1. **Orthogonal vectors**:- Two vectors u & v in R^n are orthogonal to each other if $u \cdot v = 0$.

ex:- $u = (1,3)$ & $v = (6,-2)$ are orthogonal in R^2 since $u \cdot v = 0$.

2. **Orthogonal set**:- A set of vectors $\{u_1, u_2, \ldots, u_p\}$ in R^n is said to be an orthogonal set if each pair of distinct vectors from the set is orthogonal.

i.e. if $u_i \cdot u_j = 0, i \neq j$

Ex:- The set $\{u_1, u_2, u_3\}$ such that $u_1 = (1, \sqrt{2}, 1)$
$u_2 = (1, 0, -1)$ & $u_3 = (1, -\sqrt{2}, 1)$

$$u_1 \cdot u_2 = 0, \quad u_1 \cdot u_3 = 0, \quad u_2 \cdot u_3 = 0$$

3. **Orthonormal set**:- The set $(u_1, u_2 - - u_p)$ is an orthonormal set if it is an orthogonal set of unit vector

Ex:- $\{e_1, e_2 - - e_n\}$, standard basis for R^n, is an orthonormal set.

4. **Orthogonal Basis**:- An orthogonal basis for a subspace W of R^n is a basis for W i.e also an orthogonal set.

ex:- $u_1 = (3,1,1)$, $u_2 = (-1,2,1)$ $u_3 = (-1/2, -2, 7/2)$
is an orthogonal basis for R^3. since the set S is $\{(3,1,1), (-1,2,1), (-1/2, -2, 7/2)\}$ is linearly independent

(i) $\begin{vmatrix} 3 & 1 & 1 \\ -1 & 2 & 1 \\ -1/2 & -2 & 7/2 \end{vmatrix} \neq 0$ & S forms a basis for R^3

(ii) S is an orthogonal ...

5. Orthonormal Basis: An orthonormal basis for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthonormal set.

Ex:- Show that $\{v_1, v_2, v_3\}$ is a orthonormal basis of $\mathbb{R}^3$ where $v_1 = \left(\frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}\right)$, $v_2 = \left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

$$v_3 = \left(-\frac{1}{\sqrt{66}}, -\frac{4}{\sqrt{66}}, \frac{7}{\sqrt{66}}\right)$$

$$\rightarrow \begin{vmatrix} 3/\sqrt{11} & 1/\sqrt{11} & 1/\sqrt{11} \\ -1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{66} & -4/\sqrt{66} & 7/\sqrt{66} \end{vmatrix} \neq 0$$

∴ Linearly Independent

$$|v_1| = \sqrt{\frac{9}{11} + \frac{2}{11}} = \sqrt{1} = 1$$

$$|v_2| = \sqrt{+\frac{1}{6} + \frac{4}{6} + \frac{1}{6}} = 1$$

$$|v_3| = \sqrt{\frac{1}{66} + \frac{16}{\sqrt{66}} + \frac{49}{66}} = 1$$

$$v_1 \cdot v_2 = \left(-\frac{3}{\sqrt{66}} + \frac{2}{\sqrt{66}} + \frac{1}{\sqrt{66}}\right) = 0$$

$$v_2 \cdot v_3 = \left(\frac{1}{\sqrt{6}\sqrt{6}} - \frac{8}{\sqrt{76}} + \frac{7}{\sqrt{6}\sqrt{6}}\right) = 0$$

$$v_1 \cdot v_3 = 0.$$

∴ It is an orthonormal basis.

Orthogonal Matrix :- A [illegible] real entries and satisfying the condition $A^{-1} = A^T$ is called an orthogonal Matrix.

ex:- $$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$Q^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

2×2 MAT,

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1}\begin{bmatrix} a & \cdots \\ -c & d \end{bmatrix}$

$$Q^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$\therefore \boxed{Q^{-1} = Q^T}$

$\therefore$ Q is an orthogonal Matrix.

Q: Show that the matrix $A = \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix}$ is orthogonal.

$\rightarrow$ $$A^T = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$|A| = \frac{1}{3}\left(-\frac{1}{9} + \frac{4}{9}\right) + \frac{2}{3}\left(\frac{2}{9} + \frac{4}{9}\right) + \frac{2}{3}\left(\frac{4}{9} + \cdots\right.$$

$$= \frac{1}{\cancel{3}}\left(\frac{\cancel{3}}{9}\right) + \frac{2}{\cancel{3}}\left(\frac{\cancel{6}}{9}\right) + \frac{2}{\cancel{3}}\left(\frac{6}{9}\right)^2 = \frac{1}{9} + \frac{4}{9} + \frac{4}{9}$$

$\text{Adj}\,A = \frac{1}{3}\left(\frac{1}{9}+\frac{4}{9}\right)\quad \frac{2}{3}\left(\frac{-2}{9}+\frac{4}{9}\right)\quad \frac{2}{3}\left(\frac{4}{9}+\frac{2}{9}\right)$

$-\frac{2}{3} \quad ! \quad = \quad AA^T = I \quad \frac{1}{3}\left(\frac{1}{3}\right) \quad \frac{4}{34}$

$$\begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} = \frac{1}{9}+\frac{4}{9}+\frac{4}{9} \quad \frac{2}{9}+\frac{2}{9}-\frac{4}{9} \quad \frac{2}{9}$$

A is an orthogonal Matrix. The row vectors of A

$\left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right), \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$

are orthnormal

$u_1 \cdot u_2 = 0 \qquad u_1 \cdot u_1 = 1$

$u_1 \cdot u_3 = 0 \qquad u_2 \cdot u_2 = 1$

$u_2 \cdot u_3 = 0 \qquad u_3 \cdot u_3 = 1$

The columns of the vector is also orthonormal

Note:- suppose A is $n \times n$ matrix with real entries then

(i) matrix A is orthogonal if and only if the row vectors of a form an orthonormal basis of R^n

(ii) mat A is orthogonal if and only if the column vector of A form an orthonormal basis of R^n.

Q) Verify whether the set ~~$\left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)\right\}$~~

$\left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(-\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}\right), \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) \right\}$ is an orthonormal basis for R^3.

$\rightarrow$

$|v_1| = \sqrt{\frac{1}{2}+\frac{1}{2}} = \sqrt{\frac{2}{2}} = 1$

$|v_2| = \sqrt{\frac{2}{36}+\frac{2}{36}+\frac{8}{9}} = \sqrt{\frac{2+2+32}{36}} = 1$

$|v_3| = \sqrt{\frac{4}{9}+\frac{4}{9}+\frac{1}{9}} = 1$

$$\begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{\sqrt{2}}{6} & \frac{\sqrt{2}}{6} & \frac{2\sqrt{2}}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{18} + \frac{4\sqrt{2}}{9}\right) + \frac{1}{\sqrt{2}}\left(-\frac{\sqrt{2}}{18} - \dots\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{9\sqrt{2}}{18}\right) + \frac{1}{2}\left(-\frac{9\sqrt{2}}{18}\right)$$

$$= \frac{1}{2} + \frac{1}{4} \neq 0.$$

$\therefore$ S is linearly Independent.

$\therefore$ S forms a basis for R_3

$$u_1 \cdot u_2 = -\frac{1}{6} + \frac{1}{6} + 0 = 0$$

$$u_1 \cdot u_3 = \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} + 0 = 0$$

$$u_2 \cdot u_3 = -\frac{2\sqrt{2}}{18} - \frac{2\sqrt{2}}{18} + \frac{2\sqrt{2}}{9} = 0$$

$$u_1 \cdot u_1 = 1$$

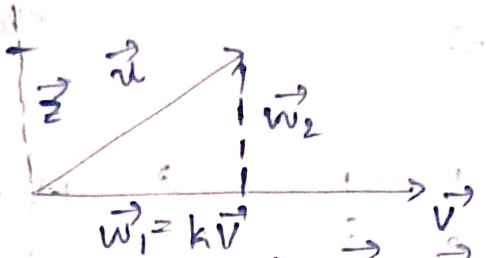
$$u_2 \cdot v_2 = 1$$

$$v_3 \cdot v_3 = 1$$

Orthogonal Projection: Let vector $\vec{u}$ & vector $\vec{v}$ be the vectors in inner product space V such that vector $\vec{v} \neq 0$ then the orthogonal projection of $\vec{u}$ onto $\vec{v}$ is equal to

$$\text{Proj}_{\vec{v}}\, \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\right) \cdot \vec{v}$$

Given a non zero vector $\vec{v}$ in R^n, Consider the problem of decomposing the vector $\vec{u}$ in R^n into the sum of two vectors, one a multiple of $\vec{v}$ and the other orthogonal to $\vec{v}$.



$\vec{u} = \vec{w_1} + \vec{w_2}$

We wish to write $\vec{u} = K\vec{v} + \vec{w_2}$

$\vec{z}$ is some vector orthogonal to vector $\vec{v}$

Given any scalar K, $\vec{z} = \vec{u} - K\vec{v}$

$\vec{z}$ is orthogonal to $\vec{v}$ if and only if $\vec{z} \cdot \vec{v} = 0$.

$z = w_2$

Consider $\vec{w_2} \cdot \vec{v} = 0$ $\therefore$

$$(\vec{u} - K\vec{v}) \cdot \vec{v} = 0$$

$$\vec{u} \cdot \vec{v} - K(\vec{v} \cdot \vec{v}) = 0$$

$$\Rightarrow K = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$$

Hence, $\vec{w_1} = K\vec{v} = \left(\frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{v}}\right)\vec{v}$.

Here $\vec{w_1}$ is called orthogonal projection of $\vec{u}$ onto $\vec{v}$ and the $\vec{z}$ is called the component of ~~ve~~ $\vec{u}$ orthogonal to $\vec{v}$

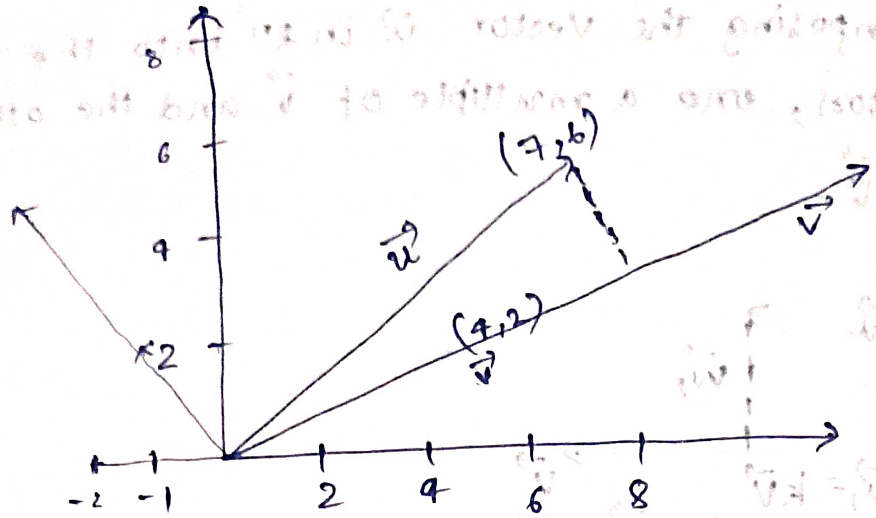
1) Let $\vec{u} = (7, 6)$ & $\vec{v} = (4, 2)$. Find the orthogonal projection of $\vec{u}$ onto $\vec{v}$

$28+12$

$$\text{Proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\right) \cdot \vec{v} = \left(\frac{\cancel{40}^{2}}{\cancel{20}}\right) \cdot (4, 2)$$

$$= (8, 4)$$

orthogonal components



$\vec{w_2} = \vec{u} - \vec{w_1}$

$= (7,6) - (8,4)$

$\vec{w_2} = (-1, 2)$

$\vec{w_1} = 8, 4$

$\vec{w}_1 \cdot \vec{w}_2 = 0$

Q) In R^2, the orthogonal projection of $u = (4,2)$ onto $(3,4)$ is ______

$$\text{Proj}_{\vec{v}} \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\right) \cdot \vec{v}$$

$$= \left(\frac{12+8}{9+16}\right)(3,4)$$

$$= \left(\frac{20}{25}\right)(3,4)$$

orthogonal proj $\longrightarrow$ $= \left(\frac{12}{5}, \frac{16}{5}\right) = \vec{w_1}$

orthogonal component $\vec{w_2} = \vec{u} - \vec{w_1} = (4,2) - \left(\frac{12}{5}, \frac{16}{5}\right)$

$= \left(8/5, -\frac{6}{5}\right)$

if $\vec{w_2} \cdot v = 0$ then $w_2 \to$ orthogonal component.

$$\left(8/5, -\frac{6}{5}\right) \cdot (3,4) = 0$$

$$\vec{w_1} \cdot \vec{w_2} = 0$$

$$w_1 = kv$$

Q) Find the orthogonal projection of $u=(6,2,4)$ onto $v=(1,2,0)$ on R^3

$\rightarrow$
$$\text{Proj}_{\vec{v}}\vec{u} = \left[\frac{(6+4+0)}{(1+4+0)}\right](1,2,0)$$

$$= \frac{10}{5}\ 2(1,2,0)$$

$$= \underline{(2,4,0)}$$

orthogonal component $\vec{w_2} = \vec{u} - \vec{w_1} = (4,-2,4)$

Gram Schmidt

Gram Schmidt ~~Or~~thogonalization Process

It is a simple algorithm for producing an orthogonal or orthonormal basis for any non-zero subspace of R^n.

Ex:- Let ~~W is equal~~ $W = \text{span}\{x_1, x_2\}$ where $x_1 = (3,6,0)$ and $x_2 = (1,2,2)$ we construct the orthogonal basis v_1, v_2 by taking the projection of $\vec{x_2}$ onto $\vec{x_1}$ and the component of $\vec{x_2}$ orthogonal to ~~component~~ $\vec{x_1}$ which is $\vec{x_2} - \vec{P}$ where p is projection of $\vec{x_2}$ onto $\vec{x_1}$.

$\rightarrow$ Let $u_1 = x_1$

$$u_2 = \vec{x_2} - \left(\frac{\vec{x_2}\cdot\vec{u_1}}{\vec{u_1}\cdot\vec{u_1}}\right)\vec{u_1} = (1,2,2) - \frac{(1,2,2)\cdot(3,6,0)}{(3,6,0)\cdot(3,6,0)}(3,6,0)$$

$$= (1,2,2) - \frac{15}{45}(3,6,0)$$

$$= (1,2,2) - (1,2,0)$$

$$u_2 = (0,0,2)$$

$\{u_1, u_2\} = \{(1,2,2), (0,0,2)\}$ $\quad v_1 \cdot v_2 \neq 0$ not an orthogonal basis

$(v_1, v_2) = \{(3,6,0), (0,0,2)\}$.

Gram Schmidt Algorithm :-

Given a basic $\{x_1, x_2 \cdots x_p\}$ for a subspace of W of R^n, we check $\vec{v_1} = \vec{x_1}$, $\vec{v_2} = \vec{x_2} - \left(\frac{\vec{x_2} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}}\right) \vec{v_1}$

, $\vec{v_3} = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$, $\cancel{v_p = x_p - \frac{x_p \cdot v_1}{v_1 \cdot v_1}}$

$v_p = x_p - \frac{x_p \cdot v_1}{v_1 \cdot v} v_1 - \frac{x_p \cdot v_2}{v_2 \cdot v_2} v_2 - \cdots - \frac{x_p \cdot v_{p-1}}{v_{p-1} \cdot v_{p-1}} v_{p-1}$

Then the set $\{v_1, v_2 \cdots v_p\}$ is an orthogonal basis for W

Q) Let W = span$\{x_1, x_2, x_3\}$ where $x_1 = (0,1,2)$, $x_2 = (1,1,2)$, $x_3 = (1,0,1)$. Construct an orthogonal basis $\{v_1, v_2, v_3\}$ for W.

→
$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} = 0 - 1(1-2) + 2(-1)$$
$$= +1 - 2$$
$$= -1 \neq 0$$

~~Linear~~

forms a basis for R_3

by Gram schmidth process,

$v_1 = x_1 = (0,1,2)$

$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 = (1,1,2) - \frac{(1,1,2)(0,1,2)}{(0,1,2)(0,1,2)}$

$= (1,1,2) - \frac{0+1+4}{1+4} (0,1,2)$

$= (1,1,0)$ ∴ Check if $v_1 \cdot v_2 = 0$

$$v_3 = x_3 - \frac{x_3 v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

$$= (1,0,1) - \frac{(1,0,1)(0,1,2)}{(0,1,2)(0,1,2)}(0,1,2) - \frac{(1,0,1)(1,0,0)}{(1,0,0)(1,0,0)}(1,0,0)$$

$$= (1,0,1) - \frac{2}{5}(0,1,2) - \frac{1}{1}(1,0,0)$$

$$= (1,0,1) - \left(0, \frac{2}{5}, \frac{4}{5}\right) - (1,0,0)$$

$$v_3 = \left(0, -\frac{2}{5}, \frac{1}{5}\right)$$

∴ orthogonal basis for W is,

$$= \left\{ (0,1,2), (1,1,2), \left(0, -\frac{2}{5}, \frac{1}{5}\right) \right\}$$
$$\{v_1, v_2, v_3\}$$

a) Find the orthogonal basis for the column space of the matrix

$$\begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$$

already forms a basis

if $x > R^n$

if no of columns < no of rows don't check

$-8(5) + (-10)(-2)$ $-40 + 20$

$8(6) + 6(-8)$ $8(-1) + 6(-2)$

$48 -$ $-8 - 12$

→

No need to check

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & -5 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 6 & -1 \\ 0 & -10 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & -20 \\ 0 & 0 & -20 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & -20 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C.S = \left\{ \underset{x_1}{\begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}}, \underset{x_2}{\begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix}}, \underset{x_3}{\begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix}} \right\}$$

$$v_1 = x_1 = (3,1,-1,3)$$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$= (-5,1,5,-7) - \frac{(-5,1,5,-7)(3,1,-1,3)}{(3,1,-1,3)(3,1,-1,3)} (3,1,-1,3)$$

$$= (-5,1,5,-7) - \frac{(-15+1-5-21)}{(9+1+1+9)} (3,1,-1,3)$$

$$(-5,1,5,-7) - \frac{-\cancel{40}^{2}}{\cancel{20}} (3,1,-1,3)$$

$$(-5,1,5,-7) - (-6,-2,+2,-6)$$

$$v_2 = (1,3,3,-1)$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

$$= (1,1,-2,8) - \frac{(1,1,-2,8)(3,1,-1,3)}{(3,1,-1,3)(3,1,-1,3)} (3,1,-1,3) - \frac{(1,1,-2,8)(1,3,3,-1)}{(1,3,3,-1)(1,3,3,-1)} (1,3,3,-1)$$

$$= (1,1,-2,8) - \frac{\overset{30}{(3+1+2+24)}}{(9+1+1+9)} (3,1,-1,3) - \frac{\overset{4-6-8}{(1+3-6-8)}}{(1+9+9+1)} (1,3,3,-1)$$

$$= (1,1,-2,8) - \frac{30}{20} (3,1,-1,3) - \left(\frac{-1\cancel{0}}{2\cancel{0}}\right) (1,3,3,-1)$$

$$= (1,1,-2,8) - \left(\frac{9}{2}, \frac{3}{2}, -\frac{3}{2}, \frac{9}{2}\right) - \left(-\frac{1}{2}, -\frac{3}{2}, -\frac{3}{2}, \frac{1}{2}\right)$$

$$(-3,1,1,3)$$

$1-\frac{9}{2}+\frac{1}{2}$

$\frac{-8}{2}$

$8-\frac{9}{2}-\frac{1}{2}$

$\frac{16-9-1}{2}$

$-2+\frac{3}{2}+\frac{3}{2}$

$\frac{-4+6}{2}$

∴ orthog...

$$\left\{ \begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 3 \\ -1 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \\ 1 \\ 3 \end{pmatrix} \right\}$$

Q) Construct an orthonormal basis for the subspace $W = \text{span}\{x_1, x_2, x_3\}$ of R^4 where $x_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$,

$x_2 = \begin{bmatrix} 5 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $x_3 = \begin{bmatrix} 2 \\ 3 \\ 4 \\ -1 \end{bmatrix}$

(Apply Gram Schmidt process for A (column))

$$A = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 1 & 3 \\ 1 & 1 & 4 \\ -1 & 1 & -1 \end{bmatrix}$$

→ Applying Gram Schmidt process,

$v_1 = x_1 = (1, -1, 1, -1)$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$= (5,1,1,1) - \frac{(5,1,1,1)(1,-1,1,-1)}{(1,-1,1,-1)(1,-1,1,-1)}(1,-1,1,-1)$$

$$= (5,1,1,1) - \frac{(5-\cancel{1}+\cancel{1}-1)}{(1+1+1+1)}(1,-1,1,-1)$$

$$= (5,1,1,1) - \frac{4}{4}(1,-1,1,-1)$$

$$= (5,1,1,1) - (1,-1,1,-1)$$

$$= (4, 2, 0, 2)$$

$$v_3 = x_3 - \frac{x_3 v_1}{v_1 v_1} v_1 - \frac{x_3 v_2}{v_2 v_2} v_2 =$$

$$= (2,3,4,-1) - \frac{(2,3,4,-1)(1,-1,1,-1)}{(1,-1,1,-1)(1,-1,1,-1)}(1,-1,1,-1) - \frac{(2,3,4,-1)(4,2,0,2)}{(4,2,0,2)(4,2,0,2)}(4,2,0,2)$$

$= (2,3,4,-1) - \frac{(2-3+4+1)}{4}(1,-1,1,-1) - \frac{8+6+0-2}{24}(\ldots)$

$= (2,3,4,-1) - (1,-1,1,-1) - (2,4,0,4)$

$= (-1,3,3,-1)$

To get orthonormal basis

$u_1 = \frac{V_1}{\sqrt{4}} = \cancel{\frac{1}{2}, \frac{-1}{\sqrt{4}}}, \left(\frac{1}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{-1}{2}\right)$

$u_2 = \frac{V_2}{\sqrt{24}} = \frac{4}{\sqrt{24}}, \frac{4}{2\sqrt{6}} \left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}\right)$

$u_3 = \frac{V_3}{\sqrt{20}} = \left(\frac{-1}{\sqrt{20}}, \frac{3}{\sqrt{20}}, \frac{3}{\sqrt{20}}, \frac{-1}{\sqrt{20}}\right) = \left(\frac{-1}{2\sqrt{5}}, \frac{3}{2\sqrt{5}}, \frac{3}{2\sqrt{5}}, \frac{-1}{2\sqrt{5}}\right)$

$\{u_1, u_2, u_3\}$ forms an orthonormal basis

Q) Find the orthogonal basis for the column space of the matrix $\begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$

$\rightarrow x_1 = V_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$

$V_2 = x_1 - \frac{x_1 \cdot V_1}{V_1 \cdot V_1} V_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} -$

QR Factorization

If A is an $n \times n$ matrix with Linearly Independent columns, then A can be factored as $A = QR$, where Q is an $m \times n$ matrix whose columns form an orthonormal basis for column space of A and R is an $n \times n$ ~~m~~ upper triangular invertible matrix with positive entries on its diagonal

Q) Find the QR factorization of $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$\{x_1, x_2, x_3\}$ where $x_1 = (1,0,1)$ or $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$x_2 = (2,1,0)$ $x_3 = (0,1,1)$

By Gram's schmidt process we have

$$\boxed{v_1 = x_1 = (1,0,1)}$$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 = (2,1,0) - \frac{(2,1,0)(1,0,1)}{(1,0,1)(1,0,1)} (1,0,1)$$

$$= (2,1,0) - \frac{2}{2}(1,0,1)$$

$$\boxed{v_2 = (1,1,-1)}$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

$$v_3 = (0,1,1) - \frac{(0,1,1)(1,0,1)}{2}(1,0,1) - \frac{(0,1,1)(1,1,-1)}{(1,1,-1)(1,1,-1)}(1,1,-1)$$

$\cancel{1} - \cancel{1}$

$$= (0,1,1) - \frac{1}{2}(1,0,1) - 0$$

$v_3 \cdot v_1 = 0$
$v_2 \cdot v_3 = 0$ (To check)

$$= (0,1,1) - (1/2, 0, 1/2)$$

$$\boxed{v_3 = (-1/2, 1, 1/2)}$$

$$u_1 = \left(\frac{1}{\sqrt{2}}, 0, 1/\sqrt{2}\right)$$

$$u_2 = \frac{1}{\sqrt{3}}(1,1,-1) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$$

$\frac{1}{4} + \frac{1}{4} + 1$

$\frac{1+1+4}{4}$

$3/2$

$$u_3 = \frac{2}{\sqrt{3}}(-1/2, 1, 1/2) = \cancel{\frac{-1}{\sqrt{3}}, 2/\sqrt{3}, 1/3}$$

$$= \left(-1/\sqrt{6}, \sqrt{\frac{2}{3}}, 1/\sqrt{6}\right)$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}_{3\times 3}$$

$$R = Q^T A$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ -1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{matrix} 1/2 + 1/\sqrt{2} & 2/\sqrt{2} \; \cancel{1/\sqrt{2}} & 1/\sqrt{2} \\ 1/\sqrt{3} - 1/\sqrt{3} & 2/\sqrt{3} + \frac{1}{\sqrt{3}} & 1/\sqrt{3} - 1/\sqrt{3} \\ -1/\sqrt{6} + 1/\sqrt{6} & -2/\sqrt{6} + 2/\sqrt{6} & 2/\sqrt{6} + 1/\sqrt{6} \end{matrix}$$

$$= \begin{bmatrix} \sqrt{2} & \sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 3/\sqrt{6} \end{bmatrix}$$

Q) Find the QR Factorization of the matrix

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

$\cancel{x_1 = (1,1,1)}$ $x_1 = (1,1,1,1)$

$x_2 = (-1,4,4,-1)$

$x_3 = (4,-2,2,0)$

$\sqrt{1+16+16+1}$

$2+3$

$\sqrt{34}$

$v_1 = x_1 = (1,1,1,1)$

$$v_2 = x_2 - \frac{x_2 v_1}{v_1 v_1} v_1$$

$$= (-1,4,4,-1) - \frac{(-1,4,4,-1)(1,1,1,1)}{1+1+1+1}(1,1,1,1)$$

$$= (-1,4,4,-1) - \frac{\overset{-2+8}{(-1+4+4-1)}}{4}(1,1,1,1)$$

$-1-3/2$

$$= (-1,4,4,-1) - (3/2, 3/2, 3/2, 3/2)$$

$-1+3/2$

$-2+$

$-1-3/2$

$4-3/2$

$\sqrt{v_2}$

$$= (-5/2, 5/2, 5/2, -5/2)$$

$\frac{25}{4}+\frac{25}{4}+\frac{25}{4}+\frac{25}{4}$

$\frac{-10}{100/25}$

$-15+5$

$-10, -5, 5, 0$

$$v_3 = (4,-2,2,0) - \frac{\overset{4-2+2}{(4,-2,2,0)(1,1,1,1)}}{4}(1,1,1,1) - \frac{(4,-2,2,0)(-5/2, 5/2, 5/2, -5/2)}{\cancel{\sqrt{100}}}(-5/2, \ldots)$$

$$= (4,-2,2,0) - (1,1,1,1) + \frac{10}{25}(-5/2, 5/2, 5/2, 5/2)$$

$5/5$

$\frac{10}{25} \times \frac{5}{2}$

$$= (3,-3,1,-1) - \cancel{(1/4, 1/4, 1/4)} \; (-1,1,1,-1)$$

$$= (4,-4,0,0) \quad \sqrt{16+16}$$

$$u_1 = \frac{v_1}{2} = (1/2, 1/2, 1/2, 1/2)$$

$u_2 = \frac{v_2}{\sqrt{25}} = \frac{v_2}{|v_2|_5} = \frac{v_2}{5} = (-1/2, 1/2, 1/2, -1/2)$

$u_3 = \frac{v_3}{4\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{0}{2}, \frac{0}{2}\right)$ $1/2, -1/2, 1/2, -1/2$

$$Q = \begin{bmatrix} 1/2 & -1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & -1/\sqrt{2} \\ 1/2 & 1/2 & 0/2 \\ 1/2 & -1/2 & 0/2 \end{bmatrix}$$

$R = Q^T A$

$$= \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0/2 & 0/2 \end{bmatrix}_{3\times 4} \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{matrix} 2 & \overset{4-1}{-1/2 + 2 + 2 - 1/2} & 2 \cancel{-1} \cancel{+1} \\ 0 & \underset{4+2/2}{+1/2 + 2 + 2 + 1/2} & -2 \cancel{-1} \cancel{+1} = 0 \\ 0 & -1/\sqrt{2} - 4/\sqrt{2} & 4/\sqrt{2} + 2/\sqrt{2} \end{matrix}$$

$\frac{6}{\sqrt{2}}$ $\frac{3 \times \sqrt{2} \times \sqrt{2}}{\sqrt{2}}$

$$= \begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

Q) Find the QR factorization of the matrix

$$A = \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}$$

→ Let $\{x_1, x_2, x_3\}$ forms a basis for col(A), where
column space

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 2 \\ 1 \\ 4 \\ -4 \\ +2 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 5 \\ -4 \\ -3 \\ 7 \\ 1 \end{bmatrix}$$

$v_1 = x_1 = (1, -1, -1, 1, 1)$

$u_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$

$= (2, 1, 4, -4, -2) - \frac{(2 - 1 - 4 - 4 + 2)}{(1+1+1+1+1)} (1, -1, -1, 1, 1)$

$= (2, 1, 4, -4, -2) - \left(\frac{-5}{5}\right)(1, -1, -1, 1, 1)$

$= (2, 1, 4, -4, +2) + (1, -1, -1, 1, 1)$

$= (3, 0, 3, -3, 3)$

$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$

6 $\quad$ 9-21

$= (5, -4, -3, 7, 1) - \frac{(5+4+3+7+1)}{(5)} {}^{20/4} (1, -1, -1, 1, 1) - \frac{(15 - 9 - 21 + 3)}{(9+9+9+9)} (3, 0, \ldots$

$= (5, -4, -3, 7, 1) - (4, -4, -4, 4, 4) + \frac{12}{36}(3, 0, 3, -3, 3)$

$= (1, 0, 1, 3, -3) + (1, 0, 1, -1, 1)$

$= (2, 0, 2, 2, -2)$

$\sqrt{36}$

$\therefore$ the orthogonal basis =

$\{(1,-1,-1,1,1), (3,0,3,-3,3), (2,0,2,2,-2)\}$

$4+4+4+4$

16

orthonormal basis

$u_1 = (1/\sqrt{5}, -1/\sqrt{5}, -1/\sqrt{5}, 1/\sqrt{5}, 1/\sqrt{5})$

$u_2 = (1/2, 0, 1/2, -1/2, 1/2)$

$u_3 = (1/2, 0, 1/2, 1/2, -1/2)$

$$Q = \begin{bmatrix} 1/\sqrt{5} & 1/2 & 1/2 \\ -1/\sqrt{5} & 0 & 0 \\ -1/\sqrt{5} & 1/2 & 1/2 \\ 1/\sqrt{5} & -1/2 & 1/2 \\ 1/\sqrt{5} & 1/2 & -1/2 \end{bmatrix}$$

$5/2$

$1/2 - 1/2 + 1/2$

$1/2 - 1/2$

$1+2+2+1$

$$R = Q^T A = \begin{bmatrix} 1/\sqrt{5} & -1/\sqrt{5} & -1/\sqrt{5} & 1/\sqrt{5} & 1/\sqrt{5} \\ 1/2 & 0 & 1/2 & -1/2 & 1/2 \\ 1/2 & 0 & 1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}$$

$6 - 10$

$5/2 - 3/2 - 7/2 + 1/2$

$1/2 - 1/2 - 1/2 + 1/2$

$1/\sqrt{5} + 1/\sqrt{5} + 1/\sqrt{5} + 1$

$$\begin{bmatrix} \frac{5}{\sqrt{5}} & -\frac{5}{\sqrt{5}} & \frac{20}{4\sqrt{5}} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

$1+2+2+1$

$2/\sqrt{5} - 1/\sqrt{5} - 4/\sqrt{5} - 4/\sqrt{5} + 2/\sqrt{5}$

$5/\sqrt{5} - 4/\sqrt{5}$ $5+4+3+7+1$

$20/\sqrt{5}$

$Q \times R = I$

Eigenvalue & Eigenvector :- only squ

An eigen vector of $m \times n$ matrix A is a non zero vector X such that, $\boxed{AX = \lambda X}$ for some scalar λ. A scalar 'λ' is called an eigen value of A. If there is a non-trivial solution X of $AX = \lambda X$ such an X is called an ~~eigen~~ eigenvector corresponding to λ.

Ex:- $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ and $X = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$

$$AX = \begin{bmatrix} 6-15 \\ 30-10 \end{bmatrix} = \begin{bmatrix} -24 \\ -20 \end{bmatrix} = \underbrace{-4}_{\text{eigen value}} \underbrace{\begin{bmatrix} 6 \\ -5 \end{bmatrix}}_{\text{eigen vector}}$$

If diagonal A is a square matrix of order 'n' we can find the matrix $\boxed{A - \lambda I}$ where I is the nth ordered unit matrix. The Determinant of this matrix is equated to 0 ie

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn}-\lambda \end{vmatrix}$$

is called characteristic eqn of matrix A.

On expanding Determinant the characteristic

A

... takes the ...

$$(-1)^n \lambda^m + K_1 \lambda^{n-1} + K_2 \lambda^{n-2} + \dots + K_H = 0$$

where ~~K's is a linear expon~~ K is expressible in terms of the elements of A_{ij}.

The roots of this eqn are called the characteristic roots of ~~linear~~ latent roots or eigen values of the matrix

Properties of eigenvalues

1. Any square matrix A & its transpose A^T have the same eigen values.

2. The eigen values of triangular matrix are just the diagonal elements of the matrix

3. The eigen values of diagonal matrix are just the diagonal elements of the matrix.

4. The sum of the eigen values of the matrix is the sum of the elements of the principle diagonal (trace)

5. The product of eigen values of matrix is equal to its determinant.

6. If λ is an eigen value of a matrix A then $1/\lambda$ is the eigen value of A^{-1}.

7. If $\lambda_1, \lambda_2 \dots \lambda_n$ are the eigen values of a matrix A, then (A^m) has the eigen values $\lambda_1^m, \lambda_2^m, \dots \lambda_n^m$ (m is +ve integer)

8. If $\lambda_1, \lambda_2 \dots \lambda_n$ are the eigen values of a matrix A, then eigen values of KA are $K\lambda_1, K\lambda_2, \dots K\lambda_n$.

Note - For a given eigen vector, there corresponds only one eigen value.

- For a given eigen value, there corresponds infinitely many eigen vectors.

- Corresponding to n distinct eigen values, we get n independent eigen vectors. When two or more

Impt eigen values are equal it may or may not be possible to get linearly independent eigen vectors corresponding to the repeated roots.

Diagonalization of a matrix :-

Suppose a $n \times n$ matrix has n linearly independent eigen vectors. If these eigen vectors are the columns of a matrix (P), then $\boxed{P^{-1}AP}$ is a is a digonal matrix D. The eigen values of matrix A are on the diagonal of the matrix D i.e. $D = P^{-1}AP = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

Criteria for Diagonizability / Spectral Decomposition

Impt An $n \times n$ matrix A is diagonizable if and only if A has ~~En~~ n linearly independent ~~vectors~~ eigen vectors

- Any matrix with distinct eigen value can be diagonalized.
- The diagonalization matrix is not unique
- Not all matrices possesses n linearly independent eigen vectors so that not all matrices are ~~dig~~ diagonalizable

Find eigen vectors of matrix $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ and also decompose the matrix A into P, D, P^{-1}. Also find A^5.

The characteristic equation is given by ~~determina~~

$|A - \lambda I| = 0$

trace = sum of principal diagonal elements

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow 10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$\lambda = 1, 6$$

If $X = \begin{bmatrix} x \\ y \end{bmatrix}$ is an eigen vector corresponding to the eigen value λ, then $[A - \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\cancel{(5x - \lambda x) + 4y}$

For $\lambda = 1$ we have, $\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

4 ~~4~~

0

$$\begin{bmatrix} 4 & 4 & : & 0 \\ 1 & 1 & : & 0 \end{bmatrix} \quad R_2 \to R_1 - 4R_2$$

$$\sim \begin{bmatrix} 4 & 4 & : & 0 \\ 0 & 0 & : & 0 \end{bmatrix}$$

$$\Rightarrow 4x + 4y = 0$$

$y = K$

$$\boxed{x = -y}$$

$\therefore$ The eigen vector is $X = \begin{bmatrix} -K \\ K \end{bmatrix} = K \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

for $\lambda = 6$, $\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}$

$-x + 4y = 0$

$x + 4y = 0$

$X = \begin{bmatrix} 4k \\ k \end{bmatrix} = k \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Consider the modal matrix $P = \begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix}$

$P^{-1} = \frac{1}{-5} \begin{bmatrix} 1 & -4 \\ -1 & -1 \end{bmatrix}$

$= \begin{bmatrix} -1/5 & 4/5 \\ 1/5 & 1/5 \end{bmatrix}$

$D = P^{-1} A P$

$= \begin{bmatrix} -1/5 & 4/5 \\ 1/5 & 1/5 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} -1 + 4/5 & -4/5 + 8/5 \\ 1 + 1/5 & 4/5 + 2/5 \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix}$

$= \begin{bmatrix} -1/5 & 4/5 \\ 6/5 & 6/5 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix} = \begin{matrix} +1/5 + 4/5 & -4/5 + 4/5 \\ -6/5 + 6/5 & \frac{24}{5} + \frac{6}{5} \end{matrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$

D → Not unique → $\begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$

If $P = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}$

$A = \ldots$ $\qquad D = P^{-1}AP$

$= \begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix}$ $\qquad PD = \cancel{PP^{-1}}AP$

$PDP^{-1} = APP^{-1}$

$= \begin{bmatrix} -1 & 4 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1^5 & 0 \\ 0 & 6^5 \end{bmatrix} \begin{bmatrix} -1/5 & 4/5 \\ 1/5 & 1/5 \end{bmatrix}$ $\qquad PDP^{-1} = A$

$A^2 = AA$

$= PD^2P^{-1}$

$= \begin{bmatrix} -1 & 31104 \\ 4 & 7776 \end{bmatrix} \begin{bmatrix} 1/5 & 4/5 \\ 1/5 & 1/5 \end{bmatrix}$

$= \quad 1/5 + \frac{31104}{5} \qquad -4/5 + \frac{31104}{5}$

$= \quad -4/5 + \frac{7776}{5} \qquad 16/5 + \frac{7776}{5}$

$= \begin{bmatrix} 6221 & 6220 \\ 1555 & 1556 \end{bmatrix}$

Q) Find the eigen value & eigen vector of matrix

$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$\rightarrow |A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)[(5-\lambda)(3-\lambda)] - 2(6-2\lambda) + 0$

$(2-\lambda)(15 - 5\lambda - 3\lambda + \lambda^2) - 2(6-2\lambda)$

$(2-\lambda)(15 - 5\lambda - 3\lambda + \lambda^2 - 12 + 4\lambda)$

$(2-\lambda)(15 - \cancel{5\lambda} - 3\lambda + \lambda^2) - 12 + 4\lambda$

$(2-\lambda)(3 - 4\lambda + \lambda^2)$

$30 - 16\lambda + 2\lambda^2 - 15\lambda + 8\lambda^2 - \lambda^3 - 12 + 4\lambda$

$-\lambda^3 + 10\lambda^2 - 27\lambda + 18 = 0$

$6 - 8\lambda + 2\lambda^2 - 3\lambda + 4\lambda^2 - \lambda^3$

$\lambda^3 \ldots - 10\lambda^2 + 2\ldots$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 2-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda = 6$,

$$\begin{bmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda^2 - S_1\lambda + S_2 = 0$

$S_1 \to$ sum of diagonal

$S_2 \to$ determinant

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 \to$ sum of princi diag

$S_2 \to$ sum of minors of principal diagonal

$S_3 \to$ Determinant

$$\left[\begin{array}{ccc|c} -4 & 2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-4x + 2y = 0$

$-3z = 0$

$\boxed{z = 0}$

$-4x = -2y$

$x = \frac{y}{2}$

$$X = \begin{bmatrix} 2k \\ k \\ 0 \end{bmatrix} = k \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} \to \text{eigen vector}$$

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 2 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$-x + 2y = 0$$

$$4y = 0$$

$$y = 0 \quad x = 0 \quad z$$

$$X = z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

for $\lambda = 1$ $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$2z = 0$$

$$z = 0$$

$$x + 2y = 0$$

$$x = -2y$$

$$X = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Consider the modal matrix $P = \begin{bmatrix} 1/2 & 0 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$\frac{1}{2}(0-1) - 2(1)$

$-\frac{1}{2} - 2$

$$P^{-1} = \frac{1}{(-5/2)} \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & -1/2 \\ 0 & -5/2 & 0 \end{bmatrix}$$

(or)

$\begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \left| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right]$ Gauss Jordann

find minor

(-ve)

(-ve) (-ve)

(-ve)

transpose

$$= -\frac{2}{5} \begin{bmatrix} -1 & -2 & 0 \\ 0 & 0 & -5/2 \\ 1 & -1/2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 & 0.8 & 0 \\ 0 & 0 & 1 \\ -0.4 & 0.2 & 0 \end{bmatrix}$$

~~$D = P^{-1}AP$~~

$D = P^{-1}AP$

$$= \begin{bmatrix} 0.4 & 0.8 & 0 \\ 0 & 0 & 1 \\ -0.4 & 0.2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Diagonalise the matrix $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$2+6$

$|A - \lambda I| = 0$

$6+3-4$

5

$$\begin{vmatrix} 1-\lambda & 6 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$1(6) - 6(3) + 10$

$6 - 18$

$$\lambda^3 - 6\lambda^2 + 5\lambda + 12 = 0$$

$$\lambda = 4, -1, 3$$

For $\lambda = 4$,

$$\begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} -3 & 6 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$-3x + 6y + z = 0$

$-z = 0$

$\boxed{z = 0}$

$\cancel{+3}x = \cancel{+6}^{2}y$

$x = 2y$

$$X = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda = -1$, $\begin{bmatrix} 2 & 6 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 6 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\boxed{z = 0}$ $2x + 6y = 0$

$x = -3y$

$$X = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda = 3$,

$$\left[\begin{array}{ccc|c} -2 & 6 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left(\begin{array}{ccc|c} -2 & 6 & 1 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$-2+6$

$$4y + z = 0 \qquad 4y = -z$$

$$-2x + 6y + z = 0 \qquad \boxed{y = -z/4}$$

$$-2x - \frac{6z}{4} + z = 0$$

$$-2x - \frac{2z}{4} = 0$$

$$-2x = z/2$$

$$x = -z/4$$

$$X = \begin{bmatrix} -1/4 \\ -1/4 \\ 1 \end{bmatrix}$$

Increasing

$$\text{Modal Matrix } P = \begin{bmatrix} 2 & -3 & -1/4 \\ 1 & 1 & -1/4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ -0.2 & 0.4 & 0.05 \\ 0 & 0 & 1 \end{bmatrix}$$

$D = P^{-1} A P$

$$= \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ -0.2 & 0.4 & 0.05 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & -3 & -1/4 \\ 1 & 1 & -1/4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow \text{not unique}$$

Gagana Achar

Shortcut

$$\lambda = -1 \quad \begin{pmatrix} 2 & 6 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$\begin{vmatrix} 6 & 1 \\ 3 & 0 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 2 & 6 \\ 1 & 3 \end{vmatrix}$$

$$\frac{x}{-3} = \frac{y}{0} = \frac{z}{0}$$

If all three numbers below are 0, then don't take that

4) Decompose matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ into P, D, P^{-1} also find A^5.

5) ~~find the diag~~ Decompose $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

symmetric matrix

real roots

4) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ $9-1$ $1(9-1)$ 8

$8+3+3$

$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$

$\underline{\lambda = 4, 1, 2}$

For $\lambda = 1$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

4
$-2+2$
$4-1$

$\sim \left[\begin{array}{ccc|c} 0 & -1 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow z = 0$

$y = 0$

x

$X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

For $\lambda = 2$, $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \sim \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

~~$z = 0$~~ $x = 0$

~~$x = 0$~~ $y - z = 0$

~~$y = 0$~~ $y = z$

$\lambda = 4,\quad \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \sim \left(\begin{array}{ccc|c} -3 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$x = 0 \quad -y - z = 0$

$-y = z$

$y = -z \qquad X = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Model matrix $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

$P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 16 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

$1(5-1) - 1(\overset{-2}{1-3}) + 3(\overset{14}{1-15})$

$4 + 2 - 42$

-40

$(5-1) + (1-9)$

$4 + -8 + 4$

$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$

(36)

$\lambda^3 - 7\lambda^2 + 36 = 0$

$\lambda = 6, -2, 3.$

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda = \cancel{2}, -2$

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & 3 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & 1 & 3 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow 5y = 0$$

$$\boxed{y = 0}$$

$$3x + y + 3z = 0$$

$$3x = -3z$$

$$x = -z$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

For $\lambda = 3$,

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \sim \begin{matrix} -2 & 1 & 3 \\ 0 & 5 & 5 \end{matrix} \quad \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 3 \\ 3 & 1 & -2 \end{bmatrix}$$

4

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & -5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x+2y+z=0$

$5y+5z=0$

$5y=-5z$

$\boxed{y=-z}$

$x-2z+z=0$

$x=z$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ -z \\ z \end{bmatrix} = z\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

for $\lambda=6$

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ -5 & 1 & 3 \\ 3 & 1 & -5 \end{bmatrix}$$

$1+3$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & -4 & 8 \\ 0 & 4 & -8 \end{bmatrix} \quad \left.\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -4 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right.$$

$-4y+8z=0$

$4y=8z$

$y=2z$

$x-y+z=0$

$x-2z+z$

$x=z$

$$\begin{matrix} z \\ 2z \\ z \end{matrix} \quad \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

~~-0.5 -0.5 -0.5~~

$$\begin{bmatrix} -0.5 & 0 & 0.5 \\ 0.3333 & -0.333 & 0.3333 \\ 0.1666 & 0.1333 & 0.166 \end{bmatrix}$$

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

6) $\begin{bmatrix} 1 & 3 & -7 \\ 4 & 2 & 1 \end{bmatrix}$ Decompose this matrix. Singular Value Decomp.