Module-4

Turing Machine

Turing Machine Model

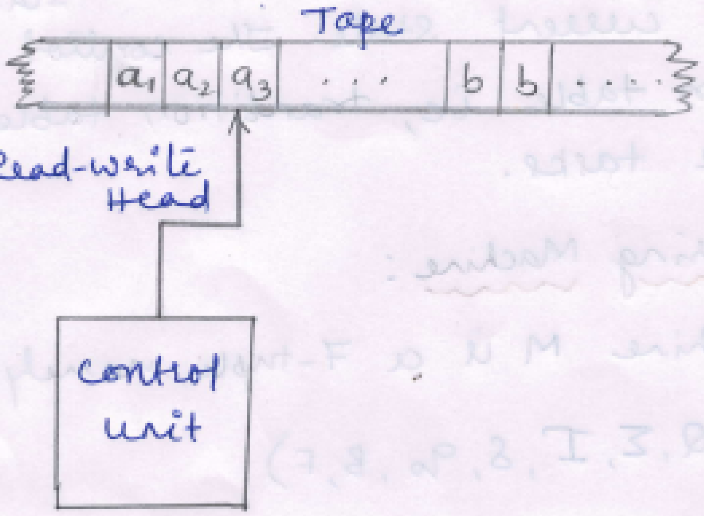


Fig: Turing machine model.

The turing machine model is shown in figure above. It is a finite automaton connected to read-write head with the following components:

* Tape
* Read-write head
* control unit.

Tape: Tape is used to store the information and the tape is divided into cells. Each cell can store the information of only one symbol. The string to be scanned will be stored from the leftmost position on the tape. The string to be scanned should end with blanks. The tape is assumed to be infinite both on left side & right side of the string.

Read-write head: The read-write head can read a symbol from where it is pointing to & it can write into the tape to where it points to.

Control unit: The reading from the tape or writing into the tape is determined by the control unit. The different moves performed by the machine depends on the current scanned symbol & the current state. The control unit consults action table i.e, transition table & carry out the tasks.

Definition of Turing Machine:

A Turing machine M is a 7-tuple, namely

$$M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$$

where

* Q is a finite nonempty set of states.
* Σ is a nonempty set of input symbols & is a subset of Γ & $b \in \Sigma$
* Γ is a finite nonempty set of tape symbols.
* δ is a transition function mapping from $Q \times \Gamma$ to $Q \times \Gamma \times \{L, R\}$
* $q_0 \in Q$ is the start state
* B is a special symbol indicating blank character.
* $F \subseteq Q$ is the set of final states.

Representation of Turing Machine

The turing machine can be represented using three different ways:

i) Transition table

ii) Instantaneous description

iii) Transition diagram.

(i) Transition table.

Definition of transition table

The transition (δ) of turing machine is represented using table is called transition table of a turing machine.

Consider, for example, a turing machine with five states $q_0, q_1, \ldots q_4$ where q_0 is the start state & q_4 is the final state.

δ	Tape symbols (Γ)				
States	a	b	X	Y	B
q_0	(q_1, X, R)	-	-	(q_3, Y, R)	-
q_1	(q_1, a, R)	(q_2, Y, L)	-	(q_1, Y, R)	-
q_2	(q_2, a, L)	-	(q_0, X, R)	(q_2, Y, L)	-
q_3	-	-	-	(q_3, Y, R)	(q_3, B, R)
q_4	-	-	-	-	-

q_0 is start state, there are corresponding entry for the symbol in Γ for each of the states from Q. The symbols a, b, X, Y & B are denoted by Γ & $\Sigma \subseteq \Gamma$. The symbol B indicates a blank character & usually the string ends with infinite number of B's i.e. blank characters.

The undefined entries in the table indicate that there are no-transitions defined or there can be transition to dead state. When there is a transition to the dead state, the machine halts & the input string is rejected by the machine.

In general, δ can be defined as follows:

$$\delta : Q \times I \text{ to } (Q \times I \times \{L, R\})$$

(iii) Representation by Instantaneous Description

The ID of a turing machine is defined in terms of the entire input string & the current state.

Definition: An ID of TM is defined as a string: $\alpha q \beta$ where

* q is the current state of TM
* given string is divided into two substring α & β such that given string is obtained by concatenating α & β. The read-write head points to the first character of the substring β.
* The initial ID is denoted by $q \alpha \beta$ where q is the start state & the read-write head points to the first symbol of α from left.
* The final ID is denoted by the string $\alpha \beta q_i \beta$ where $q \in F$ is the final state & the read-write head points to the blank character denoted by B.

Eg Given the following ID of TM:

$$a_1 a_2 a_3 a_4 q_2 a_5 a_6 a_7 a_8$$

and $\delta(q_2, a_5) = (q_3, b_1, R)$ is the transition, what is the next ID?

Soln

$$\underbrace{a_1 a_2 a_3 a_4}_{\alpha} \underbrace{q_2}_{q} \underbrace{a_5 a_6 a_7 a_8 \ldots}_{\beta}$$ (General form of ID)

The ID indicates that
* The machine is in state q_2
* The next symbol to be scanned is a_5.

When the transition $\delta(q_2, a_5) = (q_3, b_1, R)$ is applied the following actions are performed
* The machine enters into state q_3
* The current input symbol a_5 is replaced by b_1.
* R indicates that the read-write head is moved one symbol right & the following ID is obtained.

$$a_1 a_2 a_3 a_4 b_1 q_3 a_6 a_7 a_8 \ldots$$

This can be represented by a move as shown below:

$$a_1 a_2 a_3 a_4 q_2 a_5 a_6 a_7 a_8 \ldots \vdash a_1 a_2 a_3 a_4 b_1 q_3 a_6 a_7 a_8 \ldots$$

Eg Given the following ID of a TM

$$a_1 a_2 a_3 a_4 q_2 a_5 a_6 a_7 a_8 \ldots$$

and $\delta(q_2, a_5) = (q_3, b_1, L)$ is the transition, what is the next ID?

Soln

$$a_1 a_2 a_3 a_4 q_2 a_5 a_6 a_7 a_8$$

$$\vdash a_1 a_2 a_3 a_4 q_3 b_1 a_6 a_7 a_8 \ldots$$

Moves in Turing Machine

Let $M = (Q, \Sigma, I, \delta, q_0, B, F)$ be a turing machine. Let the ID of M be $a_1 a_2 a_3 \cdots a_{k-1} q a_k a_{k+1} \cdots a_n$ where $a_j \in I$ for $1 \le j \le n$, $q \in Q$ is the current state and a_k as the next symbol to be scanned. If there is a transition

$$\delta(q, a_k) = (p, b, R)$$

then the move of machine M will be

$$a_1 a_2 a_3 \cdots a_{k-1} q a_k a_{k+1} \cdots a_n$$
$$\vdash a_1 a_2 a_3 \cdots a_{k-1} b p a_{k+1} \cdots a_n$$

If there is a transition

$$\delta(q, a_k) = (p, b, L)$$

then the move of machine M will be

$$a_1 a_2 a_3 \cdots a_{k-1} q a_k a_{k+1} \cdots a_n \vdash a_1 a_2 a_3 - a_{k-2} p a_{k-1} b a_{k+1} \cdots a_n$$

Representation by transition diagram

The transition diagram for a TM

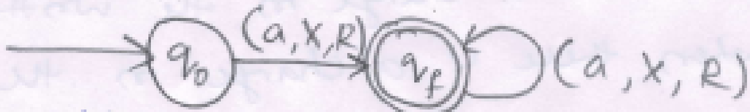
$$M = (Q, \Sigma, I, \delta, q_0, B, F)$$

is defined as a graph with circles, arrows, arcs with labels & two concentric circles where

* Each state is called as vertex, denoted using a circle or two circles
* Each edge is labelled with $(a, X, L/R)$ where a is the current input symbol read from a tape which has to be replaced by X on the tape moving the read/write head to left or right.

* The transition from one state to another state is indicated by a directed edge. Let $\delta(q_i, a) = q_j$, this indicates that there is a directed edge from q_i to q_j & the edge is labeled a.
* The start state is a state which has an arrow not originating from any node and entering into the state. This is labelled with "$\rightarrow$".
* The final states or accepting states which are in F are represented by double circles. The states which are not in F are represented by a single circle.

Eg



The machine changes the state from q_0 to q_f
Each edge is labelled with (a, X, R)
a is the current tape symbol which is replaced by X, R indicates that the read write head is moved towards right by one position.
L indicates that the read write head is moved towards left by one position.

Language acceptability by turing machines

Let us consider the TM $M = (Q, \Sigma, \Gamma, \delta, q_0, b, F)$.
A string w in Σ^* is said to be accepted by M if $q_0 w \overset{*}{\vdash} \alpha_1 p \alpha_2$ for some $p \in F$ & $\alpha_1, \alpha_2 \in \Gamma^*$.

M does not accept w if the machine M either halts in a nonaccepting state or does not halt.

Design of Turing Machines

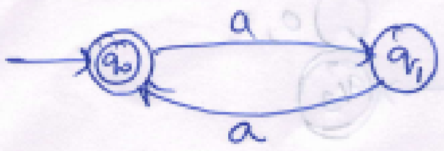
The basic guidelines for designing a TM are

(i) After scanning the symbol below the read/write head is to 'know' what activity has to be done in future. The machine must remember all the scanned symbols and the symbols to be scanned. The machine can remember the scanned symbol by moving on to next state.

(ii) The number of states must be minimized. This can be achieved by changing the states only when there is a change in the written symbol or when there is a change in the movement of the R/W head.

89) Design a TM to accept even number of a's. Show the sequence of moves made by TM for the strings aa and aaa.

Sol) DFA



Transition table

δ	Tape symbol(Γ)	
States	a	B
q_0	(q_1, B, R)	(q_f, B, R)
q_1	(q_0, B, R)	-
q_f	-	-

$\delta(q_0, B) = (q_f, B, R)$

$\delta(q_0, a) = (q_1, B, R)$

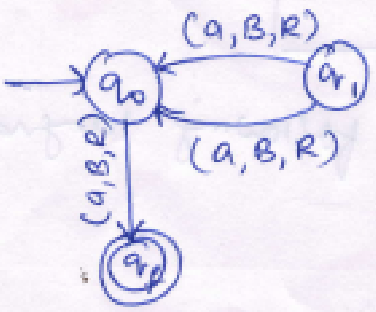
Moves for string aa:

$q_0aaB \vdash Bq_1aB \vdash BBq_0B \vdash q_f BR$ (accept).

Moves for string aaa

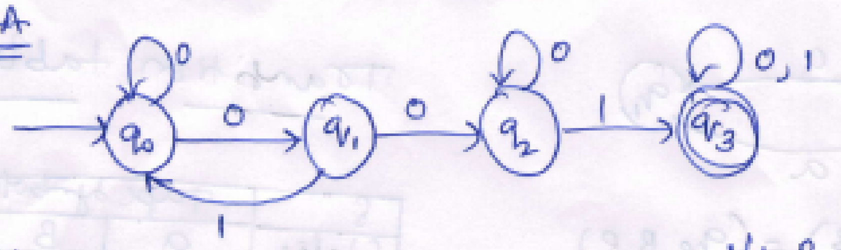
$q_0aaaB \vdash Bq_1aaB \vdash BBq_0aB \vdash BBBq_1B$ (reject)

Transition diagram

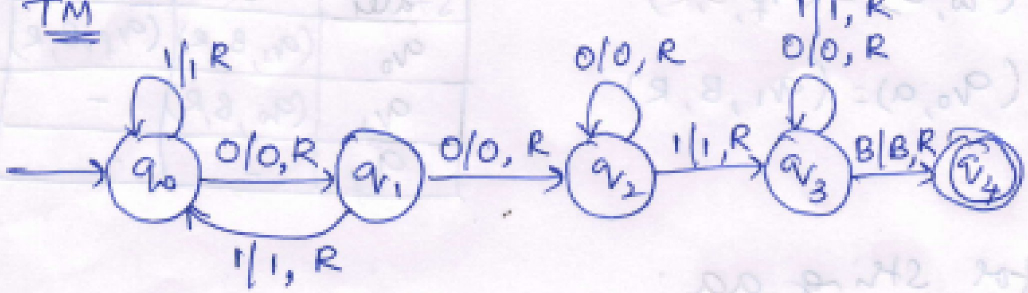


Eg Design a TM to accept the following language:
$L = \{W / W \in (0+1)^*\}$ containing the substring 001.

Sol: DFA



TM



States	0	1	B
q_0	$q_1, 0, R$	$q_0, 1, R$	
q_1	$q_2, 0, R$	$q_0, 1, R$	
q_2	$q_2, 0, R$	$q_3, 1, R$	
q_3	$q_3, 0, R$	$q_3, 1, R$	q_4, B, R
q_4	–	–	–

Eg Design a TM to accept the following language
$L = \{0^n 1^n \mid n \geq 1\}$

Sol:

$\delta(q_0, 0) = (q_1, X, R)$

$\delta(q_1, 0) = (q_1, 0, R)$

$\delta(q_1, 1) = (q_2, Y, L)$

$\delta(q_2, Y) = (q_2, Y, L)$

$\delta(q_2, 0) = (q_2, 0, L)$

0011
X011
↑
q_1
X011
X0Y1
↑
q_2
X0Y1

$\delta(q_2, X) = (q_0, X, R)$

$\delta(q_1, Y) = (q_1, Y, R)$

$\delta(q_2, X) = (q_0, X, R)$

$\delta(q_0, Y) = (q_3, Y, R)$

$\delta(q_3, Y) = (q_3, Y, R)$

$\delta(q_3, B) = (q_4, B, R)$

X 0 Y 1
↑ ↑
q_2 q_2

X 0 Y 1
↑
q_0

X X Y 1
↑ ↑ ↑
q_1 q_1 q_1

X X Y Y
↑
q_2

X X Y Y q_2
↑
q_2

Turing Machine to accept the language $L = \{0^n 1^n \mid n \geq 1\}$ is given by

$M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$

where

$Q = \{q_0, q_1, q_2, q_3, q_4\}$

$\Sigma = \{0, 1\}$

$\Gamma = \{0, 1, X, Y, B\}$

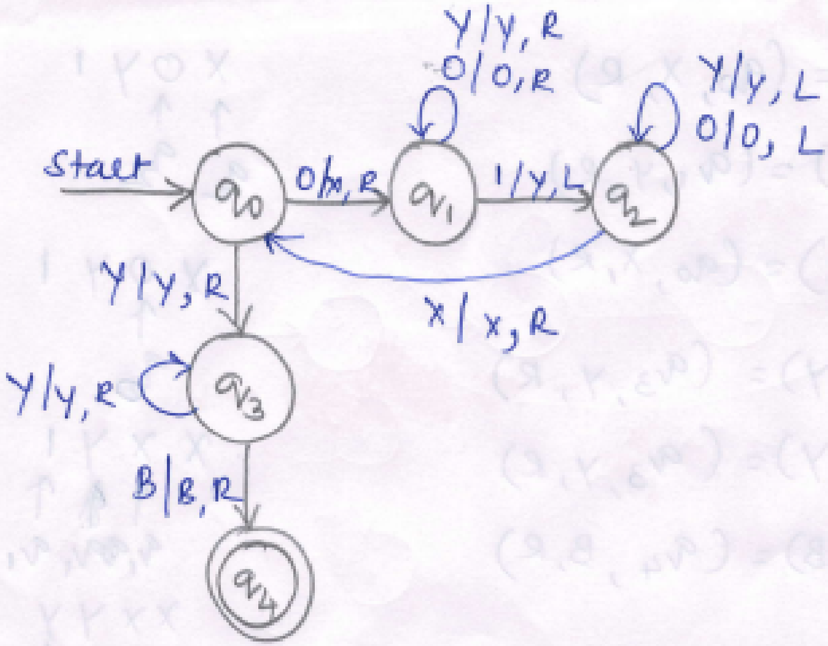
δ is shown in the form of transition table

δ	Tape Symbol (Γ)				
States	0	1	X	Y	B
q_0	(q_1, X, R)	–	–	(q_3, Y, R)	–
q_1	$(q_1, 0, R)$	(q_2, Y, L)	–	(q_1, Y, R)	
q_2	$(q_2, 0, L)$	–	(q_0, X, R)	(q_2, Y, L)	
q_3	–	–	–	(q_3, Y, R)	(q_4, B, R)
q_4	–	–	–	–	–

q_0 is the start state

B blank character

$F = \{q_4\}$ is the final state



Sequence of moves or computations (IDs) for the string 0011

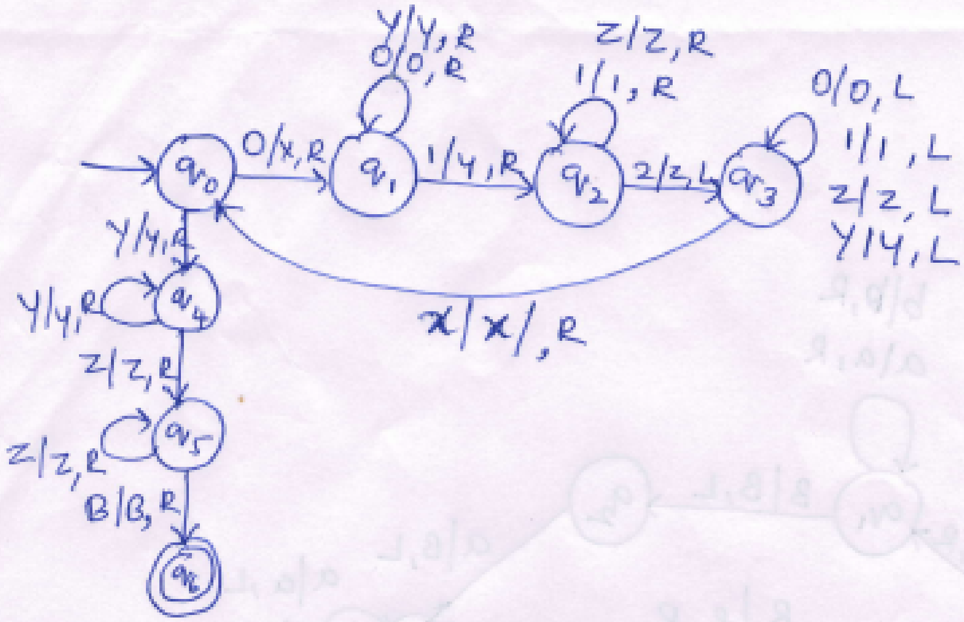
$q_0 0011 \vdash X q_1 011 \vdash X0 q_1 11 \vdash X q_2 0Y1 \vdash q_2 X0Y1$
$\vdash X q_0 0Y1 \vdash XX q_1 Y1 \vdash XXY q_1 1 \vdash XX q_2 YY$
$\vdash X q_2 XYY \vdash XX q_0 YY \vdash XXY q_3 Y \vdash XXYY q_3$
$\vdash XXYYB q_4$ (Final ID).

Eg: Design a turing machine to accept the following language:

$L = \{0^n 1^n 2^n \mid n \geq 1\}$

Sol)

States	0	1	2	Z	Y	X	B
q_0	q_1, X, R				q_4, Y, R		
q_1	q_1, 0, R	q_2, Y, R			q_1, Y, R		
q_2		q_2, 1, R	q_3, Z, L	q_2, Z, R			
q_3	q_3, 0, L	q_3, 1, L		q_3, Z, L	q_3, Y, L	q_0, X, R	
q_4				q_5, Z, R	q_4, Y, R		
q_5				q_5, Z, R			(q_6, B, R)



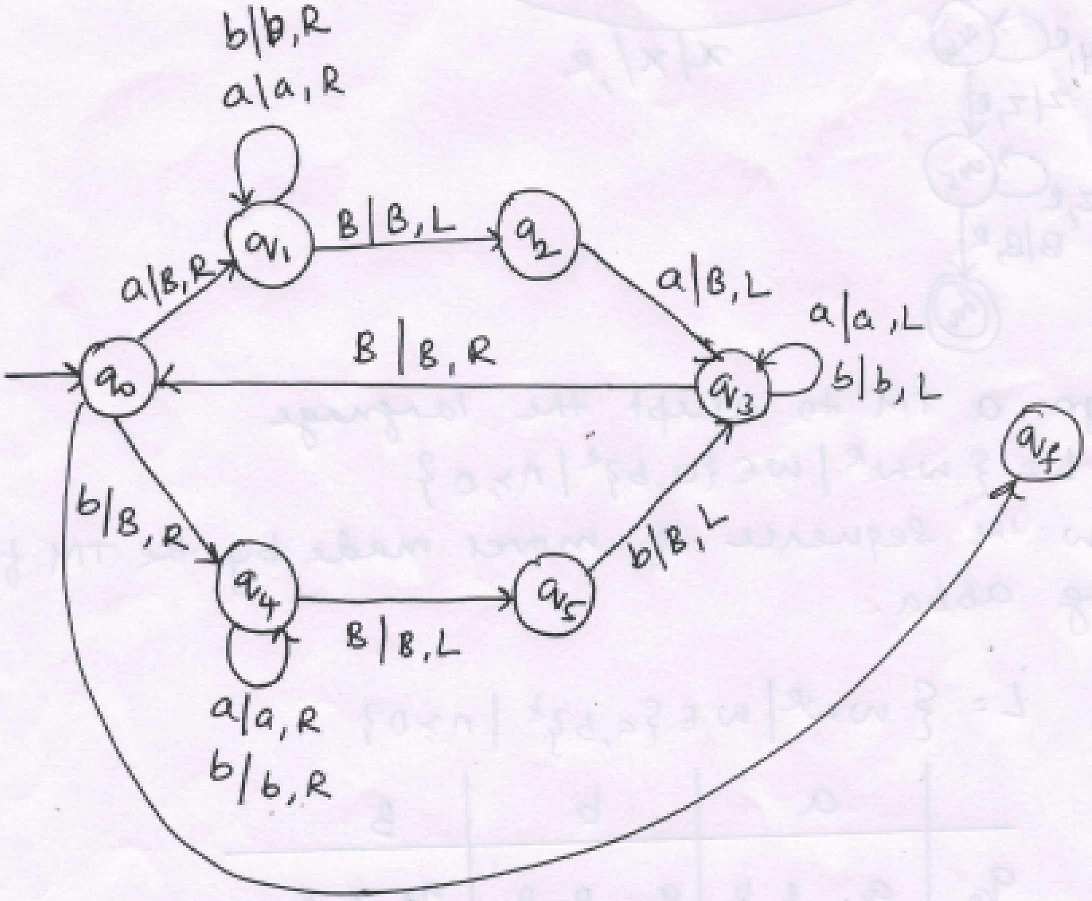
Eg: Design a TM to accept the language $L = \{ww^R \mid w \in \{a,b\}^* \mid n \geq 0\}$

Show the sequence of moves made by the TM for the string abba.

Sol:

$L = \{ww^R \mid w \in \{a,b\}^* \mid n \geq 0\}$.

	a	b	B
q_0	q_1, B, R	q_4, B, R	q_f, B, R.
q_1	q_1, a, R	q_1, b, R	q_2, B, L
q_2	q_3, B, L		
q_3	q_3, a, L	q_3, b, L	q_0, B, R
q_4	q_4, a, R	q_4, b, R	q_5, B, L
q_5		q_3, B, L	
q_f	-	-	-



Sequence of moves made for the string abba

$q_0abbaB \vdash Bq_1bbaB \vdash Bbq_1baB$

$\vdash Bbbq_1aB \vdash Bbbaq_1B \vdash Bbbaq_2B$

$\vdash Bbbq_3BB \vdash Bbq_3bBB \vdash Bq_3bbBB$

$\vdash Bq_0bbBB \vdash BBq_4bBB \vdash BBbq_4BB$

$\vdash BBbq_5BB \vdash BBq_3BBB \vdash BBBq_0BB$

$\vdash BBBBq_f$ (accept).

Techniques for TM construction

The various techniques used for constructing a TM are

i) Turing machine with stationary head
ii) Storage in the state
iii) Multiple track turing Machine
iv) Subroutines.

i) Turing Machine with Stationary Head

In standard turing we defined δ as

$$\delta(q,a) = (q', Y, D)$$

where D stands for direction. so D can be left or Right denoted by ~~[illegible]~~ L or R. so the head moves to the left or right after reading an input symbol. If we want to include the option that head can continue to be in the same cell for some input symbol. Then we can define

$$\delta(q,a) = (q', Y, S)$$

This means that the TM on reading the input symbol a, changes the state to q' & writes Y in the current cell in place of a & continues to remain in the same cell.

The defn of TM with stay-option is given by

$$M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$$

where

Q - Set of finite states

Σ - Set of input alphabet

I - Set of tape symbols

δ - transition fnc from $Q \times I$ to $Q \times I \times \{L, R, S\}$ indicating TM can be move towards left or right or stay in the same position after updating the symbol on the tape.

q_0 - Start state

B - Special symbol indicating blank character

$F \subseteq Q$ is set of final states.

ii) Storage in the state:

In all the different types of machines we have studied such as: FSM or PDA or TM, we used states to remember things. we can use a state to store a symbol as well. So the state becomes a pair (q, a) where q is the state & a is the tape symbol stored in (q, a). So the new set of states is given by: $Q \times I$.

iii) Multiple track turing machine

In a multiple track TM, a single tape is assumed to be divided into several tracks. If a single tape is divided into k tracks, then the tape alphabets consist of k-tuples of tape symbols. The only difference between standard TM & TM with multiple tracks is the set of tape symbols. In case of std TM, tape symbols are denoted by symbol I whereas in case of TM with multiple tracks, the tape symbols are denoted by symbol I^k. The nothing remains same

* Subroutines: we know that subroutines are used in programming languages whenever a task has to be done repeatedly. The same facility can be used in TM & complicated TM's can be built using subroutines. A TM subroutine is a set of states that performs some pre-defined task. The TM subroutine has a start state & a state without any moves. This state which has no moves serves as the return state & passes the control to the state which calls the subroutine.