt{2}) = a_1 + b_1\sqrt{2} = \alpha$

Thus V is a vector space over Q.

3. Let V be the set of all polynomials of degree $\leq n$, with coefficients in the field F, together with zero polynomial. Then Show that V is a vector space under addition of polynomials and scalar multiplication of polynomials with the scalar $c \in F$ defined by

$$c(a_0 + a_1x + \cdots + a_nx^n) = ca_0 + ca_1x + \cdots + ca_nx^n.$$

(i) sum of polynomials is again a polynomial.

(ii) sum of polynomials will be associative.

(iii) ~~The~~ '0' is the additive identity.

(iv) if $\alpha = a_0 + a_1x + \cdots + a_nx^n$, then $-\alpha = -a_0 - a_1x - \cdots - a_nx^n$ is the additive inverse.

(v) sum of polynomials is commutative.

(vi) $c(\alpha + \beta) = c.\alpha + c.\beta$ will hold

(vii) $(c + c')\alpha = c.\alpha + c'.\alpha$ will hold

(viii) $(c.c').\alpha = c.(c'.\alpha)$ will hold.

(ix) $1.\alpha = 1.(a_0 + a_1x + \cdots + a_nx^n) = a_0 + a_1x + \cdots + a_nx^n = \alpha$.

Thus V is a vector space over F.

4. Let $V = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \middle| \, x, y \in \mathbb{C} \right\}$, under usual addition and scalar multiplication, with field $\mathbb{C}$ of complex numbers. Show that V is a vector space

Let $\alpha = \begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{pmatrix}$, $\beta = \begin{pmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{pmatrix}$, $\gamma = \begin{pmatrix} x_3 & y_3 \\ -y_3 & x_3 \end{pmatrix} \in V$

$$c_1 = a_1 + b_1 i, \quad c_2 = a_2 + b_2 i \in \mathbb{C}.$$

(i) $\alpha + \beta \in V$.

(ii) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

(iii) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \alpha = \alpha + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\therefore \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is the additive identity

(iv) $-\alpha = \begin{pmatrix} -x_1 & -y_1 \\ y_1 & -x_1 \end{pmatrix}$ is the additive inverse.

(v) $\alpha + \beta = \beta + \alpha$.

(vi) $c.(\alpha + \beta) = c.\alpha + c.\beta$

(vii) $(c + c')\alpha = c.\alpha + c'.\alpha$

(viii) $(c \cdot c')\alpha = c.(c'.\alpha)$

(ix) $1.\alpha = 1.\begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{pmatrix} = \begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{pmatrix} = \alpha$

Thus V is a vector space over $\mathbb{C}$.

5. Let R^+ be the set of all positive ~~integers~~ real numbers. Define the operations of addition and scalar multiplication as below:

$$\alpha + \beta = \alpha\beta \quad \forall \quad \alpha, \beta \in R^+$$

$$c.\alpha = \alpha^c \quad , \quad \alpha \in R^+ \text{ and } c \in R.$$

Show that R^+ is a vector space over the real field.

(i) $\alpha + \beta = \alpha\beta \in R^+$

(ii) $(\alpha+\beta)+\gamma = (\alpha\beta)+\gamma = (\alpha\beta)\gamma = \alpha(\beta\gamma) = \alpha + \beta\gamma = \alpha + (\beta+\gamma)$

(iii) $\alpha + 1 = \alpha.1 = \alpha = 1.\alpha = 1 + \alpha$

$\therefore$ 1 is the additive identity

(iv) $\alpha + \frac{1}{\alpha} = \alpha.\frac{1}{\alpha} = 1 = \frac{1}{\alpha}.\alpha = \frac{1}{\alpha} + \alpha$

$\therefore \frac{1}{\alpha}$ is the additive inverse of α.

(v) $\alpha + \beta = \alpha\beta = \beta\alpha = \beta + \alpha$

(vi) $c.(\alpha+\beta) = c.(\alpha\beta) = (\alpha\beta)^c = \alpha^c\beta^c = \alpha^c + \beta^c = c.\alpha + c.\beta$

(vii) $(c+c')\alpha = \alpha^{(c+c')} = \alpha^c.\alpha^{c'} = \alpha^c + \alpha^{c'} = c.\alpha + c'.\alpha$

(viii) $(c.c')\alpha = \alpha^{(c.c')}$ $\cancel{= (\alpha^c)^{c'}}$ $\cancel{= c'(\alpha^c) = c}$

$= \alpha^{(c'.c)} = (\alpha^{c'})^c = c(\alpha^{c'}) = c.(c'\alpha)$

(ix) $1.\alpha = \alpha^1 = \alpha$, where 1 is the unit element of R^+.

Exercises

1. Consider the set $V = C[0,1]$, the set of all continuous functions defined over the interval $[0,1]$. The sum of any two elements $f, g \in V$ is defined by $(f+g)(x) = f(x) + g(x)$, $\forall x \in [0,1]$, and scalar multiplication is defined by

$$(c.f)(x) = c.f(x) \quad \forall x \in [0,1], \forall c \in R.$$

Show that V is a vector space.

2. Let $V = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \middle| x, y \in \mathbb{R} \right\}$ with field $\mathbb{R}$.
Show that V is a vector space over the field $\mathbb{R}$, under usual addition and scalar multiplication.

3. Let $V = \{(x,y,z) \mid x,y,z \in \mathbb{Q} \text{ and } x+2y = 3z\}$, with field $\mathbb{Q}$, under component wise addition and scalar multiplication.
Show that V is a vector space.

4. Let V be a set of all odd functions from $\mathbb{R}$ to $\mathbb{R}$, with field $\mathbb{R}$, under usual addition and scalar multiplication. Show that V is a vector space.

5. Let V be the set of all convergent sequences $\{a_n\}$ of real numbers. Then show that V is a vector space over the field $\mathbb{R}$ of real numbers, with the addition and scalar multiplication defined by $\{a_n\} + \{b_n\} = \{a_n + b_n\}$ and $c\{a_n\} = \{ca_n\}$, $\forall c \in \mathbb{R}$.

Subspace :-

A non empty subset W of a vector space V over a field F is called a subspace of V, if W is itself a vector space over F, under the same operations of addition and scalar multiplication as defined in V.

examples

(i) The set $\{0\}$ consisting of zero vector of V, is a subspace of V.

(ii) The whole vector space V, itself is a subspace of V.

These two subspaces are called trivial or improper subspaces of V.

Any subspace W of V different from $\{0\}$ and V is called a proper subspace of V.

Theorem 1. A non empty subset W of a vector space V over a field F is a subspace of V, if and only if (i) $\forall\ \alpha, \beta \in W$, $\alpha + \beta \in W$ (ii) $\forall\ c \in F, \alpha \in W$, $c.\alpha \in W$.

Proof: Suppose W is a subspace of V.

The W is a vector space over F under the same operation of addition and scalar multiplication as defined in V. Hence the conditions (i) & (ii) hold good.

Conversely, suppose W satisfies (i) & (ii).

We shall show that W is a subspace of V.

(i) $\Rightarrow$ '+' is a binary operation on W.

as '+' is asscociative in V, so is in W.

As $W \neq \phi$, $\exists\ \alpha \in W$.

By (ii), $\forall\ c \in F$ & $\alpha \in W \Rightarrow c.\alpha \in W$

In particular $0 \in F$, $\alpha \in W \Rightarrow 0.\alpha = \mathbf{0} \in W$, which acts as the identity element w.r.t addition.

Again, $-1 \in F$, $\alpha \in W \Rightarrow -1.\alpha = -\alpha \in W$, which acts as the additive inverse of α.

As '+' is commutative in V, so is in W.

Thus $(W, +)$ is an abelian group.

The other axioms (2, 3, 4) of the vector space hold in W, as they hold in the whole space V.

Hence W is a vector space over F and therefore a subspace of V.

* Verify whether $W = \{ f(x) \mid 2f(0) = f(1) \}$ over $0 \le x \le 1$, is a subspace of $V = \{\text{all functions}\}$ over the field $\mathbb{R}$.

Sol^n: Let $f_1, f_2 \in W$. To show $f_1 + f_2 \in W$

Thus $2f_1(0) = f_1(1)$ & $2f_2(0) = f_2(1)$.

Consider,
$$2(f_1+f_2)(0) = 2[f_1(0) + f_2(0)]$$
$$= 2f_1(0) + 2f_2(0)$$
$$= f_1(1) + f_2(1)$$
$$= (f_1+f_2)(1)$$

Thus, $f_1 + f_2 \in W$. i.e, W is closed under vector addition.

To show $cf_1 \in W$

Consider,
$$2(cf_1)(0) = (2c).f_1(0)$$
$$= c.2f_1(0)$$
$$= c.f_1(1)$$
$$= (cf_1)(1)$$

Thus $cf_1 \in W$. i.e, W is closed under scalar multiplication.

Hence W is a subspace.

* Is the subset $W = \{ (x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 \le 1 \}$ of $V_3(\mathbb{R})$ is a subspace of $V_3(\mathbb{R})$?

H.W.

* Verify whether $W=\{(x,y,z) \mid \sqrt{2}\,x=\sqrt{3}\,y\}$ is a subspace of $\mathbb{R}^3$.

* Verify whether $W=\{(a+2b,\,0,\,2a-b,\,b) \mid a,b\in\mathbb{R}\}$ is a subspace of $\mathbb{R}^4$.

* Show that the subset

$$W = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0\}$$

of the vector space $V_3(R)$ is a subspace of $V_3(R)$.

Sol:

Let $\alpha = (x_1, x_2, x_3)$, $\beta = (y_1, y_2, y_3)$ be any two elements of W.

$\therefore$ $x_1 + x_2 + x_3 = 0$ and $y_1 + y_2 + y_3 = 0$.

Consider,

$$c_1 \cdot \alpha + c_2 \cdot \beta = c_1(x_1, x_2, x_3) + c_2(y_1, y_2, y_3)$$

$$= (c_1 x_1, c_1 x_2, c_1 x_3) + (c_2 y_1, c_2 y_2, c_2 y_3)$$

$$= (c_1 x_1 + c_2 y_1, c_1 x_2 + c_2 y_2, c_1 x_3 + c_2 y_3)$$

To show that $c_1\alpha + c_2\beta \in W$, we have to show that the sum of the components of $c_1\alpha + c_2\beta$ is zero.

$\therefore$ consider $c_1 x_1 + c_2 y_1 + c_1 x_2 + c_2 y_2 + c_1 x_3 + c_2 y_3$

$$= c_1(x_1 + x_2 + x_3) + c_2(y_1 + y_2 + y_3)$$

$$= c_1 \cdot 0 + c_2 \cdot 0$$

$$= 0$$

$\therefore$ $c_1\alpha + c_2\beta \in W$, hence W is a subspace of $V_3(R)$.

* Verify whether $W = \{$polynomials of degree three$\}$ defined on $0 \le x \le 1$ is a subspace of the vector space $V = \{$all polynomials$\}$ over $\mathbb{R}$. ~~is a subspace~~

Soln:

The set of all polynomials of degree three is not a subspace, as the sum of two polynomials of degree three need not to be of degree three.

$$\because \quad f_1(x) = 3x^3 - 4x^2 + 2x + 1, \quad f_2(x) = -3x^3 + 3x^2 + 2x + 5$$

$$\Rightarrow \quad f_1(x) + f_2(x) = -x^2 + 4x + 6$$

which is not a polynomial of degree three.

Thus the set is not closed under vector addition.

* Verify whether $W = \{$polynomials of degree less than five$\}$ defined on $0 \le x \le 1$ is a subspace of the vector space $V = \{$all polynomials$\}$ over $\mathbb{R}$. ~~is a subspace~~

Corollary:

A non empty subset W is a subspace of a vector space V over F, if and only if

$$c_1\alpha + c_2\beta \in W, \ \forall\ \alpha, \beta \in W,\ c_1, c_2 \in F.$$

Proof:

Let W be a subspace of V.

Let $c_1, c_2 \in F$ and $\alpha, \beta \in W$.

By Theorem 1, $c_1\alpha,\ c_2\beta \in W$ and hence $c_1\alpha + c_2\beta \in W$.

Conversely, let $c_1\alpha + c_2\beta \in W$, $\forall\ \alpha, \beta \in W$, $c_1, c_2 \in W$.

let $c_1 = 1$, $c_2 = 1$, then $1.\alpha + 1.\beta = \alpha + \beta \in W$, $\forall\ \alpha, \beta \in W$,

$\therefore$ W is closed under vector addition.

Now take $\beta = 0$, then $c_1.\alpha + c_2.0 = c_1\alpha \in W$, $\forall\ \alpha \in W$ and $c_1 \in F$.

$\therefore$ W is closed under scalar multiplication.

Hence W is a subspace of V.

* Let $V = \mathbb{R}^3$, the vectorspace of all ordered triplets of real numbers, over the field of real numbers. Show that the subset $W = \{(x,0,0) \mid x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

Sol^n: The element $0 = (0,0,0) \in W$.

Thus W is non empty.

Let $\alpha_1 = (x_1, 0, 0)$ & $\alpha_2 = (x_2, 0, 0)$ be any two elements of W.

Then $\alpha_1 + \alpha_2 = (x_1, 0, 0) + (x_2, 0, 0) = (x_1 + x_2, 0, 0) \in W$.

$\therefore$ W is closed under addition.

Again, for any scalar $c \in \mathbb{R}$,

$$c \cdot \alpha_1 = c(x_1, 0, 0) = (cx_1, 0, 0) \in W$$

$\therefore$ W is closed under scalar multiplication.

Hence W is a subspace of $\mathbb{R}^3$.

* Similarly $W = \{(0, x, 0) \mid x \in \mathbb{R}\}$

$W = \{(0,0,x) \mid x \in \mathbb{R}\}$

$W = \{(x_1, x_2, 0) \mid x_1, x_2 \in \mathbb{R}\}$

$W = \{(0, x_2, x_3) \mid x_2, x_3 \in \mathbb{R}\}$

$W = \{(x_1, 0, x_3) \mid x_1, x_3 \in \mathbb{R}\}$

are subspaces of $\mathbb{R}^3$.

Linear Combination

Let V be a vector space over the field F and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be any n vectors of V. The vector of the form,

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n$$

where $c_1, c_2, \ldots, c_n \in F$, is called a linear combination of the vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Consider the vectors $\alpha_1 = (-1, 3, -1)$, $\alpha_2 = (-1, 2, 3)$ and $\alpha_3 = (1, 0, 1)$ of the vector space $\mathbb{R}^3$.

Then the vector $\alpha = 2\alpha_1 - 3\alpha_2 - \alpha_3$

$$= 2(-1, 3, -1) - 3(-1, 2, 3) - (1, 0, 1)$$

$$= (-2, 6, -2) - (-3, 6, 9) - (1, 0, 1)$$

$$\alpha = (0, 0, -12)$$

is a linear combination of the vectors α_1, α_2 and α_3.

By choosing different set of scalars, different linear combinations of $\alpha_1, \alpha_2, \alpha_3$ can be formed.

Linear span of S

Let S be a non empty subset of a vectorspace $V(F)$. The set of all linear combinations of finite number of elements of S is called the linear span of S and is denoted by $L[S]$.

i.e, $L[S] = \{c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n \,/\, c_i \in F,\ \alpha_i \in S,\ i = 1, 2, \dots n$ & n is any positive integer$\}$

If $\alpha \in L[S]$, then α is of the form,

$\alpha = c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n$, for some scalars $c_1, c_2 \dots c_n \in F$.

Theorem :-

Let S be a nonempty subset of a vectorspace $V[F]$.

Then (i) $L[S]$ is a subspace of V

(ii) $S \subseteq L[S]$

(iii) $L[S]$ is the smallest subspace of V containing S.

* Show that the vector $(2,-5,3) \in V_3(R)$ is not in $L[S]$, where $S=\{(1,-3,2),(2,-4,-1),(1,-5,7)\}$

Sol^n: If $(2,-5,3) \in L[S]$, then

$$(2,-5,3) = c_1(1,-3,2) + c_2(2,-4,-1) + c_3(1,-5,7)$$
$$= (c_1+2c_2+c_3,\ -3c_1-4c_2-5c_3,\ 2c_1-c_2+7c_3)$$

$$c_1+2c_2+c_3=2$$
$$-3c_1-4c_2-5c_3=-5$$
$$2c_1-c_2+7c_3=3$$

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ -3 & -4 & -5 & -5 \\ 2 & -1 & 7 & 3 \end{array}\right]$$

$$\begin{array}{l} R_2+3R_1 \\ R_3-2R_1 \end{array}\left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & -5 & 5 & -1 \end{array}\right]$$

$$2R_3+5R_2\left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

The system is inconsistent.
Hence has no solution.

$\therefore$ $(2,-5,3)$ cannot be expressed as linear combination of the elements of S and therefore $(2,-5,3) \notin L[S]$.

* Determine whether the polynomial $3x^2+x+5$ is the linear span of the set $S=\{x^3, x^2+2x, x^2+2, 1-x\}$ of the vector space of all polynomials over the field R.

Sol^n $3x^2+x+5 = c_1x^3 + c_2(x^2+2x) + c_3(x^2+2) + c_4(1-x)$
$$= c_1x^3 + (c_2+c_3)x^2 + (2c_2-c_4)x + (2c_3+c_4)$$

$\Rightarrow \boxed{c_1=0}$, $c_2+c_3=3$, $2c_2-c_4=1$, $2c_3+c_4=5$

$\Rightarrow \boxed{c_1=0}$, $\boxed{c_2=3}$, $\boxed{c_3=0}$, $\boxed{c_4=5}$

$\therefore\ 3x^2+x+5 = 0x^3 + 3(x^2+2x) + 0(x^2+2) + 5(1-x)$

$\therefore\ 3x^2+x+5 \in L[S]$.

* Find the subspace spanned by the set $S = \{(2,0,0),\ (0,0,-2)\}$ in the vector space $V_3(R)$.

Sol: The subspace spanned by S is $L[S]$.

Any element $\alpha \in L[S]$ is of the form

$$\alpha = c_1(2,0,0) + c_2(0,0,-2)$$

$$\alpha = (2c_1, 0, -2c_2)$$

$\therefore L[S] = \{\alpha / \alpha = (2c_1, 0, -2c_2);\ c_1, c_2 \in R\}$.

* In $V_3(R)$ show that the plane $x_3 = 0$ may be spanned by the pair of vectors $(2,2,0)$ & $(4,1,0)$.

Sol:

Linear Dependence and Independence

A set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of vectors of a vector space $V[F]$ is said to be linearly dependent if there exists scalars $c_1, c_2, \ldots, c_n \in F$, not all zero such that $c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n = 0$.

A set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of vectors of a vector space $V[F]$ is said to be linearly independent if ~~it is not be~~

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = 0 \Rightarrow c_1 = c_2 = \cdots = c_n = 0$$

* Show that the vectors $e_1 = (1, 0, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, $e_n = (0, 0, 0, \ldots, 0)$ of the vector space $V_n(\mathbb{R})$ are linearly independent.

Sol^n let $c_1, c_2, \ldots, c_n \in \mathbb{R}$

Consider $c_1 e_1 + c_2 e_2 + \cdots + c_n e_n = 0$.

$$\Rightarrow c_1(1,0,0,\ldots,0) + c_2(0,1,0,\ldots,0) + \cdots + c_n(0,0,0,\ldots,1) = (0,0..0)$$

$$\Rightarrow (c_1, c_2, \ldots, c_n) = (0, 0, 0, \ldots, 0)$$

$$\Rightarrow c_1 = 0,\quad c_2 = 0,\quad \ldots,\quad c_n = 0$$

ie $e_1, e_2, e_3, \ldots, e_n$ are linearly independent.

* Show that the set $S = \{(1,0,1), (1,1,0), (-1,0,-1)\}$ is linearly dependent in $V_3(\mathbb{R})$.

Sol^n: Consider $c_1(1,0,1) + c_2(1,1,0) + c_3(-1,0,-1) = (0,0,0)$

$\Rightarrow (c_1 + c_2 - c_3,\ c_2,\ c_1 - c_3) = (0,0,0)$

$$\Rightarrow \left.\begin{aligned} c_1 + c_2 - c_3 &= 0 \\ c_2 &= 0 \\ c_1 - c_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= c_3 \\ c_2 &= 0 \end{aligned}$$

let $c_3 = 1$ then $c_1 = 1$

Thus there exists, not all zero, scalars, such that

$c_1(1,0,1) + c_2(1,1,0) + c_3(-1,0,-1) = (0,0,0)$

$\therefore$ S is linearly dependent.

* The set $\{(x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3)\}$ of vectors of the vector space $V_3(\mathbb{R})$ is linearly dependent iff

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0$$

* Two vectors $\alpha, \beta \in V_2(\mathbb{R})$ are linearly dependent iff $\alpha = k\beta$ for some non zero $k \in \mathbb{R}$.

* A set of vectors of V, containing the zero vector is linearly dependent.

* The set consisting of a single vector α of V is linearly independent iff $\alpha \neq 0$.

Basis :-

A subset B of a vector space V[F] is called a basis of V if

(i) B is a linearly independent set

(ii) $L[B] = V$.

That is a basis of a vector space V[F] is a linearly independent subset which spans the whole space.

* Finite dimensional space :

A vector space V[F] is said to be a finite dimensional space if it has a finite basis.

*Note:- The zero vector 0 can not be an element of a basis of a vector space because a set of vectors with zero vector is always linearly dependent.

example:- Show that the vectors $e_1 = (1,0,0,\dots,0)$, $e_2 = (0,1,0,\dots,0)$, $e_n = (0,0,0,\dots,1)$ of the vector space $V_n(\mathbb{R})$ form a basis of $V_n(\mathbb{R})$.

Sol^n:- ~~The~~ Consider, $S = \{e_1, e_2, \dots, e_n\}$

$$c_1 e_1 + c_2 e_2 + \dots + c_n e_n = \mathbf{0}$$

$$\Rightarrow c_1(1,0,\dots,0) + c_2(0,1,\dots,0) + \dots + c_n(0,0,\dots,1) = (0,0,\dots,0)$$

$$\Rightarrow (c_1, c_2, \dots c_n) = (0,0,\dots,0)$$

$$\Rightarrow c_1 = 0, \quad c_2 = 0, \quad c_3 = 0 \quad \dots, c_n = 0.$$

$\therefore$ S is linearly independent.

Further, any vector $(x_1, x_2, \dots, x_n) \in V_n(\mathbb{R})$ can be expressed as a linear combination of the elements of S, as

$$(x_1, x_2, \dots, x_n) = x_1 e_1 + x_2 e_2 + \dots + x_n e_n.$$

Hence $L[S] = V_n(\mathbb{R})$.

$\therefore$ S is a basis of $V_n(\mathbb{R})$.

* Standard basis

The basis $S = \{e_1, e_2, \dots, e_n\}$ of the vector space $V_n(\mathbb{R})$ is called the standard basis.

eg The vectors $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ of $V_3(\mathbb{R})$ form a basis of $V_3(\mathbb{R})$, and is called the standard basis.

Example: Show that the set $B=\{(1,1,0),(1,0,1),(0,1,1)\}$ is a basis of the vectorspace $V_3(\mathbb{R})$.

Sol^n: Let $B=\{(1,1,0),(1,0,1),(0,1,1)\}$

Consider, $c_1(1,1,0)+c_2(1,0,1)+c_3(0,1,1)=(0,0,0)$

$\Rightarrow (c_1+c_2, c_1+c_3, c_2+c_3)=(0,0,0)$

$\Rightarrow c_1+c_2=0, \; c_1+c_3=0, \; c_2+c_3=0$

$\Rightarrow c_1=0, \; c_2=0, \; c_3=0$.

$\therefore$ B is linearly independent.

Let $(x_1, x_2, x_3) \in V_3(\mathbb{R})$ be arbitrary

let $c_1, c_2, c_3 \in \mathbb{R}$, such that

$(x_1, x_2, x_3) = c_1(1,1,0)+c_2(1,0,1)+c_3(0,1,1)$

$(x_1, x_2, x_3) = (c_1+c_2, c_1+c_3, c_2+c_3)$

$$\Rightarrow \left.\begin{aligned} x_1 &= c_1+c_2 \\ x_2 &= c_1+c_3 \\ x_3 &= c_2+c_3 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1-x_2 &= c_2-c_3 \\ x_3 &= c_2+c_3 \end{aligned}$$

$$\Rightarrow \boxed{\frac{x_1-x_2+x_3}{2} = c_2}$$

$$\Rightarrow x_1 = c_1 + \frac{x_1-x_2+x_3}{2} \Rightarrow \boxed{c_1=\frac{x_1+x_2-x_3}{2}}$$

$$\Rightarrow x_2 = \frac{x_1+x_2-x_3}{2} + c_3 \Rightarrow \boxed{c_3 = \frac{-x_1+x_2+x_3}{2}}$$

$$\therefore (x_1, x_2, x_3) = \frac{x_1+x_2-x_3}{2}(1,1,0) + \frac{x_1-x_2+x_3}{2}(1,0,1) + \frac{-x_1+x_2+x_3}{2}(0,1,1)$$

$\therefore L[B] = V_3(\mathbb{R})$

Theorem :-

Any two bases of a finite dimensional vector space V have the same finite number of elements.

Dimension of a vector space V

The dimension of a finite dimensional vector space V over F is the number of elements in any basis of V and is denoted by d[V].

eg $V_n(R)$ is a n dimensional space.

$V_3(R)$ is a three dimensional space.

~~Theo~~

* A vector space which is not finitely generated may be called an infinite dimensional space.

Theorem :-

In an n dimensional vector space V(F)

(i) any n+1 elements of V are linearly dependent.

(ii) no set of n-1 elements can span V.

Theorem :-

In a n dimensional vector space V(F) any set of n linearly independent vector is a basis.

Theorem :-

Any linearly independent set of elements of a finite dimensional vector space V is a part of a basis

Theorem :-

For n vectors of n-dimensional vector space V, to be a basis, it is sufficient that they span V or that they are L.I.

Example:

Let $A = \{(1,-2,5),\ (2,3,1)\}$ be a linearly independent subset of $V_3(\mathbb{R})$. Extend this to a basis of $V_3(\mathbb{R})$

Sol^n let $\alpha_1 = (1,-2,5)$, $\alpha_2 = (2,3,1)$

Let S be the subspace spanned by $\{\alpha_1, \alpha_2\}$

$\therefore S = \{ c_1\alpha_1 + c_2\alpha_2 \mid c_1, c_2 \in \mathbb{R}\}$

$$c_1\alpha_1 + c_2\alpha_2 = c_1(1,-2,5) + c_2(2,3,1)$$
$$= (c_1 + 2c_2,\ -2c_1 + 3c_2,\ 5c_1 + c_2)$$

$\therefore S = \{(c_1+2c_2,\ -2c_1+3c_2,\ 5c_1+c_2) \mid c_1, c_2 \in \mathbb{R}\}$

Choose a vector of $V_3(\mathbb{R})$ outside of S.

$(1,0,0) \notin S$.

$\therefore$ the set $A = \{(1,-2,5), (2,3,1), (1,0,0)\}$ is a basis of $V_3(\mathbb{R})$.

Example:

Given two linearly independent vectors $(2,1,4,3)$ & $(2,1,2,0)$, find a basis of $V_4(\mathbb{R})$ that includes these two vectors.

Sol^n $\alpha_1 = (2,1,4,3)$, $\alpha_2 = (2,1,2,0)$

$S = \{c_1\alpha_1 + c_2\alpha_2 \mid c_1, c_2 \in \mathbb{R}\}$

Choose $\alpha_3 = (1,0,0,0)$ & $\alpha_4 = (0,1,0,0) \notin S$

$\therefore \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a basis of $V_4(\mathbb{R})$.

* The non-zero rows of a row-reduced echelon form of a matrix are linearly independent.
* Let A be a matrix of the given vectors. E be the row reduced echelon matrix of A.

* The

Example:
Test the following set of vectors for linear dependence in $V_3(\mathbb{R})$. $\{(1,0,1), (0,2,2), (3,7,1)\}$. Do they form a basis?

Sol^n:- Consider the matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 3 & 7 & 1 \end{pmatrix}$

$|A| = 1(2-14) - 0(0-6) + 1(0-6) = -18 \neq 0.$

$\therefore$ The given set is linearly independent.
Any three vectors which are linearly independent, is a basis of $V_3(\mathbb{R})$.

Example:
Does the set $S = \{(1,2,3), (3,1,0), (-2,1,3)\}$ form a basis of $\mathbb{R}^3$.

Consider $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ -2 & 1 & 3 \end{bmatrix}$,

$|A| = 1(3-0) - 2(9+0) + 3(3+2)$
$\Rightarrow 0$

$\therefore S$ is linearly dependent and hence ~~doesnot~~ is not a basis of $\mathbb{R}^3$.

Example :-
Show that the vectors $(1,1,2,4)$, $(2,-1,-5,2)$, $(1,-1,-4,0)$ and $(2,1,1,6)$ are linearly dependent in $\mathbb{R}^4$ and extract a linearly independent subset. Also find the dimension and a basis of the subspace spanned by them.

Sol^n:

Consider $A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{pmatrix}$

$R_2 - 2R_1$; $R_3 - R_1$; $R_4 - 2R_1$

$$A \approx \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \\ 0 & -1 & -3 & -2 \end{bmatrix}$$

$R_2 \div -3$ $\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & -2 & -6 & -4 \\ 0 & -1 & -3 & -2 \end{bmatrix}$

$R_3 + 2R_2$
$R_4 + R_2$ $\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The final matrix is in echelon form, and the rank of A is 2. $\therefore$ The given vectors are linearly dependent.
The corresponding non-zero rows of the initial matrix are $(1,1,2,4)$ & $(2,-1,-5,2)$ which are L.I.
The dimension of the subspace spanned by these vectors is $\underline{2}$.
These two vectors form a basis of the subspace.

* Let S be the subspace of $\mathbb{R}^3$ defined by $S = \{(a,b,c) \mid a+b+c=0\}$. Find a basis and dimension of S.

Sol^n:-

$S \neq \mathbb{R}^3$. [$\because$ $(1,2,3) \in \mathbb{R}^3$ but $(1,2,3) \notin S$]
(as $1+2+3 \neq 0$)

$\alpha = (1,0,-1)$ & $\beta = (1,-1,0) \in S$,

and further they are independent.

$\therefore$ $d[S] = 2$ & hence $\{\alpha, \beta\}$ form a basis of S.

* Show that the field C of complex numbers is a vector space over the field $\mathbb{R}$ of reals. What is its dimension?

Sol^n. ~~For $c_1, c_2 \in$~~ $\mathbb{C} = \{a+ib \mid a,b \in \mathbb{R}\}$.

$\mathbb{C}$ is closed under '+'.
$\mathbb{C}$ is associative under '+'
$0+0i$ is the identity w.r.t '+'.
$-a-ib$ is the inverse of $a+ib$.
$\mathbb{C}$ is commutative

$(\mathbb{C}, +)$ is an abelian gp'.

$c(a_1+ib_1 + a_2+ib_2) \in \mathbb{C}$
$(c_1+c_2)(a_1+ib_1) \in \mathbb{C}$
$c.(a_1+ib_1) \in \mathbb{C}$
'1' is the unity

$\therefore$ $\mathbb{C}$ is a vector space, over $\mathbb{R}$.

Let $\alpha \in \mathbb{C}$, $\alpha = a+ib$. $\ni a,b \in \mathbb{R}$.

$\therefore$ $\alpha = 1.a + i.b = a.1 + b.i$

i.e, every element of $\mathbb{C}$ is a linear combination of the element $1, i$. That is $\{1, i\}$ generates $\mathbb{C}$.

Further $c_1.1 + c_2.i = 0 \Rightarrow c_1 = 0$ & $c_2 = 0$. $\therefore \{1, i\}$ is L.I.

$\therefore$ $\{1, i\}$ is a basis of $\mathbb{C}$. & $d[\mathbb{C}] = 2$.

* Let V be the vector space of 2×2 symmetric matrices over the field F. Show that $d[V]=3$.

Sol^n let $A=\begin{bmatrix} a & b \\ b & c \end{bmatrix} \in V, \quad a,b,c \in F.$

set $a=1,\ b=0,\ c=0;\ a=0,\ b=1,\ c=0;\ a=0,\ b=0,\ c=1$

We get three matrices:

$$E_1=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad E_2=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad E_3=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

We shall show that these elements of V form a basis. Let $A=\begin{pmatrix} a & b \\ b & c \end{pmatrix} \in V$ be arbitrary.

Then,
$$A=\begin{pmatrix} a & b \\ b & c \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus $\{E_1, E_2, E_3\}$ generates V.

Suppose $c_1E_1 + c_2E_2 + c_3E_3 = 0, \quad c_1, c_2, c_3 \in F$

$$\Rightarrow c_1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c_3\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} c_1 & c_2 \\ c_2 & c_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow c_1=c_2=c_3=0$$

$\therefore \{E_1, E_2, E_3\}$ is linearly independent.

Hence $\{E_1, E_2, E_3\}$ is a basis of V.

and $d[V]=3$.

* Show that the set

$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ form a basis of the vector space V of all 2×2 matrices over $\mathbb{R}$.

* Find the basis and dimension of the subspace spanned by the vectors $(1,2,0), (1,1,1), (2,0,1)$ of the vector space $V_3(Z_3)$.

* Find the basis and dimension of the subspace spanned by the subset.

$S = \left\{ \begin{pmatrix} 1 & -5 \\ -4 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 5 \end{pmatrix}, \begin{pmatrix} 2 & -4 \\ -5 & 7 \end{pmatrix} \begin{pmatrix} 1 & -7 \\ -5 & 1 \end{pmatrix} \right\}$ of the vector space of all 2×2 matrices over $\mathbb{R}$.

Soln: Let $\alpha, \beta, \gamma, \delta$ are the matrices of S.

Then the coordinates of α, β, γ & δ w.r.t standard basis are $(1,-5,-4,2)$, $(1,1,-1,5)$, $(2,-4,-5,7)$, $(1,-7,-5,1)$.

Consider $\begin{pmatrix} 1 & -5 & -4 & 2 \\ 1 & 1 & -1 & 5 \\ 2 & -4 & -5 & 7 \\ 1 & -7 & -5 & 1 \end{pmatrix}$

$\begin{array}{l} R_2-R_1 \\ R_3-2R_1 \\ R_4-R_1 \end{array} \begin{pmatrix} 1 & -5 & -4 & 2 \\ 0 & 6 & 3 & 3 \\ 0 & 6 & 3 & 3 \\ 0 & -2 & -1 & -1 \end{pmatrix}$

$\begin{array}{l} R_3-R_2 \\ 3R_4+R_2 \end{array} \begin{pmatrix} 1 & -5 & -4 & 2 \\ 0 & 6 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

The final matrix has two nonzero rows.

$\therefore$ d(subspace) $= 2$.

Further the matrices corresponding to the non-zero rows, in the final matrix are $\begin{pmatrix} 1 & -5 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} 0 & 6 \\ 3 & 3 \end{pmatrix}$

* In a vector space $V_3(\mathbb{R})$, let $\alpha = (1,2,1)$, $\beta = (3,1,5)$ & $\gamma = (-1,3,-3)$. Show that the subspace spanned by $\{\alpha, \beta\}$ & $\{\alpha, \beta, \gamma\}$ are the same.

Sol^n Consider, $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 5 \\ -1 & 3 & -3 \end{pmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 5 \\ -1 & 3 & -3 \end{vmatrix} = 1(-3-15) - 2(-9+5) + 1(9+1)$$
$$= -18 + 8 + 10 = 0$$

$\therefore \{\alpha, \beta, \gamma\}$ is L.D.

let $\gamma = c_1 \alpha + c_2 \beta$

$(-1,3,-3) = c_1(1,2,1) + c_2(3,1,5)$

$\Rightarrow (-1,3,-3) = (c_1 + 3c_2, 2c_1 + c_2, c_1 + 5c_2)$

$\Rightarrow c_1 + 3c_2 = -1, \; 2c_1 + c_2 = 3, \; c_1 + 5c_2 = -3.$

Solving these equ^s, we get $c_1 = 2$, $c_2 = -1$.

$\therefore \gamma \in$ subspace spanned by $\{\alpha, \beta\}$.

$\therefore$ the subspace spanned by $\{\alpha, \beta\}$ & $\{\alpha, \beta, \gamma\}$ are same.

Null space

The null space of a $m \times n$ matrix A, written as $\text{Nul}\, A$, is the set of all solutions to the homogeneous equation $Ax = 0$.

Theorem :-
The null space of a $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $Ax = 0$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

example :-
Let $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$ and $u = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$.

Determine if u belongs to the null space of A.

soln : $Au \neq 0$

$$\Rightarrow \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5-9+4 \\ -25+27-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore$ u is in $\text{Nul}\, A$.

example

Find a spanning set for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Sol^n: Consider $Ax = 0$

Reducing A to echelon form

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

$$\begin{array}{l} 3R_2 + R_1 \\ 3R_3 + 2R_1 \end{array} \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & 13 & 26 & -26 \end{bmatrix}$$

$$\begin{array}{l} R_2 \div 5 \\ R_3 \div 13 \end{array} \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix}$$

$$R_3 - R_2 \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\Rightarrow \begin{cases} -3x_1 + 6x_2 - x_3 + x_4 - 7x_5 = 0 \\ x_3 + 2x_4 - 2x_5 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = 2x_2 - \frac{1}{3}x_3 + \frac{1}{3}x_4 - \frac{7}{3}x_5 \\ \boxed{x_3 = -2x_4 + 2x_5} \end{cases}$$

$$\Rightarrow x_1 = 2x_2 - \frac{1}{3}(-2x_4 + 2x_5) + \frac{1}{3}x_4 - \frac{7}{3}x_5$$

$$\Rightarrow \boxed{x_1 = 2x_2 + x_4 - 3x_5}$$

with x_2, x_4, x_5 as free variables.

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= x_2 u + x_4 v + x_5 w$$

Every linear combination of u, v and w is an element of Nul A. Thus $\{u, v, w\}$ is a spanning set for Nul A.

Column space:-

The Column space of a $m \times n$ matrix A, written as $\operatorname{Col} A$, is the set of all linear combinations of the columns of A.
If $A = [a_1, \dots, a_n]$, then $\operatorname{Col} A = \operatorname{span}\{a_1, \dots, a_n\}$

Theorem:-
The column space of a $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

example:-
Find a matrix A such that $W = \operatorname{Col} A$.

$$W = \left\{ \begin{bmatrix} 6a-b \\ a+b \\ -7a \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

soln:- W can be written as

$$W = \left\{ a\begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left\{ \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Using the vectors in the spanning set as the columns of A, w get $A = \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}$.

Then $W = \operatorname{Col} A$ as desired.

Note:- The column space of a $m \times n$ matrix A is all of $\mathbb{R}^m$ iff the equation $Ax = b$ has a solution for each b in $\mathbb{R}^m$.

example:

let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}_{3\times 4}$

(a) If the column space of A is a subspace of $\mathbb{R}^k$, what is k?

(b) If the null space of A is a subspace of $\mathbb{R}^k$, what is k?

sol^n: (a) $m = 3$, Col A is a subspace of $\mathbb{R}^m$, where $m=3$.

(b) $n=4$, Nul A is a subspace of $\mathbb{R}^n$, where $n=4$.

example:-

let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$, find a nonzero vector in Col A and a nonzero vector in Nul A.

sol^n: any column of $A \in$ Col A, eg $\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} \in$ Col A

Consider $Ax = 0$.

$A = \left[\begin{array}{cccc:c} 2 & 4 & -2 & 1 & 0 \\ -2 & -5 & 7 & 3 & 0 \\ 3 & 7 & -8 & 6 & 0 \end{array}\right]$

$\begin{array}{l} R_2+R_1 \\ 2R_3-3R_1 \end{array} \left[\begin{array}{cccc:c} 2 & 4 & -2 & 1 & 0 \\ 0 & -1 & 5 & 4 & 0 \\ 0 & 2 & -10 & 9 & 0 \end{array}\right]$

$R_3+2R_2 \left[\begin{array}{cccc:c} 2 & 4 & -2 & 1 & 0 \\ 0 & -1 & 5 & 4 & 0 \\ 0 & 0 & 0 & 17 & 0 \end{array}\right]$

$\Rightarrow \begin{array}{r} 2x_1 + 4x_2 - 2x_3 + x_4 = 0 \\ -x_2 + 5x_3 + 4x_4 = 0 \\ 17x_4 = 0 \end{array}$

$x_4 = 0$

$\Rightarrow \cancel{2}x_1 = -\cancel{2}x_2 + x_3$

$\boxed{x_2 = 5x_3}$

$\Rightarrow \boxed{x_1 = -9x_3}$

x_3 is a free variable.

let $x_3 = 1$, then $x_1 = -9$, $x_2 = 5$, $x_4 = 0$

the vector $x = (-9, 5, 1, 0) \in$ Nul A.

$x_1 = \cancel{-2x_2+x_3}\ -9x_3$
$x_2 = 5x_3$
$x_3 = x_3$
$x_4 = 0$

$x_3 \begin{pmatrix} -9 \\ 5 \\ 1 \\ 0 \end{pmatrix}$

* Determine if $w = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$ is in Nul A, where

$$A = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix}.$$

* Determine if $w = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$ is in Nul A, where

$$A = \begin{bmatrix} 5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{bmatrix}.$$

* Find an explicit description of Nul A, by listing vectors that span the null space.

  * $A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$

  * $A = \begin{bmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

* Find A such that the given set is Col$A$.

  * $\left\{ \begin{bmatrix} 2s+3t \\ r+s-2t \\ 4r+s \\ 3r-s-t \end{bmatrix} : r, s, t \text{ real} \right\}$ $\quad A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -2 \\ 4 & 1 & 0 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$

  * $\left\{ \begin{bmatrix} b-c \\ 2b+c+d \\ 5c-4d \\ d \end{bmatrix} : b, c, d \text{ real} \right\}$

* Find (a) k such that Nul A is a subspace of $\mathbb{R}^k$

(b) k such that Col A is a subspace of $\mathbb{R}^k$.

$Ax=0$

* $A = \begin{bmatrix} 2 & -6 \\ -1 & 3 \\ -4 & 12 \\ +3 & -9 \end{bmatrix}$ 4×2 * $A = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 0 & -5 \\ 0 & -5 & 7 \\ -5 & 7 & -2 \end{bmatrix}$.

* $A = \begin{bmatrix} 4 & 5 & -2 & 6 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$ * $A = \begin{bmatrix} 1 & -3 & 9 & 0 & -5 \end{bmatrix}$

* With $A = \begin{bmatrix} 2 & -6 \\ -1 & 3 \\ -4 & 12 \\ +3 & -9 \end{bmatrix}$, find a nonzero vector in Nul A and a nonzero vector in Col A.

* With $A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$, find a nonzero vector in Nul A and a nonzero vector in Col A

* Let $A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$ and $w = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Determine if w is in Col A. Is w in Nul A?

$Ax=0$

$x_1 - 2x_2 = 0$

$x_1 = -2x_2$

* Let $A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}$ and $w = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$. Determine if w is in Col A. Is w in Nul A?

$x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

example :-

With $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}_{3\times 4}$ $u = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$

(a) Determine if u is in Nul A. $\mathbb{R}^4$
Could u be in Col A? $\mathbb{R}^3$

(b) Determine if v is in Col A.
Could v be in Nul A? $\mathbb{R}^4$

Sol^n:

(a) Consider $Au = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore$ $u \notin$ Nul A.

$\because$ u has 4 entries, and Col A is subspace of $\mathbb{R}^3$, $u \notin$ Col A.

(b) Consider $[A : v]$

$$\begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{bmatrix}$$

$$\begin{array}{l} R_2+R_1 \\ 2R_3-3R_1 \end{array} \left[\begin{array}{cccc:c} 2 & 4 & -2 & 1 & 3 \\ 0 & -1 & 5 & 4 & 2 \\ 0 & 2 & -10 & 9 & -6 \end{array}\right]$$

$$R_3+2R_2 \left[\begin{array}{cccc:c} 2 & 4 & -2 & 1 & 3 \\ 0 & -1 & 5 & 4 & 2 \\ 0 & 0 & 0 & 17 & 0 \end{array}\right]$$

$\Rightarrow$ $Ax = v$ is consistent.

$\therefore$ v is in Col A.

$\because$ v has 3 entries and Nul A is a subspace of $\mathbb{R}^4$, $v \notin$ Nul A.

* Find the bases for the null spaces of the matrices.

$$A = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 3 & -2 & 1 & -2 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 & -5 & 1 & 4 \\ -2 & 1 & 6 & -2 & -2 \\ 0 & 2 & -8 & 1 & 9 \end{bmatrix}$$

* Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane $x+2y+z=0$. [$Ax=0$] $A=\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}_{1\times 3}$

* Find a basis for the set of vectors in $\mathbb{R}^2$ on the line $y=5x$. [$Ax=0$] $A=\begin{bmatrix} 5 & -1 \end{bmatrix}_{1\times 2}$

* Find the bases for Nul A and Col A. Assume that A is row equivalent to B.

$$A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix}. \qquad B = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A^{ech} $\begin{bmatrix} \\ \\ \end{bmatrix}$

$(1, 0, 0)$ $(1, 0, 0)$

$(0, 1, 0)$ $(3, 0, 0)$

$(0, 0, 1)$

Row Space

If A is a $m \times n$ matrix, each row of A has n entries and thus can be identified with a vector in $\mathbb{R}^n$. The set of all linear combinations of the row vectors is called the row space of A and is denoted by Row A. Each row has n entries, so Row A is a subspace of $\mathbb{R}^n$.

Since the rows of A are identified with the columns of A^T, we could also write Col A^T in place of Row A.

example:-

let
$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}_{4\times5}$$

Sol^n
$$r_1 = (-2, -5, 8, 0, -17)$$
$$r_2 = (1, 3, -5, 1, 5)$$
$$r_3 = (3, 11, -19, 7, 1)$$
$$r_4 = (1, 7, -13, 5, -3)$$

The row space of A is the subspace of $\mathbb{R}^5$ spanned by $\{r_1, r_2, r_3, r_4\}$. That is, Row A = Span $\{r_1, r_2, r_3, r_4\}$

Theorem:
If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, then nonzero rows of B form a basis for the row space of A as well as for that of B.

Find the basis for the row space, ~~the the column space~~, ~~and the null space~~ and ~~the left null space of the~~ matrix:

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

Solⁿ

To find the basis for the row space, reduce A to echelon form.

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ -2 & -5 & 8 & 0 & -17 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

$$\begin{array}{l} R_2+2R_1 \\ R_3-3R_1 \\ R_4-R_1 \end{array} \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 2 & -4 & 4 & -14 \\ 0 & 4 & -8 & 4 & -8 \end{bmatrix}$$

$$\begin{array}{l} R_3-2R_2 \\ R_4-4R_2 \end{array} \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 20 \end{bmatrix}$$

$$R_3 \leftrightarrow R_4 \quad B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The first three rows of B form a basis for the row space of A ~~(as well as for the rowsspace of B)~~.

Basis for Row A

$= \{(1,3,-5,1,5), (0,1,-2,2,-7), (0,0,0,-4,20)\}$

* Find the basis for the ~~matrix~~ row space of

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}$$

Sol^n

$$\begin{array}{l} R_2+R_1 \\ R_3-2R_1 \\ R_4+R_1 \\ \end{array} \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & -3 & 1 & -1 \\ 0 & 0 & 9 & -2 & 6 \end{bmatrix}$$

$$\begin{array}{l} R_3+R_2 \\ R_4-3R_2 \\ \\ \end{array} \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{array}{l} R_3 \leftrightarrow R_4 \\ \\ B \sim \\ \end{array} \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The first 3 rows of B form the basis for the rowspace of A.

Row A $= \{(2, -3, 6, 2, 5), (0, 0, 3, -1, 1), (0, 0, 0, 1, 3)\}$

Left null space

The left null space of a $m \times n$ matrix A, written as $\text{Nul}(A^T)$, is the set of all solutions to the homogeneous equation $A^T y = 0$.

Theorem:-
The left null space of a $m \times n$ matrix A is a subspace of $\mathbb{R}^m$. Equivalently, the set of all solutions to a system $A^T y = 0$ of n homogeneous linear equations in m unknowns is a subspace of $\mathbb{R}^m$.

example:-
let $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ and $v = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

Determine if v belongs to the left null space of A.

sol^n: $A^T v = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3+3 \\ -6+6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$\therefore$ v is in $\text{Nul}\, A^T$.

Find a spanning set for the left null space of the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Sol?:- $A^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$A^T y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $y_1 = 0$

y_2 is a free variable.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ y_2 \end{bmatrix} = y_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = y_2 u.$$

$\therefore$ Every linear combination of u is an element of Nul A^T. Thus $\{u\}$ is a spanning set of Nul A^T.

Find the dimension and basis for the four fundamental subspaces of the matrix. (27)

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \quad 3\times 4$$

Sol^n

$$\begin{array}{l} R_2 - 2R_1 \\ R_3 + R_1 \end{array} \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$Ax = 0 \Rightarrow$

$x_1 + 3x_2 + 3x_3 + 2x_4 = 0$

$3x_3 + 3x_4 = 0$

$\cancel{3x_4 = 0}$

$\Rightarrow \cancel{x_4 = 0}$ $\quad \cancel{6x_3 + 0 = 0}$

$x_3 = -x_4$ $\quad \cancel{x_3 = 0}$

$\cancel{x_1 + 3x_2 = 0}$ $\quad \cancel{x_1 = -3x_2 + 5x_4}$

$\cancel{x_1 = -3x_2}$ $\quad x_1 = -3x_2 + x_4$

x_2, x_4 are a free variable

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_2 + x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore \{(-3, 1, 0, 0), (1, 0, -1, 1)\}$ form a basis of Nul A

and $\dim(\text{Nul} A) = 2$.

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{l} R_2 \div 3 \\ \cancel{R_3 \div 3} \end{array} \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\{(1, 2, -1), (3, 9, 3), \cancel{(2, 7, 4)}\}$ form a basis of Col A

and $\dim(\text{Col} A) = 2$

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\{(1, 3, 3, 2)\ (0, 0, 3, 3), \cancel{(0, 0, 0, 0)}\}$ form a basis of Row A

$\dim(\text{Row } A) = 2$

$\longrightarrow$

$$A^T = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ 3 & 9 & 3 \\ 2 & 7 & 4 \end{bmatrix}$$

$$\begin{array}{l} R_2 - 3R_1 \\ R_3 - 3R_1 \\ R_4 - 2R_1 \end{array} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 3 & 6 \\ 0 & 3 & 6 \end{bmatrix}$$

$$R_4 - R_3 \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^T y = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow y_1 + 2y_2 - y_3 = 0$$

$$3y_2 + 6y_3 = 0$$

$$\Rightarrow y_2 = -2y_3$$

$$\Rightarrow y_1 = -2y_2 + y_3$$

$$= 4y_3 + y_3$$

$$y_1 = 5y_3$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5y_3 \\ -2y_3 \\ y_3 \end{bmatrix}$$

$$= y_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore \{(5, -2, 1)\}$ forms a basis of $\mathrm{Nul}(A^T)$.

$\dim(\mathrm{Nul}\, A^T) = 1$.

* ~~The rank of A is~~ The dimension of the column space of A, is called the rank of A.
* Since Row A is the same as Col A^T, The dimension of the row space of A is the rank of A^T.
* The dimension of the null space is called the nullity of A.

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

<u>The Rank-Nullity Theorem.</u>

For a $m \times n$ matrix A,

$$\text{rank } A + \text{nullity } A = n.$$

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

<u>example!</u>

* If A is a 7×9 matrix with two dimensional null space, what is the rank of A?

$$\text{rank} + \text{nullity} = 9$$
$$\text{rank} + 2 = 9$$
$$\Rightarrow \underline{\text{rank} = 7}$$

* ~~If~~ Could a 6×9 matrix have a two-dimensional null space?

$$\text{rank} + \text{nullity} = 9 \qquad \text{Col } A \subseteq \mathbb{R}^6$$
$$\text{rank} + 2 = 9$$
$$\Rightarrow \underline{\text{rank} = 7}$$ which contradicts that

basis of Col A is a subspace of $\mathbb{R}^6$.

$\therefore$ a 6×9 matrix cannot have a two-dimensional null space

Linear Transformations

Consider the matrix equation $Ax = b$.
where A ~~xxxx~~ is $m \times n$ matrix, x - $n \times 1$ matrix, and b is $m \times 1$ matrix.
In other words x is a vector in $\mathbb{R}^n$ and b is a vector in $\mathbb{R}^m$.

Solving the equation $Ax = b$ amounts to finding all vectors x in $\mathbb{R}^n$ that are transformed into the vector b in $\mathbb{R}^m$ under the action of multiplication by A.
The correspondence from x to Ax is a function from one set of vectors to another.
This concept generalizes the common notation of a function as a rule that transforms one real number into another.
A transformation (or function or mapping) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector x in $\mathbb{R}^n$ a vector $T(x)$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the domain of T and $\mathbb{R}^m$ is called the codomain of T. The notation $T: \mathbb{R}^n \to \mathbb{R}^m$ indicates that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$.
For x in $\mathbb{R}^n$, the vector $T(x)$ in $\mathbb{R}^m$ is called the image of x. The set of all images $T(x)$ is called the range of T.

Matrix Transformations

For each x in $\mathbb{R}^n$, $T(x)$ is computed as Ax, where A is an $m\times n$ matrix. It is also denoted ~~by~~ the matrix transformation $x \mapsto Ax$.

Observe that the domain of T is $\mathbb{R}^n$ when A has n columns and codomain of T is $\mathbb{R}^m$ when each column of A has m entries.

The range of T is the set of all linear combinations of the columns of A, because each image $T(x)$ is of the form Ax.

ex Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$ $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $c = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ and define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(x) = Ax$,

So that $T(x) = Ax = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$.

a. Find $T(u)$, the image of u under the Transformation T.

b. Find an x in $\mathbb{R}^2$ whose image under T is b.

c. Is there more than one ~~vec~~ x whose image under T is b?

d. Determine if c is in the range of the transformation T.

Sol^n

a. $T(u) = Au = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$

b. $T(x) = b \Rightarrow Ax = b \Rightarrow \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$

$\longrightarrow$

which can be written in matrix form as

$$\begin{bmatrix} 1 & -3 & : & 3 \\ 3 & 5 & : & 2 \\ -1 & 7 & : & -5 \end{bmatrix}$$

Reducing to echelon form as below:

$R_2 = R_2 - 3R_1$, $R_3 = R_3 + R_1$ $\sim \begin{bmatrix} 1 & -3 & : & 3 \\ 0 & 14 & : & -7 \\ 0 & 4 & : & -2 \end{bmatrix}$ $R_3 = 14R_3 - 4R_2$ $\sim \begin{bmatrix} 1 & -3 & : & 3 \\ 0 & 14 & : & -7 \\ 0 & 0 & : & 0 \end{bmatrix}$

$\Rightarrow \left.\begin{aligned} x_1 - 3x_2 &= 3 \\ 14x_2 &= -7 \end{aligned}\right\} \Rightarrow \begin{aligned} x_2 &= -1/2 \\ x_1 &= 3/2 \end{aligned}$ Hence $x = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$

c. From b. we can see that, the vector x is unique.

d. Let $T(x) = c$ i.e, $Ax = c$ i.e, $\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$.

$\Rightarrow \begin{bmatrix} 1 & -3 & : & 3 \\ 3 & 5 & : & 2 \\ -1 & 7 & : & 5 \end{bmatrix}$

$R_2 = R_2 - 3R_1$, $R_3 = R_3 + R_1$ $\sim \begin{bmatrix} 1 & -3 & : & 3 \\ 0 & 14 & : & -7 \\ 0 & 4 & : & 8 \end{bmatrix}$ $R_3 = 14R_3 - 4R_2$ $\sim \begin{bmatrix} 1 & -3 & : & 3 \\ 0 & 14 & : & -7 \\ 0 & 0 & : & 140 \end{bmatrix}$

The third row show that $0 = 140$. (which is invalid)

Hence the system is inconsistent.

Hence c is not in the range of the transformation.

ex. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = Ax$.

Find the images under T of $u = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $v = \begin{bmatrix} a \\ b \end{bmatrix}$

Sol^n $T(u) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$

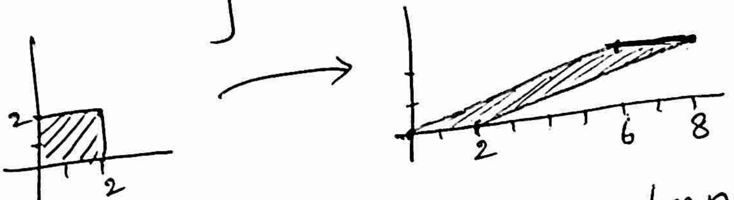
$T(v) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \end{bmatrix}$

ex. Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

Then the transformation $T(x) = Ax$,
transforms the square with vertices
$(0,0)$, $(2,0)$, $(2,2)$, $(0,2)$ to.

$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

sketching the above transformation, we see

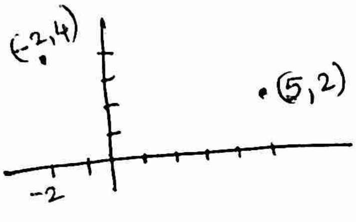


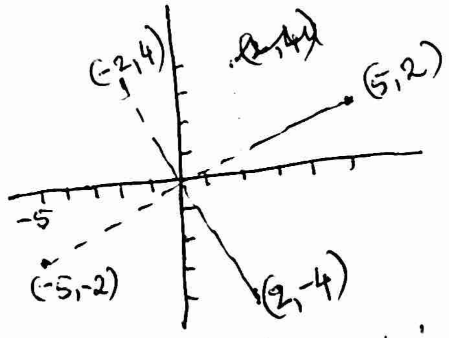
that the square has been transformed to a parallelogram. [In other words, the square has been stretched to a parallelogram, keeping the base intact].

ex. Let $u = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ $v = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

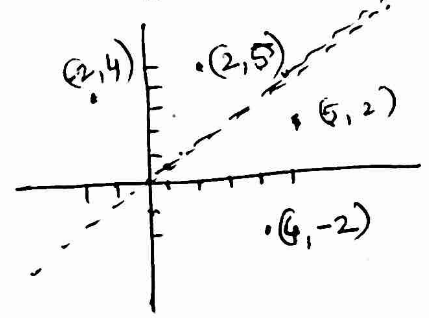
with $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $Au = \begin{bmatrix} -5 \\ -2 \end{bmatrix}$ $Av = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$

with $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. $Au = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ $Av = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$





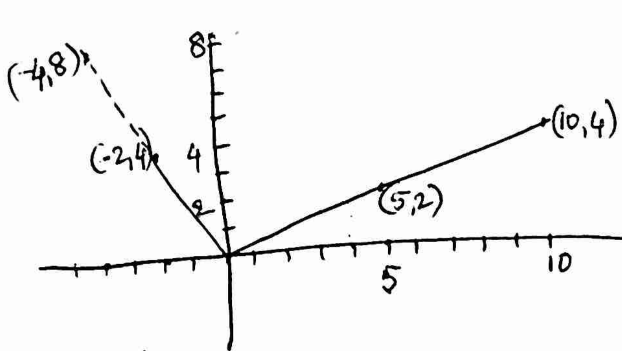
reflected about origin



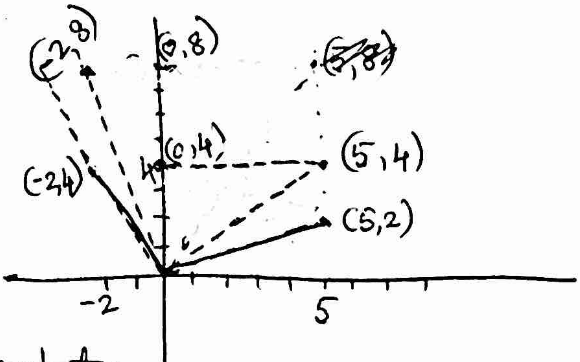
reflects about $y=x$.

with $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $Au = \begin{bmatrix} 10 \\ 4 \end{bmatrix}$, $Av = \begin{bmatrix} -4 \\ 8 \end{bmatrix}$

with $A = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ $Au = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$, $Av = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$



stretches the vector



~~stretches~~ rotates about origin and projects onto y-axis

If A is an $m \times n$ matrix, then the transformation $x \mapsto Ax$ has the properties $A(u+v) = Au + Av$ and $A(cu) = cAu$ for all u, v in $\mathbb{R}^n$ and all scalars c.

These properties, written in function notation, identify the most important class of transformations in Linear Algebra.

The transformation (or mapping) $T : V \to W$ is linear if:

(i) $T(u+v) = T(u) + T(v)$ for all u, v in the domain of T;

(ii) $T(cu) = cT(u)$ for all u and all scalars c.

Note: * Every matrix transformation is a linear transformation.

* Linear transformations preserve the operations of vector addition and scalar multiplication.

Q. If T is a mapping from $V_3(\mathbb{R})$ into $V_3(\mathbb{R})$ defined by $T(x_1, x_2, x_3) = (0, x_2, x_3)$, show that T is a linear transformation.

Sol: Let $\alpha = (x_1, x_2, x_3)$, $\beta = (y_1, y_2, y_3)$,

Such that $T(\alpha) = (0, x_2, x_3)$, $T(\beta) = (0, y_2, y_3)$.

Consider (i) $T(\alpha+\beta) = T(x_1+y_1, x_2+y_2, x_3+y_3)$

$$= (0, x_2+y_2, x_3+y_3)$$

$$= (0, x_2, x_3) + (0, y_2, y_3)$$

$$= T(\alpha) + T(\beta)$$

(ii) $T(c\alpha) = T(cx_1, cx_2, cx_3)$

$$= (0, cx_2, cx_3)$$

$$= c(0, x_2, x_3)$$

$$= cT(\alpha)$$

From (i) & (ii) T is a linear Transformation.

ex. If T is a mapping from $V_2(\mathbb{R})$ into $V_2(\mathbb{R})$ defined by $T(x, y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$, show that T is a linear Transformation.

Sol^n: Let $\alpha = (x_1, y_1)$, $\beta = (x_2, y_2) \in V_2(\mathbb{R})$, such that $T(\alpha) = (x_1\cos\theta - y_1\sin\theta, x_1\sin\theta + y_1\cos\theta)$

$$T(\beta) = (x_2\cos\theta - y_2\sin\theta, x_2\sin\theta + y_2\cos\theta).$$

Consider,

(i) $T(\alpha + \beta) = T(x_1 + x_2, y_1 + y_2)$

$$= ((x_1 + x_2)\cos\theta - (y_1 + y_2)\sin\theta, (x_1 + x_2)\sin\theta + (y_1 + y_2)\cos\theta)$$

$$= (x_1\cos\theta - y_1\sin\theta, x_1\sin\theta + y_1\cos\theta) + (x_2\cos\theta - y_2\sin\theta, x_2\sin\theta + y_2\cos\theta)$$

$$= T(\alpha) + T(\beta).$$

(ii) $T(c\alpha) = T(cx_1, cy_1)$

$$= (cx_1\cos\theta - cy_1\sin\theta, cx_1\sin\theta + cy_1\cos\theta)$$

$$= c(x_1\cos\theta - y_1\sin\theta, x_1\sin\theta + y_1\cos\theta)$$

$$= c\,T(\alpha)$$

From (i) & (ii) T is a linear transformation

* If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $T(1, 0) = (1, 1)$ & $T(0, 1) = (-1, 2)$, show that T maps the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ & $(0,1)$, into a parallelogram.

* Let $M(\mathbb{R})$ be the vector space of all 2×2 matrices over $\mathbb{R}$ and B be a fixed non-zero element of $M(\mathbb{R})$. Show that the mapping $T : M(\mathbb{R}) \to M(\mathbb{R})$, defined by $T(A) = AB - BA$, $\forall A \in M(\mathbb{R})$ is a linear map.

* If $T: V_1(\mathbb{R}) \to V_3(\mathbb{R})$ is defined by $T(x) = (x, x^2, x^3)$, verify whether T is linear or not.

Sol^n let $x, y \in V_1(\mathbb{R})$

Then $T(x) = (x, x^2, x^3)$, $T(y) = (y, y^2, y^3)$.

Consider $T(x+y) = (x+y, (x+y)^2, (x+y)^3)$

$$= (x+y, x^2+y^2+2xy, x^3+y^3+3x^2y+3xy^2)$$

$$\neq (x, x^2, x^3) + (y, y^2, y^3)$$

$$\neq T(x) + T(y).$$

$\therefore$ T is not a linear transformation.

* Find a linear transformation $f: \mathbb{R}^2 \to \mathbb{R}^2$, such that $f(1,0) = (1,1)$ and $f(0,1) = (-1,2)$.

Sol^n let $(x, y) \in \mathbb{R}^2$.

Then $(x,y) = x(1,0) + y(0,1)$

and $T(x,y) = x\,T(1,0) + y\,T(0,1)$

$$= x(1,1) + y(-1,2)$$

$$T(x,y) = (x-y, x+2y)$$

* Find the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1,1) = (0,1,2)$ and $T(-1,1) = (2,1,0)$.

Sol^n let $(x,y) \in \mathbb{R}^2$

Then $(x,y) = c_1(1,1) + c_2(-1,1) \Rightarrow (x,y) = (c_1 - c_2, c_1 + c_2)$

$\Rightarrow c_1 - c_2 = x,\ c_1 + c_2 = y \Rightarrow c_1 = \frac{x+y}{2}$ & $c_2 = \frac{y-x}{2}$.

Hence $(x,y) = \left(\frac{x+y}{2}\right)(1,1) + \left(\frac{y-x}{2}\right)(-1,1)$

Then $T(x,y) = \left(\frac{x+y}{2}\right)T(1,1) + \left(\frac{y-x}{2}\right)T(-1,1)$

$$= \left(\frac{x+y}{2}\right)(0,1,2) + \left(\frac{y-x}{2}\right)(2,1,0)$$

$$T(x,y) = (y-x,\ y,\ x+y)$$

Range and kernel of a Linear Transformation

Definition :-

Let $T : V \to W$ be a linear transformation.
The range of T is the set $R(T) = \{T(\alpha) \mid \alpha \in V\}$

Definition :-

Let $T : V \to W$ be a linear transformation.
The Kernel (or null space) of T is the set
$N(T) = \{\alpha \in V \mid T(\alpha) = 0\}$, where 0 is the zero vector of W.

Note :-

* For the identity map $I : V \to V$ the range is the entire space V and the kernel is the zero subspace of V.
* For the zero linear map $T : V \to W$ defined by $T(\alpha) = 0$ $\forall \alpha \in V$, the range $R(T) = \{0\}$ = zero space of V and the null space $N(T) = V$.

Theorem :-

Let $T : V \to W$ be a linear transformation.
Then (a) $R(T)$ is a subspace of W.
(b) $N(T)$ is a subspace of V
(c) T is one-one iff $N(T) = \{0\}$, where 0 is the zero vector of W.

Definition :-

Let $T : V \to W$ be a linear transformation. The dimension of the range space $R(T)$ is called the rank of the linear transformation T and is denoted by $r(T)$. The dimension of the nullspace $N(T)$ is called the nullity of the linear transformation T and is denoted by $n(T)$.

Theorem :-
Let $T : V \to W$ be a linear transformation.
If the vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ generates V, then the vectors $T\alpha_1, T\alpha_2, \ldots, T\alpha_n$ generates $R(T)$.

Theorem : (Rank - nullity theorem)
Let $T : V \to W$ be a linear transformation and V be a finite dimensional vector space.
Then $r(T) + n(T) = d[V]$
or $d[R(T)] + d[N(T)] = d[V]$
or rank + nullity = dimension of the domain.

example :-
Let $T : V \to W$ be a linear transformation defined by $T(x, y, z) = (x+y,\ x-y,\ 2x+z)$.
Find the range, null space, rank, nullity and hence verify the rank-nullity theorem.

Sol^n

$$T(e_1) = T(1,0,0) = (1,1,2) = \alpha_1$$
$$T(e_2) = T(0,+1,0) = (1,-1,0) = \alpha_2$$
$$T(e_3) = T(0,0,1) = (0,0,1) = \alpha_3$$

$\{\alpha_1, \alpha_2, \alpha_3\}$ generates $R[T]$.

Consider $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$|A| = -2 \neq 0$

$\therefore \{\alpha_1, \alpha_2, \alpha_3\}$ is L.I, thus it is a basis of $R[T]$.

$$d[R(T)] = 3$$

Let $\alpha \in R(T)$

Then $\alpha = c_1(\alpha_1) + c_2(\alpha_2) + c_3(\alpha_3)$

$= c_1(1,1,2) + c_2(1,-1,0) + c_3(0,0,1)$

$= (c_1 + c_2,\ c_1 - c_2,\ 2c_1 + c_3)$

$\therefore R(T) = \{ (c_1 + c_2,\ c_1 - c_2,\ 2c_1 + c_3) / c_1, c_2, c_3 \in \mathbb{R} \}$

Suppose $T(x,y,z) = (0,0,0)$

$\Rightarrow (x+y,\ x-y,\ 2x+z) = (0,0,0)$

$\Rightarrow x+y=0,\quad x-y=0,\quad 2x+z=0$

$\Rightarrow x=0,\quad y=0,\quad z=0$

$\therefore N(T) = \{(0,0,0)\}$

$\therefore d[N(T)] = 0$

rank + nullity $= 3+0 = 3 = d[V_3(\mathbb{R})]$.

* Find the range, nullspace, rank and nullity of the linear transformation $T: V_3(\mathbb{R}) \to V_2(\mathbb{R})$ defined by $T(x,y,z) = (y-x,\ y-z)$ and also verify Rank-nullity theorem.

Soln:

$T(1,0,0) = (-1,0) = \alpha_1$

$T(0,1,0) = (1,1) = \alpha_2$

$T(0,0,1) = (0,-1) = \alpha_3$

$R(T) = L\{\alpha_1, \alpha_2, \alpha_3\}$

Consider
$$\begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\therefore d[R(T)] = 2 \qquad$ Basis of $R(T) = \{(-1,0), (1,1)\}$

Let $\alpha \in R(T)$

$\alpha = c_1\alpha_1 + c_2\alpha_2$

$= c_1(-1,0) + c_2(1,1)$

$= \{(-c_1+c_2, c_2)\}$

$\therefore R(T) = \{(-c_1+c_2, c_2) \mid c_1, c_2 \in \mathbb{R}\}$.

Suppose $\cancel{N(T)} = T(x,y,z) = (0,0)$

$\Rightarrow (y-x, y-z) = (0,0)$

$\Rightarrow y = x, \; y = z \Rightarrow x = y = z$.

$\therefore N(T) = \{(a,a,a) \mid a \in \mathbb{R}\}$.

Basis of $N(T) = \{(1,1,1)\}$.

$\therefore d[N(T)] = 1$.

Rank + Nullity $= 2 + 1 = 3 = d[V_3(\mathbb{R})]$

* If T is a linear transformation from $V_3(\mathbb{R})$ into $V_4(\mathbb{R})$ defined by $T(1,0,0) = (0,1,0,2)$, $T(0,1,0) = (0,1,1,0)$, $T(0,0,1) = (0,1,-1,4)$. Find the range, nullspace, rank, nullity of T.

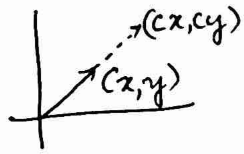
* Find the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by, $T(e_1) = e_1 - e_2$, $T(e_2) = 2e_1 + e_3$, $T(e_3) = e_1 + e_2 + e_3$. Also find the range, null space, rank and nullity of T.

Suppose x is an n-dimensional vector. when A multiplies x, it transforms that vector into a new vector Ax. This happens at every point x of the n dimensional space $\mathbb{R}^n$. The whole space is transformed or "mapped into itself" by the matrix A.

Stretch:

A multiple of the identity matrix, $A = cI = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$ stretches every ~~no~~ vector by the same factor c. The whole space expands $(c>1)$ or contracts $(0<c<1)$ or goes through the origin and out the opposite side $(c<0)$.

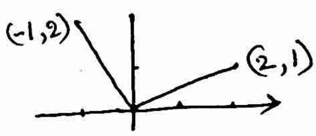
$$Ax = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$$



Rotation:

A rotation matrix turns the whole space around the origin.

$$Q_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$



$\theta = 90° = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \rightarrow$ Rotation about $90°$

$Q_{-\theta} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \rightarrow$ Rotation in backwards through θ.

$$Q_{-\theta} = (Q_\theta)^T$$

$$Q_\theta \cdot Q_{-\theta} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$Q_\theta \cdot Q_\theta = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix} = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} = Q_{2\theta}$$

$$Q_\theta \cdot Q_\phi = Q_{\theta+\phi} \; ; \; Q_\theta^{-1} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} = (Q_\theta)^T$$

Projection:-

A projection matrix takes the whole space onto a lower-dimensional subspace (not invertible).

$$P = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$$

$|P| = 0 \Rightarrow$ inverse doesnot exist.

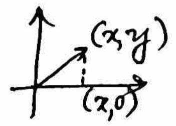
$$P^2 = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}\begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} = P$$

$\Rightarrow$ projection twice on the θ-line is the same as projecting once on θ-line.

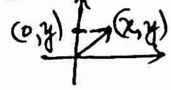
points on the Y-axis is projected to $(0,0)$.

points on θ-line projected onto itself.

$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ transforms each vector $\begin{bmatrix} x \\ y \end{bmatrix}$ in the plane to nearest point $\begin{bmatrix} x \\ 0 \end{bmatrix}$ on the horizontal axis. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$



$P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ transforms $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} 0 \\ y \end{bmatrix}$

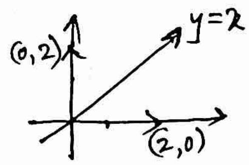


Reflection:-

A reflection matrix transforms every vector into its image on the opposite side of a mirror.

$$H = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

$$H^2 = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$



$\Rightarrow$ Two reflections bring back the original.

$H^{-1} = H \Rightarrow$ self inverse.

$\theta = 45^\circ$ $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ~~transfo~~ gives the reflection through $y = x$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$

$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ gives the reflection through x axis $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$.