STRENGTH OF MATERIALS AND STRUCTURES

FOURTH EDITION

JOHN CASE
LORD CHILVER
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Strength of Materials and Structures

Fourth edition

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CTFR, 1999

"Only when you climb the highest mountain, will you be aware of the vastness that lies around you."

Oscar Wilde, 1854—1900.

Chinese Proverb — *It is better to ask a question and look a fool for five minutes, than not to ask a question at all and be a fool for the rest of your life.*

Heaven and Hell — *In heaven you are faced with an infinite number of solvable problems and in hell you are faced with an infinite number of unsolvable problems.*

Principal notation

a	length	A	area
b	breadth	C	complementary energy
c	wave velocity, distance	D	diameter
d	diameter	E	young's modulus
h	depth	F	shearing force
j	number of joints	G	shearing modulus
l	length	H	force
m	mass, modular ratio, number of numbers	I	second moment of area
		J	torsion constant
n	frequency, load factor, distance	K	bulk modulus
p	pressure	L	length
q	shearing force per unit length	M	bending moment
r	radius	P	force
s	distance	Q	force
t	thickness	R	force, radius
u	displacement	S	force
v	displacement, velocity	T	torque
w	displacement, load intensity, force	U	strain energy
		V	force, volume, velocity
x	coordinate	W	work done, force
y	coordinate	X	force
z	coordinate	Y	force
		Z	section modulus, force

α	coefficient of linear expansion	ρ	density
γ	shearing strain	σ	direct stress
δ	deflection	τ	shearing stress
ε	direct strain	ω	angular velocity
η	efficiency	Δ	deflection
θ	temperature, angle of twist	Φ	step-function
ν	Poisson's ratio		

[k]	element stiffness matrix	**[K]**	system stiffness matrix
[m]	elemental mass matrix	**[M]**	system mass matrix

Note on SI units

The units used throughout the book are those of the Système Internationale d'Unités; this is usually referred to as the SI system. In the field of the strength of materials and structures we are concerned with the following basic units of the SI system:

length	metre (m)
mass	kilogramme (kg)
time	second (s)
temperature	kelvin (K)

There are two further basic units of the SI system – electric current and luminous intensity – which we need not consider for our present purposes, since these do not enter the field of the strength of materials and structures. For temperatures we shall use conventional degrees centigrade (°C), since we shall be concerned with temperature changes rather than absolute temperatures. The units which we derive from the basic SI units, and which are relevant to out fielf of study, are:

force	newton (N)	$kg.m.s^{-2}$
work, energy	joule (J)	$kg.m^2.s^{-2}$ = Nm
power	watt (W)	$kg.m^2.s^{-3} = Js^{-1}$
frequency	hertz (Hz)	cycle per second
pressure	Pascal (Pa)	$N.m^{-2} = 10^{-5}$ bar

The acceleration due to gravity is taken as:

$$g = 9.81 ms^{-2}$$

Linear distances are expressed in metres and multiples or divisions of 10^3 of metres, i.e.

Kilometre (km)	10^3 m
metre (m)	1 m
millimetre (mm	10^{-3} m

In many problems of stress analysis these are not convenient units, and others, such as the centimetre (cm), which is 10^{-2} m, are more appropriate.

The unit of force, the newton (N), is the force required to give unit acceleration (ms^{-2}) to unit mass kg). In terms of newtons the common force units in the foot-pound-second-system (with g = 9.81 ms^{-2}) are

$$1 \text{ lb.wt} = 4.45 \text{ newtons (N)}$$

$$1 \text{ ton.wt} = 9.96 \times 10^3 \text{ newtons (N)}$$

In general, decimal multiples in the SI system are taken in units of 10^3. The prefixes we make most use of are:

kilo	k	10^3
mega	M	10^6
giga	G	10^9

Thus:

$$1 \text{ ton.wt} = 9.96 \text{ kN}$$

The unit of force, the newton (N), is used for external loads and internal forces, such as shearing forces. Torques and bending of moments are expressed in newton-metres (Nm).

An important unit in the strength of materials and structures is stress. In the foot-pound-second system, stresses are commonly expressed in lb.wt/in^2, and tons/in^2. In the SI system these take the values:

$$1 \text{ lb.wt/in}^2 = 6.89 \times 103 \text{ N/m}^2 = 6.89 \text{ kN/m}^2$$

$$1 \text{ ton.wt/in}^2 = 15.42 \times 106 \text{ N/m}^2 = 15.42 \text{ MN/m}^2$$

Yield stresses of the common metallic materials are in the range:

$$200 \text{ MN/m}^2 \text{ to } 750 \text{ MN/m}^2$$

Again, Young's modulus for steel becomes:

$$E_{\text{steel}} = 30 \times 106 \text{ lb.wt/in}^2 = 207 \text{ GN/m}^2$$

Thus, working and yield stresses will usually be expressed in MN/m^2 units, while Young's modulus will usually be given in GN/m^2 units.

Preface

This new edition is updated by Professor Ross, and while it retains much of the basic and traditional work in Case & Chilver's *Strength of Materials and Structures*, it introduces modern numerical techniques, such as matrix and finite element methods.

Additionally, because of the difficulties experienced by many of today's students with basic traditional mathematics, the book includes an introductory chapter which covers in some detail the application of elementary mathematics to some problems involving simple statics.

The 1971 edition was begun by Mr. John Case and Lord Chilver but, because of the death of Mr. John Case, it was completed by Lord Chilver.

Whereas many of the chapters are retained in their 1971 version, much tuning has been applied to some chapters, plus the inclusion of other important topics, such as the plastic theory of rigid jointed frames, the torsion of non-circular sections, thick shells, flat plates and the stress analysis of composites.

The book covers most of the requirements for an engineering undergraduate course on strength of materials and structures.

The introductory chapter presents much of the mathematics required for solving simple problems in statics.

Chapter 1 provides a simple introduction to direct stresses and discusses some of the fundamental features under the title: Strength of materials and structures.

Chapter 2 is on pin-jointed frames and shows how to calculate the internal forces in some simple pin-jointed trusses. Chapter 3 introduces shearing stresses and Chapter 4 discusses the modes of failure of some structural joints.

Chapter 5 is on two-dimensional stress and strain systems and Chapter 6 is on thin walled circular cylindrical and spherical pressure vessels.

Chapter 7 deals with bending moments and shearing forces in beams, which are extended in Chapters 13 and 14 to include beam deflections. Chapter 8 is on geometrical properties.

Chapters 9 and 10 cover direct and shear stresses due to the bending of beams, which are extended in Chapter 13. Chapter 11 is on beam theory for beams made from two dissimilar materials. Chapter 15 introduces the plastic hinge theory and Chapter 16 introduces stresses due to torsion. Chapter 17 is on energy methods and, among other applications, introduces the plastic design of rigid-jointed plane frames.

Chapter 18 is on elastic buckling.

Chapter 19 is on flat plate theory and Chapter 20 is on the torsion of non-circular sections. Chapter 21 is on thick cylinders and spheres.

Chapter 22 introduces matrix algebra and Chapter 23 introduces the matrix displacement method.

Chapter 24 introduces the finite element method and in Chapter 25 this method is extended to cover the vibrations of complex structures.

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Contents

Introduction

I.1 Introduction

Stress analysis is an important part of engineering science, as failure of most engineering components is usually due to stress. The component under a stress investigation can vary from the legs of an integrated circuit to the legs of an offshore drilling rig, or from a submarine pressure hull to the fuselage of a jumbo jet aircraft.

The present chapter will commence with elementary trigonometric definitions and show how elementary trigonometry can be used for analysing simple pin-jointed frameworks (or trusses). The chapter will then be extended to define couples and show the reader how to take moments.

I.2 Trigonometrical definitions

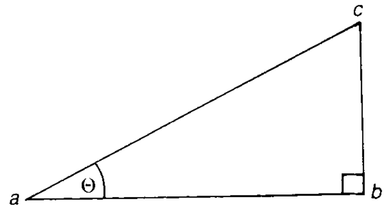


Figure I.1 Right-angled triangle.

With reference to Figure I.1,

$$\begin{aligned} \sin\theta &= bc/ac \\ \cos\theta &= ab/ac \\ \tan\theta &= bc/ab \end{aligned} \qquad \text{(I.1)}$$

For a triangle without a right angle in it, as shown in Figure I.2, the *sine* and *cosine* rules can be used to determine the lengths of unknown sides or the value of unknown angles.

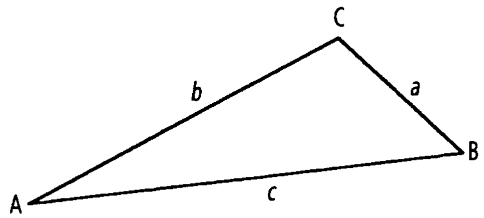


Figure I.2. Triangle with no right angle.

The *sine rule* states that:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \tag{I.2}$$

where

a = length of side BC; opposite the angle A

b = length of side AC; opposite the angle B

c = length of side AB; opposite the angle C

The cosine rule states that:

$$a^2 = b^2 + c^2 - 2bc \cos \mathrm{A}$$

I.3 Vectors and scalars

A scalar is a quantity which has magnitude but no direction, such as a mass, length and time. A vector is a quantity which has magnitude and direction, such as weight, force, velocity and acceleration.

NB It is interesting to note that the moment of a couple, (Section I.6) and energy (Chapter 17), have the same units; but a moment of a couple is a vector quantity and energy is a scalar quantity.

I.4 Newton's laws of motion

These are very important in engineering mechanics, as they form the very fundamentals of this topic.

Newton's three laws of motion were first published by Sir Isaac Newton in *The Principia* in 1687, and they can be expressed as follows:

(1) Every body continues in its state of rest or uniform motion in a straight line, unless it is compelled by an external force to change that state.

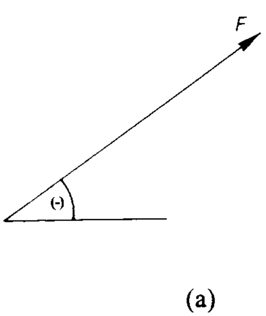
(2) The rate of change of momentum of a body with respect to time, is proportional to the resultant force, and takes place in a direction of which the resultant force acts.

(3) Action and reaction are equal and opposite.

I.5 Elementary statics

The trigonometrical formulae of I.2 can be used in statics. Consider the force F acting on an angle θ to the horizontal, as shown by Figure I.3(a). Now as the force F is a vector, (i.e. it has magnitude and direction), it can be represented as being equivalent to its horizontal and vertical components, namely F_H and F_V, respectively, as shown by Figure I.3(b). These horizontal and vertical components are also vectors, as they have magnitude and direction.

NB If F is drawn to scale, it is possible to obtain F_H and F_V from the scaled drawing.



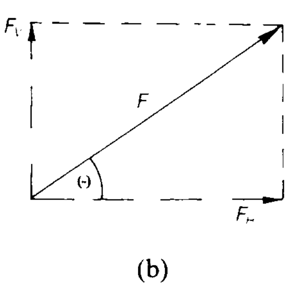


Figure I.3 Resolving a force.

From elementary trigonometry

$$\frac{F_H}{F} = \cos\theta$$

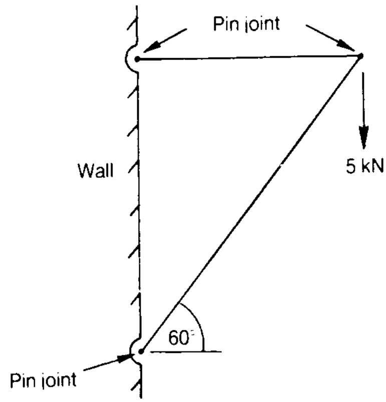
$$\therefore F_H = F\cos\theta\text{—horizontal component of } F$$

Similarly,

$$\frac{F_V}{F} = \sin\theta$$

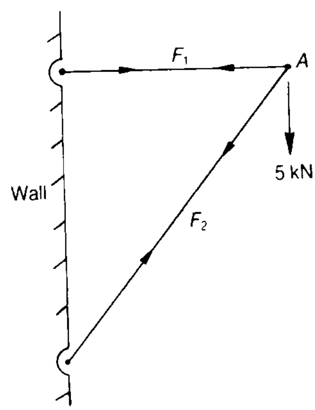
$$\therefore F_V = F\sin\theta\text{—vertical component of } F$$

Problem I.1 Determine the forces in the plane pin-jointed framework shown below.

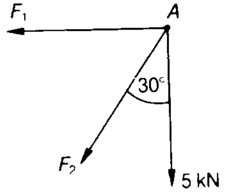


Solution

Assume all unknown forces in each member are in tension, i.e. the internal force in each member is pulling away from its nearest joint, as shown below.



Isolate joint A and consider equilibrium around the joint,



Resolving forces vertically
From Section I.7

$$\text{upward forces} = \text{downward forces}$$
$$0 = 5 + F_2 \cos 30$$

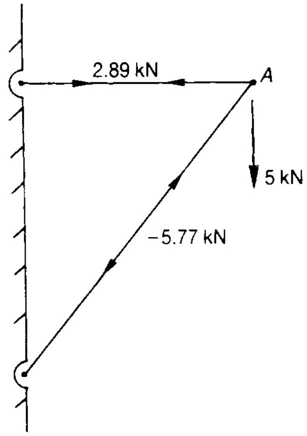
or
$$F_2 = -\frac{5}{\cos 30} = -5.77 \text{ kN (compression)}$$

The negative sign for F_2 indicates that this member is in compression.

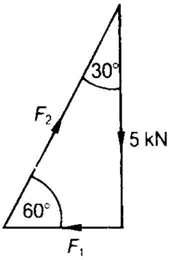
Resolving forces horizontally
From Section I.7

$$\text{forces to the left} = \text{forces to the right}$$
$$F_1 + F_2 \sin 30 = 0$$
$$F_1 = -F_2 \sin 30 = 5.77 \sin 30$$
$$F_1 = 2.887 \text{ kN (tension)}$$

The force diagram is as follows:



Another method of determining the internal forces in the truss shown on page 4 is through the use of the triangle of forces. For this method, the magnitude and the direction of the known force, namely the 5kN load in this case, must be drawn to scale.



To complete the triangle, the directions of the unknown forces, namely F_1 and F_2 must be drawn, as shown above. The directions of these forces can then be drawn by adding the arrowheads to the triangle so that the arrowheads are either all in a clockwise direction or, alternatively, all in a counter-clockwise direction.

Applying the sine rule to the triangle of forces above,

$$\frac{5}{\sin 60} = \frac{F_1}{\sin 30}$$

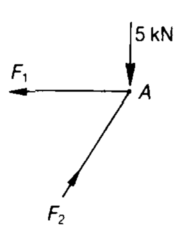
$$\therefore F_1 = \frac{5 \times 0.5}{0.866} = 2.887 \text{ kN}$$

Similarly by applying the sine rule:

$$\frac{5}{\sin 60} = \frac{F_2}{\sin 90}$$

$$\therefore F_2 = \frac{5}{0.866} = 5.77 \text{ kN}$$

These forces can now be transferred to the joint A of the pin-jointed truss below, where it can be seen that the member with the load F_1 is in tension, and that the member with the load F_2 is in compression.



This is known as a free body diagram.

I.6 Couples

A couple can be described as the moment produced by two equal and opposite forces acting together, as shown in Figure I.4 where,

the moment at the couple = $M = F \times l$ (N.m)

F = force (N)

l = lever length (m)

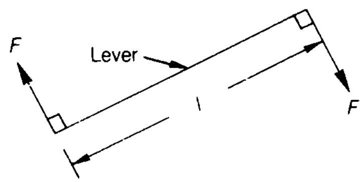


Figure I.4 A clockwise couple.

For the counter-clockwise couple of Figure I.5,

$$M = F \cos \theta \times l$$

where $F \cos \theta$ = the force acting perpendicularly to the lever of length l.

NB The components of force $F \sin \theta$ will simply place the lever in tension, and will not cause a moment.

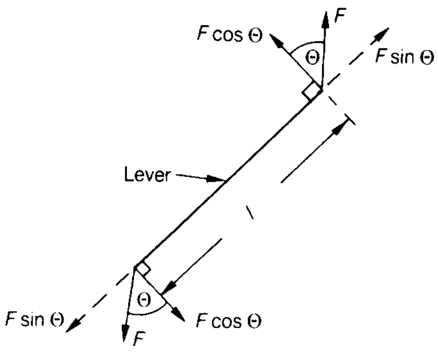


Figure I.5 A counter-clockwise couple.

It should be noted from Figure I.4 that the lever can be described as the perpendicular distance between the line of action of the two forces causing the couple.

Furthermore, in Figure I.5, although the above definition still applies, the same value of couple can be calculated, if the lever is chosen as the perpendicular distance between the components of the force that are perpendicular to the lever, and the forces acting on this lever are in fact those components of force.

1.7 Equilibrium

This section will be limited to one- or two-dimensional systems, where all the forces and couples will be acting in on plane; such a system of forces is called a coplanar system.

In two dimensions, equilibrium is achieved when the following laws are satisfied:

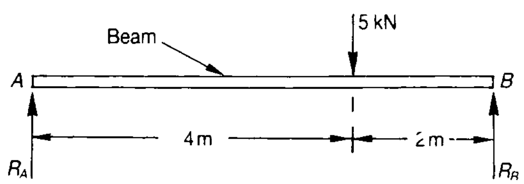
(1) upward forces = downward forces

(2) forces to the left = forces to the right

(3) clockwise couples = counter-clockwise couples.

To demonstrate the use of these two-dimensional laws of equilibrium, the following problems will be considered.

Problem 1.2 Determine the values of the reactions R_A and R_B, when a beam is simply-supported at its ends and subjected to a downward force of 5 kN.



<u>*Solution*</u>

For this problem, it will be necessary to take moments. By taking moments, it is meant that the values of the moments must be considered about a suitable position.

Suitable positions for taking moments on this beam are A and B. This is because, if moments are taken about A, the unknown section R_A will have no lever and hence, no moment about A, thereby simplifying the arithmetic. Similarly, by taking moments about B, the unknown R_B will have no lever and hence, no moment about B, thereby simplifying the arithmetic.

Taking moments about B

clockwise moments = counter-clockwise moments

$$R_A \times (4 + 2) = 5 \times 2$$

or $$R_A = 10/6$$

$$R_A = 1.667 \text{ kN}$$

Resolving forces vertically

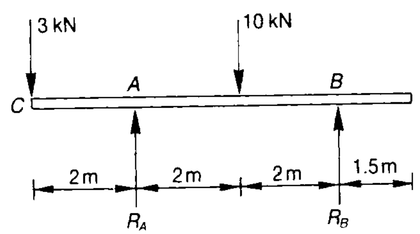
$$\text{upward forces} = \text{downward forces}$$

$$R_A + R_B = 5$$

or $$R_B = 5 - R_A = 5 - 1.667$$

$$R_B = 3.333 \text{ kN}$$

Problem I.3 Determine the values of the reactions of R_A and R_B for the simply-supported beam shown.



<u>*Solution*</u>

Taking moments about B

$$\text{clockwise couples} = \text{counter-clockwise couples}$$

$$R_A \times 4 = 3 \times 6 + 10 \times 2$$

$$R_A = \frac{18 + 20}{4}$$

$$R_A = 9.5 \text{ kN}$$

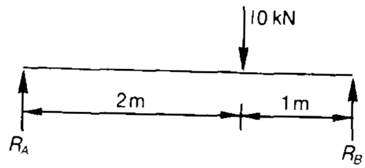
Resolving forces vertically

$$R_A + R_B = 3 + 10$$

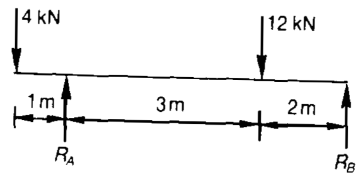
or $$R_B = 13 - 9.5 = 3.5 \text{ kN}$$

Further problems ***(answers on page 691)***

Problem I.4 Determine the reactions R_A and R_B for the simply-supported beams.



(a)



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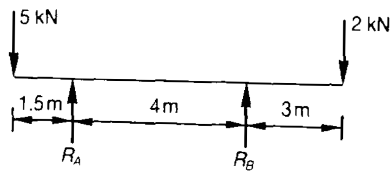
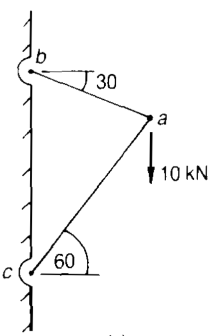
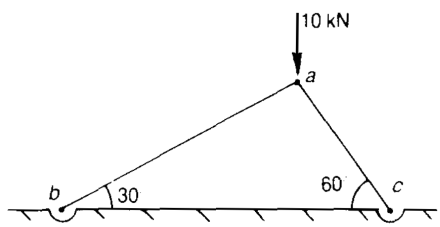


Problem I.5 Determine the forces the pin-jointed trusses shown.



(a)



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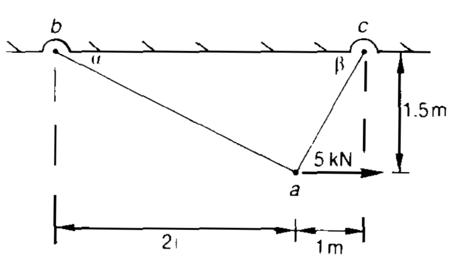

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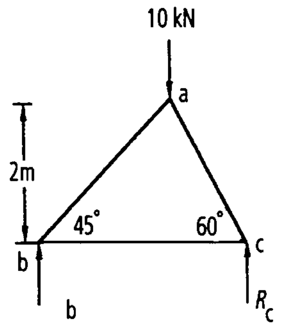

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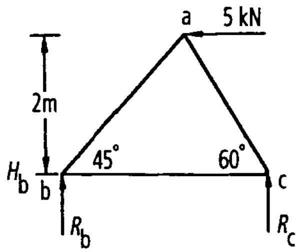


(e)

1 Tension and compression: direct stresses

1.1 Introduction

The strength of a material, whatever its nature, is defined largely by the internal stresses, or intensities of force, in the material. A knowledge of these stresses is essential to the safe design of a machine, aircraft, or any type of structure. Most practical structures consist of complex arrangements of many component members; an aircraft fuselage, for example, usually consists of an elaborate system of interconnected sheeting, longitudinal stringers, and transverse rings. The detailed stress analysis of such a structure is a difficult task, even when the loading conditions are simple. The problem is complicated further because the loads experienced by a structure are variable and sometimes unpredictable. We shall be concerned mainly with stresses in materials under relatively simple loading conditions; we begin with a discussion of the behaviour of a stretched wire, and introduce the concepts of direct stress and strain.

1.2 Stretching of a steel wire

One of the simplest loading conditions of a material is that of *tension*, in which the fibres of the material are stretched. Consider, for example, a long steel wire held rigidly at its upper end, Figure 1.1, and loaded by a mass hung from the lower end. If vertical movements of the lower end are observed during loading it will be found that the wire is stretched by a small, but measurable, amount from its original unloaded length. The material of the wire is composed of a large number of small crystals which are only visible under a microscopic study; these crystals have irregularly shaped boundaries, and largely random orientations with respect to each other; as loads are applied to the wire, the crystal structure of the metal is distorted.

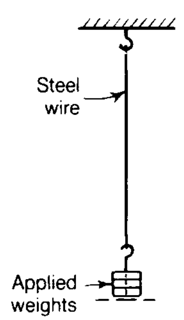


Figure 1.1 Stretching of a steel wire under end load.

For small loads it is found that the extension of the wire is roughly proportional to the applied load, Figure 1.2. This linear relationship between load and extension was discovered by Robert Hooke in 1678; a material showing this characteristic is said to obey *Hooke's law*.

As the tensile load in the wire is increased, a stage is reached where the material ceases to show this linear characteristic; the corresponding point on the load–extension curve of Figure 1.2 is known as the *limit of proportionality*. If the wire is made from a high-strength steel then the load–extension curve up to the *breaking point* has the form shown in Figure 1.2. Beyond the limit of proportionality the extension of the wire increases non-linearly up to the elastic limit and, eventually, the breaking point.

The elastic limit is important because it divides the load–extension curve into two regions. For loads up to the elastic limit, the wire returns to its original unstretched length on removal of the loads; this property of a material to recover its original form on removal of the loads is known as *elasticity*; the steel wire behaves, in fact, as a still elastic spring. When loads are applied above the elastic limit, and are then removed, it is found that the wire recovers only part of its extension and is stretched permanently; in this condition the wire is said to have undergone an *inelastic*, or *plastic*, extension. For most materials, the limit of proportionality and the elastic limit are assumed to have the same value.

In the case of elastic extensions, work performed in stretching the wire is stored as *strain energy* in the material; this energy is recovered when the loads are removed. During inelastic extensions, work is performed in making permanent changes in the internal structure of the material; not all the work performed during an inelastic extension is recoverable on removal of the loads; this energy reappears in other forms, mainly as heat.

The load–extension curve of Figure 1.2 is not typical of all materials; it is reasonably typical, however, of the behaviour of *brittle* materials, which are discussed more fully in Section 1.5. An important feature of most engineering materials is that they behave elastically up to the limit of proportionality, that is, all extensions are recoverable for loads up to this limit. The concepts of linearity and elasticity[1] form the basis of the theory of small deformations in stressed materials.

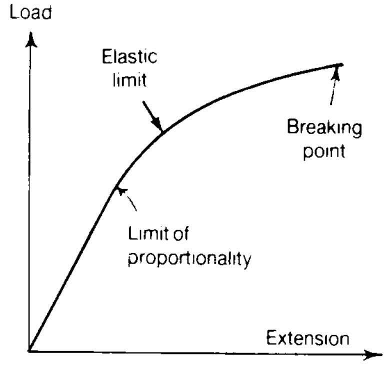


Figure 1.2 Load–extension curve for a steel wire, showing the limit of linear-elastic behaviour (or limit of proportionality) and the breaking point.

[1]The definition of elasticity requires only that the extensions are recoverable on removal of the loads; this does not preclude the possibility of a non-linear relation between load and extension .

1.3 Tensile and compressive stresses

The wire of Figure 1.1 was pulled by the action of a mass attached to the lower end; in this condition the wire is in *tension*. Consider a cylindrical bar *ab*, Figure 1.3, which has a uniform cross-section throughout its length. Suppose that at each end of the bar the cross-section is divided into small elements of equal area; the cross-sections are taken normal to the longitudinal axis of the bar. To each of these elemental areas an equal tensile load is applied normal to the cross-section and parallel to the longitudinal axis of the bar. The bar is then uniformly stressed in tension.

Suppose the total load on the end cross-sections is P; if an imaginary break is made perpendicular to the axis of the bar at the section c, Figure 1.3, then equal forces P are required at the section c to maintain equilibrium of the lengths ac and cb. This is equally true for any section across the bar, and hence on any imaginary section perpendicular to the axis of the bar there is a total force P.

When tensile tests are carried out on steel wires of the same material, but of different cross-sectional area, the breaking loads are found to be proportional approximately to the respective cross-sectional areas of the wires. This is so because the tensile strength is governed by the intensity of force on a normal cross-section of a wire, and not by the total force. This intensity of force is known as *stress*; in Figure 1.3 the *tensile stress* σ at any normal cross-section of the bar is

$$\sigma = \frac{P}{A} \tag{1.1}$$

where P is the total force on a cross-section and A is the area of the cross-section.

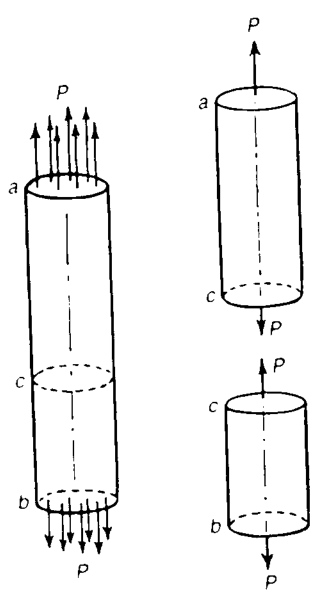


Figure 1.3 Cylindrical bar under uniform tensile stress; there is a similar state of tensile stress over any imaginary normal cross-section.

In Figure 1.3 uniform stressing of the bar was ensured by applying equal loads to equal small areas at the ends of the bar. In general we are not dealing with equal force intensities of this type, and a more precise definition of stress is required. Suppose δA is an element of area of the cross-section of the bar, Figure 1.4; if the normal force acting on this element is δP, then the tensile stress at this point of the cross-section is defined as the limiting value of the ratio $(\delta P/\delta A)$ as δA becomes infinitesimally small. Thus

$$\sigma = \underset{\delta A \to 0}{\text{Limit}} \frac{\delta P}{\delta A} = \frac{dP}{dA} \tag{1.2}$$

This definition of stress is used in studying problems of non-uniform stress distribution in materials.

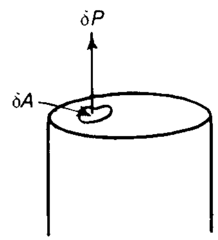
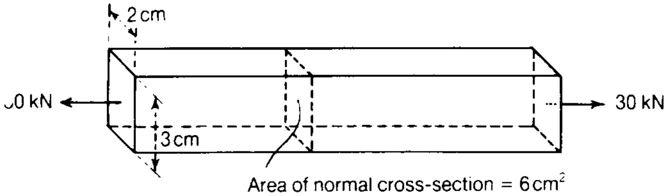


Figure 1.4 Normal load on an element of area of the cross-section.

When the forces P in Figure 1.3 are reversed in direction at each end of the bar they tend to *compress* the bar; the loads then give rise to *compressive stresses*. Tensile and compressive stresses are together referred to as *direct* (or *normal*) *stresses*, because they act perpendicularly to the surface.

Problem 1.1 A steel bar of rectangular cross-section, 3 cm by 2 cm, carries an axial load of 30 kN. Estimate the average tensile stress over a normal cross-section of the bar.



Solution

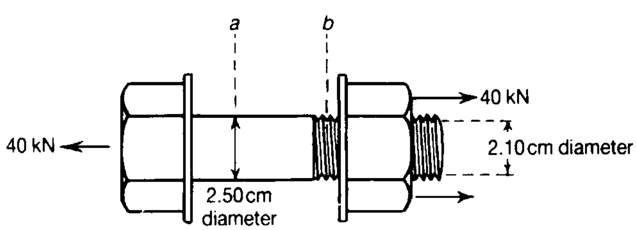
The area of a normal cross-section of the bar is

$$A = 0.03 \times 0.02 = 0.6 \times 10^{-3}\,\mathrm{m}^2$$

The average tensile stress over this cross-section is then

$$\sigma = \frac{P}{A} = \frac{30 \times 10^3}{0.6 \times 10^{-3}} = 50\mathrm{MN/m}^2$$

Problem 1.2 A steel bolt, 2.50 cm in diameter, carries a tensile load of 40 kN. Estimate the average tensile stress at the section *a* and at the screwed section *b*, where the diameter at the root of the thread is 2.10 cm.



Solution

The cross-sectional area of the bolt at the section *a* is

$$A_a = \frac{\pi}{4}\,(0.025)^2 = 0.491 \times 10^{-3}\,\mathrm{m}^2$$

The average tensile stress at A is then

$$\sigma_a = \frac{P}{A_a} = \frac{40 \times 10^3}{0.491 \times 10^{-3}} = 81.4\,\mathrm{MN/m}^2$$

The cross-sectional area at the root of the thread, section *b*, is

$$A_b = \frac{\pi}{4}\,(0.021)^2 = 0.346 \times 10^{-3}\,\mathrm{m}^2$$

The average tensile stress over this section is

$$\sigma_b = \frac{P}{A_b} = \frac{40 \times 10^3}{0.346 \times 10^{-3}} = 115.6\,\mathrm{MN/m}^2$$

1.4 Tensile and compressive strains

In the steel wire experiment of Figure 1.1 we discussed the extension of the whole wire. If we measure the extension of, say, the lowest quarter-length of the wire we find that for a given load it is equal to a quarter of the extension of the whole wire. In general we find that, at a given load, the ratio of the extension of any length to that length is constant for all parts of the wire; this ratio is known as the *tensile strain*.

Suppose the initial unstrained length of the wire is L_0, and the e is the extension due to straining; the tensile strain ε is defined as

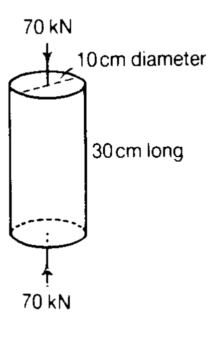
$$\varepsilon = \frac{e}{L_0} \tag{1.3}$$

This definition of strain is useful only for small distortions, in which the extension e is small compared with the original length L_0; this definition is adequate for the study of most engineering problems, where we are concerned with values of ε of the order 0.001, or so.

If a material is compressed the resulting strain is defined in a similar way, except that e is the contraction of a length.

We note that strain is a *non-dimensional* quantity, being the ratio of the extension, or contraction, of a bar to its original length.

Problem 1.3 A cylindrical block is 30 cm long and has a circular cross-section 10 cm in diameter. It carries a total compressive load of 70 kN, and under this load it contracts by 0.02 cm. Estimate the average compressive stress over a normal cross-section and the compressive strain.



<u>*Solution*</u>

The area of a normal cross-section is

$$A = \frac{\pi}{4}(0.10)^2 = 7.85 \times 10^{-3}\,\mathrm{m}^2$$

The average compressive stress over this cross-section is then

$$\sigma = \frac{P}{A} = \frac{70 \times 10^3}{7.85 \times 10^{-3}} = 8.92\ \mathrm{MN/m^2}$$

The average compressive strain over the length of the cylinder is

$$\varepsilon = \frac{0.02 \times 10^{-2}}{30 \times 10^{-2}} = 0.67 \times 10^{-3}$$

1.5 Stress–strain curves for brittle materials

Many of the characteristics of a material can be deduced from the tensile test. In the experiment of Figure 1.1 we measured the extensions of the wire for increasing loads; it is more convenient to compare materials in terms of stresses and strains, rather than loads and extensions of a particular specimen of a material.

The tensile *stress–strain* curve for a high-strength steel has the form shown in Figure 1.5. The stress at any stage is the ratio of the load of the *original* cross-sectional area of the test specimen; the strain is the elongation of a unit length of the test specimen. For stresses up to about 750 $\mathrm{MN/m^2}$ the stress–strain curve is linear, showing that the material obeys Hooke's law in this range; the material is also elastic in this range, and no permanent extensions remain after removal of the stresses. The ratio of stress to strain for this linear region is usually about 200 $\mathrm{GN/m^2}$ for steels; this ratio is known as *Young's modulus* and is denoted by *E*. The strain at the limit of proportionality is of the order 0.003, and is small compared with strains of the order 0.100 at fracture.

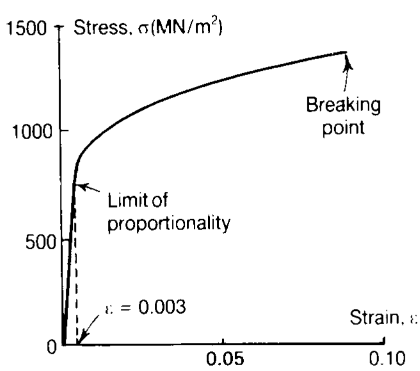


Figure 1.5 Tensile stress–strain curve for a high-strength steel.

We note that *Young's modulus* has the units of a stress; the value of E defines the constant in the linear relation between stress and strain in the elastic range of the material. We have

$$E = \frac{\sigma}{\varepsilon} \tag{1.4}$$

for the linear-elastic range. If P is the total tensile load in a bar, A its cross-sectional area, and L_0 its length, then

$$E = \frac{\sigma}{\varepsilon} = \frac{P / A}{e / L_0} \tag{1.5}$$

where e is the extension of the length L_0. Thus the expansion is given by

$$e = \frac{PL_0}{EA} \tag{1.6}$$

If the material is stressed beyond the linear-elastic range the limit of proportionality is exceeded, and the strains increase non-linearly with the stresses. Moreover, removal of the stress leaves the material with some permanent extension; this range is then both non-linear and inelastic. The maximum stress attained may be of the order of 1500 MN/m^2, and the total extension, or *elongation*, at this stage may be of the order of 10%.

The curve of Figure 1.5 is typical of the behaviour of *brittle* materials—as, for example, area characterized by small permanent elongation at the breaking point; in the case of metals this is usually 10%, or less.

When a material is stressed beyond the limit of proportionality and is then unloaded, permanent deformations of the material take place. Suppose the tensile test-specimen of Figure 1.5 is stressed beyond the limit of proportionality, (point a in Figure 1.6), to a point b on the stress–strain diagram. If the stress is now removed, the stress–strain relation follows the curve bc; when the stress is completely removed there is a residual strain given by the intercept $0c$ on the ε-axis. If the stress is applied again, the stress–strain relation follows the curve cd initially, and finally the curve df to the breaking point. Both the unloading curve bc and the reloading curve cd are approximately parallel to the elastic line $0a$; they are curved slightly in opposite directions. The process of unloading and reloading, bcd, had little or no effect on the stress at the breaking point, the stress–strain curve being interrupted by only a small amount bd, Figure 1.6.

The stress–strain curves of brittle materials for tension and compression are usually similar in form, although the stresses at the limit of proportionality and at fracture may be very different for the two loading conditions. Typical tensile and compressive stress–strain curves for concrete are shown in Figure 1.7; the maximum stress attainable in tension is only about one-tenth of that in compression, although the slopes of the stress–strain curves in the region of zero stress are nearly equal.

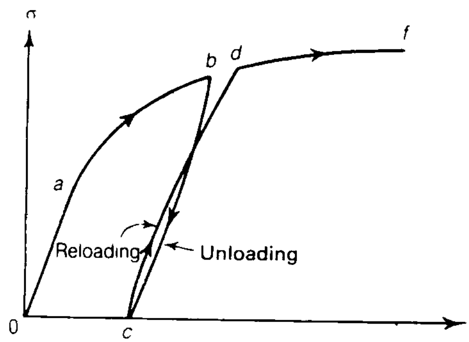


Figure 1.6 Unloading and reloading of a material in the inelastic range; the paths *bc* and *cd* are approximately parallel to the linear-elastic line *oa*.

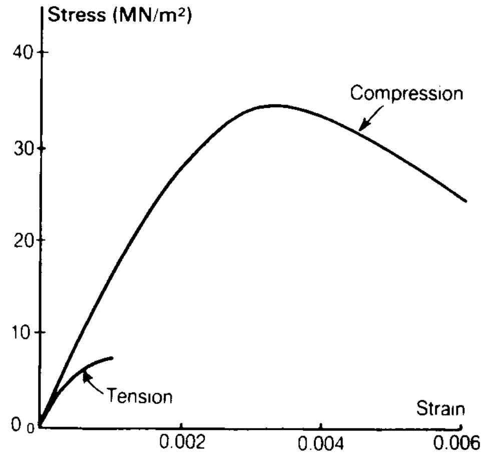


Figure 1.7 Typical compressive and tensile stress–strain curves for concrete, showing the comparative weakness of concrete in tension.

1.6 Ductile materials *(see Section 1.8)*

A brittle material is one showing relatively little elongation at fracture in the tensile test; by contrast some materials, such as mild steel, copper, and synthetic polymers, may be stretched appreciably before breaking. These latter materials are *ductile* in character.

If tensile and compressive tests are made on a mild steel, the resulting stress–strain curves are different in form from those of a brittle material, such as a high-strength steel. If a tensile test specimen of mild steel is loaded axially, the stress–strain curve is linear and elastic up to a point *a*, Figure 1.8; the small strain region of Figure 1.8. is reproduced to a larger scale in Figure 1.9. The ratio of stress to strain, or Young's modulus, for the linear portion 0*a* is usually about 200 GN/m^2, ie, 200 $\times 10^9$ N/m^2. The tensile stress at the point *a* is of order 300 MN/m^2, i.e. 300 $\times$ 10^6 N/m^2. If the test specimen is strained beyond the point *a*, Figures 1.8 and 1.9, the stress must be reduced almost immediately to maintain equilibrium; the reduction of stress, *ab*, takes place rapidly, and the form of the curve *ab* is difficult to define precisely. Continued straining proceeds at a roughly constant stress along *bc*. In the range of strains from *a* to *c* the material is said to *yield*; *a* is the *upper yield point*, and *b* the *lower yield point*. Yielding at constant stress along *bc* proceeds usually to a strain about 40 times greater than that at *a*; beyond the point *c* the material *strain-hardens*, and stress again increases with strain where the slope from *c* to *d* is about 1/50th that from 0 to *a*. The stress for a tensile specimen attains a maximum value at *d* if the stress is evaluated on the basis of the original cross-sectional area of the bar; the stress corresponding to the point *d* is known as the *ultimate stress*, σ_{ult}, of the material. From *d* to *f* there is a reduction in the nominal stress until fracture occurs at *f*. The ultimate stress in tension is attained at a stage when *necking* begins; this is a reduction of area at a relatively weak cross-section of the test specimen. It is usual to measure the diameter of the neck after fracture, and to evaluate a true stress at fracture, based on the breaking load and the reduced cross-sectional area at the neck. Necking and considerable elongation before fracture are characteristics of ductile materials; there is little or no necking at fracture for brittle materials.

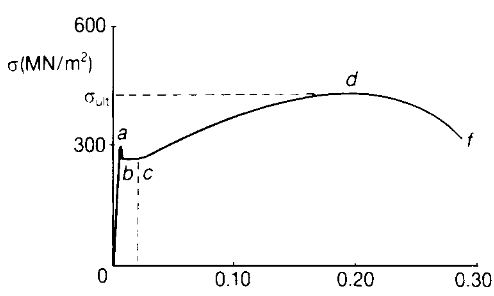


Figure 1.8 Tensile stress–strain curve for an annealed mild steel, showing the drop in stress at yielding from the upper yield point *a* to the lower yield point *b*.

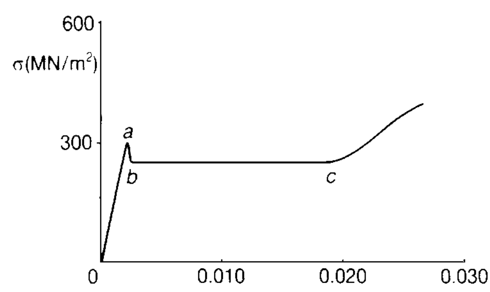


Figure 1.9 Upper and lower yield points of a mild steel.

Compressive tests of mild steel give stress–strain curves similar to those for tension. If we consider tensile stresses and strains as positive, and compressive stresses and strains as negative, we can plot the tensile and compressive stress–strain curves on the same diagram; Figure 1.10 shows the stress–strain curves for an annealed mild steel. In determining the stress–strain curves experimentally, it is important to ensure that the bar is loaded axially; with even small eccentricities of loading the stress distribution over any cross-section of the bar is non-uniform, and the upper yield point stress is not attained in all fibres of the material simultaneously. For this reason the lower yield point stress is taken usually as a more realistic definition of yielding of the material.

Some ductile materials show no clearly defined upper yield stress; for these materials the limit of proportionality may be lower than the stress for continuous yielding. The term *yield stress* refers to the stress for continuous yielding of a material; this implies the lower yield stress for a material in which an upper yield point exists; the yield stress is denoted by σ_Y.

Tensile failures of some steel bars are shown in Figure 1.11; specimen (ii) is a brittle material, showing little or no necking at the fractured section; specimens (i) and (iii) are ductile steels showing a characteristic necking at the fractured sections. The tensile specimens of Figure 1.12 show the forms of failure in a ductile steel and a ductile light-alloy material; the steel specimen (i) fails at a necked section in the form of a 'cup and cone'; in the case of the light-alloy bar, two 'cups' are formed. The compressive failure of a brittle cast iron is shown in Figure 1.13. In the case of a mild steel, failure in compression occurs in a 'barrel-like' fashion, as shown in Figure 1.14.

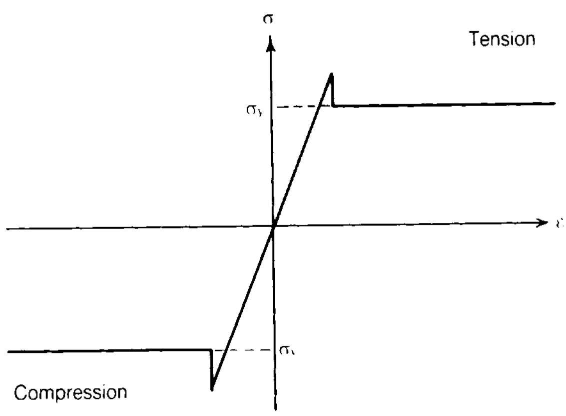


Figure 1.10 Tensile and compressive stress–strain curves for an annealed mild steel; in the annealed condition the yield stresses in tension and compression are approximately equal.

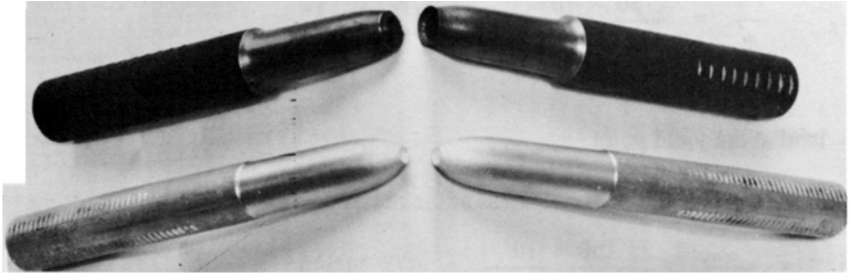
The stress–strain curves discussed in the preceding paragraph refer to static tests carried out at negligible speed. When stresses are applied rapidly the yield stress and ultimate stresses of metallic materials are usually raised. At a strain rate of 100 per second the yield stress of a mild steel may be twice that at negligible speed.

(i)

(ii)

(iii)

Figure 1.11 Tensile failures in steel specimens showing necking in mild steel, (i) and (iii), and brittle fracture in high-strength steel, (ii).



(i)

(ii)

Figure 1.12 Necking in tensile failures of ductile materials.
(i) Mild-steel specimen showing 'cup and cone' at the broken section.
(ii) Aluminium-alloy specimen showing double 'cup' type of failure.

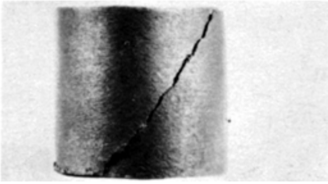


Figure 1.13 Failure in compression of a circular specimen of cast iron, showing fracture on a diagonal plane.

Figure 1.14 Barrel-like failure in a compressed specimen of mild steel.

Problem 1.4 A tensile test is carried out on a bar of mild steel of diameter 2 cm. The bar yields under a load of 80 kN. It reaches a maximum load of 150 kN, and breaks finally at a load of 70 kN.

Estimate:

(i) the tensile stress at the yield point;
(ii) the ultimate tensile stress;
(iii) the average stress at the breaking point, if the diameter of the fractured neck is 1 cm.

Solution

The original cross-section of the bar is

$$A_0 = \frac{\pi}{4}(0.020)^2 = 0.314 \times 10^{-3} \text{ m}^2$$

(i) The average tensile stress at yielding is then

$$\sigma_Y = \frac{P_Y}{A_0} = \frac{80 \times 10^3}{0.314 \times 10^{-3}} = 254 \text{ MN/m}^2,$$

where P_Y = load at the yield point

(ii) The ultimate stress is the nominal stress at the maximum load, i.e.,

$$\sigma_{\text{ult}} = \frac{P_{\max}}{A_0} = \frac{150 \times 10^3}{0.314 \times 10^{-3}} = 477 \text{ MN/m}^2$$

where $P_{\max}$ = maximum load

(iii) The cross-sectional area in the fractured neck is

$$A_f = \frac{\pi}{4}(0.010)^2 = 0.0785 \times 10^{-3} \text{ m}^2$$

The average stress at the breaking point is then

$$\sigma_f = \frac{P_f}{A_f} = \frac{70 \times 10^3}{0.0785 \times 10^{-3}} = 892 \text{ MN/m}^2,$$

where P_f = final breaking load.

Problem 1.5 A circular bar of diameter 2.50 cm is subjected to an axial tension of 20 kN. If the material is elastic with a Young's modulus E = 70 GN/m^2, estimate the percentage elongation.

Solution

The cross-sectional area of the bar is

$$A = \frac{\pi}{4}(0.025)^2 = 0.491 \times 10^{-3}\ \mathrm{m}^2$$

The average tensile stress is then

$$\sigma = \frac{P}{A} = \frac{20 \times 10^3}{0.491 \times 10^{-3}} = 40.7\ \mathrm{MN/m}^2$$

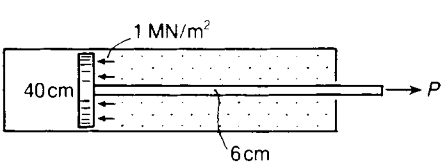
The longitudinal tensile strain will therefore be

$$\varepsilon = \frac{\sigma}{E} = \frac{40.7 \times 10^6}{70 \times 10^9} = 0.582 \times 10^{-3}$$

The percentage elongation will therefore be

$$(0.582 \times 10^{-3})\,100 = 0.058\%$$

Problem 1.6 The piston of a hydraulic ram is 40 cm diameter, and the piston rod 6 cm diameter. The water pressure is 1 MN/m^2. Estimate the stress in the piston rod and the elongation of a length of 1 m of the rod when the piston is under pressure from the piston-rod side. Take Young's modulus as E = 200 GN/m^2.



<u>*Solution*</u>

The pressure on the back of the piston acts on a net area

$$\frac{\pi}{4}\left[(0.40)^2 - (0.06)^2\right] = \frac{\pi}{4}(0.46)(0.34) = 0.123 \text{ m}^2$$

The load on the piston is then

$$P = (1)(0.123) = 0.123 \text{ MN}$$

Area of the piston rod is

$$A = \frac{\pi}{4}(0.060)^2 = 0.283 \times 10^{-2} \text{ m}^2$$

The average tensile stress in the rod is then

$$\sigma = \frac{P}{A} = \frac{0.123 \times 10^6}{0.283 \times 10^{-2}} = 43.5 \text{ MN/m}^2$$

From equation (1.6), the elongation of a length $L = 1$ m is

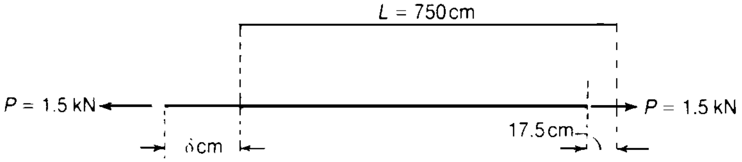
$$e = \frac{PL}{EA} = \frac{P}{A}\left(\frac{L}{E}\right) = \frac{\sigma L}{E}$$

$$= \frac{(43.5 \times 10^6)(1)}{200 \times 10^9}$$

$$= 0.218 \times 10^{-3} \text{ m}$$

$$= 0.0218 \text{ cm}$$

Problem 1.7 The steel wire working a signal is 750 m long and 0.5 cm diameter. Assuming a pull on the wire of 1.5 kN, find the movement which must be given to the signal-box end of the wire if the movement at the signal end is to be 17.5 cm. Take Young's modulus as 200 GN/m^2.



Solution

If δ(cm) is the movement at the signal-box end, the actual stretch of the wire is $e = (\delta - 17.5)$cm

The longitudinal strain is then

$$\varepsilon = \frac{(\delta - 17.5)\ 10^{-2}}{750}$$

Now the cross-sectional area of the wire is

$$A = \frac{\pi}{4}(0.005)^2 = 0.0196 \times 10^{-3}\ \mathrm{m}^2$$

The longitudinal strain can also be defined in terms of the tensile load, namely,

$$\varepsilon = \frac{e}{L} = \frac{P}{EA} = \frac{1.5 \times 10^3}{\left(200 \times 10^9\right)\left(0.0196 \times 10^{-3}\right)}$$

$$= 0.383 \times 10^{-3}$$

On equating these two values of ε,

$$\frac{(\delta - 17.5)\ 10^{-2}}{750} = 0.383 \times 10^{-3}$$

The equation gives

$$\delta = 46.2\ \mathrm{cm}$$

Problem 1.8 A circular, metal rod of diameter 1 cm is loaded in tension. When the tensile load is 5kN, the extension of a 25 cm length is measured accurately and found to be 0.0227 cm. Estimate the value of Young's modulus, E, of the metal.

<u>*Solution*</u>

The cross-sectional area is

$$A = \frac{\pi}{4}(0.01)^2 = 0.0785 \times 10^{-3}\ \mathrm{m}^2$$

The tensile stress is then

$$\sigma = \frac{P}{A} = \frac{5 \times 10^3}{0.0785 \times 10^{-3}} = 63.7\ \mathrm{MN/m}^2$$

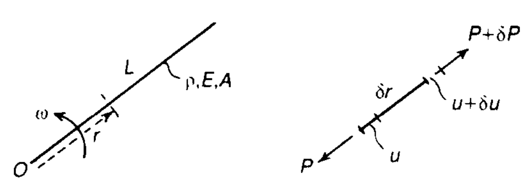
The measured tensile strain is

$$\varepsilon = \frac{e}{L} = \frac{0.0227 \times 10^{-2}}{25 \times 10^{-2}} = 0.910 \times 10^{-3}$$

Then Young's modulus is defined by

$$E = \frac{\sigma}{\varepsilon} = \frac{63.7 \times 10^6}{0.91 \times 10^{-3}} = 70\ \mathrm{GN/m}^2$$

Problem 1.9 A straight, uniform rod of length L rotates at uniform angular speed ω about an axis through one end and perpendicular to its length. Estimate the maximum tensile stress generated in the rod and the elongation of the rod at this speed. The density of the material is ρ and Young's modulus is E.



<u>*Solution*</u>

Suppose the radial displacement of any point a distance r from the axis of rotation is u. The radial displacement a distance $r + \delta r$) from 0 is then $(u + \delta u)$, and the elemental length δr of the rod is stretched therefore an amount δu. The longitudinal strain of this element is therefore

$$\varepsilon = \underset{\delta r \to 0}{\text{Limit}} \frac{\delta u}{\delta r} = \frac{\mathrm{du}}{\mathrm{dr}}$$

The longitudinal stress in the elemental length is then

$$\sigma = E\varepsilon = E \frac{\mathrm{du}}{\mathrm{dr}}$$

If A is the cross-sectional area of the rod, the longitudinal load at any radius r is then

$$P = \sigma A = EA \frac{\mathrm{du}}{\mathrm{dr}}$$

The centrifugal force acting on the elemental length δr is

$$(\rho A \, \delta r) \, \omega^2 r$$

Then, for radial equilibrium of the elemental length,

$$\delta P + \rho A \omega^2 \, r \, \delta r = 0$$

This gives

$$\frac{dP}{dr} = -\rho A \omega^2 r$$

On integrating, we have

$$P = -\frac{1}{2} \rho A \omega^2 r^2 + C$$

where C is an arbitrary constant; if $P = 0$ at the remote end, $r = L$, of the rod, then

$$C = \frac{1}{2} \rho A \omega^2 L^2$$

and

$$P = \frac{1}{2}\rho A\omega^2 L^2\left(1-\frac{r^2}{L^2}\right)$$

The tensile stress at any radius is then

$$\sigma = \frac{P}{A} = \frac{1}{2}\rho\omega^2 L^2\left(1-\frac{r^2}{L^2}\right)$$

This is greatest at the axis of rotation, $r = 0$, so that

$$\sigma_{max} = \frac{1}{2}\rho\omega^2 L^2$$

The longitudinal stress, σ, is defined by

$$\sigma = E\frac{du}{dr}$$

so

$$\frac{du}{dr} = \frac{\sigma}{E} = \frac{\rho\,\omega^2 L^2}{2E}\left(1 - \frac{r^2}{L^2}\right)$$

On integrating,

$$u = \frac{\rho\,\omega^2 L^2}{2E}\left(r - \frac{r^3}{3L^2} + D\right)$$

where D is an arbitrary constant; if there is no radial movement at 0, then $u = 0$ at $= r = 0$, and we have $D = 0$.

Thus

$$u = \frac{\rho\,\omega^2 L^2}{2E}\left[r\left(1 - \frac{r^2}{3L^2}\right)\right]$$

At the remote end, $r = L$,

$$u_L = \frac{\rho\,\omega^2 L^2}{2E}\left[L\left(\frac{2}{3}\right)\right] = \frac{\rho\,\omega^2 L^3}{3E}$$

1.7 Proof stresses

Many materials show no well-defined yield stresses when tested in tension or compression. A typical stress–strain curve for an aluminium alloy is shown in Figure 1.15.

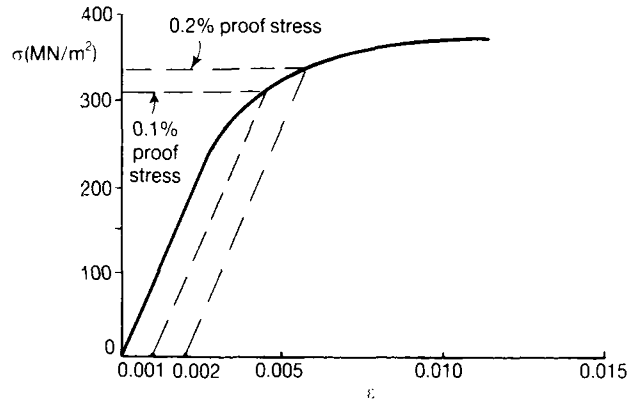


Figure 1.15 Proof stresses of an aluminium-alloy material; the proof stress is found by drawing the line parallel to the linear-elastic line at the appropriate proof strain.

The limit of proportionality is in the region of 300 MN/m^2, but the exact position of this limit is difficult to determine experimentally. To overcome this problem a *proof stress* is defined; the 0.1% proof stress required to produce a permanent strain of 0.001 (or 0.1%) on removal of the stress. Suppose we draw a line from the point 0.001 on the strain axis, Figure 1.15, parallel to the elastic line of the material; the point where this line cuts the stress–strain curve defines the proof stress. The 0.2% proof stress is defined in a similar way.

1.8 Ductility measurement

The Ductility value of a material can be described as the ability of the material to suffer plastic deformation while still being able to resist applied loading. The more ductile a material is the more it is said to have the ability to deform under applied loading.

The ductility of a metal is usually measured by its percentage reduction in cross-sectional area or by its percentage increase in length, i.e.

$$\text{percentage reduction in area} = \frac{(A_I - A_F)}{A_I} \times 100\%$$

and

$$\text{percentage increase in length} = \frac{(L_I - L_F)}{L_I} \times 100\%$$

where

A_I = initial cross-sectional area of the tensile specimen

A_F = final cross-sectional area of the tensile specimen

L_I = initial gauge length of the tensile specimen

L_F = final gauge length of the tensile specimen

It should be emphasised that the shape of the tensile specimen plays a major role on the measurement of the ductility and some typical relationships between length and character for tensile specimens i.e. given in Table 1.1

Materials such as copper and mild steel have high ductility and brittle materials such as bronze and cast iron have low ductility.

Table 1.1 Circular cylindrical tensile specimens

Place	L_I	L_I/D_I*
UK	4√area	3.54
USA	4.51√area	4.0
Europe	5.65√area	5.0

area = cross-sectional area

* D_I = initial diameter of the tensile specimen

1.9 Working stresses

In many engineering problems the loads sustained by a component of a machine or structure are reasonably well-defined; for example, the lower stanchions of a tall building support the weight of material forming the upper storeys. The stresses which are present in a component, under normal working conditions, are called the *working stresses*; the ratio of the yield stress, σ_Y, of a material to the largest working stress, σ_W, in the component is the *stress factor* against yielding. The stress factor on yielding is then

$$\frac{\sigma_Y}{\sigma_W} \tag{1.7}$$

If the material has no well-defined yield point, it is more convenient to use the *proof stress*, σ_p; the stress factor on proof stress is then

$$\frac{\sigma_P}{\sigma_W} \tag{1.8}$$

Some writers refer to the stress factor defined above as a 'safety factor'. It is preferable, however, to avoid any reference to 'safe' stresses, as the degree of safety in any practical problem is difficult to define. The present writers prefer the term 'stress factor' as this defines more precisely that the working stress is compared with the yield, or proof stress of the material. Another reason for using 'stress factor' will become more evident after the reader has studied Section 1.10.

1.10 Load factors

The *stress factor* in a component gives an indication of the working stresses in relation to the yield, or proof, stress of the material. In practical problems working stresses can only be estimated approximately in stress calculations. For this reason the stress factor may give little indication of the degree of safety of a component.

A more realistic estimate of safety can be made by finding the extent to which the working loads on a component may be increased before collapse or fracture occurs. Consider, for example, the continuous beam in Figure 1.16, resting on three supports. Under working conditions the beam carries lateral loads P_1, P_2 and P_3, Figure 1.16(i). If all these loads can be increased simultaneously by a factor n before collapse occurs, the load factor against collapse is n. In some complex structural systems, as for example continuous beams, the collapse loads, such as nP_1, nP_2 and nP_3, can be estimated reasonably accurately; the value of the load factor can then be deduced to give working loads P_1, P_2 and P_3.

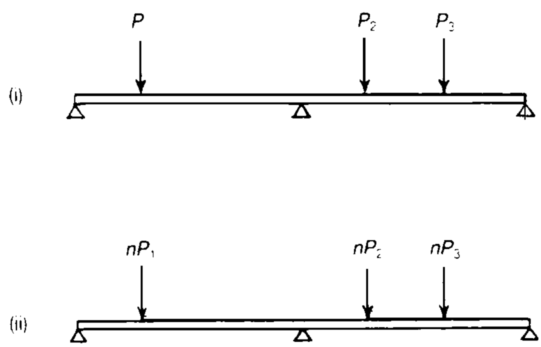


Figure 1.16 Factored loads on a continuous beam.
(i) Working loads. (ii) Factored working loads leading to collapse.

1.11 Lateral strains due to direct stresses

When a bar of a material is stretched longitudinally—as in a tensile test—the bar extends in the direction of the applied load. This longitudinal extension is accompanied by a lateral contraction of the bar, as shown in Figure 1.17. In the linear-elastic range of a material the lateral strain is proportional to the longitudinal strain; if ε_x is the longitudinal strain of the bar, then the lateral strain is

$$\varepsilon_y = v\varepsilon_x \tag{1.9}$$

The constant v in this relationship is known as *Poisson's ratio*, and for most metals it has a value of about 0.3 in the linear-elastic range; it cannot exceed a value of 0.5. For concrete it has a value of about 0.1. If the longitudinal strain is tensile, the lateral strain is a contraction; for a compressed bar there is a lateral expansion.

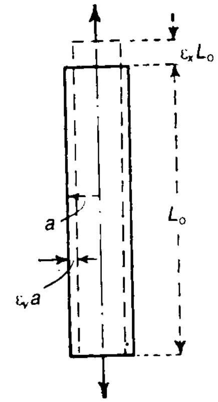


Figure 1.17 The Poisson ratio effect leading to lateral contraction of a bar in tension.

With a knowledge of the lateral contraction of a stretched bar it is possible to calculate the change in volume due to straining. The bar of Figure 1.17 is assumed to have a square cross-section of side a; L_0 is the unstrained length of the bar. When strained longitudinally an amount ε_x, the corresponding lateral strain of contractions is ε_y. The bar extends therefore an amount $\varepsilon_x L_0$, and each side of the cross-section contracts an amount $\varepsilon_y a$. The volume of the bar before stretching is

$$V_0 = a^2 L_0$$

After straining the volume is

$$V = (a - \varepsilon_y a)^2 \; (L_0 + \varepsilon_x L_0)$$

which may be written

$$V = a^2 L_0 (1 - \varepsilon_y)^2 (1 + \varepsilon_x) = V_0 (1 - \varepsilon_y)^2 (1 + \varepsilon_y)$$

If ε_x and ε_y are small quantities compared to unit, we may write

$$(1 - \varepsilon_y)^2 (1 + \varepsilon_x) = (1 - 2\,\varepsilon_y)(1 + \varepsilon_x) = 1 + \varepsilon_x - 2\,\varepsilon_y$$

ignoring squares and products of ε_x and ε_y. The volume after straining is then

$$V = V_0 (1 + \varepsilon_x - 2\,\varepsilon_y)$$

The *volumetric strain* is defined as the ratio of the change of volume to the original volume, and is therefore

$$\frac{V - V_0}{V_0} = \varepsilon_x - 2\,\varepsilon_y \tag{1.10}$$

If $\varepsilon_y = \nu \varepsilon_{xy}$ then the volumetric strain is $\varepsilon_x (1 - 2\nu)$. Equation (1.10) shows why ν cannot be greater than 0.5; if it were, then under *compressive* hydrostatic stress a *positive* volumetric strain will result, which is impossible.

Problem 1.10 A bar of steel, having a rectangular cross-section 7.5 cm by 2.5 cm, carries an axial tensile load of 180 kN. Estimate the decrease in the length of the sides of the cross-section if Young's modulus, E, is 200 GN/m^2 and Poisson's ratio, ν, is 0.3.

Solution

The cross-sectional area is

$$A = (0.075)(0.025) = 1.875 \times 10^{-3}\ \mathrm{m}^2$$

The average longitudinal tensile stress is

$$\sigma = \frac{P}{A} = \frac{180 \times 10^3}{1.875 \times 10^{-3}} = 96.0\ \mathrm{MN/m}^2$$

The longitudinal tensile strain is therefore

$$\varepsilon = \frac{\sigma}{E} = \frac{96.0 \times 10^6}{200 \times 10^9} = 0.48 \times 10^{-3}$$

The lateral strain is therefore

$$\nu\varepsilon = 0.3(0.48 \times 10^{-3}) = 0.144 \times 10^{-3}$$

The 7.5 cm side then contracts by an amount

$$\begin{aligned}(0.075)(0.144 \times 10^{-3}) &= 0.0108 \times 10^{-3}\ \mathrm{m} \\ &= 0.00108\ \mathrm{cm}\end{aligned}$$

The 2.5 cm side contracts by an amount

$$\begin{aligned}(0.025)(0.144 \times 10^{-3}) &= 0.0036 \times 10^{-3}\ \mathrm{m} \\ &= 0.00036\ \mathrm{cm}\end{aligned}$$

1.12 Strength properties of some engineering materials

The mechanical properties of some engineering materials are given in Table 1.2. Most of the materials are in common engineering use, including a number of relatively new and important materials; namely glass-fibre composites, carbon-fibre composites and boron composites. In the case of some brittle materials, such as cast iron and concrete, the ultimate stress in tension is considerably smaller than in compression.

Composite materials, such as glass fibre reinforced plastics, (GRP), carbon-fibre reinforced plastics (CFRP), boron-fibre reinforced plastics, 'Kevlar' and metal-matrix composites are likely to revolutionise the design and construction of many structures in the 21st century. The glass fibres used in GRP are usually made from a borosilicate glass, similar to the glass used for cooking utensils. Borosilicate glass fibres are usually produced in 'E' glass or glass that has good electrical resistance. A very strong form of borosilicate glass fibre appears in the form of 'S' glass which is much more expensive than 'E' glass.

Some carbon fibres, namely high modulus (HM) carbon fibres, have a tensile modulus much larger than high strength steels, whereas other carbon fibres have a very high tensile strength (HS) much larger than high tensile steels.

Currently 'S' glass is some eight times more expensive than 'E' glass and HS carbon is about 50 times more expensive than 'E' glass. HM carbon is some 250 times more expensive than 'E' glass while 'Kevlar' is some 15 times more expensive than 'E' glass.

1.13 Weight and stiffness economy of materials

In some machine components and structures it is important that the weight of material should be as small as possible. This is particularly true of aircraft, submarines and rockets, for example, in which less structural weight leads to a larger pay-load. If σ_{ult} is the ultimate stress of a material in tension and ρ is its density, then a measure of the strength economy is the ratio

$$\frac{\sigma_{ult}}{\rho}$$

The materials shown in Table 1.2 are compared on the basis of strength economy in Table 1.3 from which it is clear that the modern fibre-reinforced composites offer distinct savings in weight over the more common materials in engineering use.

In some engineering applications, stiffness rather than strength is required of materials; this is so in structures likely to buckle and components governed by deflection limitations. A measure of the stiffness economy of a material is the ratio

$$\frac{E}{\rho},$$

some values of which are shown in Table 1.2. Boron composites and carbon-fibre composites show outstanding stiffness properties, whereas glass-fibre composites fall more into line with the best materials already in common use.

Table 1.2 Approximate strength properties of some engineering materials

Material	Limit of proportionality (MN/m^2)	Ultimate stress σ_{ult} (MN/m^2)	Elongation at tensile fracture (as a fraction of the original length)	Young's modulus E (GN/m^2)	Density ρ (kg/m^3)	σ_{ult}/ρ (m/s)^2	E/ρ (m/s)^2	Coefficient of linear expansion α (per °C)
Medium-strength mild steel	280	370	0.30	200	7840	47×10^3	25×10^6	1.2×10^{-5}
High-strength steel	770	1550	0.10	200	7840	198×10^3	25×10^6	1.3×10^{-5}
Medium-strength aluminium alloy	230	430	0.10	70	2800	154×10^3	25×10^6	2.3×10^{-5}
Titanium alloy	385	690	0.15	120	4500	153×10^3	27×10^6	0.9×10^{-5}
Magnesium alloy	155	280	0.08	45	1800	156×10^3	25×10^6	2.7×10^{-5}
Wrought iron	185	310	—	190	7670	40×10^3	25×10^6	1.2×10^{-5}
Cast iron } tension	—	155	—	140	7200	—	20×10^6	1.1×10^{-5}
} compression	—	700	—	140	7200	$97^* \times 10^3$	20×10^6	1.1×10^{-5}
Concrete } tension	—	3.0	—	14	2410	—	6×10^6	1.2×10^{-5}
} compression	—	30.0	—	14	2410	$12^* \times 10^3$	6×10^6	1.2×10^{-5}
Nylon (polyamide)	77	90	1.00	2		79×10^3	1.8×10^6	10×10^{-5}
Polystyrene	46	60	0.03	3.5	1050	57×10^3	3.3×10^6	10×10^{-5}
Fluon (tetrafluoroethylene)	8	15	2.00	0.4	2220	7×10^3	0.2×10^6	11×10^{-5}
Polythene (ethylene)	6	12	5.00	0.2	915	13×10^3	0.2×10^6	28×10^{-5}
High-strength glass-fibre composite	—	1600	—	60	2000	800×10^3	30×10^6	—
Carbon-fibre composite	—	1400	—	170	1600	875×10^3	105×10^6	—
Boron composite	—	1300	—	270	2000	650×10^3	135×10^6	—

* Evaluate on the compressive value of σ_{ult}.

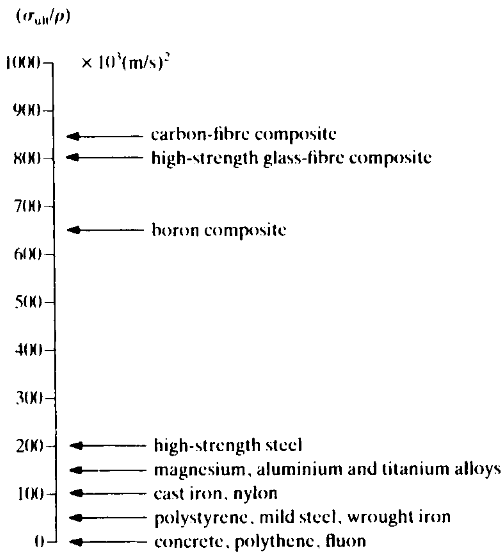


Table 1.3 Strength economy of some engineering materials

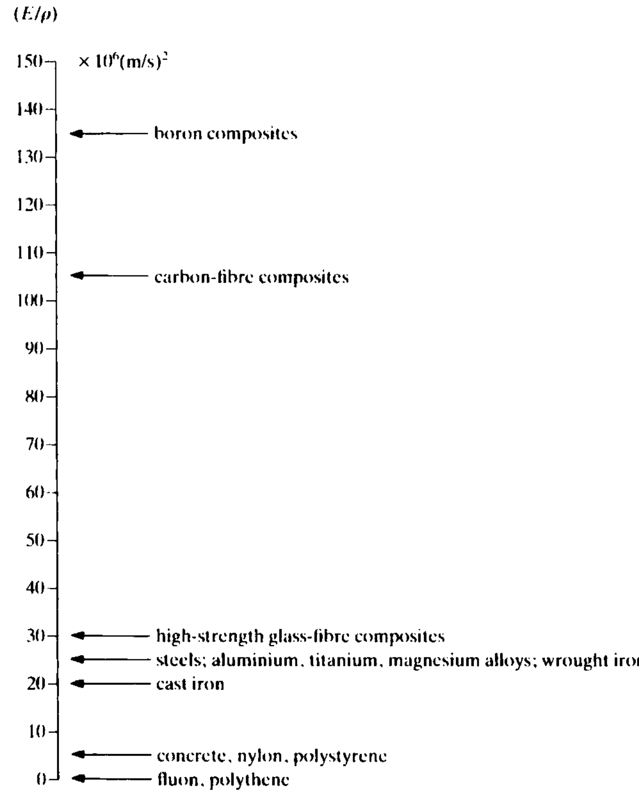


Table 1.4 Stiffness economy of some engineering materials

1.14 Strain energy and work done in the tensile test

As a tensile specimen extends under load, the forces applied to the ends of the test specimen move through small distances. These forces perform work in stretching the bar. If, at a tensile load P, the bar is stretched a small additional amount δe, Figure 1.18, then the work done on the bar is approximately $P\delta e$

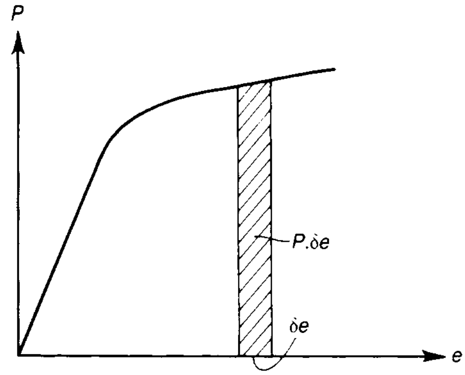


Figure 1.18 Work done in stretching a bar through a small extension, δe.

The total work done in extending the bar to the extension e is then

$$W = \int_0^e P\,de, \tag{1.11}$$

which is the area under the P–e curve up to the stretched condition. If the limit of proportionality is not exceeded, the work done in extending the bar is stored as *strain energy*, which is directly recoverable on removal of the load. For this case, the strain energy, U, is

$$U = W = \int_0^e P\,de \tag{1.12}$$

But in the linear-elastic range of the material, we have from equation (1.6) that

$$e = \frac{PL_0}{EA}$$

where L_0 is the initial length of the bar, A is its cross-sectional area and E is Young's modulus. Then equation (1.12) becomes

$$U = \int_0^e \frac{EA}{L_0}\,e\,de = \frac{EA}{2L_0}\left(e^2\right) \tag{1.13}$$

In terms of P

$$U = \frac{EA}{2L_0}\left(e^2\right) = \frac{L_0}{2EA}\left(P^2\right) \tag{1.14}$$

Now (P/A) is the tensile stress σ in the bar, and so we may write

$$U = \frac{AL_0}{2E}\left(\sigma^2\right) = \frac{\sigma^2}{2E} \times \text{ the volume} \tag{1.15}$$

Moreover, as AL_0 is the original volume of the bar, the strain energy per unit volume is

$$\frac{\sigma^2}{2E} \tag{1.16}$$

When the limit of proportionality of a material is exceeded, the work done in extending the bar is still given by equation (1.11); however, not all this work is stored as strain energy; some of the work done is used in producing permanent distortions in the material, the work reappearing largely in the form of heat. Suppose a mild-steel bar is stressed beyond the yield point, Figure 1.19, and up to the point where strain-hardening begins; the strain at the limit of proportionality is small compared with this large inelastic strain; the work done per unit volume in producing a strain ε is approximately

$$W = \sigma_Y \, \varepsilon \tag{1.17}$$

in which σ_Y is the yield stress of the material. This work is considerably greater than that required to reach the limit of proportionality. A ductile material of this type is useful in absorbing relatively large amounts of work before breaking.

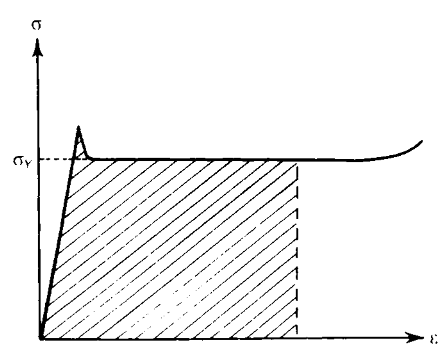
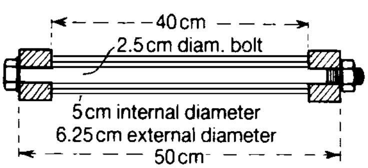


Figure 1.19 Work done in stretching a mild-steel bar; the work done during plastic deformation is very considerable compared with the elastic strain energy.

1.15 Initial stresses

It frequently happens that, before any load is applied to some part of a machine or structure, it is already in a state of stress. In other words, the component is *initially stressed* before external forces are applied. Bolted joints and connections, for example, involve bolts which are pre-tensioned; subsequent loading may, or may not, affect the tension in a bolt. Most forms of welded connections introduce initial stresses around the welds, unless the whole connection is stress relieved by a suitable heat treatment; in such cases, the initial stresses are not usually known with any real accuracy. Initial stresses can also be used to considerable effect in strengthening certain materials; for example, concrete can be made a more effective material by precompression in the form of prestressed concrete. The problems solved below are *statically indeterminate* (see Chapter 2) and therefore require *compatibility* considerations as well as *equilibrium* considerations.

Problem 1.11 A 2.5 cm diameter steel bolt passes through a steel tube 5 cm internal diameter, 6.25 cm external diameter, and 40 cm long. The bolt is then tightened up onto the tube through rigid end blocks until the tensile force in the bolts is 40 kN. The distance between the head of the bolt and the nut is 50 cm. If an external force of 30 kN is applied to the end blocks, tending to pull them apart, estimate the resulting tensile force in the bolt.



<u>*Solution*:</u>

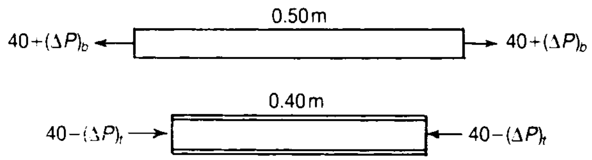
The cross-sectional area of the bolt is

$$\frac{\pi}{4}(0.025)^2 = 0.491 \times 10^{-3}\ \mathrm{m}^2$$

The cross-sectional are of the tube is

$$\frac{\pi}{4}\left[(0.0625)^2 - (0.050)^2\right] = \frac{\pi}{4}(0.1125)(0.0125) = 0.110 \times 10^{-2}\ \mathrm{m}^2$$

Before the external load of 30 kN is applied, the bolt and tube carry internal loads of 40 kN. When the external load of 30 kN is applied, suppose the tube and bolt are each stretched by amounts δ; suppose further that the *change* of load in the bolt is $(\Delta P)_b$, tensile, and the *change* of load in the tube is $(\Delta P)_t$, tensile.



Then for *compatibility*, the elastic stretch of each component due to the additional external load of 30 kN is

$$\delta = \frac{(\Delta P)_b\,(0.50)}{\left(0.491 \times 10^{-3}\right) E} = \frac{(\Delta P)_t\,(0.40)}{\left(0.110 \times 10^{-2}\right) E}$$

where E is Young's modulus. Then

$$(\Delta P)_b = 0.357\,(\Delta P)_t$$

But for *equilibrium* of internal and external forces,

$$(\Delta P)_b + (\Delta P)_t = 30 \text{ kN}$$

These two equations give

$$(\Delta P)_b = 7.89 \text{ kN}, \qquad (\Delta P)_t = 22.11 \text{ kN}$$

The resulting tensile force in the bolt is

$$40 + (\Delta P)_b = 47.89 \text{ kN}$$

1.16 Composite bars in tension or compression

A composite bar is one made of two materials, such as steel rods embedded in concrete. The construction of the bar is such that constituent components extend or contract equally under load. To illustrate the behaviour of such bars consider a rod made of two materials, 1 and 2, Figure 1.20; A_1, A_2 are the cross-sectional areas of the bars, and E_1, E_2 are the values of Young's modulus. We imagine the bars to be rigidly connected together at the ends; then for *compatibility*, the longitudinal strains to be the same when the composite bar is stretched we must have

$$\varepsilon = \frac{\sigma_1}{E_1} = \frac{\sigma_2}{E_2} \tag{1.18}$$

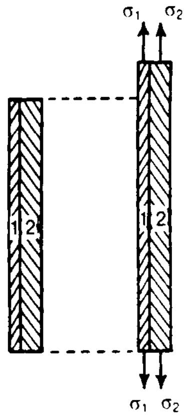


Figure 1.20 Composite bar in tension; if the bars are connected rigidly at their ends, they suffer the same extensions.

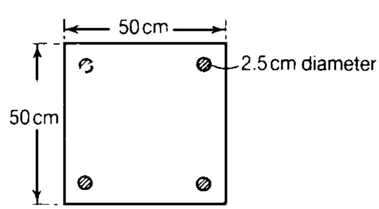
where σ_1 and σ_2 are the stresses in the two bars. But from *equilibrium* considerations,

$$P = \sigma_1 A_1 + \sigma_2 A_2 \tag{1.19}$$

Equations (1.18) and (1.19) give

$$\sigma_1 = \frac{PE_1}{A_1 E_1 + A_2 E_2}, \quad \sigma_2 = \frac{PE_2}{A_1 E_1 + A_2 E_2} \tag{1.20}$$

Problem 1.12 A concrete column, 50 cm square, is reinforced with four steel rods, each 2.5 cm in diameter, embedded in the concrete near the corners of the square. If Young's modulus for steel is 200 GN/m^2 and that for concrete is 14 GN/m^2, estimate the compressive stresses in the steel and concrete when the total thrust on the column is 1 MN.



Solution

Suppose subscripts c and s refer to concrete and steel, respectively. The cross-sectional area of steel is

$$A_s = 4\left[\frac{\pi}{4}(0.025)^2\right] = 1.96 \times 10^{-3}\ m^2$$

and the cross-sectional area of concrete is

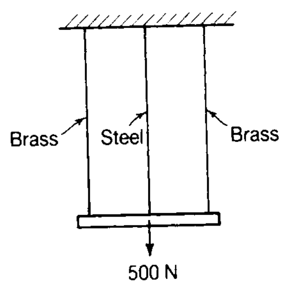
$$A_c = (0.50)^2 - A_s = 0.248 \text{ m}^2$$

Equations (1.20) then give

$$\sigma_c = \frac{10^6}{(0.248) + \left(1.96 \times 10^{-3}\right)\left(\dfrac{200}{14}\right)} = 3.62 \text{ MN/m}^2$$

$$\sigma_s = \frac{10^6}{(0.248)\left(\dfrac{14}{200}\right) + \left(1.96 \times 10^{-3}\right)} = 51.76 \text{ MN/m}^2$$

Problem 1.13 A uniform beam weighing 500 N is held in a horizontal position by three vertical wires, one attached to each end of the beam, and one at the mid-length. The outer wires are brass of diameter 0.125 cm, and the central wire is of steel of diameter 0.0625 cm. If the beam is rigid and the wires are of the same length, and unstressed before the beam is attached, estimate the stresses in the wires. Young's modulus for brass is 85 GN/m^2 and for steel is 200 GN/m^2.



Solution

On considering the two outer brass wires together, we may take the system as a composite one consisting of a single brass member and a steel member. The area of the steel member is

$$A_s = \frac{\pi}{4}\left(0.625 \times 10^{-3}\right)^2 = 0.306 \times 10^{-6} \text{ m}^2$$

The total area of the two brass members is

$$A_b = 2\left[\frac{\pi}{4}\left(1.25 \times 10^{-3}\right)^2\right] = 2.45 \times 10^{-6} \text{ m}^2$$

Equations (1.20) then give, for the steel wire

$$\sigma_s = \frac{500}{(0.306 \times 10^{-6}) + (2.45 \times 10^{-6})\left(\frac{85}{200}\right)} = 370 \text{ MN/m}^2$$

and for the brass wires

$$\sigma_b = \frac{500}{(0.306 \times 10^{-6})\left(\frac{200}{85}\right) + (2.45 \times 10^{-6})} = 158 \text{ MN/m}^2$$

1.17 Temperature stresses

When the temperature of a body is raised, or lowered, the material expands, or contracts. If this expansion or contraction is wholly or partially resisted, stresses are set up in the body. Consider a long bar of a material; suppose L_0 is the length of the bar at a temperature θ_0, and that α is the coefficient of linear expansion of the material. The bar is now subjected to an increase θ in temperature. If the bar is completely free to expand, its length increases by $\alpha L_0 \theta$, and the length becomes $L_0(1 + \alpha\theta)$ were compressed to a length L_0; in this case the compressive strain is

$$\varepsilon = \frac{\alpha L_0 \theta}{L_0 (1 + \alpha\theta)} = \alpha\theta$$

since $\alpha\theta$ is small compared with unity; the corresponding stress is

$$\sigma = E\varepsilon = \alpha\theta E \tag{1.21}$$

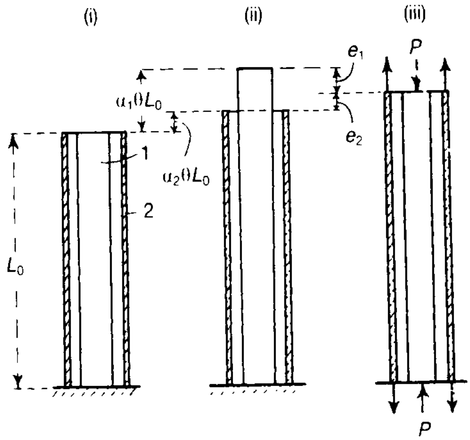
By a similar argument the tensile stress set up in a constrained bar by a fall θ in temperature is $\alpha\theta E$. It is assumed that the material remains elastic.

In the case of steel $\alpha = 1.3 \times 10^{-5}$ per °C; the product αE is approximately 2.6 MN/m^2 per °C, so that a change in temperature of 4°C produces a stress of approximately 10 MN/m^2 if the bar is completely restrained.

1.18 Temperature stresses in composite bars

In a component or structure made wholly of one material, temperature stresses arise only if external restraints prevent thermal expansion or contraction. In composite bars made of materials with different rates of thermal expansion, internal stresses can be set up by temperature changes; these stresses occur independently of those due to external restraints.

Consider, for example, a simple composite bar consisting of two members—a solid circular bar, 1, contained inside a circular tube, 2, Figure 1.21. The materials of the bar and tube have different coefficients of linear expansion, α_1 and α_2, respectively. If the ends of the bar and tube are attached rigidly to each other, longitudinal stresses are set up by a change of temperature. Suppose firstly, however, that the bar and tube are quite free of each other; if L_0 is the original length of each bar, Figure 1.21, the extensions due to a temperature increase θ are $\alpha_1 \theta L_0$ and $\alpha_2 \theta L_0$, Figure 1.21(ii). The difference in lengths of the two members is $(\alpha_1 - \alpha_2)\theta L_0$; this is now eliminated by compressing the inner bar with a force P, and pulling the outer tube with an equal force P, Figure 1.21(iii).



(i) (ii) (iii)

Figure 1.21 Temperature stress in a composite bar.

If A_1 and E_1 are the cross-sectional area and Young's modulus, respectively, of the inner bar, and A_2 and E_2 refer to the outer tube, then the contraction of the inner bar to P is

$$e_1 = \frac{PL_0}{E_1 A_1}$$

and the extension of the outer tube due to P is

$$e_2 = \frac{PL_0}{E_2 A_2}$$

Then from *compatibility* considerations, the difference in lengths $(\alpha_1 - \alpha_2)\theta L$, is eliminated completely when

$$(\alpha_1 - \alpha_2)\theta L_0 = e_1 + e_2$$

On substituting for $e_1 + e_2$, we have

$$(\alpha_1 - \alpha_2)\theta L_0 = PL\left(\frac{1}{E_1 A_1} + \frac{1}{E_2 A_2}\right) \tag{1.22}$$

The force P is induced by the temperature change θ if the ends of the two members are attached rigidly to each other; from equation (1.22), P has the value

$$P = \frac{(\alpha_1 - \alpha_2)\theta}{\left(\frac{1}{E_1 A_1} + \frac{1}{E_2 A_2}\right)} \tag{1.23}$$

An internal load is only set up if α_1 is different from α_2.

Problem 1.14 An aluminium rod 2.2 cm diameter is screwed at the ends, and passes through a steel tube 2.5 cm internal diameter and 0.3 cm thick. Both are heated to a temperature of 140°C, when the nuts on the rod are screwed lightly on to the ends of the tube. Estimate the stress in the rod when the common temperature has fallen to 20°C. For steel, E = 200 GN/m^2 and α = 1.2×10^{-5} per °C, and for aluminium, E = 70 GN/m^2 and α = 2.3×10^{-5} per °C, where E is Young's modulus and α is the coefficient of linear expansion.

Solution

Let subscript a refer to the aluminium rod and subscript s to the steel tube. The problem is similar to the one discussed in Section 1.17, except that the composite rod has its temperature lowered, in this case from 140°C to 20°C. From equation (1.23), the common force between the two components is

$$P = \frac{(\alpha_a - \alpha_s)\,\theta}{\dfrac{1}{(EA)_a} + \dfrac{1}{(EA)_s}}$$

The stress in the rod is therefore

$$\frac{P}{A_a} = \frac{(\alpha_a - \alpha_s)\,\theta}{\dfrac{1}{E_a} + \dfrac{A_a}{E_s\,A_s}} = \frac{(\alpha_a - \alpha_s)\,E_a\,\theta}{1 + \dfrac{E_a\,A_a}{E_s\,A_s}}$$

Now

$$(EA)_a = (70 \times 10^9)\left[\frac{\pi}{4}(0.022)^2\right] = 26.6 \text{ MN}$$

Again

$$(EA)_s = (200 \times 10^9)\left[\pi\,(0.028)\,(0.003)\right] = 52.8\ \text{MN}$$

Then

$$\frac{P}{A_a} = \frac{\left[(2.3 - 1.2)\,10^{-5}\right](70 \times 10^9)\,(120)}{1 + \left(\dfrac{26.6}{52.8}\right)} = 61.4\ \text{MN/m}^2$$

1.19 Circular ring under radial pressure

When a thin circular ring is loaded radially, a circumferential force is set up in the ring; this force extends the circumference of the ring, which in turn leads to an increase in the radius of the ring. Consider a thin ring of mean radius r, Figure 1.22(i), acted upon by an internal radial force of intensity p per unit length of the boundary. If the ring is cut across a diameter, Figure 1.22(ii), circumferential forces P are required at the cut sections of the ring to maintain equilibrium of the half-ring. For equilibrium

$$2P = 2pr$$

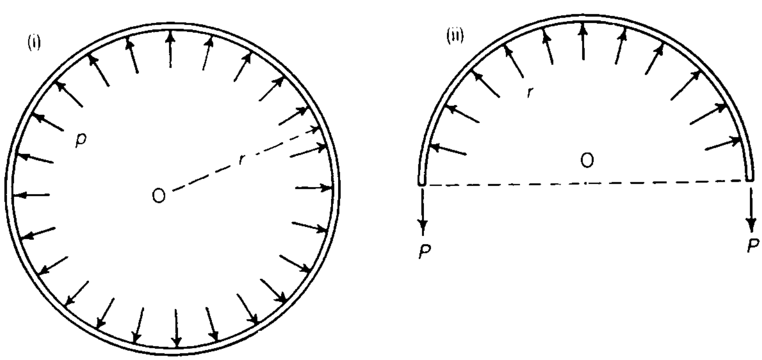
so that

$$P = pr \tag{1.24}$$

A section may be taken across any diameter, leading to the same result; we conclude, therefore, that P is the circumferential tension in all parts of the ring.

If A is the cross-sectional area of the ring at any point of the circumference, then the tensile circumferential stress in the ring is

$$\sigma = \frac{P}{A} = \frac{pr}{A} \tag{1.25}$$



(i) (ii)

Figure 1.22 Thin circular ring under uniform radial loading, leading to a uniform circumferential tension.

If the cross-section is a rectangle of breadth b, (normal to the plane of Figure 1.22), and thickness t, (in the plane of Figure 1.22), then

$$\sigma = \frac{pr}{bt} \tag{1.26}$$

Circumferential stresses of a similar type are set up in a circular ring rotating about an axis through its centre. We suppose the ring is a uniform circular one, having a cross-sectional area A at any point, and that it is rotating about its central axis at uniform angular velocity ω. If ρ is the density of the material of the ring, then the centrifugal force on a unit length of the circumference is

$$\rho A \omega^2 r$$

In equation (1.25) we put this equal to p; thus, the circumferential tensile stress in the ring is

$$\sigma = \frac{pr}{A} = \rho \omega^2 r^2 \tag{1.27}$$

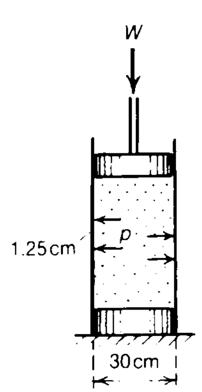
which we see is independent of the actual cross-sectional area. Now, ωr is the circumferential velocity, V (say), of the ring, so

$$\sigma = \rho V^2 \tag{1.28}$$

For steel we have $\rho = 7840\ \mathrm{kg/m^3}$; to produce a tensile stress of $10\ \mathrm{MN/m^2}$, the circumferential velocity must be

$$V = \sqrt{\frac{\sigma}{\rho}} = \sqrt{\frac{(10 \times 10^6)}{7840}} = 35.7\ \mathrm{m/s}$$

Problem 1.15 A circular cylinder, containing oil, has an internal bore of 30 cm diameter. The cylinder is 1.25 cm thick. If the tensile stress in the cylinder must not exceed $75\ \mathrm{MN/m^2}$, estimate the maximum load W which may be supported on a piston sliding in the cylinder.



Solution

A load W on the piston generates an internal pressure p given by

$$W = \pi r^2 p$$

where r is the radius of the cylinder. In this case

$$p = \frac{W}{\pi r^2} = \frac{W}{\pi (0.150)^2}$$

A unit length of the cylinder is equivalent to a circular ring subjected to an internal load of p per unit length of circumference. The circumferential load set up by p in this ring is, from equation (1.24),

$$P = pr = p\,(0.150)$$

The circumferential stress is, therefore,

$$\sigma = \frac{P}{1 \times t} = \frac{P}{0.0125} = 80P$$

where t is the thickness of the wall of the cylinder. If σ is limited to 75 MN/m^2, then

$$80P = 75 \times 10^6$$

But

$$80P = 80\,[p\,(0.150)] = 12p = \frac{12W}{\pi (0.150)^2}$$

Then

$$\frac{12W}{\pi (0.150)^2} = 75 \times 10^6$$

giving

$$W = 441 \text{ kN}$$

Problem 1.16 An aluminium-alloy cylinder of internal diameter 10.000 cm and wall thickness 0.50 cm is shrunk onto a steel cylinder of external diameter 10.004 cm and wall thickness 0.50 cm. If the values of Young's modulus for the alloy and the steel are 70 GN/m^2 and 200 GN/m^2, respectively, estimate the circumferential stresses in the cylinders and the radial pressure between the cylinders.

Solution

We take unit lengths of the cylinders as behaving like thin circular rings. After the shrinking operation, we suppose p is the radial between the cylinders. The mean radius of the steel tube is

$$\tfrac{1}{2}\,[10.004 - 0.50] = 4.75 \text{ cm}$$

The compressive circumferential stress in the steel tube is then

$$\sigma_s = \frac{pr}{t} = \frac{p\,(0.0475)}{0.0050} = 9.5p$$

The circumferential strain in the steel tube is then

$$\varepsilon_s = \frac{\sigma_s}{E_s} = \frac{9.50p}{200 \times 10^9}$$

The mean radius of the alloy tube is

$$\tfrac{1}{2}\,[10.000 + 0.50] = 5.25 \text{ cm}$$

The tensile circumferential stress in the alloy tube is then

$$\sigma_a = \frac{pr}{t} = \frac{p(0.0525)}{(0.0050)} = 10.5p$$

The circumferential strain in the alloy tube is then

$$\varepsilon_a = \frac{\sigma_a}{E_a} = \frac{10.5p}{70 \times 10^9}$$

The circumferential expansion of the alloy tube is

$$2\pi\, r\, \varepsilon_a$$

so the mean radius increases effectively by an amount

$$\delta_a = r\,\varepsilon_a = 0.0525\,\varepsilon_a$$

Similarly, the mean radius of the steel tube contracts by an amount

$$\delta_s = r\,\varepsilon_s = 0.0475\,\varepsilon_s$$

For the shrinking operation to be carried out we must have that the initial lack of fit, δ, is given by

$$\delta = \delta_a + \delta_s$$

Then

$$\delta_a + \delta_s = 0.002 \times 10^{-2}$$

On substituting for δ_a and δ_s, we have

$$0.0525 \left[\frac{10.5p}{70 \times 10^9}\right] + 0.0475 \left[\frac{9.50p}{200 \times 10^9}\right] = 0.002 \times 10^{-2}$$

This gives

$$p = 1.97 \text{ MN/m}^2$$

The compressive circumferential stress in the steel cylinder is then

$$\sigma_s = 9.50p = 18.7 \text{ MN/m}^2$$

The tensile circumferential stress in the alloy cylinder is

$$\sigma_a = 10.5p = 20.7 \text{ MN/m}^2$$

1.20 Creep of materials under sustained stresses

At ordinary laboratory temperatures most metals will sustain stresses below the limit of proportionality for long periods without showing additional measurable strains. At these temperatures metals deform continuously when stressed above the elastic range. This process of continuous inelastic strain is called *creep*. At high temperatures metals lose some of their elastic properties, and creep under constant stress takes place more rapidly.

When a tensile specimen of a metal is tested at a high temperature under a constant load, the strain assumes instantaneously some value ε_0, Figure 1.23. If the initial strain is in the inelastic range of the material then creep takes place under constant stress. At first the creep rate is fairly rapid, but diminishes until a point *a* is reached on the strain–time curve, Figure 1.23; the point *a* is a point of inflection in this curve, and continued application of the load increases the creep rate until fracture of the specimen occurs at *b*.

At ordinary temperatures concrete shows creep properties; these may be important in prestressed members, where some of the initial stresses in the concrete may be lost after a long period due to creep. Composites are also vulnerable to creep and this must be considered when using them for construction.

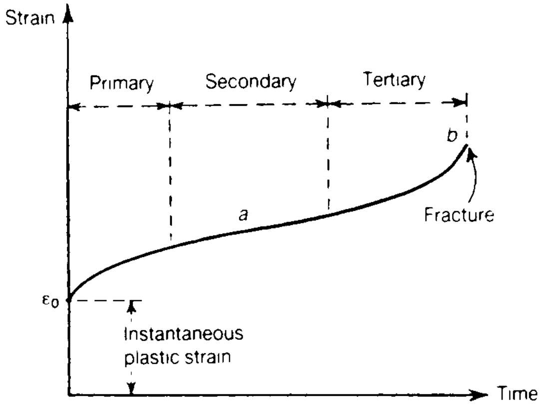


Figure 1.23 Creep curve for a material in the inelastic range; ε_0 is the instantaneous plastic strain.

1.21 Fatigue under repeated stresses

When a material is subjected to repeated cyclic loading, it can fail at a stress which may be much less than the material's yield stress. The problem that occurs here, is that the structure might have minute cracks in it or other stress raisers. Under repeated cyclic loading the large stresses that occur at these stress concentrations cause the cracks to grow, until fracture eventually occurs. Materials likely to suffer fatigue include aluminium alloys and composites; see Figure 1.24.

Failure of a material after a large number of cycles of tensile stress occurs with little, or no, permanent set; fractures show the characteristics of brittle materials. Fatigue is primarily a problem of repeated tensile stresses; this is due probably to the fact that microscopic cracks in a material can propagate more easily when the material is stressed in tension. In the case of steels it is found that there is a critical stress—called the *endurance limit*—below which fluctuating stresses cannot cause a fatigue failure; titanium alloys show a similar phenomenon. No such endurance limit has been found for other non-ferrous metals and other materials.

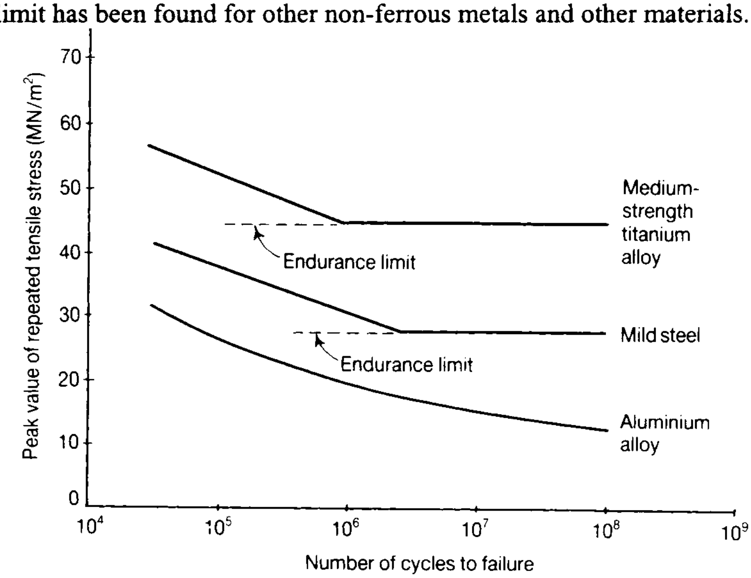


Figure 1.24 Comparison of the fatigue strengths of metals under repeated tensile stresses.

Further problems *(answers on page 691)*

1.17 The piston rod of a double-acting hydraulic cylinder is 20 cm diameter and 4 m long. The piston has a diameter of 40 cm, and is subjected to 10 MN/m^2 water pressure on one side and 3 MN/m^2 on the other. On the return stroke these pressures are interchanged. Estimate the maximum stress occurring in the piston-rod, and the change of length of the rod between two strokes, allowing for the area of piston-rod on one side of the piston. Take $E = 200\ GN/m^2$. *(RNC)*

1.18 A uniform steel rope 250 m long hangs down a shaft. Find the elongation of the first 125 m at the top if the density of steel is 7840 kg/m^3 and Young's modulus is 200 GN/m^2. *(Cambridge)*

1.19 A steel wire, 150 m long, weighs 20 N per metre length. It is placed on a horizontal floor and pulled slowly along by a horizontal force applied to one end. If this force measures 600 N, estimate the increase in length of the wire due to its being towed, assuming a uniform coefficient of friction. Take the density of steel as 7840 kg/m^3 and Young's modulus as 200 GN/m^2. *(RNEC)*

1.20 The hoisting rope for a mine shaft is to lift a cage of weight W. The rope is of variable section so that the stress on every section is equal to σ when the rope is fully extended. If ρ is the density of the material of the rope, show that the cross-sectional area A at a height z above the cage is

$$A = \left(\frac{W}{\sigma}\right) e^{\rho g z/\sigma}$$

1.21 To enable two walls, 10 m apart, to give mutual support they are stayed together by a 2.5 cm diameter steel tension rod with screwed ends, plates and nuts. The rod is heated to 100°C when the nuts are screwed up. If the walls yield, relatively, by 0.5 cm when the rod cools to 15°C, find the pull of rod at that temperature. The coefficient of linear expansion of steel is $\alpha = 1.2 \times 10^{-5}$ per °C, and Young's modulus $E = 200\ GN/m^2$. *(RNEC)*

1.22 A steel tube 3 cm diameter, 0.25 cm thick and 4 m long, is covered and lined throughout with copper tubes 0.2 cm thick. The three tubes are firmly joined together at their ends. The compound tube is then raised in temperature by 100°C. Find the stresses in the steel and copper, and the increase in length of the tube, will prevent its expansion? Assume $E = 200\ GN/m^2$ for steel and $E = 110\ GN/m^2$ for copper; the coefficients of linear expansion of steel and copper are 1.2×10^{-5} per °C and 1.9×10^{-5} per °C, respectively.

2 Pin-jointed frames or trusses

2.1 Introduction

In problems of stress analysis we discriminate between two types of structure; in the first, the forces in the structure can be determined by considering only its statical equilibrium. Such a structure is said to be *statically determinate*. The second type of structure is said to be *statically indeterminate*. In the case of the latter type of structure, the forces in the structure cannot be obtained by considerations of statical equilibrium alone. This is because there are more unknown forces than there are simultaneous equations obtained from considerations of statical equilibrium alone. For statically indeterminate structures, other methods have to be used to obtain the additional number of the required simultaneous equations; one such method is to consider compatibility, as was adopted in Chapter 1. In this chapter, we will consider statically determinate frames and one simple statically indeterminate frame.

Figure 2.1 shows a rigid beam BD supported by two vertical wires BF and DG; the beam carries a force of $4W$ at C. We suppose the wires extend by negligibly small amounts, so that the geometrical configuration of the structure is practically unaffected; then for equilibrium the forces in the wires must be $3W$ in BF and W in DG. As the forces in the wires are known, it is a simple matter to calculate their extensions and hence to determine the displacement of any point of the beam. The calculation of the forces in the wires and structure of Figure 2.1 is said to be *statically determinate*. If, however, the rigid beam be supported by three wires, with an additional wire, say, between H and J in Figure 2.1, then the forces in the three wires cannot be solved by considering statical equilibrium alone; this gives a second type of stress analysis problem, which is discussed more fully in Section 2.5; such a structure is *statically indeterminate*.

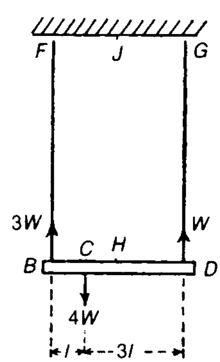


Figure 2.1 Statically determinate system of a beam supported by two wires.

2.2 Statically determinate pin-jointed frames

By a *frame* we mean a structure which is composed of straight bars joined together at their ends. A *pin-jointed frame or truss* is one in which no bending actions can be transmitted from one bar to another as described in the introductory chapter; ideally this could be achieved if the bars were joined together through pin-joints. If the frame has just sufficient bars or rods to prevent collapse without the application of external forces, it is said to be *simply-stiff*; when there are more bars or rods than this, the frame is said to be *redundant*. A redundant framework is said to contain one or more redundant members, where the latter are not required for the framework to be classified as a framework, as distinct from being a mechanism. It should be emphasised, however, that if a redundant member is removed from the framework, the stresses in the remaining members of the framework may become so large that the framework collapses. A redundant member of a framework does not necessarily have a zero internal force in it. Definite relations exist which must be satisfied by the numbers of bars and joints if a frame is said to be simply-stiff, or statically determinate.

In the plane frame of Figure 2.2, BC is one member. To locate the joint D relative to BC requires two members, namely, BD and CD; to locate another joint F requires two further members, namely, CF and DF. Obviously, for each new joint of the frame, two new members are required. If m be the total number of members, including BC, and j is the total number of joints, we must have

$$m = 2j - 3, \tag{2.1}$$

if the frame is to be simply-stiff or statically determinate.

When the frame is rigidly attached to a wall, say at B and C, BC is not part of the frame as such, and equation (2.1) becomes, omitting member BC, and joints B and C,

$$m = 2j \tag{2.2}$$

These conditions must be satisfied, but they may not necessarily ensure that the frame is simply-stiff. For example, the frames of Figures 2.2 and 2.3 have the same numbers of members and joints; the frame of Figure 2.2 is simply-stiff. The frame of Figure 2.3 is *not* simply-stiff, since a mechanism can be formed with pivots at D, G, J, F. Thus, although a frame having j joints must have at least $(2j - 3)$ members, the mode of arrangement of these members is important.

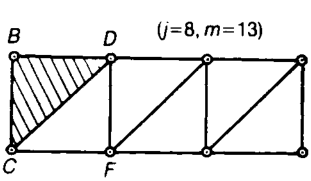


Figure 2.2 Simply-stiff plane frame built up from a basic triangle BCD.

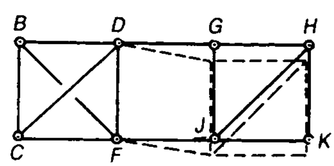


Figure 2.3 Rearrangement of the members of Figure 2.2 to give a mechanism.

For a pin-jointed space frame attached to three joints in a rigid wall, the condition for the frame to be simply-stiff is

$$m = 3j \tag{2.3}$$

where m is the total number of members, and j is the total number of joints, exclusive of the three joints in the rigid wall. When a space frame is not rigidly attached to a wall, the condition becomes

$$m = 3j - 6, \tag{2.4}$$

where m is the total number of members in the frame, and j the total number of joints.

2.3 The method of joints

This method can only be used to determine the internal forces in the members of statically determinate pin-jointed trusses. It consists of isolating each joint of the framework in the form of a *free-body diagram* and then by considering equilibrium at each of these joints, the forces in the members of the framework can be determined. Initially, all *unknown forces* in the members of the framework are assumed to be in *tension*, and before analysing each joint it should be ensured that each joint does not have more than two unknown forces.

To demonstrate the method, the following example will be considered.

Problem 2.1 Using the method of joints, determine the member forces of the plane pin-jointed truss of Figure 2.4.

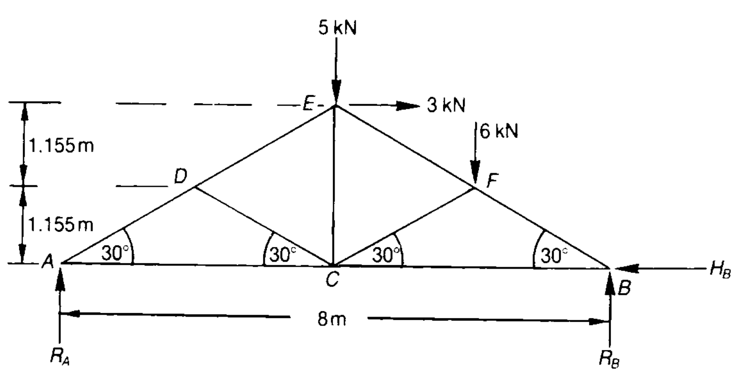


Figure 2.4 Pin-jointed truss.

Solution

Assume all unknown internal forces are in tension, because if they are in compression, their signs will be negative.

As each joint must only have two unknown forces acting on it, it will be necessary to determine the values of R_A, R_B and H_B, prior to using the method of joints.

Resolving the forces horizontally

$$\text{forces to the left} = \text{forces to the right}$$

$$3 = H_B$$

$$\therefore H_B = 3 \text{ kN}$$

Taking moments about B

$$\text{clockwise moments} = \text{counter-clockwise moments}$$

$$R_A \times 8 + 3 \times 2.311 = 5 \times 4 + 6 \times 2$$

$$\therefore R_A = 25.07/8 = 3.13 \text{ kN}$$

Resolving forces vertically

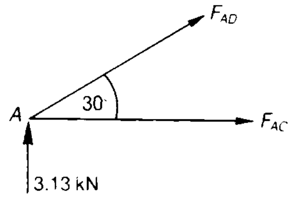
$$\text{upward forces} = \text{downward forces}$$

$$R_A + R_B = 5 + 6$$

or

$$R_B = 11 - 3.13 = 7.87 \text{ kN}$$

Isolate *joint A* and consider equilibrium, as shown by the following free-body diagram.



Resolving forces vertically

$$\text{upward forces} = \text{downward forces}$$

$$3.13 + F_{AD} \sin 30 = 0$$

or

$$F_{AD} = -6.26 \text{ kN (compression)}$$

NB The *negative sign* for this force denotes that this member is in *compression*, and such a member is called a *strut*.

Resolving forces horizontally

$$\text{forces to the right} = \text{forces to the left}$$

$$F_{AC} + F_{AD} \cos 30 = 0$$

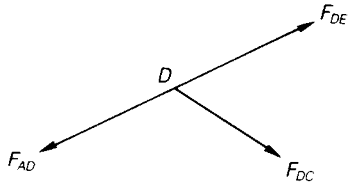
or

$$F_{AC} = 6.26 \times 0.866$$

$$F_{AC} = 5.42 \text{ kN (tension)}$$

NB The *positive sign* for this force denotes that this member if in *tension*, and such a member is called a *tie*.

It is possible now to analyse *joint D*, because F_{AD} is known and therefore the joint has only two unknown forces acting on it, as shown by the free-body diagram.



Resolving vertically

$$\text{upward forces} = \text{downward forces}$$

$$F_{DE} \sin 30 = F_{AD} \sin 30 + F_{DC} \sin 30$$

or

$$F_{DE} = -6.26 + F_{DC} \tag{2.5}$$

Resolving horizontally

$$\text{forces to the left} = \text{forces to the right}$$

$$F_{AD}\cos 30 = F_{DE}\cos 30 + F_{DC}\cos 30$$

or
$$F_{DE} = -6.26 - F_{DC} \tag{2.6}$$

Equating (2.5) and (2.6)

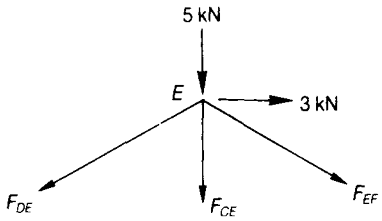
$$-6.26 + F_{DC} = -6.26 - F_{DC}$$

or
$$F_{DC} = 0 \tag{2.7}$$

Substituting equation (2.7) into equation (2.5)

$$F_{DE} = -6.26 \text{ kN (compression)}$$

It is now possible to examine *joint E*, as it has two unknown forces acting on it, as shown:



Resolving horizontally

$$\text{forces to the left} = \text{forces to the right}$$

$$F_{DE}\cos 30 = F_{EF}\cos 30 + 3$$

or
$$F_{EF} = -6.26 - 3/0.866$$

$$F_{EF} = -9.72 \text{ kN (compression)}$$

Resolving vertically

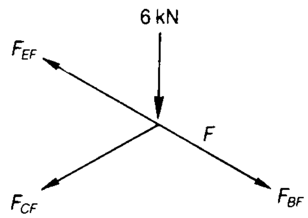
$$\text{upward forces} = \text{downward forces}$$

$$0 = 5 + F_{DE}\sin 30 + F_{CE} + F_{EF}\sin 30$$

$$F_{CE} = -5 + 6.26 \times 0.5 + 9.72 \times 0.5$$

$$F_{CE} = 3 \text{ kN (tension)}$$

It is now possible to analyse either *joint F* or *joint C*, as each of these joints has only got two unknown forces acting on it. Consider *joint F*,



Resolving horizontally

$$\text{forces to the left} = \text{forces to the right}$$

$$F_{EF}\cos 30 + F_{CF}\cos 30 = F_{BF}\cos 30$$

$$\therefore F_{BF} = -9.72 + F_{CF} \tag{2.8}$$

Resolving vertically

$$\text{upward forces} = \text{downward forces}$$

$$F_{EF}\sin 30 = F_{CF}\sin 30 + F_{BF}\sin 30 + 6$$

or

$$F_{BF} \times 0.5 = -9.72 \times 0.5 - 0.5\,F_{CF} - 6$$

$$\therefore F_{BF} = -21.72 - F_{CF} \tag{2.9}$$

Equating (2.8) and (2.9)

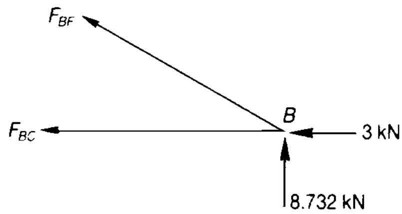
$$-9.72 + F_{CF} = -21.72 - F_{CF}$$

$$\therefore F_{CF} = -6\text{ kN (compression)} \tag{2.10}$$

Substituting equation (2.10) into equation (2.8)

$$F_{BF} = -9.72 - 6 = -15.72\text{ kN (compression)}$$

Consider *joint B* to determine the remaining unknown force, namely F_{BC},



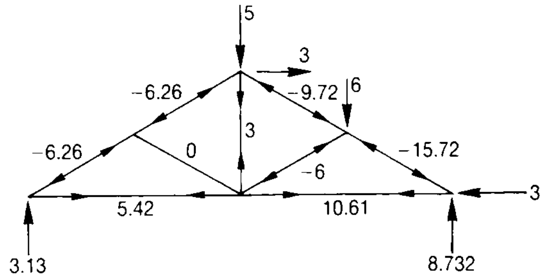
Resolving horizontally

$$\text{forces to the left} = \text{forces to the right}$$

$$F_{BF}\cos 30 + F_{BC} + 3 = 0$$

$$\therefore F_{BC} = -3 + 15.72 \times 0.866 = \text{kN (tension)}$$

Here are the magnitudes and 'directions' of the internal forces in this truss:



2.4 The method of sections

This method is useful if it is required to determine the internal forces in only a few members. The process is to make an imaginary cut across the framework, and then by considering equilibrium, to determine the internal forces in the members that lie across this path. In this method, it is only possible to examine a section that has a maximum of three unknown internal forces, and here again, it is convenient to assume that all unknown forces are in tension.

To demonstrate the method, an imaginary cut will be made through members DE, CD and AC of the truss of Figure 2.4, as shown by the free-body diagram of Figure 2.5

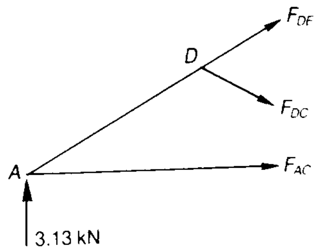


Figure 2.5 Free-body diagram.

Taking moments about D

$$\text{counter-clockwise couples} = \text{clockwise couples}$$

$$F_{AC} \times 1.55 = 3.13 \times 2$$

$$\therefore F_{AC} = 5.42 \text{ kN}$$

NB It was convenient to take moments about *D*, as there were two unknown forces acting through this point and therefore, the arithmetic was simplified.

Resolving vertically

$$\text{upward forces} = \text{downward forces}$$

$$F_{DE} \sin 30 + 3.13 = F_{DC} \sin 30$$

$$\therefore F_{DC} = 6.26 + F_{DE} \qquad (2.11)$$

Resolving horizontally

$$\text{forces to the right} = \text{forces to the left}$$

$$F_{DE} \cos 30 + F_{DC} \cos 30 + F_{AC} = 0$$

$$\therefore F_{DC} = -5.42/0.866 - F_{DE}$$

or

$$F_{DC} = -6.26 - F_{DE} \qquad (2.12)$$

Equating (2.11) and (2.12)

$$F_{DE} = -6.26 \text{ kN} \qquad (2.13)$$

Substituting equation (2.13) into equation (2.11)

$$F_{DC} = 0 \text{ kN}$$

These values can be seen to be the same as those obtained by the method of joints.

2.5 A statically indeterminate problem

In Section 2.1 we mentioned a type of stress analysis problem in which internal stresses are not calculable on considering statical equilibrium alone; such problems are *statically indeterminate*. Consider the rigid beam *BD* of Figure 2.6 which is supported on three wires; suppose the tensions in the wires are T_1, T_2 and T_3. Then by resolving forces vertically, we have

$$T_1 + T_2 + T_3 = 4W \qquad (2.14)$$

and by taking moments about the point C, we get

$$T_1 - T_2 - 3T_3 = 0 \tag{2.15}$$

From these equilibrium equations alone we cannot derive the values of the three tensile forces T_1, T_2, T_3; a third equation is found by discussing the extensions of the wires or considering compatibility. If the wires extend by amounts e_1, e_2, e_3, we must have from Figure 2.6(ii) that

$$e_1 + e_3 = 2e_2 \tag{2.16}$$

because the beam BD is rigid. Suppose the wires are all of the same material and cross-sectional area, and that they remain elastic. Then we may write

$$e_1 = \lambda T_1 , \qquad e_2 = \lambda T_2 , \qquad e_3 = \lambda T_3 , \tag{2.17}$$

where λ is a constant common to the three wires. Then equation (2.16) may be written

$$T_1 + T_3 = 2 T_2 \tag{2.18}$$

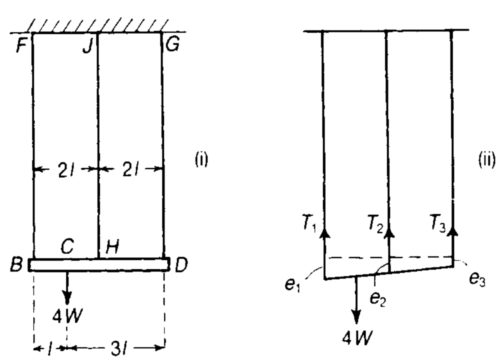


Figure 2.6 A simple statically indeterminate system consisting of a rigid beam supported by three extensible wires.

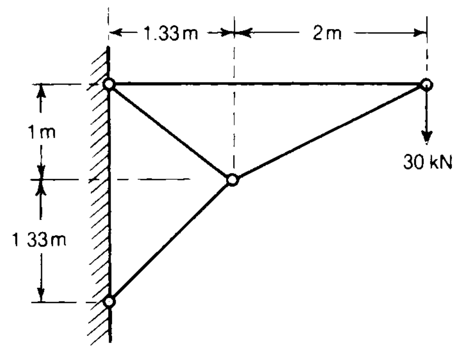
The three equations (2.14), (2.15) and (2.18) then give

$$T_1 = \frac{7W}{12} \qquad T_2 = \frac{4W}{12} \qquad T_3 = \frac{W}{12} \tag{2.19}$$

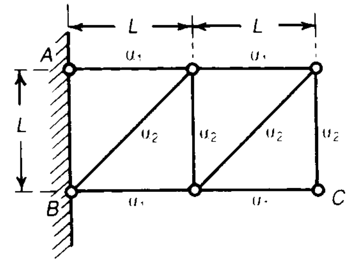
Equation (2.16) is a condition which the extensions of the wires must satisfy; it is called a *strain compatibility* condition. Statically indeterminate problems are soluble if strain compatibilities are considered as well as statical equilibrium.

Further problems *(answers on page 691)*

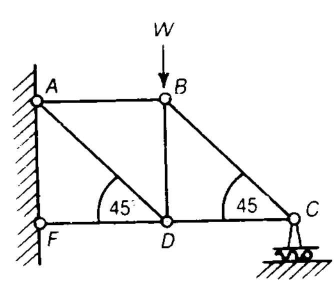
2.2 Determine the internal forces in the plane pin-jointed trusses shown below:



(a)



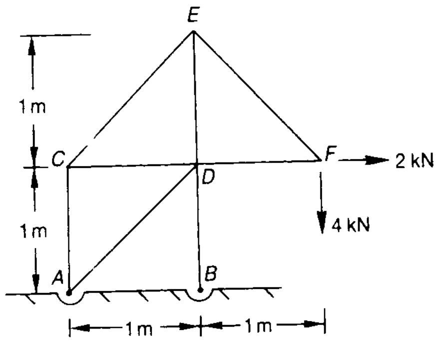
(b)



(c)

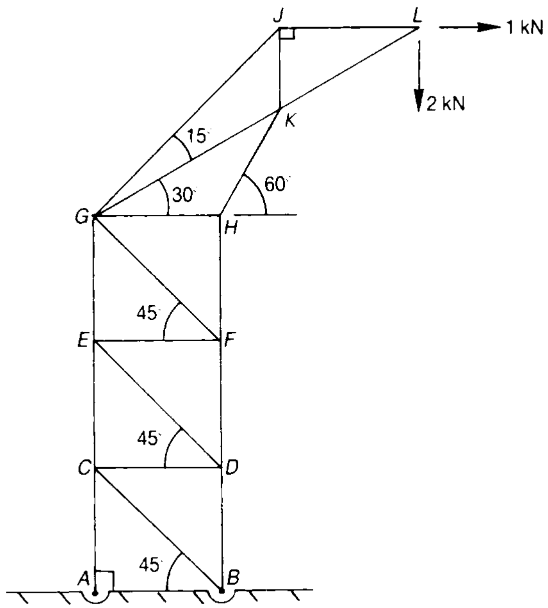
2.3 The plane pin-jointed truss below is firmly pinned at A and B and subjected to two point loads at the joint F.

Using any method, determine the forces in all the members, stating whether they are tensile or compressive. *(Portsmouth 1982)*

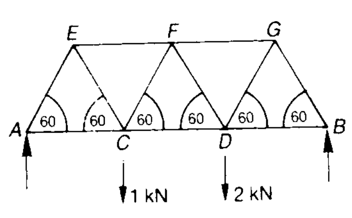


2.4 A plane pin-jointed truss is firmly pinned at its base, as shown below.

Determine the forces in the members of this truss, stating whether they are in tension or compression. *(Portsmouth 1980)*



2.5 Determine the internal forces in the pin-jointed truss, below, which is known as a Warren girder.



3 Shearing stress

3.1 Introduction

In Chapter 1 we made a study of tensile and compressive stresses, which we called direct stresses. There is another type of stress which plays a vital role in the behaviour of materials, especially metals.

Consider a thin block of material, Figure 3.1, which is glued to a table; suppose a thin plate is now glued to the upper surface of the block. If a horizontal force F is applied to the plate, the plate will tend to slide along the top of the block of material, and the block itself will tend to slide along the table. Provided the glued surfaces remain intact, the table resists the sliding of the block, and the block resists the sliding of the plate on its upper surface. If we consider the block to be divided by any imaginary horizontal plane, such as ab, the part of the block above this plane will be trying to slide over the part below the plane. The material on each side of this plane will be trying to slide over the part below the plane. The material on each side of this plane is said to be subjected to a *shearing* action; the stresses arising from these actions are called *shearing stresses*. Shearing stresses act *tangential* to the surface, unlike direct stresses which act perpendicular to the surface.

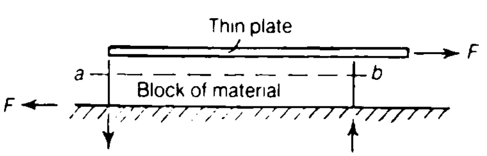


Figure 3.1 Shearing stresses caused by shearing forces.

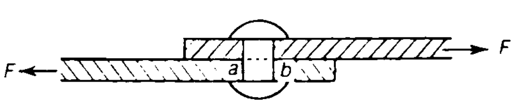


Figure 3.2 Shearing stresses in a rivet; shearing forces F is transmitted over the face ab of the rivet.

In general, a pair of garden shears cuts the stems of shrubs through shearing action and not bending action. Shearing stresses arise in many other practical problems. Figure 3.2 shows two flat plates held together by a single rivet, and carrying a tensile force F. We imagine the rivet divided into two portions by the plane ab; then the upper half of the rivet is tending to slide over the lower half, and a shearing stress is set up in the plane ab. Figure 3.3 shows a circular shaft a, with a collar c, held in bearing b, one end of the shaft being pushed with a force F; in this case there is, firstly, a tendency for the shaft to be pushed bodily through the collar, thereby inducing shearing stresses over the cylindrical surfaces d of the shaft and the collar; secondly, there is a tendency for the collar to push through the bearing, so that shearing stresses are set up on cylindrical surfaces such as e in the bearing. As a third example, consider the case of a steel bolt in the end of a bar of wood, Figure 3.4, the bolt being pulled by forces F; suppose the grain of the wood runs parallel to the length of the bar; then if the forces F are large enough the block *abcd* will be pushed out, shearing taking place along the planes *ab* and *cd*.

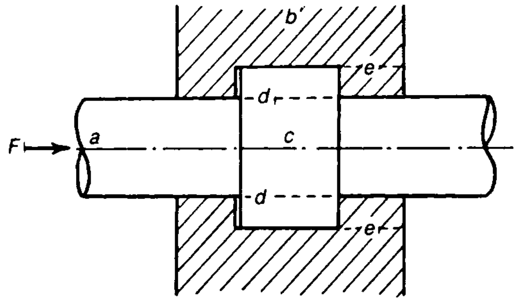


Figure 3.3 Thrust on the collar of a shaft, generating shearing stress over the planes *d*.

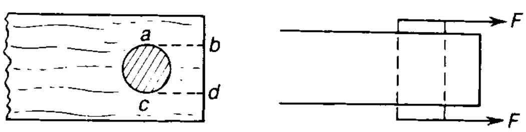


Figure 3.4 Tearing of the end of a timber member by a steel bolt, generating a shearing action on the planes *ab* and *cd*.

3.2 Measurement of shearing stress

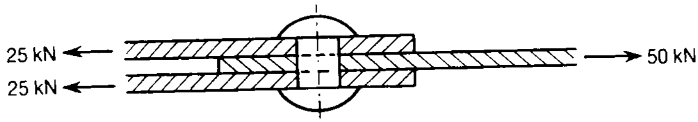
Shearing stress on any surface is defined as the intensity of shearing force tangential to the surface. If the block of material of Figure 3.1 has an area A over any section such as *ab*, the average shearing stress τ over the section *ab* is

$$\tau = \frac{F}{A} \tag{3.1}$$

In many cases the shearing force is not distributed uniformly over any section; if δF is the shearing force on any elemental area δA of a section, the shearing stress on that elemental area is

$$\tau = \operatorname*{Limit}_{\delta A \to 0} \frac{\delta F}{\delta A} = \frac{dF}{dA} \tag{3.2}$$

Problem 3.1 Three steel plates are held together by a 1.5 cm diameter rivet. If the load transmitted is 50 kN, estimate the shearing stress in the rivet.



<u>Solution</u>

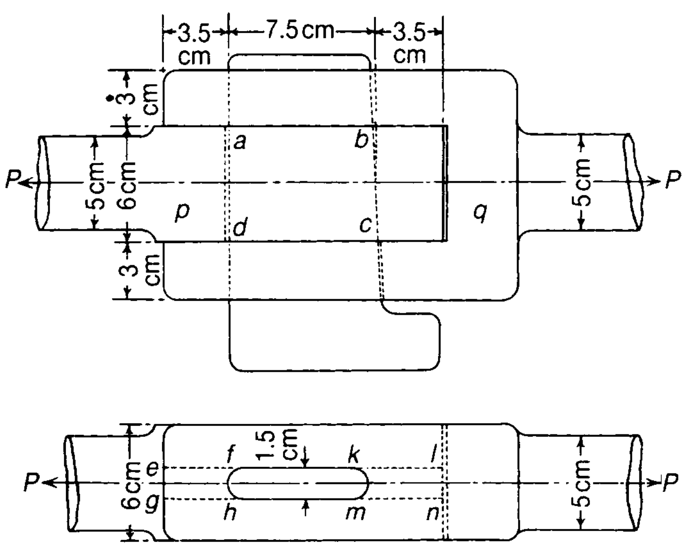
There is a tendency to shear across the planes in the rivet shown by broken lines. The area resisting shear is twice the cross-sectional area of the rivet; the cross-sectional area of the rivet is

$$A = \frac{\pi}{4}(0.015)^2 = 0.177 \times 10^{-3} \text{ m}^2$$

The average shearing stress in the rivet is then

$$\tau = \frac{F}{A} = \frac{25 \times 10^3}{0.177 \times 10^{-3}} = 141 \text{ MN/m}^2$$

Problem 3.2 Two steel rods are connected by a cotter joint. If the shearing strength of the steel used in the rods and the cotter is 150 MN/m^2, estimate which part of the joint is more prone to shearing failure.



<u>Solution</u>

Shearing failure may occur in the following ways:

(i) Shearing of the cotter in the planes *ab* and *cd*.
The area resisting shear is 2(*fkmh*) = 2(0.075 (0.015) = 2.25×10^{-3} m^2

For a shearing failure on these planes, the tensile force is

$$P = \tau A = (150 \times 10^6)(2.25 \times 10^{-3}) = 338 \text{ kN}$$

(ii) By the cotter tearing through the ends of the socket q, i.e. by shearing the planes ef and gh. The total area resisting shear is

$$A = 4(0.030)(0.035) = 4.20 \times 10^{-3} \text{ m}^2$$

For a shearing failure on these planes

$$P = \tau A = (150 \times 10^6)(4.20 \times 10^{-3}) = 630 \text{ kN}$$

(iii) By the cotter tearing through the ends of the rod p, i.e. by shearing in the planes kl and mn. The total area resisting shear is

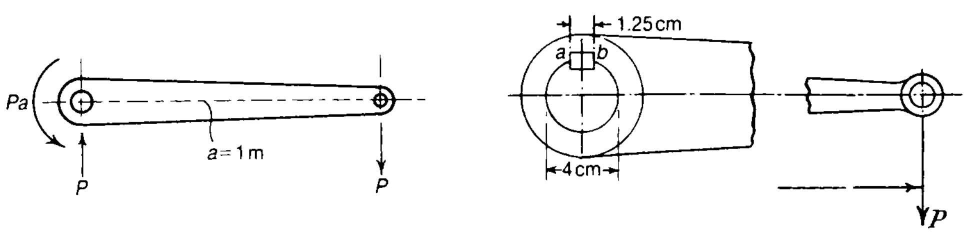
$$A = 2(0.035)(0.060) = 4.20 \times 10^{-3} \text{ m}^2$$

For a shearing failure on these planes

$$P = \tau A = (150 \times 10^6)(4.20 \times 10^{-3}) = 630 \text{ kN}$$

Thus, the connection is most vulnerable to shearing failure in the cotter itself, as discussed in (i); the tensile load for shearing failure is 338 kN.

Problem 3.3 A lever is keyed to a shaft 4 cm in diameter, the width of the key being 1.25 cm and its length 5 cm. What load P can be applied at an arm of a = 1 m if the average shearing stress in the key is not to exceed 60 MN/m^2?



Solution

The torque applied to the shaft is Pa. If this is resisted by a shearing force F on the plane ab of the key, then

$$Fr = Pa$$

where r is the radius of the shaft. Then

$$F = \frac{Pa}{r} = \frac{P(1)}{(0.02)} = 50P$$

The area resisting shear in the key is

$$A = 0.0125 \times 0.050 = 0.625 \times 10^{-3} \text{ m}^2$$

The permissible shearing force on the plane *ab* of the key is then

$$F = \tau A = (60 \times 10^6)(0.625 \times 10^{-3}) = 37.5 \text{ kN}$$

The permissible value of P is then

$$P = \frac{F}{50} = 750 \text{ N}$$

3.3 Complementary shearing stress

Let us return now to the consideration of the block shown in Figure 3.1. We have seen that horizontal planes, such as *ab*, are subjected to shearing stresses. In fact the state of stress is rather more complex than we have supposed, because for rotational equilibrium of the whole block an external couple is required to balance the couple due to the shearing forces F. Suppose the material of the block is divided into a number of rectangular elements, as shown by the full lines of Figure 3.5. Under the actions of the shearing forces F, which together constitute a couple, the elements will tend to take up the positions shown by the broken lines in Figure 3.5. It will be seen that there is a tendency for the vertical faces of the elements to slide over each other. Actually the ends of the elements do not slide over each other in this way, but the tendency to so do shows that the shearing stress in horizontal planes is accompanied by shearing stresses in vertical planes perpendicular to the applied shearing forces. This is true of all cases of shearing action: a given shearing stress acting on one plane is always accompanied by a *complementary shearing stress* on planes at right angles to the plane on which the given stress acts.

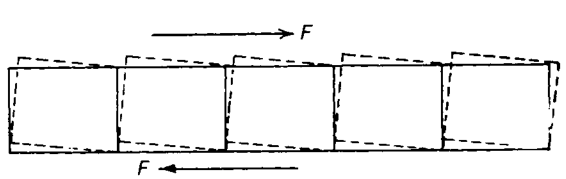


Figure 3.5 Tendency for a set of disconnected blocks to rotate when shearing forces are applied.

Consider now the equilibrium of one of the elementary blocks of Figure 3.5. Let τ_{xy} be the shearing stress on the horizontal faces of the element, and τ_{yx} the complementary shearing stress[2]

[2]Notice that the first suffix x shows the direction, the second the plane on which the stress acts; thus τ_{xy} acts in direction of x axis on planes y = constant.

on vertical faces of the element, Figure 3.6. Suppose a is the length of the element, b its height, and that it has unit thickness. The total shearing force on the upper and lower faces is then

$$\tau_{xy} \times a \times 1 = a\tau_{xy}$$

while the total shearing force on the end faces is

$$\tau_{yx} \times b \times 1 = b\tau_{yx}$$

For rotational equilibrium of the element we then have

$$(a\tau_{xy}) \times b = (b\tau_{yx}) \times a$$

and thus

$$\tau_{xy} = \tau_{yx}$$

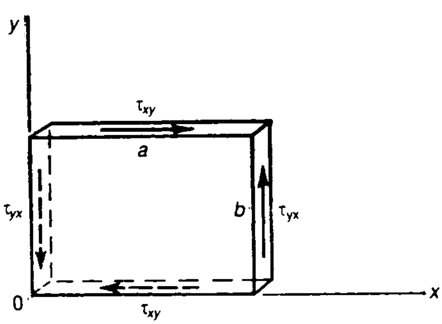


Figure 3.6 Complementary shearing stresses over the faces of a block when they are connected.

We see then that, whenever there is a shearing stress over a plane passing through a given line, there must be an *equal* complementary shearing stress on a plane perpendicular to the given plane, and passing through the given line. The directions of the two shearing stresses must be either both towards, or both away from, the line of intersection of the two planes in which they act.

It is extremely important to appreciate the existence of the complementary shearing stress, for its necessary presence has a direct effect on the maximum stress in the material, as we shall see later in Chapter 5.

3.4 Shearing strain

Shearing stresses in a material give rise to *shearing strains*. Consider a rectangular block of material, Figure 3.7, subjected to shearing stresses τ in one plane. The shearing stresses distort the rectangular face of the block into a parallelogram. If the right-angles at the corners of the face change by amounts γ, then γ is the shearing strain. The angle γ is measured in radians, and is non-dimensional therefore.

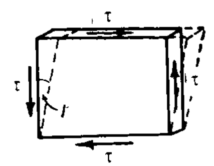


Figure 3.7 Shearing strain in a rectangular block; small values of γ lead to a negligible change of volume in shear straining.

For many materials shearing strain is linearly proportional to shearing stress within certain limits. This linear dependence is similar to the case of direct tension and compression. Within the limits of proportionality

$$\tau = G_\gamma \ , \tag{3.3}$$

where G is the *shearing modulus* or modulus of rigidity, and is similar to Young's modulus E, for direct tension and compression. For most materials E is about 2.5 times greater than G.

It should be noted that no volume changes occur as a result of shearing stresses acting alone. In Figure 3.7 the volume of the strained block is approximately equal to the volume of the original rectangular prism if the angular strain γ is small.

3.5 Strain energy due to shearing actions

In shearing the rectangular prism of Figure 3.7, the forces acting on the upper and lower faces undergo displacements. Work is done, therefore, during these displacements. If the strains are kept within the elastic limit the work done is recoverable, and is stored in the form of strain energy. Suppose all edges of the prism of Figure 3.7 are of unit length; then the prism has unit volume, and the shearing forces on the sheared faces are τ. Now suppose τ is increased by a small amount, causing a small increment of shearing strain $\delta\gamma$. The work done on the prism during this small change is $\tau\delta\gamma$, as the force τ moves through a distance $\delta\gamma$. The total work done in producing a shearing strain γ is then

$$\int_0^{y} \tau \, d\gamma$$

While the material remains elastic, we have from equation (3.3) that $\tau = G\gamma$, and the work done is stored as strain energy; the strain energy is therefore

$$\int_0^y \tau d\gamma = \int_0^y G\gamma\, d\gamma = \frac{1}{2}G\gamma^2 \tag{3.4}$$

per unit volume. In terms of τ this becomes

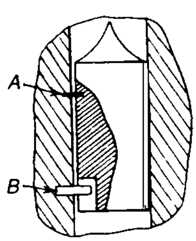
$$\frac{1}{2}G\gamma^2 = \frac{\tau^2}{2G} \quad = \text{shear strain energy per unit volume} \tag{3.5}$$

Further problems *(answers on page 691)*

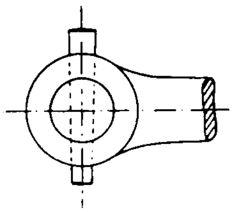
3.4 Rivet holes 2.5 cm diameter are punched in a steel plate 1 cm thick. The shearing strength of the plate is 300 MN/m^2. Find the average compressive stress in the punch at the time of punching.

3.5 The diameter of the bolt circle of a flanged coupling for a shaft 12.5 cm in diameter is 37.5 cm. There are six bolts 2.5 cm diameter. What power can be transmitted at 150 rev/min if the shearing stress in the bolts is not to exceed 60 MN/m^2?

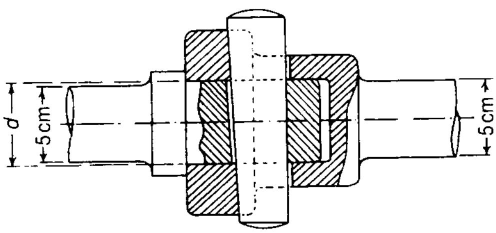
3.6 A pellet carrying the striking needle of a fuse has a mass of 0.1 kg; it is prevented from moving longitudinally relative to the body of the fuse by a copper pin A of diameter 0.05 cm. It is prevented from turning relative to the body of the fuse by a steel stud B. A fits loosely in the pellet so that no stress comes on A due to rotation. If the copper shears at 150 MN/m^2, find the retardation of the shell necessary to shear A. *(RNC)*



3.7 A lever is secured to a shaft by a taper pin through the boss of the lever. The shaft is 4 cm diameter and the mean diameter of the pin is 1 cm. What torque can be applied to the lever without causing the average shearing stress in the pin to exceed 60 MN/m^2.



3.8 A cotter joint connects two circular rods in tension. Taking the tensile strength of the rods as 350 MN/m^2, the shearing strength of the cotter 275 MN/m^2, the permissible bearing pressure between surfaces in contact 700 MN/m^2, the shearing strength of the rod ends 185 MN/m^2, calculate suitable dimensions for the joint so that it may be equally strong against the possible types of failure. Take the thickness of the cotter = $d/4$, and the taper of the cotter 1 in 48.



3.9 A horizontal arm, capable of rotation about a vertical shaft, carries a mass of 2.5 kg, bolted to it by a 1 cm bolt at a distance 50 cm from the axis of the shaft. The axis of the bolt is vertical. If the ultimate shearing strength of the bolt is 50 MN/m^2, at what speed will the bolt snap? *(RNEC)*

3.10 A copper disc 10 cm in diameter and 0.0125 cm thick, is fitted in the casing of an air compressor, so as to blow and safeguard the cast-iron case in the event of a serious compressed air leak. If pressure inside the case is suddenly built up by a burst cooling coil, calculate at what pressure the disc will blow out, assuming that failure occurs by shear round the edges of the disc, and that copper will normally fail under a shearing stress of 120 MN/m^2. *(RNEC)*

4 Joints and connections

4.1 Importance of connections

Many engineering structures and machines consist of components suitably connected through carefully designed joints. In metallic materials, these joints may take a number of different forms, as for example welded joints, bolted joints and riveted joints. In general, such joints are stressed in complex ways, and it is not usually possible to calculate stresses accurately because of the geometrical discontinuities in the region of a joint. For this reason, good design of connections is a mixture of stress analysis and experience of the behaviour of actual joints; this is particularly true of connections subjected to repeated loads.

Bolted joints are widely used in structural steel work and recently the performance of such joints has been greatly improved by the introduction of high-tensile, friction-grip bolts. Welded joints are widely used in steel structures, as for example, in ship construction. Riveted joints are still widely used in aircraft-skin construction in light-alloy materials. Epoxy resin glues are often used in the aeronautical field to bond metals.

4.2 Modes of failure of simple bolted and riveted joints

One of the simplest types of joint between two plates of material is a bolted or riveted lap joint, Figure 4.1.

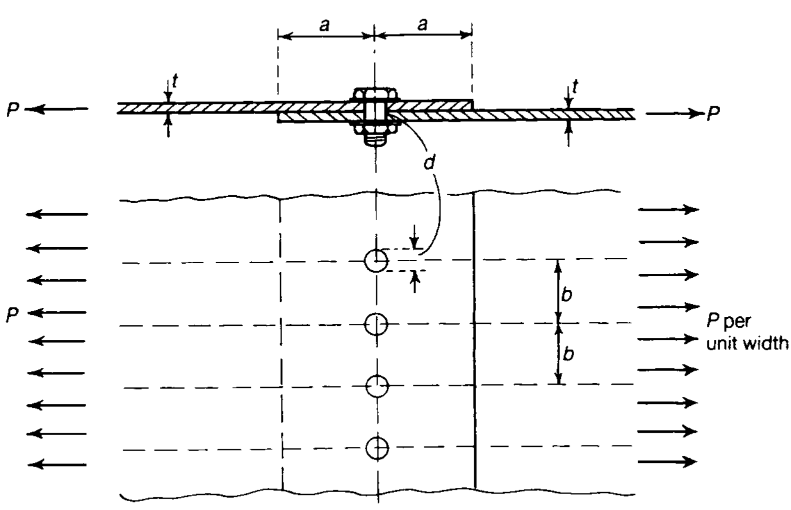


Figure 4.1 Single-bolted lap joint under tensile load.

We shall discuss the forms of failure of the joint assuming it is bolted, but the analysis can be extended in principle to the case of a riveted connection. Consider a joint between two wide plates, Figure 4.1; suppose the plates are each of thickness t, and that they are connected together with a single line of bolts, giving a total overlap of breadth $2a$. Suppose also that the bolts are each of diameter d, and that their centres are a distance b apart along the line of bolts; the line of bolts is a distance a from the edge of each plate. It is assumed that a bolt fills a hole, so that the holes in the plates are also of diameter d.

We consider all possible simple modes of failure when each plate carries a tensile load of P per unit width of plate:

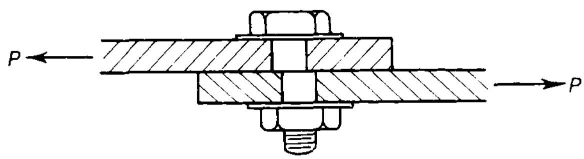


Figure 4.2 Failure by shearing of the bolts.

(1) The bolts may fail by shearing, as shown in Figure 4.2; if τ_1 is the maximum shearing stress the bolts will withstand, the total shearing force required to shear a bolt is

$$\tau_1 \times \left(\frac{\pi d^2}{4}\right)$$

Now, the load carried by a single bolt is Pb, so that a failure of this type occurs when

$$Pb = \tau_1 \left(\frac{\pi d^2}{4}\right)$$

This gives

$$P = \frac{\pi d^2 \tau_1}{4b} \tag{4.1}$$

(2) The bearing pressure between the bolts and the plates may become excessive; the total bearing load taken by a bolt is Pb, Figure 4.3, so that the average bearing pressure between a bolt and its surrounding hole is

$$\frac{Pb}{td}$$