Joint probability density function

Let x and y be two continuous random variables. Suppose that there exists a real-valued function $p(x,y)$ of x and y such that the following conditions hold:

(i) $p(x,y) \geq 0$ for all x, y

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\,dx\,dy$ exists and is equal to 1.

Then $p(x,y)$ is called the joint probability density function of x and y.

If $[a,b]$ and $[c,d]$ are two intervals, then the probability that $x \in [a,b]$ and $y \in [c,d]$, denoted by $P(a \leq x \leq b, c \leq y \leq d)$, is defined by the formula

$$P(a \leq x \leq b,\ c \leq y \leq d) = \int_a^b \int_c^d p(x,y)\,dy\,dx$$

Further, the function

$$p_1(x) = \int_{-\infty}^{\infty} p(x,y)\,dy$$

is called the marginal density function of x, and the function

$$p_2(y) = \int_{-\infty}^{\infty} p(x,y)\,dx$$

is called the marginal density function of y.

Further $p_1(x)$ is the density function of x and $p_2(y)$ is the density function of y.

Consequently,

$$P(a \le x \le b) = \int_a^b p_1(x)\,dx = \int_{x=a}^{b} \int_{y=-\infty}^{\infty} p(x,y)\,dy\,dx$$

$$P(c \le y \le d) = \int_c^d p_2(y)\,dy = \int_{y=c}^{d} \int_{x=-\infty}^{\infty} p(x,y)\,dx\,dy$$

The variables x and y are said to be stochastically independent (or just independent) if

$$p_1(x)\,p_2(y) = p(x,y)$$

If $\phi(x,y)$ is a function of x and y, then the expectation of $\phi(x,y)$ is defined by

$$E[\phi(x,y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x,y)\, p(x,y)\,dx\,dy$$

As particular cases, we get

$$E[x] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, p(x,y)\,dy\,dx = \int_{-\infty}^{\infty} x\, p_1(x)\,dx$$

$$E[y] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y\, p(x,y)\,dy\,dx = \int_{-\infty}^{\infty} y\, p_2(y)\,dy$$

$$E[xy] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, p(x,y)\,dy\,dx$$

The covariance between x and y is $\mathrm{Cov}(x,y) = E[xy] - E[x]E[y]$

* Find the constant k so that (8)

$$p(x,y) = \begin{cases} k(x+1)e^{-y}, & 0<x<1, y>0 \\ 0, & \text{elsewhere} \end{cases}$$

is a joint probability density function.
Are x and y independent?

Sol^n: The function $p(x,y) = k(x+1)e^{-y} \geq 0 \;\forall\; x, y,$ if $k \geq 0$.

Further, $\displaystyle\int_{y=0}^{\infty}\int_{x=0}^{1} k(x+1)e^{-y}\,dx\,dy = 1$

$\Rightarrow \displaystyle\int_{y=0}^{\infty} ke^{-y}\left(\frac{x^2}{2}+x\right)\Big]_0^1 dy = 1$

$\Rightarrow \displaystyle\int_{y=0}^{\infty} ke^{-y} \times \frac{3}{2}\,dy = 1$

$\Rightarrow \displaystyle\frac{3k}{2} \times \frac{e^{-y}}{-1}\Big]_0^{\infty} = 1$

$\Rightarrow 0 + \dfrac{3k}{2} = 1$

$\Rightarrow k = \dfrac{2}{3}$

$\longrightarrow$

with $k = \frac{2}{3}$, the marginal density functions are,

$$p_1(x) = \int_{-\infty}^{\infty} p(x,y)\,dy$$

$$= \frac{2}{3}(x+1)\int_0^{\infty} e^{-y}\,dy$$

$$= \frac{2}{3}(x+1)\left.\frac{e^{-y}}{-1}\right]_0^{\infty}$$

$$= \frac{2}{3}(x+1), \qquad 0 < x < 1$$

$$p_2(y) = \int_{-\infty}^{\infty} p(x,y)\,dx$$

$$= \frac{2}{3}e^{-y}\int_0^1 (x+1)\,dx$$

$$= \frac{2}{3}e^{-y}\left[\frac{x^2}{2} + x\right]_0^1$$

$$= \cancel{x}e^{-y}, \quad y > 0.$$

$$p_1(x)\cdot p_2(y) = p(x,y).$$

$\therefore$ x and y are stochastically independent

* The joint probability density function of two random variables x and y is given by

$$p(x,y) = \begin{cases} 2, & 0<x<y<1 \\ 0, & \text{otherwise} \end{cases}$$

Find the marginal density functions of x and y. Also, find the covariance between x and y.

Sol^n:

$$p_1(x) = \int_{-\infty}^{\infty} p(x,y)\,dy = \begin{cases} \int_x^1 2\,dy, & 0<x<1 \\ \int 0\,dy, & \text{elsewhere} \end{cases}$$

$$= \begin{cases} 2x\Big]_x^1, & 0<x<1 \\ 0 & \text{elsewhere} \end{cases}$$

$$= \begin{cases} 2-2x, & 0<x<1 \\ 0, & \text{elsewhere} \end{cases}$$

$$p_2(y) = \int_{-\infty}^{\infty} p(x,y)\,dx = \begin{cases} \int_0^y 2\,dx, & 0<y<1 \\ \int 0\,dx, & \text{elsewhere} \end{cases}$$

$$= \begin{cases} 2x\Big]_0^y, & 0<y<1 \\ 0, & \text{elsewhere} \end{cases}$$

$$= \begin{cases} 2y, & 0<y<1 \\ 0, & \text{elsewhere} \end{cases}$$

$$E[x] = \int_{-\infty}^{\infty} x\, p_1(x)\, dx$$

$$= \int_0^1 x[2(1-x)]\, dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{3}$$

$$E[y] = \int_{-\infty}^{\infty} y\, p_2(y)\, dy$$

$$= \int_0^1 y \times 2y\, dy$$

$$= \left.\frac{2y^3}{3}\right]_0^1$$

$$= \frac{2}{3}$$

$$E[xy] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, p(x,y)\, dx\, dy$$

$$= \int_0^1 \int_0^y xy \times 2\, dx\, dy$$

$$= \int_0^1 \left[2y \times \frac{x^2}{2}\right]_0^y dy$$

$$= \int_0^1 y^3\, dy$$

$$= \left.\frac{y^4}{4}\right]_0^1 = \frac{1}{4}$$

$\therefore\ Cov(x,y)$

$= E[xy] - E[x]E[y]$

$= \frac{1}{4} - \frac{1}{3}\cdot\frac{2}{3}$

$= \frac{1}{36}$

* If the joint probability function for (x,y) is (10)

$$f(x,y) = \begin{cases} c(x^2+y^2), & 0 \le x \le 1,\ 0 \le y \le 1,\ c \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

determine (i) the value of the constant c,
(ii) marginal density functions of x and y,
(iii) $P(x < 1/2, y > 1/2)$, (iv) $P(1/4 < x < 3/4)$, (v) $P(y < 1/2)$.

Sol^n

(i) $$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{1} c(x^2+y^2)\,dy\,dx$$

$$1 = \int_{x=0}^{1} c\left(x^2 y + \frac{y^3}{3}\right)_0^1 dx = \int_{x=0}^{1} c\left(x^2 + \frac{1}{3}\right)dx$$

$$1 = c\left[\frac{x^3}{3} + \frac{x}{3}\right]_0^1 \Longrightarrow 1 = \frac{2c}{3} \Rightarrow c = \frac{3}{2}.$$

(ii) marginal density function of x is

$$p_1(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 \frac{3}{2}(x^2+y^2)\,dy = \frac{3}{2}\left[x^2 y + \frac{y^3}{3}\right]_0^1$$

$$= \frac{1}{2}(3x^2+1)$$

marginal density function of y is

$$p_2(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 \frac{3}{2}(x^2+y^2)\,dx = \frac{3}{2}\left[\frac{x^3}{3} + y^2 x\right]_0^1$$

$$= \frac{1}{2}(3y^2+1)$$

$\longrightarrow$

(iii) $P(x < \frac{1}{2}, y > \frac{1}{2}) = \int_{x=0}^{1/2} \int_{y=1/2}^{1} \frac{3}{2}(x^2+y^2)\,dy\,dx$

$= \frac{3}{2} \int_{x=0}^{1/2} \left(x^2y + \frac{y^3}{3}\right)_{1/2}^{1} dx = \frac{3}{2} \int_{x=0}^{1/2} \left(\frac{x^2}{2} + \frac{7}{24}\right) dx$

$= \frac{3}{2} \left[\frac{x^3}{6} + \frac{7x}{24}\right]_0^{1/2} = \frac{3}{2}\left[\frac{1}{48} + \frac{7}{48}\right] = \frac{1}{4}$

(iv) $P(\frac{1}{4} < x < \frac{3}{4}) = \int_{x=1/4}^{3/4} \int_{y=0}^{1} \frac{3}{2}(x^2+y^2)\,dy\,dx$

$= \frac{3}{2} \int_{x=1/4}^{3/4} \left(x^2y + \frac{y^3}{3}\right)_0^1 dx = \frac{3}{2} \int_{x=1/4}^{3/4} \left(x^2 + \frac{1}{3}\right) dx$

$= \frac{3}{2}\left[\frac{x^3}{3} + \frac{x}{3}\right]_{1/4}^{3/4} = \frac{1}{2}\left[\frac{27}{64} - \frac{1}{64} + \frac{3}{4} - \frac{1}{4}\right] = \frac{29}{64}$

(v) $P(y < \frac{1}{2}) = \int_{x=0}^{1} \int_{y=0}^{1/2} \frac{3}{2}(x^2+y^2)\,dy\,dx$

$= \frac{3}{2}\left[\int_{x=0}^{1} \left[x^2y + \frac{y^3}{3}\right]_0^{1/2} dx\right] = \frac{3}{2}\left[\int_{x=0}^{1} \left(\frac{x^2}{2} + \frac{1}{24}\right) dx\right]$

$= \frac{3}{2}\left(\frac{x^3}{6} + \frac{x}{24}\right]_0^1 = \frac{3}{2}\left[\frac{1}{6} + \frac{1}{24}\right] = \frac{5}{16}$

* Verify that $f(x,y) = \begin{cases} e^{-(x+y)} & , x \geq 0, y \geq 0 \\ 0 & , \text{otherwise} \end{cases}$

is a density function of a joint probability distribution. The evaluate the following:

(i) $P(1/2 < x < 2, 0 < y < 4)$ (ii) $P(x<1)$, (iii) $P(x \leq y)$

(iv) $P(x>y)$ (v) $P(x+y \leq 1)$.

Sol.^{n} $f(x,y) \geq 0$ for all x and y.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^{\infty}\int_0^{\infty} e^{-(x+y)}\,dx\,dy = \int_0^{\infty} \left[e^{-y}.\frac{e^{-x}}{-1}\right]_0^{\infty} dy$$

$$= \int_0^{\infty} e^{-y} \times 1\,dy = \left.\frac{e^{-y}}{-1}\right]_0^{\infty} = 1$$

$\therefore$ $f(x,y)$ is a density function.

(i) $$P(1/2 < x < 2, 0 < y < 4) = \int_{x=1/2}^{2}\int_{y=0}^{4} e^{-x}.e^{-y}\,dy\,dx$$

$$= \int_{x=1/2}^{2} e^{-x}\left.\frac{e^{-y}}{-1}\right]_0^4 dx = \int_{x=1/2}^{2} e^{-x}\left[1-\frac{1}{e^4}\right]dx = \left(1-\frac{1}{e^4}\right)\left.\frac{e^{-x}}{-1}\right]_{1/2}^{2}$$

$$= \left(1-\frac{1}{e^4}\right)\left(e^{-1/2}-e^{-2}\right)$$

(ii) $$P(x<1) = \int_{x=0}^{1}\int_{y=0}^{\infty} e^{-x}e^{-y}\,dy\,dx = \int_{x=0}^{1} e^{-x}\left[\frac{e^{-y}}{-1}\right]_0^{\infty} dx$$

$$= \int_{x=0}^{1} e^{-x}(1)\,dx = \left.\frac{e^{-x}}{-1}\right]_0^1 = 1 - e^{-1}$$

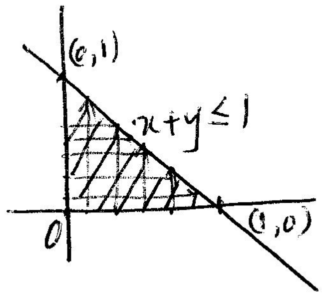
(iii) $P(x \le y) = \int_{y=0}^{\infty} \int_{x=0}^{y} e^{-x} e^{-y} \, dx \, dy$

$$= \int_{y=0}^{\infty} e^{-y} \left[\frac{e^{-x}}{-1}\right]_0^y dy = \int_{y=0}^{\infty} e^{-y}(1-e^{-y}) \, dy$$

$$= \frac{e^{-y}}{-1} - \frac{e^{-2y}}{-2}\Bigg]_0^{\infty} = 1 - \frac{1}{2} = \frac{1}{2}$$

(iv) $P(x > y) = 1 - P(x \le y) = 1 - \frac{1}{2} = \frac{1}{2}$

(v) $P(x+y \le 1) = \iint_A e^{-x} e^{-y} \, dy \, dx$



$$= \int_{x=0}^{1} \int_{y=0}^{1-x} e^{-x} e^{-y} \, dy \, dx$$

$$= \int_{x=0}^{1} e^{-x} \left[\frac{e^{-y}}{-1}\right]_0^{1-x} dx = \int_{x=0}^{1} e^{-x}(1 - e^{-1+x}) \, dx$$

$$= \int_{x=0}^{1} (e^{-x} - e^{-1}) \, dx = \frac{e^{-x}}{-1} - e^{-1}x\Bigg]_0^1 = -e^{-1} - e^{-1} + 1$$

$$= 1 - 2e^{-1}$$

* For the distribution given by the density function $f(x,y) = \begin{cases} \frac{1}{96}xy, & 0<x<4,\ 1<y<5 \\ 0, & \text{otherwise} \end{cases}$

evaluate (i) $P(1<x<2,\ 2<y<3)$ (ii) $P(x \geq 3,\ y \leq 2)$ (iii) $P(y \leq x)$, (iv) $P(y > x)$ (v) $P(x+y \leq 3)$ (vi) $P(x+y > 3)$

Sol^n

(i) $P(1<x<2,\ 2<y<3) = \int_{x=1}^{2}\int_{y=2}^{3} \frac{1}{96} xy \, dy\, dx$

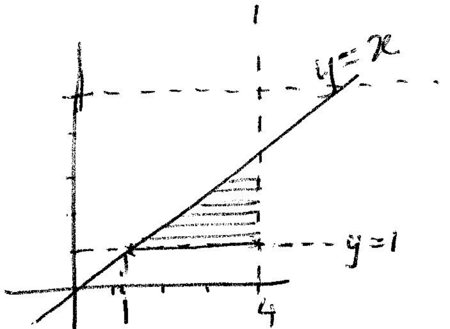
$$= \frac{1}{96}\int_{x=1}^{2} x \left[\frac{y^2}{2}\right]_2^3 dx = \frac{1}{96}\int_{x=1}^{2} x \times \frac{5}{2}\, dx = \frac{5}{192\times 2}\left[\frac{x^2}{2}\right]_1^2$$

$$= \frac{5\times 3}{96\times 4} = \frac{5}{128}$$

(ii) $P(x \geq 3,\ y \leq 2) = \int_{x=3}^{4}\int_{y=1}^{2} \frac{1}{96} xy\, dy\, dx$

$$= \frac{1}{96}\left[\frac{x^2}{2}\right]_{x=3}^{4} \times \left[\frac{y^2}{2}\right]_{y=1}^{2} = \frac{1}{96} \times \frac{7}{2} \times \frac{3}{2} = \frac{7}{128}$$

(iii) $P(y \leq x) = \int_{x=1}^{4}\int_{y=1}^{y=x} \frac{1}{96} xy\, dy\, dx$

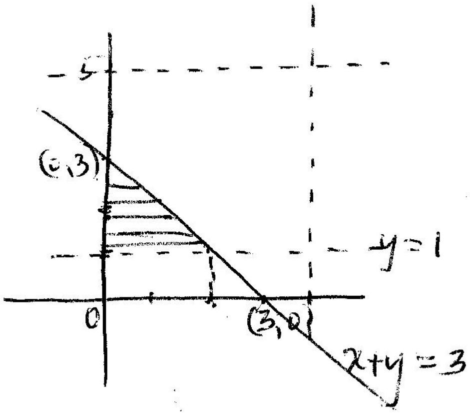


$$= \frac{1}{96}\int_{x=1}^{4} \left[\frac{xy^2}{2}\right]_1^x dx = \frac{1}{192}\int_{x=1}^{4} (x^3 - x)\, dx$$

$$= \frac{1}{192}\left[\frac{x^4}{4} - \frac{x^2}{2}\right]_1^4 = \frac{1}{192}\left(\frac{256}{4} - \frac{1}{4} - \frac{16}{2} + \frac{1}{2}\right) = \frac{225}{768}$$

(iv) $P(y > x) = 1 - P(y \leq x) = 1 - \frac{225}{768} = \frac{543}{768}$

(v) $P(x+y \leq 3)$



$$= \int_{x=0}^{2} \int_{y=1}^{3-x} \frac{1}{96} xy \, dy \, dx$$

$$= \frac{1}{96} \int_{x=0}^{2} \left[x \frac{y^2}{2} \right]_1^{3-x} dx = \frac{1}{96 \times 2} \int_0^2 \left[x(3-x)^2 - x \right] dx$$

$$= \frac{1}{192} \int_0^2 (x^3 - 6x^2 + 8x) \, dx = \frac{1}{192} \left[\frac{x^4}{4} - \frac{6x^3}{3} + \frac{8x^2}{2} \right]_0^2$$

$$= \frac{1}{192} \left[\frac{16}{4} - \frac{48}{3} + \frac{32}{2} \right] = \frac{1}{48}$$

(vi) $P(x+y > 3) = 1 - P(x+y \leq 3)$

$$= 1 - \frac{1}{48}$$

$$= \frac{47}{48}$$

1. The joint probability density function of two random variables x and y is given by

$$p(x,y) = \begin{cases} (8/9)\,xy, & 0 \le x \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the marginal density functions of x and y.

Ans $p_1(x) = \left(\frac{4}{9}\right)x(4-x^2), \quad 1 \le x \le 2, \quad p_2(y) = \begin{cases} \left(\frac{4}{9}\right)y(y^2-1), & 1 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$

2. Find the constant k so that

$$f(x,y) = \begin{cases} kxy, & 0 \le y \le 4,\ 0 \le x \le y, \\ 0, & \text{otherwise} \end{cases}$$

is a joint probability density function. Then check whether x and y are stochastically independent or not.

Ans: $k = \frac{1}{32}$; not independent.

3. For the distribution defined by the density function

$$f(x,y) = \begin{cases} 3xy(x+y), & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

find the covariance between x and y.

Ans: $\frac{1}{2} - \left(\frac{17}{24}\right)^2$

4. For the distribution with density function

$$p(x,y) = \begin{cases} \frac{1}{8}(6-x-y), & 0 < x < 2,\ 2 < y < 4 \\ 0, & \text{otherwise,} \end{cases}$$

evaluate (i) $P(x<1, y<3)$ (ii) $P(x+y<3)$.

Ans: (i) $\frac{3}{8}$, (ii) $\frac{5}{24}$