LINEAR ALGEBRA

Number System :-

$N = \{1, 2, 3, 4, \dots\}$

$W = \{0, 1, 2, 3, \dots\}$

Z OR $I = \{\dots -3, -2, -1, 0, 1, 2, 3, \dots\}$

$Q = \{p/q, q \neq 0, p, q \in Z\}$

Q' = irrational Ex:- $\sqrt{2}, i,$

$Q' \cup Q = R$ - Real Numbers.

C - Complex Numbers $z = a + ib$

Every Real Numbers can be expressed as complex number. Ex:- $-2.8 + i(0)$.

Binary Operation :- $(*)$

	+	−	$\cdot(\times)$	/
N	✓	✗	✓	✗
W	✓	✗	✓	✗
Z	✓	✓	✓	✗
Q	✓	✓	✓	✓
$R - \{0\}$ (only for division)	✓	✓	✓	✓ → (discarding zero)
C	✓	✓	✓	✓

2 5

2 −5

$2/3$, $7/3$

$p = 0$, $q = 3/2$

Semi Group

G → Semi Group
- Closure Property: a, b $a * b \in G$
- Associative property: $(a * b) * c = a * (b * c)$

Group { Inverse Element $a * a^{-1} = e$

under Addition
- Natural Numbers are semi group. not a group
- Whole Numbers are semi Group "
- Integers are considered as Group under operation

all that 5 property

Commutative property $a * b = b * a$

↳ If this condition is satisfied then → Commutative Group / Abelian Group.

$\mathbb{Z}$ is an abelian Group under addition.

Ring-

$(S, +, \cdot)$

$(S, +) \longrightarrow$ abelian Group

$(S, \cdot) \rightarrow$ closure property. $a, b \in Z$ $a \cdot b \in S$

$(S, \times) \rightarrow$ ~~Distri~~ Associative $(a \times b) \times c = a \times (b \times c)$

$\rightarrow$ Distributive law $(a+b) \times c = ac + bc$

$a \cdot (b+c) = ab + ac$

$\mathbb{Z}$ is a Ring

Commutative Ring

$a \times b = b \times a$

commutative Ring with unity

$1 \in S$ $1a = a(1) = a$

$\mathbb{Z}$ is a commutative Ring

Integral domain { commutative Ring.
$1 \in S$ $1a = a(1) = a$
$ab = 0$ $a = 0$ or $b = 0$

These 2 props along with ...tive Ri...

Can only left

Field (F) :- An algebraic structure that satisfies all the above condition & all the non-zero elements must have a multiplicative inverse.

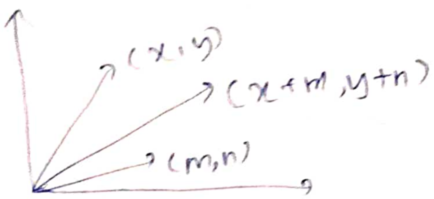
$\mathbb{Z}$ is not a Field bcz it doesn't have multiplicative inverse.

$3+0=3$ $\qquad$ $3\times 1=3$

$3+(-3)=0$ $\qquad$ $3\times(\ )=1$ $\quad \frac{1}{3}\notin \mathbb{Z}$

Example: $\mathbb{Q}, \mathbb{R}, \mathbb{C}$

Vector :→ Quantity which has both magnitude & direction.



(Two vectors can't be multiplied. vector & scalar can be multiplied)

Vector Space :-

If V is a non empty set &

The vector space has following props $\begin{cases} u+v \\ ku \end{cases}$

Addition

① closure $\quad u, v \in V, \ u+v \in V$ → under Addition

② Associative $\quad u, v, w \in V$ $\quad$ ~~under Ad~~

$$(u+v)+w = u+(v+w)$$

③ $u+(\overset{?}{e}) = u$

④ $u+(\ ) = e$

⑤ $u+v = v+u$

Multiplication

k → scalar

u & v → vector

⑥ $k\in F, c\in F$ Field

$ku \in V$

~~Only~~ Vector can be multiplied only with scalar

⑦ $k(cu) = (kc)u$

⑧ $k(u+v) = ku+kv$

⑨ $(k+c)u = ku+cu$

⑩ $1\cdot u = u \quad 1\in F$

- Polynomials

under Addition
→ satisfies closure property ✓
→ Associative ✓
→ $(-7x^3+3x^2-5x+0) + (0x^3+0x^2+0x+0=0) = -7x^3+3x^2-5$
Additive Identity ✓
→ Additive Inverse ✓
→ Commutative property ✓

under Multiplication
→ closure ✓
→ Associative ✓
→ Distributive (1) (2) ✓
→ Multiplicative Identity ✓

Hence Polynomial ≤ 3 is a vector space.

* $\{(x,y) / \overbrace{x \geq 0, y \geq 0}^{\text{I QUAD}}\}$

{ If $x>0$ & $y>0$
No identity element

Additive Inverse Doesn't Exist
Not a vector space.

Notes 18/12/23

Vector Space - Let V be a non empty set of elements called vectors we define two operations on the set V:-
Vector Addition & Scalar Multiplication.

Let u, v & w be vectors in the set V. The set V is called a vector space if it satisfies following axioms.

1) vector Addition $u+v$ is also in the vectorspace(V) $u+v \in V$

(Closure under Vector Addition)

2) Commutative Property :- $u+v = v+u$

3) Associative Property :- $(u+v)+w = u+(v+w)$

4) There is a vector called zero vector in V denoted by 0 which satisfies $u+0=u$ for every vector u in V.

(Additive Identity)

5) Additive Inverse:-

For every Vector u, there is a vector $-u$ which satisfies $u+(-u) = 0$ Additive Token

(Not simply zero)

6) Let K be a real scalar, then Ku is also in V ($Ku \in V$). We say that closure under scalar multiplication

7) Associative law for scalar multiplication:-

Let K & C be real scalars then $K(Cu)=(KC)u$

8) Distributive Law for scalars.

Let K & C be real scalars then $(K+C)u = Ku+Cu$.

9) Distributive Law for vectors.

Let K be a real scalar, then $K(u+v) = Ku+Kv$.

10) Identity element:- For every Vector u in V we have $1u=u$, $u \in V$ & $1 \in F$

No of variables − rank of matrix

Q :-

1) Let F be a field & N be a positive Integer, Let $V_n(F)$ be the set of all ordered n-tuples of the elements of the field F, $V_n(F) = \{(x_1, x_2, \dots x_n) / x_i$

Prove that the set $V_n(F)$ of all ordered n-tuples of real numbers is a vector space with usual vector Addition & scalar Multiplication.

→ Proof:-

Let $u = (x_1, x_2, x_3, \dots x_n)$

$v = (y_1, y_2 \dots \dots y_n)$

$w = (z_1, z_2, \dots \dots z_n)$

$k, c \in F, u, v, w \in R^n$

① $u+v = (x_1, x_2, \dots x_n) + (y_1, y_2, \dots y_n)$

$= (x_1+y_1, x_2+y_2, \dots x_n+y_n) \in R^n$

② $(u+v)+w = (x_1+y_1, x_2+y_2, \dots x_n+y_n)+(z_1, z_2 \dots z_n)$

$= (x_1+y_1+z_1, x_2+y_2+z_2, \dots x_n+y_n+z_n)$

$= (x_1, x_2, \dots x_n) + (y_1+z_1, y_2+z_2, \dots y_n+z_n)$

$= u+(v+w)$

③ $u+0 = (x_1, x_2, \dots x_n)+(0, 0, \dots 0)$

$= (x_1, x_2, \dots x_n)$

$u+0 = u$

④ $u+(-u) = (x_1, x_2, \dots x_n)+(-x_1, -x_2, \dots -x_n)$

$= (x_1-x_1, x_2-x_2, \dots x_n-x_n)$

$u+(-u) = (0, 0, \dots 0) = 0$

(5) $u+v = (x_1+y_1, x_2+y_2, \ldots x_n+y_n)$

$= (y_1+x_1, y_2+x_2, \ldots, y_n+x_n)$

$= (y_1, y_2, \ldots y_n) + (x_1, x_2, \ldots x_n)$

$= v+u$

(6) $k(u) = (kx_1, kx_2, kx_3 \ldots kx_n) \in \mathbb{R}^n$

(7) $k(cu) = k(cx_1, cx_2, cx_3 \ldots cx_n)$

$= kc(x_1, x_2, x_3 \ldots x_n)$

$= (kc)u$

(8) $(k+c)u = (k+c)(x_1, x_2, x_3, \ldots x_n)$

$= ((k+c)x_1, (k+c)x_2 \ldots (k+c)x_n)$

$= [(kx_1+cx_1), (kx_2+cx_2) \ldots (kx_n+cx_n)]$

$= (kx_1, kx_2, \ldots kx_n) + (cx_1, cx_2, \ldots cx_n)$

$= k(x_1, x_2, \ldots x_n) + c(x_1, x_2 \ldots x_n)$

$= ku + cu$

(9) $k(u+v) = k(x_1+y_1, x_2+y_2, \ldots x_n+y_n)$

$(kx_1+ky_1, kx_2+ky_2, \ldots kx_n+ky_n)$

$= k[(x_1, x_2, \ldots x_n) + (y_1, y_2, y_3 \ldots y_n)]$

$((kx_1, kx_2, \ldots kx_n) + (ky_1, ky_2, \ldots ky_n))$

$= k(x_1, x_2, \ldots x_n) + k(y_1, y_2, y_3 \ldots y_n)$

$= ku + ku$

(10) $1 \cdot u = 1(x_1, x_2 \ldots x_n)$

$= (x_1, x_2, \ldots x_n)$

$= \underline{u}$

$\therefore \mathbb{R}^n$ is a vector space.

Q2:- Let V be the set of all polynomials of degree $\leq n$ with coefficients in the field F, together with zero polynomial, then show that V is a vector space under Addition of polynomials & scalar multiplication of poly with the scalar $K, C \in F$

Proof:- Let $u = a_0 + a_1 x + \dots + a_n x^n$

$v = b_0 + b_1 x + \dots + b_n x^n$

$w = c_0 + c_1 x + \dots + c_n x^n$.

$K, C \in F$ & $u, v, w \in V$

① $u+v = (a_0 + a_1 x + \dots + a_n x^n) + (b_0 + b_1 x + \dots + b_n$

$= (a_0 + b_0) + (a_1 x + b_1 x) + \dots + (a_n x^n + b_n x^n) \in V$

$= (a_0 + b_0) + (a_1 + b_1) x + \dots + (a_n + b_n) x^n \in V$

~~② $(u+v)+w = (a_0+b_0) + (a_1 x + b_1 x) + \dots + a_n x^n + b_n$~~

② $(u+v)+w = (a_0+b_0) + (a_1+b_1)x + \dots + (a_n+b_n) x^n + (c_0 + c_1 x +$

$= (a_0 + b_0 + c_0) + a_1 x + (b_1 + c_1) x + \dots + a_n x^n + (b_n + c_n$

$= a_0 + a_1 x + \dots + a_n x^n + (b_0 + c_0) + (b_1 + c_1) x + \dots + (b_n + c_n)$

$= a_0 + a_1 x + \dots + a_n x^n + ((b_0 + b_1 x + \dots + b_n x^n) + (c_0 + c_1 x + \dots + c_n x^n$

$= u + (v + w)$

③ $u + 0 = (a_0 + a_1 x + \dots + a_n x^n) + (0, 0x, \dots, 0x^n)$

$= a_0 + a_1 x + \dots + a_n x^n$

$= u$

$+(-a_0-a_1x-\cdots-a_nx^n)$

$=\underline{0}\,(a_0-a_0)+(a_1x-a_1x)+\cdots+(a_nx^n-a_nx^n)$

$=\underline{(0,0,\cdots 0)}$

⑤ $u+v=(a_0+a_1x+\cdots a_nx^n)+(b_0+b_1x+\cdots+b_nx^n)$

$=(a_0+b_0)+(a_1+b_1)x+\cdots+(a_n+b_n)x^n$

$=(b_0+a_0)+\underset{b_1x+a_1x}{(b_1+a_1)x}+\cdots+\underset{b_nx^n+a_nx^n}{(b_n+a_n)x^n}$

$=(b_0+b_1x+\cdots+b_nx^n)+(a_0+a_1x+\cdots+a_nx^n)$

$=\underline{v+u}$

⑥ $K(u)=K(a_0+a_1x+\cdots+a_nx^n)$

$=Ka_0+Ka_1x+\cdots+Ka_nx^n\in V$

⑦ $(K+c)u=(K+c)(a_0+a_1x+\cdots+a_nx^n)$

$=(K+c)a_0+(K+c)a_1x+\cdots+(K+c)a_nx^n$

$=(Ka_0+ca_0)+\underset{Ka_1x+ca_1x}{(Ka_1+ca_1)x}+\cdots+\underset{Ka_nx^n+ca_nx^n}{(Ka_n+ca_n)x^n}$

$\cancel{=Ka_0+Ka_1+\cdots+Ka_nx^n+ca_0+ca}$

$=Ka_0+Ka_1x+\cdots+Ka_nx^n+ca_0+ca_1x+\cdots+ca_nx^n$

$=K(a_0+a_1x+\cdots+a_nx^n)+c(a_0+a_1x+\cdots+a_nx^n)$

$=\underline{Ku+cu}$

⑧ $K(cu)=K(ca_0+ca_1x+\cdots+ca_nx^n)$

$=Kc(a_0+a_1x+\cdots+a_nx^n)$

$=\underline{Kcu}$

9) $k(u+v) = k\left((a_0+b_0)+(a_1+b_1)x+\dots+(a_n+b_n)x^n\right)$

ka_1x+kb_1x

$= ka_0+kb_0+(ka_1+kb_1)x+\dots+(ka_n+kb_n)x^n$

$= (ka_0+ka_1x+\dots+ka_nx^n)+(kb_0+kb_1x+\dots+$

$= k(a_0+a_1x+\dots+a_nx^n)+k(b_0+b_1x+\dots+$

$= ku+kv$

10) $1\cdot u = 1(a_0+a_1x+\dots+a_nx^n)$

$= 1a_0+1a_1x+\dots+1a_nx^n$

$= a_0+a_1x+\dots+a_nx^n$

$= \underline{\underline{u}}$

$\therefore$ Set of Polynomials of degree $\leq n$ over the field F is a vector space.

3) Let R^+ be the set of all positive Reals. Define the operation of addition & scalar Multiplication as below

i) $\alpha+\beta = \alpha\beta$, $\forall \alpha,\beta \in R^+$

ii) $c\alpha = \alpha^c$, $\forall \alpha \in R^+$, $c \in R$

Check R^+ is a vector space over the real field.

→ 1) $u+v = \alpha+\beta \Rightarrow \alpha\beta \in R^+$

2) ~~Let~~ Let $\alpha =$

2) $(u+v)+w = (uv)+w$

$= uvw$

$= u(vw)$

$= u(v+w)$

$= u+(v+w)$

(3) $u+1 = u$ $\quad \alpha + 1 = \alpha(1) = \alpha$

(4) $u + u^{-1} = 1$ $\quad \alpha + \alpha^{-1} = 1$

(5) $u + v = uv$ $\quad \alpha + \beta = \alpha\beta$

$= vu$ $\quad = \beta\alpha$

$= v + u$ $\quad = \beta + \alpha$

$-3(7) = 7^{-3} \in R^{+} > 0$

(6) $K(u) = u^K \in R^+$

(7) $(K+C)u = u^{K+C} = u^K u^C = Ku \cdot Cu$ $\quad (K+C)\alpha = \alpha^{K+C}$

$Ku + Cu = Ku + Cu$ $\quad = \alpha^K \alpha^C$

$= K\alpha \, C\alpha$

(8) $K(Cu) = K(u^C)$ $\quad = K\alpha + C\alpha$

$= (u^C)^K$

$= (u^C)^K = (u^K)^C = (CK)u$

$\cancel{= (KC)u}$

(9) $K(u+v) = K(uv)$ $\quad K(\alpha + \beta) = K(\alpha\beta)$

$= (uv)^K$ $\quad = (\alpha\beta)^K$

$= u^K v^K$ $\quad = \alpha^K \beta^K$

$= Ku \, Kv$ $\quad = K\alpha \, K\beta$

$= Ku + Kv$ $\quad = K\alpha + K\beta$

(10) $1 \cdot u = u^1 = u$ $\quad 1 \cdot \alpha = \alpha^1 = \alpha$

4) Let $V = R^2$ → 2 dimension space & defined addition & scalar multiplication operation as follows

If $u = (u_1, u_2)$

$v = (v_1, v_2)$

$u + v = (u_1 + v_1, u_2 + v_2)$

if K is any real Number

then define $Ku = (Ku_1, 0)$. Check whether V is a vector space.

1) $u+v = (u_1+v_1,\ u_2+v_2) \in \mathbb{R}^2$

2) $(u+v)+w = (u_1+v_1,\ u_2+v_2) + (w_1, w_2)$

$= (u_1+v_1+w_1,\ u_2+v_2+w_2)$

$= (u_1+(v_1+w_1),\ u_2+(v_2+w_2))$

$= (u_1, u_2) + (v_1+w_1,\ v_2+w_2)$

$= \underline{u+(v+w)}$

3) $u+0 = (u_1, u_2) + (0,0)$

$= (u_1+0,\ u_2+0)$

$= (u_1, u_2) = \underline{u}$

4) $u+(-u) = \cancel{u_1+u_2} \ (u_1, u_2) + (-u_1, -u_2)$

$= (u_1-u_1,\ u_2-u_2)$

$= (0,0)$

5) $u+v = (u_1+v_1,\ u_2+v_2)$

$= (v_1+u_1,\ v_2+u_2)$

$= \underline{v+u}$

6) $Ku = K(u_1, u_2)$ $\qquad (K+c)(x,y)$

$= \underline{(Ku_1, 0)} \in \mathbb{R}^2$ $\qquad (K+c)(x),\ y.$

7) $K, c \in \mathbb{R}$, $(K+c)u = (K+c)(u_1, u_2)$. $\qquad (Kx,y) + c(x,y)$

$= ((K+c)u_1,\ \overset{0}{\cancel{(K+c)u_2}})$ $\qquad K(x,y)+c(x,y)$

$= (Ku_1, 0) + (cu_1, 0)$

$= K(u_1, u_2) + c(u_1, u_2)$

$= \underline{Ku + cu}$

8) $K(cu) = K(cu_1, 0)$

$= Kc(u_1, u_2)$

$= (Kc)u$

9) $K(u+v) = K(u_1+v_1, u_2+v_2)$

$= K(u_1, u_2)$

$= K[(u_1, u_2)+(v_1, v_2)]$

$= (Ku_1, 0) + (Kv_1, 0)$

$= K(u_1, u_2) + K(v_1, v_2)$

$= Ku + Kv$

LHS = RHS

$[K(u_1+v_1), 0]$

RHS = $Ku + Kv$

10) $1 \cdot u = 1(u_1, u_2) = (u_1, 0) \neq u$.

$\therefore$ V is not a vector space.

$c_1 u + c_2 v \in S$

Scalar multiplication

Subspace :- A subset W of a vector space V is called a subspace of V if W is itself a vector space under the addition & scalar multiplication defined on V.

Theorem :- If W is a non-empty set in vector space V, then W is a subspace of V if and only if the following conditions are satisfied.

(1) If u & v are vectors in W, the $u+v \in W$ (is in W)

(2) If K is a scalar & u is a vector in W, the Ku is in W

- If V is any vector space and $W = \{0\}$ is the subset that consists of ~~zer~~ zero vector only the W is closed under addition & scalar multiplication since $0+0=0$ & $K(0)=0$ for any scalar K. We call W the zero subspace of V — Trivial Subspace

- Lines through the origin are the subspaces of R_2 & If W is a line through the origin of either R_2 or R_3 the adding two vectors on the line & multiplying vector on the line by a scalar produces another vector on the line So W is closed under addition & scalar Multiplication

- The sum of two symmetric n by n matrices is symm & that scalar multiple of symmetric n by n matrix symmetric. Thus the set of symmetric n by n matrice is closed under addition & scalar multiplication & hence is a subspace of $M_{n \times n}$.

- Ill'y the sets of upper triangular matrices, lower triangular matrices & diagonal matrices are subspa of $M_{n \times n}$

*Ex for subset of $M_{n \times n}$ is not a subspace

- A set W of invertible $n \times n$ matrices is not a subspace of $M_{n \times n}$, failing on two properties i.e. it is not closed under addition & not closed under scalar Multiplication.

Ex- $u = \begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix}$ $v = \begin{bmatrix} -1 & 2 \\ -2 & 7 \end{bmatrix}$

$u+v = \begin{bmatrix} 0 & 4 \\ 0 & 14 \end{bmatrix} \notin w$

Q:- Let V be the set R^2 & vector addition and scalar multiplicatn define as usual. Let S be the set of vectors of the form $\begin{bmatrix} 0 \\ y \end{bmatrix}$. S.T. S is a subspace of V.

→

$u = \begin{pmatrix} 0 \\ y_1 \end{pmatrix} \in S$

$v = \begin{pmatrix} 0 \\ y_2 \end{pmatrix} \in S$

$u+v = \begin{pmatrix} 0 \\ y_1+y_2 \end{pmatrix} \in S$

$ku = k\begin{pmatrix} 0 \\ y \end{pmatrix}$

$= \begin{pmatrix} 0 \\ ky \end{pmatrix} \in S$ ∴ it is a subspace of V

$\begin{bmatrix} x \\ 0 \end{bmatrix}$ $u = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$

$v = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$

$u+v = \begin{bmatrix} x_1+x_2 \\ 0 \end{bmatrix} \in S$

$ku = k\begin{pmatrix} x \\ 0 \end{pmatrix}$

$= \begin{bmatrix} kx \\ 0 \end{bmatrix} \in S$

Q: Let S be the subset of the vectors of the form where $x \geq 0$ in the vector space R^2. Verify whether S is a subspace

→

$$u = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \in S$$

$$v = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in S$$

$$u + v = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} \in S$$

$$ku = \begin{pmatrix} kx_1 \\ ky_1 \end{pmatrix} \notin S \text{ if } k < 0$$

If $k = -1$ $\quad u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$ku = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \notin S \notin R^2$$

∴ S is not a subspace

Q:- Let $M_{2\times2}$ be the set of 2×2 matrices. Let S be a subset of $M_{2\times2}$, containing matrices of the form $\begin{bmatrix} 1 & b \\ c & d \end{bmatrix}$. S.T. S is not a subspace of $M_{2\times2}$

→ $u = \begin{bmatrix} 1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ $\quad v = \begin{bmatrix} 1 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$$u + u = \begin{bmatrix} 2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix} \notin M_{2\times2}$$

Q:- Let W be set of all points (x, y) in $\mathbb{R}^2$ for which $x \geq 0$ & $y \geq 0$. Verify W is a subspace or not

→ W is not a subspace because it is not closed under scalar multiplication

$k = -1$ $\quad u = (2, 2)$ $\quad ku = (-2, -2) \notin W$

Q:- Given W is set of all vectors of the form $2a$

$$\begin{bmatrix} 2a+3b \\ 3b-c \\ 4b \\ a+2c \end{bmatrix}$$. S.T. W is a subspace of R^4

→ Soln: $u = \begin{bmatrix} 2a_1+3b_1 \\ 3b_1-c_1 \\ 4b_1 \\ a_1+2c_1 \end{bmatrix}$ $v = \begin{bmatrix} 2a_2+3b_2 \\ 3b_2-c_2 \\ 4b_2 \\ a_2+2c_2 \end{bmatrix}$

$$u+v = \begin{bmatrix} 2(a_1+a_2)+3(b_1+b_2) \\ 3(b_1+b_2)-(c_1+c_2) \\ 4(b_1+b_2) \\ (a_1+a_2)+2(c_1+c_2) \end{bmatrix} = \begin{bmatrix} 2A+3B \\ 3B-C \\ 4B \\ A+2C \end{bmatrix} \in W$$

$$Ku = \begin{bmatrix} 2(a_1k)+3(b_1k) \\ 3(Kb_1)-(Kc_1) \\ 4(Kb_1) \\ (Ka_1)+2(Kc_1) \end{bmatrix} \cancel{\in R^4} = \cancel{2A+3B} \quad \cancel{3B}.$$

$$= \begin{bmatrix} 2a_3+3b_3 \\ 3b_3-c_3 \\ 4b_3 \\ a_3+2c_3 \end{bmatrix} \in W$$

Q:- Verify whether $W = \{f(x) / 2f(0) = f(1)\}$ over $0 \leq x \leq 1$ is a subspace of V where V = {all functions} over the Field R

→ $f(x) = 2^x$ $2(2^0) = 2$

Let $f(x) = x+1$

$f(x) = x^2+x+2$

Let $f(x) = x+1$ $2f(0) = f(1)$

$g(x) = x^2+x+2$

$f(x)+g(x) = x^2+2x+3 \quad \in W$

$f(0) = 3 \quad f(1) = 6$

$2(f(0)) = f(1)$

$6 = 6$

$K f(x)$ Let $K = -3$

$-3(g(x)) = -3x^2 - 3x - 6$

$-3-3-6$

$-12 = -12 \quad \in W$

$\therefore$ W is a subspace.

Let f_1 & f_2 be two functions $\in W$

to show that ~~$2(f_1+f_2)(0) = (f_1+f_2)(1)$~~ $(f_1+f_2) \in W$,

we take $2f_1(0) = f_1(1)$ & $2f_2(0) = f_2(1)$

$2(f_1+f_2)(0) = 2[f_1(0)+f_2(0)]$

$= 2f_1(0)+2f_2(0)$

$= f_1(1)+f_2(1)$

$2(f_1+f_2)(0) = (f_1+f_2)(1) \quad \in W$

Consider K belongs to

$2(Kf_1)(0) = (2K)f_1(0)$

$= K(2f_1(0))$

$2(Kf_1)(0) = K f_1(1) \quad \in W$

$\therefore$ given set is a subspace

Q:- P.T. The set of all solutions A, B, C of the equation $a+b+2c=0$ is a subspace of the vectorspace $V_3(R)$ V over i.e. R^3

$\rightarrow$ Let $u = a_1+b_1+2c_1=0$

$u=(a_1, b_1, 2c_1)$ $\quad v=(a_2, b_2, 2c_2)$

$u+v = ((a_1+a_2)+(b_1+b_2), (2c_1+2c_2))$

$= (A+B+C)=0 \in R3$

$Ku = Ka_1 + Kb_1 + 2Kc_1 = 0$

~~$a+b+2c=0 \in R3$~~

$A+B+2C=0 \quad \in R3$

$\therefore$ V is a ~~vecto~~ subspace over R^3

or

$K, C \in R$ $\quad Ku+Cv$

Consider $Ku+Cv = K(a_1, b_1, c_1) + C(a_2, b_2, c_2)$

$=(Ka_1, Kb_1, Kc_1) + (Ca_2, Cb_2, Cc_2)$

$= (Ka_1+Ca_2,\ Kb_1+Cb_2,\ Kc_1+Cc_2)$

From eq^n $= (Ka_1+Ca_2+Kb_1+Cb_2+2(Kc_1+Cc_2))$

$=(Ka_1+Kb_1+2Kc_1) + Ca_2+Cb_2+2Cc_2$

$= K\overbrace{(a_1+b_1+2c_1)}^{0} + C\overbrace{(a_2+b_2+2c_2)}^{0}$

$=0$

Q:- Let $V=R3$, the vector space of all ordered triplets of real numbers over the field of real numbers. S.T. the set $W_1 = \{(x, 0, 0) / x \in R\}$ is a subspace of R^3

$u=(x_1, 0, 0)$ $\quad v=(x_2, 0, 0)$

$u+v = (x_1+x_2, 0, 0) \in R3$

$K(u) = K($

$(Kx_1, 0, 0)$

$\in R^3$

Q_2:- If $W = \{x^2+y^2 \le 1 \ / x, y \in R\}$ S.T. the subset of consisting of all points on $x^2+y^2=1$ is not a subspace

$\rightarrow$ $u = (1,0)$ $v = (0,1)$

$u+v = (1,1) \in W$ ($\because 1^2+1^2 \le 1$)

$x_1^2+y_1^2 = 1$ $x_2^2+y_2^2 = 1$

$x_1^2+x_2^2+y_1^2+y_2^2 = 1+1$

$= 2 \notin W$

Linear Combination

$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

two lines intersection

Homogeneous system → consistent. → soln will always be subspace

Inconsistence → No soln

rank(A) = rank(A : B) = n → unique

< n → infinite n → no of unknowns

rank(A) ≠ rank(A : B) → No soln

Spanning set :-

$S = \{ v_1, v_2 \dots v_n \} \in V$

$(1,1)\ (3,3)$

$c_1(1,1) + c_2(3,3)$

$c_1 + 3c_2$

$(7,8)$

$c_1 + 3c_2 = 7$

$c_1 + 3c_2 = 8$

No soln

$(1,1)\ (1,-1)$

$(3,4)$

$c_1 + c_2 = 3$

$c_1 - c_2 = 4$

Notes

Linear Combination

A vector v in a vector space V is a linear combination of the vectors $u_1, u_2, \dots u_n$ in V when v can be ~~returned~~ written in the form $v = c_1u_1 + c_2u_2 + \dots + c_nu_n$.

where $c_1, c_2, \dots c_n$ are scalars.

Examp:- Consider the vectors $u = (1, 2, -1)$ & $v = (6, 4, 2)$ in R^3.

Show that $w = (9, 2, 7)$ is a linear combination of u & v, $r = (4, -1, 8)$ is not a linear combination of u & v.

→ For w to be a linear combination of u & v, there must be a scalars c_1 & c_2 such that $w = c_1u + c_2v$. i.e.

$$(9, 2, 7) = c_1(1, 2, -1) + c_2(6, 4, 2)$$

$(9,2,7) = (c_1, 2c_1, -c_1) + (6c_2, 4c_2, 2c_2)$

$(9,2,7) = (c_1 + 6c_2, 2c_1 + 4c_2, -c_1 + 2c_2)$

Equating corresponding components

~~$c_1 + 6c_2 = 9$~~

$c_1 + 6c_2 = 9 \longrightarrow (1)$

$2c_1 + 4c_2 = 2 \longrightarrow (2)$

$-c_1 + 2c_2 = 7 \longrightarrow (3)$

$(1) + (3) \Rightarrow 8c_2 = 16 \Rightarrow c_2 = 2$

$c_1 = 9 - 12 = -3$

$\therefore \boxed{w = -3u + 2v}$

(ii) $(4,-1,8) = c_1(1,2,-1) + c_2(6,4,2)$

$= (c_1, 2c_1, -c_1) + (6c_2, 4c_2, 2c_2)$

$(4,-1,8) = (c_1 + 6c_2, 2c_1 + 4c_2, -c_1 + 2c_2)$

$c_1 + 6c_2 = 4 \longrightarrow (1)$

$2c_1 + 4c_2 = -1 \longrightarrow (2)$

$-c_1 + 2c_2 = 8 \longrightarrow (3)$

$(1) + (3) \Rightarrow 8c_2 = 12$

$c_2 = 3/2$

$c_1 + \cancel{6}^{3} \times \frac{3}{\cancel{2}} = 4$

9

$c_1 = -5$

Does not satisfy (2)

~~$v = -5(1,2,-1) + 3/2(6,4,2)$~~

$8c_2 = +7/8$ $\quad -1 + 7/4 = 8$

$$\begin{bmatrix} 1 & 6 & : & 4 \\ 2 & 4 & : & -1 \\ -1 & 2 & : & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & : & 4 \\ 0 & 8 & : & 9 \\ 0 & 8 & : & 12 \end{bmatrix} \cdot R_3 \to R_2 - R_3$$

$R_2 \to 2R_1 - R_2$
$R_3 \to R_1 + R_3$

$$\begin{array}{c} A \quad : \quad B \\ \sim \begin{bmatrix} 1 & 6 & : & 4 \\ 0 & 8 & : & 9 \\ 0 & 0 & : & -3 \end{bmatrix} \end{array}$$

since $\text{rank}(A) \neq \text{rank}(A:B)$

the system is inconsistent $\therefore$ no soln exist

r cannot be a linear combinatn of u & v

Ex2:- Write the vector $W = (1,1,1)$ are the linear combinatn of vectors in the set $S = \{ \underset{u_1}{(1,2,3)}, \underset{u_2}{(0,1,2)}, \underset{u_3}{(-1,0,1)} \}$

$\rightarrow$

$(1,1,1) = c_1 u_1 + c_2 u_2 + c_3 u_3$

$= c_1(1,2,3) + c_2(0,1,2) + c_3(-1,0,1)$

$(1,1,1) = (c_1, 2c_1, 3c_1) + (0, c_2, 2c_2) + (-c_3, 0, c_3)$

$= (c_1 - c_3,\ 2c_1 + c_2,\ 3c_1 + 2c_2 + c_3)$

$c_1 - c_3 = 1$

$2c_1 + c_2 = 1$

$3c_1 + 2c_2 + c_3 = 1$

$$\begin{bmatrix} 1 & 0 & -1 & | & 1 \\ 2 & 1 & 0 & | & 1 \\ 3 & 2 & 1 & | & 1 \end{bmatrix} \quad \begin{array}{l} R_2 = 2R_1 - R_2 \\ R_3 = 3R_1 - R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & | & 1 \\ 0 & -1 & -2 & | & 1 \\ 0 & -2 & -4 & | & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & | & 1 \\ 0 & -1 & -2 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$2R_2 - R_3$

since $rank(A) = rank(A:B) < n$ $\therefore$ infinitely the $2c_3$

soln and is consistent

If $r < n$, $(n-r)$ no of free variables for soln

$-c_2 - 2c_3 = 1$

$\boxed{c_2 = -2c_3 - 1}$

$c_1 - c_3 = 1$

$\boxed{c_1 = 1 + c_3}$

Free variable: $c_3 = k$

$\boxed{c_1 = 1 + k}$

$\boxed{c_2 = \cancel{-2k} -1 - 2k}$

$\boxed{c_3 = k}$

$\therefore$ W can be expressed as $\boxed{(1+k)u_1 + (-1-2k)u_2 + k(u_3)}$

3) If possible, write the vector $(1, -2, 2)$ as a linear combination of vectors in the set $S = \{(1,2,3), (0,1,2), (-1,0,1)\}$

$(1,-2,2) = c_1(1,2,3) + c_2(0,1,2) + c_3(-1,0,1)$

$c_1 - c_3 = 1$

$2c_1 + c_2 = -2$

$3c_1 + 2c_2 + c_3 = 2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 2 & 1 & 2 \end{array}\right]$$

$R_2 = 2R_1 - R_2$

$R_3 \Rightarrow 3R_1 - R_3$

$-3-(-1)$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & -1 & -2 & 4 \\ 0 & -2 & -2 & 1 \end{array}\right]$$

$-3+1$

$3-2$ $\quad -4+1$

$R_3 = 2R_2 - R_3$

$-4+2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & -1 & -2 & 4 \\ 0 & 0 & -2 & 7 \end{array}\right]$$

Since the system is inconsistant it is not possibl to express $(1,-2,2)$ is not a linear combinat^n of vector in set S.

*spanning set :-

Q) $S = \{v_1, v_2, \dots v_n\}$ be a subset of vectorspace V the set S is a spanning set of the vectorspace V when every vector in V can be ~~retturn~~ written as a linear combinat^n of vectors in S. Example of spanning set :-

- The set $\{(1,0),(0,1)\}$ spans R^2 bcz any vector $v = (v_1, v_2)$ in R^2 can be written as ~~$v = v_1$~~.

$v = v_1(1,0) + v_2(0,1)$

- The set $S = \{1, x, x^2\}$ spans P_2 (a polynomial of degree 2), bcz any polynomial function $P(x) = a + bx + cx^2$ in P_2 can be written as

$P(x) = a(1) + b(x) + c(x^2)$

Linear Dependence

For a given set of vectors, $S = \{v_1, v_2, \ldots, v_n\}$ in vector space V, the vector equation $c_1 v_1 + c_2 v_2 + \ldots$ always has a trivial sol^n $c_1 = 0, c_2 = 0, \ldots, c_n = 0$. However there are also non trivial sol^n. For ex:-

$$S = \{(1,3,1), (0,1,2), (1,0,-5)\}$$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

$\rho(A) <$ rows $\cdot$ independ

$$\Rightarrow c_1(1,3,1) + c_2(0,1,2) + c_3(1,0,-5) = (0,0,0)$$

as a non-trivial sol^n in which the coefficie are not all zero, $c_1 = 1, c_2 = -3, c_3 = -1$. Hence the set is Linearly Dependent.

Definatn :- A set of vectors $S = \{v_1, v_2, \ldots, v_n\}$ in a vector space V is called Linearly Independent when the vector eq^n $c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = 0$ has only triv sol^n i.e. $c_1 = 0, c_2 = 0, \ldots, c_n = 0$. If there are also non-trivial sol^{ns}, the S is called Linearly Dependent.

Ex:- $S = \{(1,2), (2,4)\}$ in R^2 is linearly Dependent since $c_1(1,2) + c_2(2,4) = (0,0)$

$$c_1 + 2c_2 = 0$$

$$2c_1 + 4c_2 = 0$$

on solving $\boxed{c_1 = -2c_2}$

$$c_1 = -2, \quad c_2 = 1$$

$\therefore$ S is Linearly Dependent set.

Ex 2:- The set $S = \{(0,0), (1,2)\}$ in R_2 is Linearly Dependent.

$$c_1(0,0) + c_2(1,2) = (0,0)$$

$$c_2 = 0$$

$$2c_2 = 0 \Rightarrow c_2 = 0$$

c_1 is any no in R_2

$(c_1, 0)$

Q2) Determine whether the set $S = \{(1,2,3), (-2,0,1), (0,1,2)\}$ is linearly dependent / Independent.

Soln:-

$$c_1(1,2,3) + c_2(-2,0,1) + c_3(0,1,2) = (0,0,0)$$

$$c_1 - 2c_2 = 0$$

$$2c_1 + c_3 = 0$$

$$3c_1 + c_2 + 2c_3 = 0$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 \end{array}\right]$$

$R_2 = 2R_1 - R_2$
$R_3 = 3R_1 - R_3$
$3(-2) - 1$
$-6 - 1$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & -4 & -1 & 0 \\ 0 & -7 & -2 & 0 \end{array}\right]$$

$7R_2 - 4R_3$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & -4 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$R(A) = R(A:B)$

Linearly Independent

$\boxed{c_3 = 0}$

$-4c_2 - c_3 = 0 \Rightarrow \boxed{c_2 = 0}$

$c_1 - 2c_2 = 0$

$\boxed{c_1 = 0}$

$\Rightarrow$ S is Linearly

OR

$$\begin{bmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix} = 1(-1)+2(4-3)+0$$

$$= -1+2$$

$$= 1 \neq 0$$

Since determinant $\neq 0$ $\therefore$ the set is Linearly Independent

$\boxed{\text{Determinant} \neq 0 \rightarrow \text{Linearly Independent}}$

Q2) Determine whether the set of vectors in P_2 is Linearly Dependent $S = \{1+x-2x^2, 2+5x-x^2, x+x$

$c_1(1+x-2x^2)+c_2(2+5x-x^2)+c_3(x+x^2) = (0,0,0)$

$(c_1+2c_2 ; c_1x+5c_2x+c_3, -2c_2x^2-c_2x^2+c_3x^2) = (0,$

$c_1+2c_2 = 0$

$c_1x+5c_2x+c_3x = 0 \quad c_1+5c_2+c_3 = 0$

$-2c_2x^2-c_2x^2+c_3x^2 = 0 \quad -2c_2-c_2+c_3 = 0$

$$\begin{vmatrix} 1 & 2 & 0 \\ 1 & 5 & 1 \\ -2 & -1 & 1 \end{vmatrix} = 1(5+1)-2(1+2)$$

$$= 6-6$$

$$= 0$$

Linearly dependent.

1) Determine the whether the set of vectors in $M_{2\times2}$ is linearly independent or dependent. $S = \left\{ \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \right\}$

$\rightarrow$ $$c_1 \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2c_1 + 3c_2 + c_3 & c_1 \\ 2c_2 + 2c_3 & c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\boxed{c_1 = 0}$ $\quad c_1 + c_2 = 0$

$\Rightarrow \boxed{c_2 = 0}$

$2c_2 + 2c_3 = 0 \Rightarrow \boxed{c_3 = 0}$

$\boxed{c_1 = c_2 = c_3 = 0}$

Set S is Linearly Independent.

2) Determine whether the set of vectors in $M_{4\times1}$ is Linearly Independent or Linearly dependent.

$$M_{4\times1} \Rightarrow S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ +1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} \right\}$$

$c_1 + c_2 = 0$

$c_2 + 3c_3 + c_4 = 0$

$-c_1 + c_3 - c_4 = 0$

$2c_2 - 2c_3 + 2c_4 = 0$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ -1 & 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 2 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 2 & -2 & 2 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 8 & 0 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 8 & 0 \end{array}\right]$$

$8c_4 = 0$ $\boxed{c_4 = 0}$ $\Rightarrow$ $2c_3 + 2c_4 = 0$

$\boxed{c_3 = 0}$

$\boxed{c_2 = 0}$

$\boxed{c_1 = 0}$

Linearly Independent.

3) Find the value of k such that the vectors $(1,-2,3)$, $(5,6,-1)$, $(k,2,1)$ are linearly dependent.

$\rightarrow$ $\begin{bmatrix} 1 & -2 & 3 \\ 5 & 6 & -1 \\ k & 2 & 1 \end{bmatrix} \Rightarrow 1(6+2) + 2(5+k) + 3(10-6k$

$48 - 16k = 0$

$\boxed{k = 3}$

Basis:- A set of vectors $S=\{v_1, v_2, \ldots, v_n\}$ in a vectorspace V is a basis for V when the conditions below are true

1) S spans V

2) S is linearly Independent.

Ex:- • The standard basis for R_2 is $S=\{(1,0),(0,1)\}$.

• The standard basis for R_3 is $S=\{(1,0,0),(0,1,0),(0,0,1)\}$

• A basis for polynomials

the vectorspace p_3 has the basis

degree 3

$$S=\{1, x, x^2, x^3\}$$

• A basis for $M_{2\times2}$, $S=\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$

Dimension of a vector space No of linearly independent vectors for basis

If a vector space V has a basis consisting of "n" vectors, then the number n is the dimension of V, denoted by $\boxed{\dim[V]=n}$ OR $\boxed{d[V]=n}$

Note:- 1) The Dimension of R^n with standard operations is 'n'.

2) The dimension of p^n with standard operations is $(n+1)$

* In a 'n' dimensional vectorspace V

1) Any $(n+1)$ elements of V are linearly Dependent

2) No set of $(n-1)$ elements can span V.

3) Any set of n linearly independent vector is a basis

Q1) Show that the set B = { (1,1,0), (1,0,1), (0,1,1) } is a basis of the vectorspace R^3.

→ Let, $c_1v_1 + c_2v_2 + c_3v_3 = 0$

$c_1(1,1,0) + c_2(1,0,1) + c_3(0,1,1) = (0,0,0)$

$(c_1, c_1, 0) + (c_2, 0, c_2) + (0, c_3, c_3) = (0,0,0)$

$c_1 + c_2 = 0$ — ①

$c_1 + c_3 = 0$ $\boxed{c_1 - c_2 = 0}$ — ②

$c_2 + c_3 = 0$

$\boxed{c_3 = -c_2}$ — ③

Solving ① & ②

$2c_1 = 0$

$\boxed{c_1 = 0} \Rightarrow \boxed{c_2 = 0}$ $\boxed{c_3 = 0}$

∴ the set B is linearly Independent.

Let $x = (x_1, x_2, x_3) \in R^3$ be arbitrary

Consider $(x_1, x_2, x_3) = c_1(1,1,0) + c_2(1,0,1) + c_3(0,1,1)$

$(x_1, x_2, x_3) = (c_1 + c_2 + 0, c_1 + 0 + c_3, 0 + c_2 + c_3)$

$c_1 + c_2 = x_1$ — ①

$c_1 + c_3 = x_2$ — ②

$c_2 + c_3 = x_3$ — ③

① − ② ⇒ $c_2 - c_3 = x_1 - x_2$ — ④

③ & ④ $2c_2 = x_1 - x_2 + x_3$

$$\boxed{c_2 = \frac{x_1 - x_2 + x_3}{2}}$$

$① \Rightarrow c_1 + \frac{x_1 - x_2 + x_3}{2} = x_1$

$c_1 = x_1 - \frac{x_1}{2} + \frac{x_2}{2} - \frac{x_3}{2}$

$c_1 = \frac{x_1}{2} + \frac{x_2}{2} - \frac{x_3}{2}$

$\boxed{c_1 = \frac{x_1 + x_2 - x_3}{2}}$ $\qquad d(B) = 3$

$② \Rightarrow \frac{x_1 + x_2 - x_3}{2} + c_3 = x_2$

$c_3 = x_2 - \frac{x_1}{2} - \frac{x_2}{2} + \frac{x_3}{2}$

$\boxed{c_3 = \frac{x_2 - x_1 + x_3}{2}}$

$\therefore$ any arbitrary vectors of R_3 is a linear combination

$(x_1, x_2, x_3) = \left(\frac{x_1 + x_2 - x_3}{2}\right)(1,1,0) + \frac{x_1 - x_2 + x_3}{2}(1,0,1)$
$+ \frac{-x_1 + x_2 + x_3}{2}(0,1,1)$

Let $(x_1, x_2, x_3) = (3, 2, -6)$

B is a basis of R_3

Q2) S.T the set $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ is forms a basis of the vectorspace V of all 2x2 matrices over ~~equa~~ ~~dual num~~ real numbers R.

$\rightarrow c_1\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + c_2\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + c_3\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + c_4\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$

$\cancel{c_1+c_2}\begin{bmatrix}c_1+ & c_2\\c_3 & c_4\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ $\quad c_1 = c_2 = c_3 \ldots$

Linearly Independent

Let $x = (x_1, x_2, x_3, \cancel{x_4}) \in R$

$$\begin{bmatrix}x_1 & x_2\\x_3 & x_4\end{bmatrix} = c_1\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + c_2\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + c_3\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + c_4\begin{bmatrix}0 & \\0 & \end{bmatrix}$$

$c_1 = x_1 \quad c_2 = x_2$

$c_3 = x_3 \quad c_4 = x_4$

$$\begin{bmatrix}x_1 & x_2\\x_3 & x_4\end{bmatrix} = x_1\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + x_2\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + x_3\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + x_4\begin{bmatrix}0 & \\0 & \end{bmatrix}$$

Any arbitrary vector of $M_{2\times2}$ can be linear combination of ~~given~~ vectors in S, $\therefore$ S is a basis $M_{2\times2}$ vector space

$\cancel{\#(B)=4}$ $\quad d(S) = 4$

3) Let $A = \{(1,-2,5),(2,3,1)\}$ be a linearly independent subset of R_3. Extend this to a basis of R_3

$$\begin{bmatrix}1 & -2 & 5\\2 & 3 & 1\\1 & 0 & 0\end{bmatrix}$$ $\quad \cancel{2, 0, 0 = 4 \ldots} \; 2 = 2$

Let $\cancel{B}$: $v_1 = (1, -2, 5)$ $\quad v_2 = (2, 3, 1)$

$\{v_1, v_2\} =$

(single ton set except one)

$\{c_1(1,-2,5) + c_2(2,3,1) / c \in \mathbb{R}\}$

~~$c_1 v_1 + c_2 v_2 + c_4(1,-2,5)$~~

Let S be the subspace spanned by v_1 & v_2

$S = \{c_1 v_1 + c_2 v_2 \mid c_1, c_2 \in \mathbb{R}\}$

$c_1 v_1 + c_2 v_2 = c_1(1,-2,5) + c_2(2,3,1)$

~~$= (c_1 + 2c_2 + 3c_1 + 4c_2, \; 5c_1 \; \& \; E$~~

$= (c_1 + 2c_2), (-2c_1 + 3c_2 + c_2 = E$

check if

$x = c_1 v_1 + c_2 v_2$

So, $(1,0,0) = (c_1 + 2c_2, -2c_1 + 3c_2, 5c_1 + c_2)$

$$\left.\begin{aligned} c_1 + 2c_2 &= 1 \\ -2c_1 + 3c_2 &= 0 \\ 5c_1 + c_2 &= 0 \end{aligned}\right\} \rightarrow \text{No sol}^n \;\therefore \text{ linearly independent.}$$

$\therefore$ The set $A = \{(1,-2,5), (2,3,1), (1,0,0)\}$ is a basis of $\mathbb{R}^3$

Q) Given 2 ... that includes them

(2,1,2,0). Find a basis for R_4

2 vectors.

$\rightarrow$ Let $w = (1,0,0,0)$

$\left\{ \begin{array}{l} (1,0,0,0) \\ (0,1,0,0) \end{array} \right. \in S$

~~$(1,0,0,0) = c_1(2,1,4,3)$~~

$(1,0,0,0)$, $(0,1,0,0)$

$(2,1,4,3)$ $(2,1,2,0)$

is a basis

Check if $w = c_1v_1 + c_2v_2$

$(1,0,0,0) = c_1(2,1,4,3) + c_2(2,1,2,0)$

$2c_1 + 2c_2 = 1$

$c_1 + c_2 = 0$ $\quad \boxed{c_1 = 0}$

$4c_1 + 2c_2 = 0$ $\quad \boxed{c_2 = 0}$

$3c_1 = 0$

But $c_1 = c_2 = 0$ does not satisfy

① $\therefore$ w, v_1, v_2 are linearly independent.

Let $x = (0,0,0,1)$

$(0,0,0,1) = c_1(2,1,4,3) + c_2(2,1,2,0)$

$2c_1 + 2c_2 = 0$

$c_1 + c_2 = 0$

$4c_1 + 2c_2 = 0$

$3c_1 = \cancel{0}\ 1$ $\quad$ ~~$c_1 = 0$~~ ~~$c_2 = 0$~~

$c_1 = 1/3$ & $c_2 = -1/3$ does not satisfy ③

$\therefore$ $(1,0,0,0)$ $(0,0,0,1)$ $(2,1,4,3)$ $(2,1,2,0)$

are linearly independent.

$\therefore S = \{(2,1,4,3), (2,1,2,0), (1,0,0,0) (0,0,0,1)\}$

is a basis for R.

(or)

Take 2 vectors & check rank with other 2

if rank = 4 $\rightarrow$ linearly independent

Ans3 - Test the following set of vectors for Linear Dependence in R_3 $\{(1,0,1), (0,2,2) (3,7,1)\}$. Do they form basis

$\rightarrow$ (1 0 1) (0,2,2) (3,7,1)

$$\text{Det}\begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 3 & 7 & 1 \end{vmatrix} = 1(2-14)+0+1(0-6)$$

$$= -12-6$$

$$= -18$$

the given vectors are linearly independent.

Yes they form basis. Any 3 linearly Independent vectors in R_3 forms a basis $\therefore$ the given set is a basis for R_3. dimension $\rightarrow$ 3

6) Does the set $S = \{(1,2,3), (3,1,0), (-2,1,3)\}$ form a b[asis] of R^3.

→
$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ -2 & 1 & 3 \end{bmatrix} = 1(3-0) - 2(9-0) + 3(3+2)$$
$$= 3 - 18 + 15$$

$\text{Det}(S) = 0$

Linearly Dependent.

∴ S does not form a basis for R_3

7) S.T. the vectors $(1,1,2,4)$, $(2,-1,5,2)$, $(1,-1,4,0)$ and $(2,1,1,6)$ are linearly dependent in R^4 an[d] extract a linearly Independent subset also find the dimension & basis of the subspace spanned by th[ese].

→

~~$c_1(1,1,2,4) + c_2(2,-1,5,2) + c_3(1,-1,4,0) + c_4($~~

If we write $c_1 v_1 + c$ we will get transpose

$$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & 5 & 2 \\ 1 & -1 & 4 & 0 \\ 2 & 1 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 3 & -1 & 6 \\ 0 & 2 & -2 & 4 \\ 0 & 1 & 3 & 2 \end{bmatrix}$$

$$= \begin{matrix} 1 & 1 & 2 & 4 \\ 0 & 3 & -1 & \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 3 & -1 & 6 \\ 0 & 2 & -2 & 4 \\ 0 & 1 & 3 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 3 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 10 & 0 \end{bmatrix} \quad 10R_3 - 4R_4$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = 3

~~∴ 3 linea~~

$\rho(A) \Rightarrow$ 3 linearly independent vectors

The final matrix is in echolan form & rank of A is 3 ∴ the given vectors are linearly dependent. The corresponding non-zero rows of the matrix

(1, 1, 2, 4) (0, 1, 3, 2) (0, 0, 4, 0) are linearly independent. The dimension of the subspace spanned by these vectors is 3.

8) Let S be the subspace of R^3 defined by $S = \{(a, b, c) / a+b+c=0\}$ find a basis & dimension of S.

→

$u_2 = a_2, b_2, c_2$

$u_3 = a_3, b_3, c_3$

Gagana

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} R_3 = R_1 + R_2 + R_3$$

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & 0 \end{bmatrix} = R_2 = b_1$$

$(1,1,1) \notin S$ $\therefore$ S cannot span R_3

Since $a = -b-c$ we have

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -b-c \\ b \\ c \end{bmatrix}$$

$$= b\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ $v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ & further v_1 & v_2 are linearly Independent.

$\therefore$ dimension of subspace spanned by the set $S = 2$.

v_1 & v_2 are basis for the subspace

... symmetric matrices in $M_{2\times 2}$, what is the dimension of W.

$\rightarrow$ $W = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} / a, b, c \in R \right\}$

$W \Rightarrow a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore$ they are linearly Independent.

Impt

$M_{2\times 2} = R^{2\times 2}$

$M_{2\times 2} = R^4$

dimension $= 3$

$d[W] = 3$

3) In a vectorspace $R^3 / V_3(R)$ $u = (1,2,1)$ $v = (3,1,5)$, $w = (-1,3,-3)$. S.T. the subspace spanned by $\{u, v\}$ & $\{u, v, w\}$ are same

$\rightarrow$

u, v are Independent

w depends on them

$$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 5 \\ -1 & 3 & -3 \end{vmatrix} = 1(-3-15) - 2\underbrace{(-9+5)}_{-8} + 1(9+1)$$

$= -18 - \cancel{8}16 + 10$

$= \underline{0}$

linearly ~~Independent~~

$c_1(1,2,1) + c_2(3,1,5) \cancel{+ c_3} = (-1,3,-3)$

$\cancel{c_1 + 3c_2 = -1}$

$\cancel{2c_2 + c_2 = 3}$

~~$c_1 + 2c_1 +$~~

$c_1 + 3c_2 = -1$

$2c_1 + c_2 = 3$

$c_1 + 5c_2 = -3$

$$\left[\begin{array}{cc|c} 1 & 3 & -1 \\ 2 & 1 & 3 \\ 1 & 5 & -3 \end{array}\right]$$

$c_1 = 2$

$c_2 = -1$

4) Find the Dim

i) $w = \{(d, c-d, c) \mid c \& d \in R\}$

ii) $w = \{(2b, b, 0) \mid b \in R\}$

5) Find the basis & dimension of the subspace spanned by $S = \left\{ \begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}, \begin{bmatrix} 2 & -4 \\ -5 & 7 \end{bmatrix} \right\}$

(i) $1(c) + (-1)d = c - d$

$\{d, (c-d), c\} \rightarrow$ linearly Dep

$\{d, c\}$ linearly independent

dim $\rightarrow 2$

(ii) $c_1(2b) + c_2 b + 0(c_3) = 0$

$c_3 = ?$

$c_3 =$ any value

$c_1 = 1$

dim $= 1$

e)

$$\begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}, \begin{bmatrix} +2 & -4 \\ -5 & 7 \end{bmatrix} \begin{bmatrix} 1 & -7 \\ -5 & 1 \end{bmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ -5 & 1 & -4 & -2 \\ 0 & 3 & 3 & -1 \\ 0 & -3 & -3 & 1 \end{pmatrix} \sim \begin{matrix} 1 & 1 & 2 & 1 \\ 0 & 3 & 3 & -1 \\ 0 & 0 & 0 & \end{matrix}$$

Four Fundamental subspaces

Nullity + rank = No of columns

For an $m \times n$ matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

$m \times n$

$RS \in R^n$

$CS \in R^m$

the vectors $r_1 = [a_{11} \; a_{12} \; \cdots \; a_{1n}]$, ~~$r_2 = a_{21}, a_2$~~

$r_2 = [a_{21} \; a_{22} \; \cdots \; a_{2n}]$

$\vdots$

$r_m = [a_{m1} \; a_{m2} \; \cdots \; a_{mn}]$ in R^n that

are formed from the Rows of A are called the row vector of A. and the vectors $c_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$, $c_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}$

$\ldots \; c_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$ in R^m formed from the colums of A

are called column vectors of A

Ex:- Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}$ of A

or $r_1 = [2 \ 1 \ 0]$, $r_2 = [3 \ -1 \ 4]$

$c_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $c_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $c_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

- If A is an $m \times n$ matrix, then the subspace $\mathbb{R}^n$ spanned by row vectors of A is called the row space of A and the subspace of $\mathbb{R}^m$ spanned by the column vectors of A is called the column space of A.

The Sol^n space of the homogeneous system of equation $AX=0$, which is a subspace of $\mathbb{R}^n$ is called the Null space of A.

- The dimension of null space is called nullity of a matrix A & is denoted by nullity (A).

Left Null Space:- The left null space of $m \times n$ matrix A is written as $\boxed{Nul(A^T)}$, is the set of all solutions to the homogeneous equation $\boxed{A^T y = 0}$. The left null space of $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

Note:- • Elementary Row operations do not change the row space of a matrix.

- Elementary Row operations do not change the null space of a matrix

- Rank of a matrix A is the dimension of the row space of A.

Rank Nullity Theorem:- If A is an $m \times n$ matrix then the rank of (A) + Nullity (A) = n [no of columns]

Note:- If A is an $m \times n$ matrix, the dimension of the row space of A = dimension of column space of A

- The system $AX = B$ is consistent if and only if B is in the column space of A.

Q1) Find the basis & dimension of 4 fundamental subspaces of the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 7 \end{bmatrix}$

$\rightarrow$ $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 7 \end{bmatrix}$ $R_2 \leftrightarrow R_2 - R_1$, $R_3 \leftrightarrow 3R_1 - R_3$

$\approx \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & +1 \end{bmatrix}$

$\approx \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{3 \times 4}$ Row space $\in \mathbb{R}^4$

$\rho(A) = 2$

Basis for the Row space $= \{(1,2,1,2)\}, (0,0,0,1)\}$
can be written from echelon form / from previous

OR

$= \{(1,2,1,2), (1,2,1,3)\}$

row space

$d(R(A)) = d[R(A)] = 2$

$d[C(A)] = 2$

To find Null space :-

$AX = 0$

Consider $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{3\times4} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}_{4\times1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{3\times1}$ no of unknowns = no of col

Nullspace ~~exist~~ $\in R^n$

$x_1 + 2x_2 + x_3 + 2\cancel{x_4}^{0} = 0$

$x_4 = 0$

$x_1 = -2x_2 - x_3$

$\rho(A) < n \quad n - r \quad \cancel{4} - 2 = \underline{2}$ unknown

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 - x_3 \\ x_2 \\ x_3 \\ 0 \end{bmatrix}$

$= x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

Basis for Nullspace $= \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$d[N(A)] = 2$

To find Left Null space, $\in R^3$

$R + d(LNS) = $ Rows - origin

$2 + (1) = 3$

$CS = \cancel{R} \; R^3$ original

$A^T y = 0$

$A^T = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 6 \\ 1 & 1 & 3 \\ 2 & 3 & 7 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{4\times3} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}_{3\times1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$y_1 + y_2 + 3y_3 = 0 \qquad y_1 = -2y_3$

$y_2 + y_3 = 0$

$y_2 = -y_3$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -2y_3 \\ -y_3 \\ y_3 \end{bmatrix} = y_3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \quad (A^T=0)$$

Basis for Left Null space $= \left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$

$\cancel{d(LN(A)) = 1}$

$$\boxed{d(N(A^T)) = 1}$$

2) Find the basis & dimension of the 4 fundamental subspaces : $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ -2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$

$\rightarrow$ $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & \cancel{7} \\ -1 & -3 & \underbrace{3 \quad 4}_{\text{either of this}} \end{bmatrix}$

$$= \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$S(A) = 2$

Basis for Row space $= \{(1,3,3,2), (0,0,3,3)\}$

$d(r(A)) = 2$ OR

Basis for column space $= \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 3 \end{bmatrix} \right\}$ either of last 2 cols

$d[c(A)] =$

To Find Null space ;

$AX = 0$

$$\underset{3\times4}{\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \underset{4\times1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 3x_2 + 3x_3 + 2x_4 = 0$

$3x_3 + 3x_4 = 0$

2 free variable

$x_3 = -x_4$

$x_1 + 3x_2 - 3x_4 + 2x_4 = 0$

$x_1 + 3x_2 - x_4 = 0$

$x_1 = -3x_2 + x_4$

$x_2 =$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_4 \\ \\ x_4 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_4$$

Basis for Null space $= \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

$d(N(A)) = 2$

To find Left Null Space:

$A^T y = 0$

2

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ 3 & 9 & 3 \\ 2 & 7 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 3 & 6 \\ 0 & 3 & 6 \end{bmatrix}$$

$-3-($

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{4\times 3} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}_{3\times 1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$y_1 + 2y_2 - y_3 = 0$

$3y_2 + 6y_3 = 0$

$2(-2)$
$-4-1$

$y_2 = -2y_3$

$y_1 = +5y_3$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} +5y_3 \\ -2y_3 \\ y_3 \end{bmatrix}$$

$$= y_3 \begin{bmatrix} +5 \\ -2 \\ 1 \end{bmatrix}$$

$2+(1)=3$

Basis for LNS $= \left\{ \begin{bmatrix} +5 \\ -2 \\ 1 \end{bmatrix} \right\}$

$d(N(A^T)) = 1$

(3) $$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix}$$

$-3-(-2)$ $\quad 0-2$

$$\rightarrow \sim \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2$

Basis of Row space $= \{(1, 2, -1, 1), (0, 0, -1, 2)\}$

$d(R(A)) = 2$

Basis for Column space $= \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} \right\}$

$d(C(A)) = 2$

To find Null space;

$AX = 0$

$$\underset{3\times4}{\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 - x_3 + x_4 = 0$$

$$-x_3 - 2x_4 = 0$$

$$-x_3 = 2x_4$$

$$x_3 = -2x_4$$

$$x_1 + 2x_2 + 2x_4 + x_4$$

$$x_1 = -2x_2 - 3x_4$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Basis for Null space $= \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

To find RNS :-

$A^T y = 0$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ -1 & -3 & 1 \\ 1 & 0 & 5 \end{bmatrix} \sim \begin{matrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 4 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

4) $\begin{bmatrix} -3 & 6 & & 3 & -1 \\ 1 & -2 & 2 & 8 & -4 \\ 2 & -4 & 5 & & \end{bmatrix}$

Also verify Rank-Nullity theorem

→ rank(A)+Nullity(A) = no of columns.

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \quad R_2+3R_1$$

$$\sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

~~Basis for~~

$R(A)=2$

To find Null space

$AX=0$

$$\cancel{\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}_{3\times5} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}_{5\times1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1-2x_2+2x_3+3x_4-x_5=0$

$x_3+2x_4-2x_5=0$

$\boxed{x_3=-2x_4+2x_5}$

$\rho(A)<n$

$5-2=3$

$x_1-2x_2+2(-2x_4+2x_5)+3x_4-x_5=0$

$x_1-2x_2-4x_4+4x_5+3x_4-x_5=0$

$x_1-2x_2-x_4+3x_5=0$

$x_1=2x_2+x_4-3x_5$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2+x_4-3x_5 \\ x_2 \\ -2x_4+2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$d(N(A))=3$

rank(A) + Nullity(A) = 2+3

= 5 = no of colums.

Basis fo

$d(R(A)) = 2$

Basis for column space $= \left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$

$d(C(A)) = 2$

To find LNS,

~~AT y~~ $A^T y = 0$

$$\cancel{\begin{bmatrix} -3 & 1 & 2 \\ 6 & -2 & -4 \\ -1 & 2 & 5 \\ 1 & 3 & 8 \\ -7 & -1 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

5×3, 3×1

$2R_1 + R_2$

$R_1 - 3(R_3)$

$R_1 + 3(R_3)$

$7(R_1) - 3(R_5)$

$21 - 21$ $14 + 12$

$7 + 3$

$$\begin{bmatrix} -3 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & -5 & -13 \\ 0 & 10 & 26 \\ 0 & 10 & 26 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & 2 \\ 0 & -5 & -13 \\ 0 & 10 & 26 \\ 0 & 10 & 26 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 1 & 2 \\ 0 & -5 & -13 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3y_1 + y_2 + 2y_3 = 0$

$-5y_2 - 13y_3 = 0$

$y_3 = \dfrac{-5y_2}{13}$

$-3y_1 + y_2 + 2\left(\dfrac{-5y_2}{13}\right) = 0$

$-3y_1 + y_2 - \dfrac{10}{13} y_2 = 0$

$\cancel{-3y_1 + \dfrac{23y_2}{13} = 0}$

$\cancel{y_1 = \dfrac{13}{3\times23} y_2}$

$\cancel{y_1 = \dfrac{13y_2}{69}}$

$-3y_1 + \dfrac{3y_2}{13} = 0$

$3y_1 = \dfrac{3y_2}{13}$

$y_1 = \dfrac{y_2}{13}$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \dfrac{y_2}{13} \\ y_2 \\ -\dfrac{5y_2}{13} \end{bmatrix} = y_2 \begin{bmatrix} \dfrac{1}{13} \\ 1 \\ \dfrac{-5}{13} \end{bmatrix}$$

$d(N(A^T)) = 1$

2)
$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -4 & 8 \\ -1 & 2 & 5 & -6 \\ -1 & -2 & -3 & 1 \\ -1 & 1 & 3 & -5 \end{bmatrix}$$

3×4

$\boxed{\rho(A) \le 3}$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -5 & -10 & 10 \\ 0 & 4 & 8 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 6 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -5 & -10 & 10 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$5R_3 + 4R_2$

$40-40 \quad -35+40$

$3+1 = 4$

Basis of row space $= \{(1,2,3,-1), (0,-5,-10,10), (0,0,0,5)\}$

$d(R(A)) = 3$

Basis for column space $= \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 8 \\ -6 \\ 1 \\ -5 \end{bmatrix} \right\}$

$d(C(A)) = 3$

To find NS, $AX = 0$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -5 & -10 & 10 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{5\times 4} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}_{4\times 1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$5x_4 = 0$

$\boxed{x_4 = 0}$

$-5x_2 - 10x_3 + 10x_4 = 0$

$-5x_2 - 10x_3 = 0$

$5x_2 = -10x_3$

$x_2 = 2x_3$

$x_1 + 2x_2 + 3x_3 - x_4 = 0$

$x_1 + 4x_3 + 3x_3 - 0 = 0$

$x_1 = -7x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -7x_3 \\ 2x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -7 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

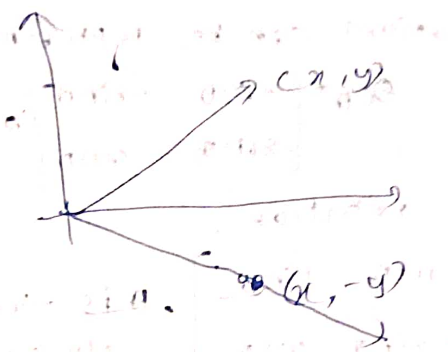
$d(N(A)) = 1$

To find LNS, $A^T y = 0$

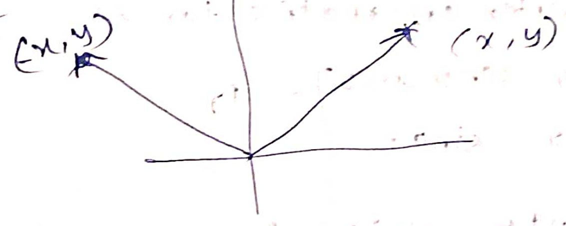
$$\begin{bmatrix} 1 & 2 & -1 & -1 & -1 \\ 2 & -1 & 2 & -2 & 1 \\ 3 & -4 & 5 & -3 & 3 \\ -1 & 8 & -6 & 1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & -1 & -1 \\ 0 & -5 & 4 & 0 & 3 \\ 0 & -10 & 8 & 0 & 6 \\ 0 & 10 & -7 & 0 & -6 \end{bmatrix}$$

$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ gives the reflection through x-a, representing

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

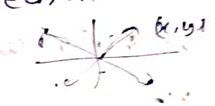


* $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ gives the reflection through y axis



- $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ is the Reflection through the line $\boxed{y=-x}$

$\begin{matrix} x & -y \\ y & -x \end{matrix}$



- $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ gives the reflection through origin

<u>Projection</u> :- A projection matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ transforms each vector (x, y) in the plane to the nearest point $(x, 0)$ on the horizontal axis

$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ transforms each vector (x, y) in the plane to the nearest point $(0, y)$ on the vertical axis

Linear

Matr

Rotation matrix

The family of Rotations can be represented in matrix form as $Q_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, θ is +ve or counter clockwise counterclockwise rotation.

$\sin(-x) = -\sin x$
$\cos(-x) = \cos x$

$Q_{-\theta} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, θ is -ve or clockwise rotation

Projection Matrix :- The family of projection can be represented in matrix for as

$$P = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \times P$$

$\cancel{P2 = \cos^4\theta \quad \sin^2\theta\cos^2\theta}$

$P2 = \cos^4\theta + \sin^2\theta + \cos^2\theta \quad \cancel{\sin\theta\cos^3\theta + \sin^4\theta}$ $\cancel{\cos^2\theta}$ $\sin\theta + \cos 3\theta$

$\cancel{\cos^2\theta + \sin^2\theta\cos^2\theta}$ $\sin$ $\quad \cancel{\sin\theta\cos^3\theta +}\ \sin^2\theta\,(1)$

$$P2 = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

$P2 = P$

Projection twice on the theta line is same as projection once on theta line.

Reflection :- The family of Reflection, can be represented as a matrix as $H = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

$H = 2P - I$

$H^2 = (2P - I)^2 = I$

$H^{-1} = H \rightarrow$ reflection of reflection

$H = 2P - I$
$H^2 = (2P-I)^2$
$= 4P^2 + I^2 - 4P$
$= 4P - 4P + I$
$H^2 = I$
$H^{-1} = H$

Q1) The position vector (2, 1) in $\mathbb{R}^2$ is first rotated through an angle of 30° clockwise and stretched by a factor of 2 unit. Give the rotation matrix & stretching matrix for this situatn also ~~give~~ identify the position vector after rotation is stretching

→ rotation → clockwise $Q_\theta = \begin{bmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$Q_{30} = \begin{bmatrix} \sqrt{3}/2 & +1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$$

Position vector ⇒ $\begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3} + 1/2 \\ -1 + \sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} \frac{2\sqrt{3}+1}{2} \\ \frac{\sqrt{3}-2}{2} \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Stretching matrix :- $\cancel{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}}$

stretched by 2 units

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{2\sqrt{3}+1}{2} \\ \frac{\sqrt{3}-2}{2} \end{bmatrix} = \begin{bmatrix} 2\sqrt{3}+1 \\ \sqrt{3}-2 \end{bmatrix}$$

∴ position vector = $\begin{bmatrix} 2\sqrt{3}+1 \\ \sqrt{3}-2 \end{bmatrix}$

Linear transformation in [illegible]

It is a function that map a vector space V onto a vector space W. This type of function is denoted by $T: V \to W$ where V is called ~~the~~ domain of T & W is called the codomain of T.

(Input) (Output)

If v is in V and w is in W such that $T(v) = w$ the w is called the image of v ~~under~~ under T. The set of all images of vectors in V is called the range of T and the set of all v in V such that $T(v) = w$ is called the pre image of w.

Q) For any vector $v = (v_1, v_2)$ in R^2 define $T: R^2 \to R^2$ by $T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$

(i) Find the image of $v = (-1, 2)$, $v = (0, 0)$

(ii) Find the preimage of $w = (-1, 11)$

Soln

→ (i) image of $(-1, 2) = (-3, 3)$

Image of $(0, 0) = (0, 0)$

(ii) pre-image

$w = (-1, 11)$

$v_1 - v_2 = -1$

$v_1 + 2v_2 = 11$

$-3v_2 = -12$

$\underline{v_2 = 4}$ $v_1 = 3$

preimage $= (3, 4)$

Let V and W be vector spaces $T: V \to W$ is called a linear transformation of V into W when the following properties are true for all u & v in V and for any scalar c, $\boxed{T(u+v) = T(u) + T(v)}$

$\boxed{T(cu) = cT(u)}$

Q) Show that the function is a linear transformation from $R2 \to R2$ defined by $T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$

→ Let $u = (u_1, u_2)$ & $v = (v_1, v_2)$

$\Rightarrow u + v = (u_1 + v_1, u_2 + v_2)$

$T(u+v) = T(u_1 + v_1, u_2 + v_2)$

$= ((u_1 + v_1) - (u_2 + v_2), u_1 + v_1 + 2(u_2 + v_2))$

$= (u_1 - u_2 + v_1 - v_2, u_1 + 2u_2 + v_1 + 2v_2)$

$= (u_1 - u_2, u_1 + 2u_2) + (v_1 - v_2, v_1 + 2v_2)$

$= T(u_1, u_2) + T(v_1, v_2)$

$T(u+v) = T(u) + T(v)$

$T(cu) = T(cu_1, cu_2) = (cu_1 - cu_2, cu_1 + 2cu_2)$

$= (c(u_1 - u_2), c(u_1 + 2u_2))$

$= c(u_1 - u_2, u_1 + 2u_2)$

$= c \cdot T(u_1, u_2)$

$= \underline{cT(u)}$

T is a linear transformation.

Note:- Some transformation.

(1) $f(x) = \cos x$ is not a linear transformation from set of Real numbers to Real numbers since

$\cos(x_1 + x_2) \neq \cos x_1 + \cos x_2$

(2) x^2 is not a linear transformation from R into R

$\therefore (x_1 + x_2)^2 \neq x_1^2 + x_2^2$

(3) $x+2$ is not a linear transformation since

$f(x_1) = x_1 + 2 \quad f(x_2) = x_2 + 2$

$f(x_1) + f(x_2) = x_1 + x_2 + 4$

$f(x_1 + x_2) = x_1 + x_2 + 2$

$\therefore f(x_1 + x_2) \neq f(x_1) + f(x_2)$

Gagana Acharya

2) If a T is a mapping from R^3 to R_3 defined by $T(x,y,z) = (0,y,z)$. Show that T is a linear transformation

Soln:- Consider, $u = (x_1, y_1, z_1)$ & $v = (x_2, y_2, z_2)$

$u + v = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

~~$T(u,v) = T($~~

$T(u+v) = T(x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$= (0, y_1 + y_2, z_1 + z_2)$

$= (0, y_1, z_1) + (0, y_2, z_2)$

$= T(x_1, y_1, z_1) + T(x_2, y_2, z_2)$

$\underline{T(u+v) = T(u) + T(v)}$

$T(cu) = T(cx_1, cy_1, cz_1)$

$= (0, cy_1, cz_1)$

$= c(0, y_1, z_1)$

$= cT(x_1, y_1, z_1)$

$\underline{T(cu) = cT(u)}$

$\therefore T$ is a linear transformation.

Impt

$\divideontimes$ Find the linear transformation R^2 into ... Hence find $T($

$T(1,1) = (0,1,2)$ & $T(-1,1) = (2,1,0)$

$\rightarrow$ Let $S = \{(1,1), (-1,1)\}$ & $\det |S| \neq 0$ $\therefore S$ is linearly independent & S form basis of

consider $(x,y) = c_1(1,1) + c_2(-1,1)$

$x = c_1 - c_2$

$y = c_1 + c_2$

$c_1 = \frac{x+y}{2}$

$T(x,y) = \frac{x+y}{2} T(1,1) + \frac{y-x}{2} T(-1,1)$

$= \frac{x+y}{2}(0,1,2) + \frac{y-x}{2}(2,1,0)$

$\boxed{T(x,y) = (y-x, y, x+y)}$

$T(1,1) = 0, 1, 2)$

Find ... and ...

Note:- Sor the linear transformation tran $T(1,1)=(0,1)$ $T(-1,1)=(3,2)$

$\rightarrow$ $(x,y) = c_1(1,1) + c_2(-1,1)$

$x = c_1 - c_2$

$y = c_1 + c_2$

$c_1 = \frac{x+y}{2}$ $c_2 = \frac{y-x}{2}$

$T(x,y) = \frac{x+y}{2}T(1,1) + \frac{y-x}{2}T(-1,1)$

$= \frac{x+y}{2}(0,1) + \frac{y-x}{2}(3,2)$

$\left(0, \frac{x+y}{2}\right) + \frac{3y-3x}{2}, \frac{2y-2x}{2}$

$= \frac{x+y}{2} + \frac{3(y-x)}{2} +$

$= \frac{3y-3x}{2}, \frac{y+x}{2} + 2y - x$

$= \left(\frac{3y-3x}{2}, \frac{3y}{2} - \frac{x}{2}\right)$

5) If $T: R \to R^3$ defined by $T(x) = (x, x^2, x^3)$ verify whether T is a linear Transformation.

$\rightarrow$ $T(x_1) = (x_1, x_1^2, x_1^3)$ $T(x_2) = (x_2, x_2^2, x_2^3)$

~~$u+v = (x_1+x_2, x_1^2+x_2^2, x_1^3+x_2^3)$~~

$T(x_1)+T(x_2) = T(x_1+x_2, x_1^2+x_2^2, x_1^3+x_2^3)$

$T(x_1+x_2) = (x_1+x_2, (x_1+x_2)^2, (x_1+x_2)^3)$

$\therefore T(x_1+x_2) \neq T(x_1)+T(x_2)$

Note: If $T(x,y) = (y-x, y, x+y)$ transformation is equal then the Matrix of linear tra. is

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}_{3\times 2}$$

$R^2 \to R^3$

$R^n \to R^m$

Rank + Nullity = dim(domain)

$m \times n \to$ matrix of linear trans

Matrix transformation:- Let A be an $m\times n$ matrix, the function T, differs by $T(V) = AV$ is a linear transformation from R^n into R^m.

Q) Consider the linear transformation T from $R_n \to R_m$ defined by $T(V) = AV$. Find the dimensions of R^n & R^m for the linear transformation represented by the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 3 & 0 \\ 4 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -3 \\ -5 & 0 \\ 0 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 0 & 0 \end{bmatrix}$$

Ans:- $A = R^3 \to R^3$ linear transformation from R^3 to R_3

$B = R^2$ to R^3 $C = R^4$ to R_2

Q) Define the function T from $T: R^2 \to R^3$ as follows.

$T(V) = AV$, $A = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix}$, $V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

(i) Find $T(V)$ where V is $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

(ii) ... linear transformation from $R^2 \to R^3$

Note:- sor th
tran

$$\begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}_{3\times 1}$$

(ii) $T(x,y) = (3x, 2x+y, -x-2y)$

$u = (x_1, y_1) \quad v = (x_2, y_2)$

$u+v = (x_1+x_2, y_1+y_2)$

$T(u+v) = (3x_1+3x_2, 2x_1+2x_2+y_1+y_2, -x_1-x_2-2y_1-2y_2)$

$T(u) = (3x_1, 2x_1+y_1, -x_1-2y_1)$

$T(v)$

Kernel of a transformation

Let $T: V \to W$ be a linear transformation. The set of all vectors v in V that are transformed to zero vector in W is called the kernel of T, denoted by $Ker(T)$. It is the set of vectors v in V such that $T(v) = 0$.

Note :- Kernel of a linear transformation $T: V \to W$ is a subspace of the domain V.

... linear transformation given by $T(X) = AX$. Then kernel of T is equal to the solution space of $AX = 0$.

Range of Linear Transformation :- Let $T: V \to W$ be a linear transformation then the range of T denoted by $R(T)$ is the set of all images of the elements of V under T i.e. $R(T) =$

$$R(T) = \{ T(v) / v \in V \}$$

Note :- Range of a Linear transformation $T: V \to W$ is a subspace of W.

Let $T: R^n \to R^m$ be the linear transformation given by $T(x) = AX$ then the column space of A is equal to Range of T

Rank of Linear Transformation

Let $T: V \to W$ be a given linear transformation then the dimension of $R(T)$ is called Rank of linear Transformation & is denoted by $r(T)$

Nullity of Linear Transformation :- Let $T: V \to W$ be a linear transformation, the dimension of solution space or null space is called the nullity of linear transformation denoted by $n(T)$.

Rank Nullity theorem :- Let $T: V \to W$ be a linear transformation & V be a finite dimensional vectorspace then the dimension

dimension of Rangespace + dimension of Nullspace = dimension of Domain

$\dim [R(T)]$

* Impt 10M

Q) Find the Range space & Null space of the linear transformation $T(x,y,z) = (y-x, y-z)$ and also verify Rank Nullity theorem.

→ $R^3 \to R^2$

Let us have the standard basis in R^3 i.e.

$T(e_1) = T(1,0,0) = (-1,0) = w_1$

$T(e_2) = T(0,1,0) = (1,1) = w_2$

$T(e_3) = T(0,0,1) = (0,-1) = w_3$

∴ the set $\{w_1, w_2, w_3\}$ generates $R(T)$

Let $A = \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix}$

$\sim \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

∴ The basis for $R(T) = \{(-1,0),(0,1)\}$

∴ $R(T) = \{c_1(-1,0) + c_2(0,1)\}$

Hence, Rank $= 2$

$\dim[R(T)] = 2$

To find the kernel of the transformation

$T(x,y,z) = (0,0)$

$y - x = 0$ $\boxed{y = x}$

$y - z = 0$ $\boxed{y = z}$

∴ The kernel of transformation ⇒

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ y \\ y \end{bmatrix} = y\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Hence $N(T) / Ker(T) = \left\{k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$

$\dim [N(T)] = 1$

$2 + 1 = 3$

Verifies Rank Nullity theorem.

$\dim[R(T)] + \dim[N(T)] = $ dimension of domain

$2 + 1 = 3$

Q) Let $T(x,y,z) = (x+y, x-y, 2x+z)$ Verify Rank Nullity theorem.

$\begin{matrix} 1 & 2 & 3 \\ -1 & -2 & 1 \\ 3 & 1 & -3 \\ 2 & 1 & \end{matrix}$

$1(6-1) - 2(-9-2) + 3(3+4)$

$= 5 + 22 + 21$