MATHEMATICS CONCEPT

IIT-JEE

(MAINS & ADVANCE)

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"Objective Mathematics for IIT-JEE" authored by Er. L.K.Sharma has also proved a great help for engineering aspirants and its e-book format can be downloaded from http://mathematicsgyan.weebly.com.

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1. BASICS of MATHEMATICS

{1} Number System :

(i) Natural Numbers

The counting numbers 1,2,3,4,..... are called Natural Numbers.
The set of natural numbers is denoted by N.
Thus N = {1,2,3,4, ...}.

(ii) Whole Numbers :

Natural numbers including zero are called whole numbers.
The set of whole numbers, is denoted by W.
Thus W = {0,1,2, ...}

(iii) Integers :

The numbers ... -3, -2, -1, 0, 1,2,3,....... are called integers and the set is denoted by I or Z.
Thus I (or Z) = {...-3,-2,-1, 0, 1, 2, 3...}

(a) Set of positive integers denoted by I^+ and consists of {1,2,3,....} called as set of Natural numbers.

(b) Set of negative integers, denoted by I^- and consists of {......, -3,-2,-1}

(c) Set of non-negative integers {0,1,2 ......}, called as set of Whole numbers.

(d) Set of non-positive integers {.........., -3, -2,-1,0]

(iv) Even Integers :

Integers which are divisible by 2 are called even integers.
e.g. 0, ± 2, ± 4.....

(v) Odd Integers :

Integers, which are not divisible by 2 are called as odd integers.
e.g. ± 1, ± 3, ± 5, ± 7......

(vi) Prime Number :

Let 'p' be a natural number, 'p' is said to be prime if it has exactly two distinct factors, namely 1 and itself.
so, Natural number which are divisible by 1 and itself only are prime numbers (except 1).
e.g. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, ...

(vii) Composite Number :

Let 'a' be a natural number, 'a' is said to be composite if, it has at least three distinct factors.

Note:

(i) '1' is neither prime nor composite.

(ii) '2' is the only even prime number.

(iii) Number which are not prime are composite numbers (except 1).

(iv) '4' is the smallest composite number.

(viii) Co-prime Number :

Two natural numbers (not necessarily prime) are coprime, if there H.C.F (Highest common factor) is one.
e.g. (1,2), (1,3), (3,4), (3, 10), (3,8), (5,6), (7,8) etc.

These numbers are also called as **relatively prime** numbers.

Note:

(a) Two prime number(s) are always co-prime but converse need not be true.

(b) Consecutive numbers are always co-prime numbers.

(ix) Twin Prime Numbers :

If the difference between two prime numbers is two, then the numbers are twin prime numbers.

e.g. {3,5}, {5,7}, (11, 13}, {17, 19}, {29, 31}

(x) Rational Numbers :

All the numbers that can be represented in the form p/q, where p and q are integers and $q \neq 0$, are called rational numbers and their set is denoted by Q.

Thus $Q = \{\frac{p}{q} : p, q \in I \text{ and } q \neq 0\}$. It may be noted that every integer is a rational number since it can be written as p/1. It may be noted that all recurring decimals are rational numbers .

(xi) Irrational Numbers :

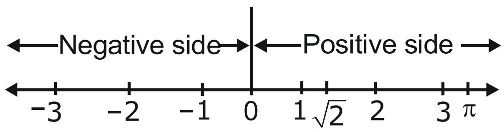
There are real numbers which can not be expressed in p/q form. These numbers are called irrational numbers and their set is denoted by Q^c. (i.e. complementary set of Q) e.g. $\sqrt{2}$, $1 + \sqrt{3}$, e, π etc. Irrational numbers can not be expressed as recurring decimals.

Note: $e \approx 2.71$ is called Napier's constant and $\pi \approx 3.14$.

(xii) Real Numbers :

The complete set of rational and irrational number is the set of real numbers and is denoted by R. Thus $R = Q \cup Q^c$.

The real numbers can be represented as a position of a point on the real number line. The real number line is the number line where in the position of a point relative to the origin (i.e. 0) represents a unique real number and vice versa.



Real line

All the numbers defined so far follow the order property i.e. if there are two distinct numbers a and b then either a < b or a > b.

Note:

(a) Integers are rational numbers, but converse need not be true.

(b) Negative of an irrational number is an irrational number.

(c) Sum of a rational number and an irrational number is always an irrational number e.g. $2 + \sqrt{3}$

(d) The product of a non zero rational number & an irrational number will always be an irrational number.

(e) If $a \in Q$ and $b \notin Q$, then ab = rational number, only if a = 0.

(f) sum, difference, product and quotient of two irrational numbers need not be a irrational number or we can say, result may be a rational number also.

(xiii) Complex Number :

A number of the form a + ib is called complex number, where $a, b \in R$ and $i = \sqrt{-1}$, Complex number is usually denoted by Z and the set of complex number is represented by C.

Note : It may be noted that $N \subset W \subset I \subset Q \subset R \subset C$.

{2} Divisibility Test :

(i) A number will be divisible by 2 iff the digit at the unit place of the number is divisible by 2.

(ii) A number will be divisible by 3 iff the sum of its digits of the number is divisible by 3.

(iii) A number will be divisible by 4 iff last two digit of the number together are divisible by 4.

(iv) A number will be divisible by 5 iff the digit of the number at the unit place is either 0 or 5.

(v) A number will be divisible by 6 iff the digits at the unit place of the number is divisible by 2 & sum of all digits of the number is divisible by 3.

(vi) A number will be divisible by 8 iff the last 3 digits of the number all together are divisible by 8.

(vii) A number will be divisible by 9 iff sum of all it's digits is divisible by 9.

(viii) A number will be divisible by 10 iff it's last digit is 0.

(ix) A number will be divisible by 11, iff the difference between the sum of the digits at even places and sum of the digits at odd places is 0 or multiple of 11.
e.g. 1298, 1221, 123321, 12344321, 1234554321, 123456654321

{3} (i) Remainder Theorem :

Let p(x) be any polynomial of degree greater than or equal to one and 'a' be any real number. If p(x) is divided by (x – a), then the remainder is equal to p(a).

(ii) Factor Theorem :

Let p(x) be a polynomial of degree greater than of equal to 1 and 'a' be a real number such that p(a) = 0, then (x – a) is a factor of p(x). Conversely, if (x – a) is a factor of p(x), then p(a) = 0.

(iii) Some Important formulae :

(1) $(a + b)^2 = a^2 + 2ab + b^2 \quad = (a - b)^2 + 4ab$

(2) $(a - b)^2 = a^2 - 2ab + b^2 \quad = (a + b)^2 - 4ab$

(3) $a^2 - b^2 = (a + b)(a - b)$

(4) $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$

(5) $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$

(6) $a^3 + b^3 = (a + b)^3 - 3ab(a + b) = (a + b)(a^2 + b^2 - ab)$

(7) $a^3 - b^3 = (a - b)^3\ 3ab(a - b) = (a - b)(a^2 + b^2 + ab)$

(8) $(a + b + c)^2 = a^2 + b^2\ c^2 + 2ab + 2bc + 2ca$

$$= a^2 + b^2 + c^2 + 2abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

(9) $a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2]$

(10) $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$$= \frac{1}{2}(a + b + c)[(a-b)^2 + (b-c)^2 + (c-a)^2]$$

(11) $a^4 - b^4 = (a + b)(a - b)(a^2 + b^2)$

(12) $a^4 + a^2 + 1 = (a^2 + 1)^2 - a^2 = (1 + a + a^2)(1 - a + a^2)$

{4} Definition of indices :

If 'a' is any non zero real or imaginary number and 'm' is the positive integer, then a^m = a.a.a....a (m times). Here a is called the base and m is the index, power or exponent.

(I) Law of indices :

(1) $a^0 = 1$, $(a \neq 0)$

(2) $a^{-m} = \frac{1}{a^m}$, $(a \neq 0)$

(3) $a^{m+n} = a^m a^n$, where m and n are rational numbers

(4) $a^{m-n} = \frac{a^m}{a^n}$, where m and n are rational numbers, $a \neq 0$

(5) $(a^m)^n = a^{mn}$

(6) $a^{p/q} = \sqrt[q]{a^p}$

{5} Ratio & proportion :

(i) Ratio :

1. If A and B be two quantities of the same kind, then their ratio is A : B; which may be denoted by the fraction $\frac{A}{B}$ (This may be an integer or fraction)

2. A ratio may represented in a number of ways e.g. $\frac{a}{b} = \frac{ma}{mb} = \frac{na}{nb}$ = ...where m, n, ... are non-zero numbers.

3. To compare two or more ratio, reduced them to common denominator.

4. Ratio between two ratios may be represented as the ratio of two integers e.g. $\frac{a}{b} : \frac{c}{d} : \frac{a/b}{c/d} = \frac{ad}{bc}$ or ad : bc. duplicate, triplicate ratio.

5. Ratios are compounded by multiplying them together i.e. $\frac{a}{b} \cdot \frac{c}{d} \cdot \frac{e}{f} \ldots\ldots = \frac{ace}{bdf} \ldots$

6. If a : b is any ratio then its duplicate ratio is $a^2 : b^2$; triplicate ratio is $a^3 : b^3$ .....etc.

7. If a : b is any ratio, then its sub-duplicate ratio is $a^{1/2} : b^{1/2}$; sub-triplicate ratio is $a^{1/3} : b^{1/3}$ etc.

(ii) Proportion :

When two ratios are equal , then the four quantities compositing them are said to be proportional. If $\frac{a}{b} = \frac{c}{d}$, then it is written as a : b = c : d or a : b :: c : d

1. 'a' and 'd' are known as extremes and 'b and c' are known as means.

2. An important property of proportion : Product of extremes = product of means.

3. If a : b = c : d, then
b : a = d : c(Invertando)

4. If a : b = c : d, then
a : c = b : d(Alternando)

5. If a : b = c : d, then
$\frac{a+b}{b} = \frac{c+d}{d}$ (Componendo)

6. If a : b = c : d, then
$\frac{a-b}{b} = \frac{c+d}{d}$ = (Dividendo)

7. If a : b = c : d, then
$\frac{a+b}{a-b} = \frac{c+d}{c-d}$ (Componendo and Dividendo)

{6} Cross Multiplication :

If two equations containing three unknown are

$a_1x + b_1y + c_1z = 0$ ....(i)

$a_2x + b_2y + c_2z = 0$ ....(ii)

Then by the rule of cross multiplication

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1} \quad ....(ii)$$

In order to write down the denominators of x, y and z in (3) apply the following rule, "write down the coefficients of x, y and z in order beginning with the coefficients of y and repeat them as in the diagram"

$$\begin{array}{cccc} b_1 & c_1 & a_1 & b_1 \\ b_2 & c_2 & a_2 & b_2 \end{array}$$

Multiply the coefficients across in the way indicated by the arrows; remembering that informing the products any one obtained by descending is positive and any one obtained by ascending is negative.

{7} Intervals :

Intervals are basically subsets of R and are commonly used in solving inequalities or in finding domains. If there are two numbers $a,b \in R$ such that a < b, we can define four types of intervals as follows :

Symbols Used

(i) Open interval : (a,b) = {x : a < x < b} i.e. end points are not included ( ) or] [

(ii) Closed interval : [a,b] = {x : $a \leq x \leq b$} i.e. end points are also included [ ]
This is possible only when both a and b are finite.

(iii) Open-closed interval :(a,b] = {x : a < $x \leq b$} ( ] or]]

(iv) Closed-open interval :[a,b) = x : $a \leq x$ < b} [ ) or [[

The infinite intervals are defined as follows :

(i) $(a, \infty) = \{x : x > a\}$ (ii) $[a, \infty) = \{x : x \geq a\}$

(iii) $(-\infty, b) = \{x : x<b\}$ (iv) $(\infty, b] = \{x : x \leq b\}$

(v) $(-\infty, \infty) = \{x : x \in R\}$

Note:

(i) For some particular values of x, we use symbol { } e.g. If x = 1, 2 we can write it as $x \in \{1,2\}$

(ii) If their is no value of x, then we say $x \in \phi$ (null set)

Various Types of Functions :

(i) Polynomial Function :
If a function f is defined by $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots\dots\dots + a_{n-1} x + a_n$ where n is a non **negative integer** and $a_0, a_1, a_2, \dots\dots\dots, a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n.

☞ There are two polynomial functions, satisfying the relation; $f(x).f(1/x) = f(x)+f(1/x)$, which are $f(x) = 1 \pm x^n$

(ii) Constant function :
A function $f : A \to B$ is said to be a constant function, if every element of A has the same f image in B. Thus $f : A \to B$; $f(x) = c$, $\forall\ x \in A$, $c \in B$ is a constant function.

(iii) Identity function :
The function $f : A \to A$ define dby, $f(x) = x \ \forall\ x \in A$ is called the identity function on A and is denoted by I_A. It is easy to observe that identity function is a bijection.

(iv) Algebraic Function :
y is an algebraic function of x, if it is a function that satisfies an algebraic equation of the form, $P_0(x)\, y^n + P_1(x)\, y^{n-1} + \dots\dots\dots + P_{n-1}(x)\, y + P_n(x) = 0$ where n is a positive integer and $P_0(x), P_1(x) \dots\dots\dots$ are polynomials in x. e.g. $y = |x|$ is an algebraic function, since it satisfies the equation $y^2 - x^2 = 0$.

☞ All polynomial functions are algebraic but not the converse.

☞ A function that is not algebraic is called Transcendental Function .

(v) Rational Function :

A rational function is a function of the form, $y = f(x) = \dfrac{g(x)}{h(x)}$, where g(x) & h(x) are polynomial functions.

(vi) Irrational Function :
An irrational function is a function y = f(x) in which the operations of additions, substraction, multiplication, division and raising to a fractional power are used

For example $y = \dfrac{x^3 + x^{1/3}}{2x + \sqrt{x}}$ is an irrational function

(a) The equation $\sqrt{f(x)} = g(x)$ is equivalent to the following system
$f(x) = g^2(x)$ & $g(x) \geq 0$

(b) The inequation $\sqrt{f(x)} < g(x)$ is equivalent to the following system f(x) < $g^2(x)$ & f(x) $\geq$ 0 & g(x) > 0

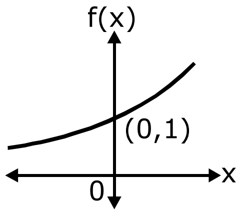
(c) The inequation $\sqrt{f(x)}$ > g(x) is equivalent to the following system

g(x) < 0 & f(x) $\geq$ 0 or g(x) $\geq$ 0 & f(x) > $g^2(x)$

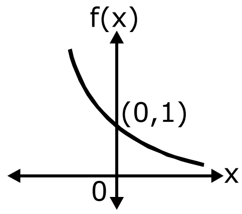
(vii) Exponential Function :

A function f(x) = a^x = $e^{x \ln a}$ (a > 0, a $\neq$ 1, x $\in$ R) is called an exponential function. Graph of exponential function can be as follows :

Case - I
For a > 1

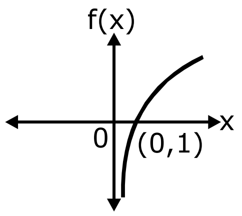


Case - II
For 0 < a < 1

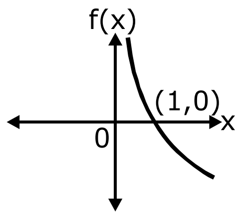


(viii) Logarithmic Function : f(x) = $\log_a x$ is called logarithmic function where a > 0 and a $\neq$ 1 and x > 0. Its graph can be as follows

Case - I
For a > 1



Case - II
For 0 < a < 1



LOGARITHM OF A NUMBER :

The logarithm of the number N to the base 'a' is the exponent indicating the power to which the base 'a' must be raised to obtain the number N. This number is designated as $\log_a N$.

Hence : $\log_a N = x \Leftrightarrow a^x = N$, $a > 0$, $a \neq 1$ & $N > 0$

If $a = 10$, then we write $\log b$ rather than $\log_{10} b$.
If $a = e$, we write $\ln b$ rather than $\log_e b$.

The existence and uniqueness of the number $\log_a N$ follows from the properties of an experimental functions .

From the definition of the logarithm of the number N to the base 'a' , we have an

identity : $a^{\log_a N} = N$, $a > 0$, $a \neq 1$ & $N > 0$

This is known as the FUNDAMENTAL LOGARITHMIC IDENTITY .

NOTE : $\log_a 1 = 0$ $(a > 0$, $a \neq 1)$
$\log_a a = 1$ $(a > 0$, $a \neq 1)$ and
$\log_{1/a} a = -1$ $(a > 0$, $a \neq 1)$

THE PRINCIPAL PROPERTIES OF LOGARITHMS :

Let M & N are arbitrary posiitive numbers, $a > 0, a \neq 1, b > 0, b \neq 1$ and α is any real number then ;

(i) $\log_a (M.N) = \log_a M + \log_a N$

(ii) $\log_a (M/N) = \log_a M - \log_a N$

(iii) $\log_a M^\alpha = \alpha . \log_a M$

(iv) $\log_b M = \dfrac{\log_a M}{\log_a b}$

NOTE : ✐ $\log_b a . \log_a b = 1 \Leftrightarrow \log_b a = 1/\log_a b$.

✐ $\log_b a . \log_c b . \log_a c = 1$

✐ $\log_y x . \log_z y . \log_a z = \log_a x$.

✐ $e^{\ln a^x} = a^x$

PROPERTIES OF MONOTONOCITY OF LOGARITHM :

(i) For $a > 1$ the inequality $0 < x < y$ & $\log_a x < \log_a y$ are equivalent.

(ii) For $0 < a < 1$ the inequality $0 < x < y$ & $\log_a x > \log_a y$ are equivalent.

(iii) If $a > 1$ then $\log_a x < p \Rightarrow 0 < x < a^p$

(iv) If $a > 1$ then $\log_a x > p \Rightarrow x > a^p$

(v) If $0 < a < 1$ then $\log_a x < p \Rightarrow x > a^p$

(vi) If $0 < a < 1$ then $\log_a x > p \Rightarrow 0 < x < a^p$

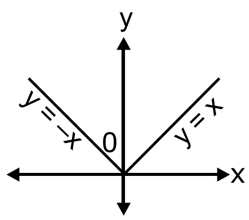
✐ If the number & the base are on one side of the unity, then the logarithm is positive ; If the number & the base are on different sides of unity, then the logarithm is negative.

✐ The base of the logarithm 'a' must not equal unity otherwise numbers not equal to unity will not have a logarithm & any number will be the logarithm of unity.

✐ For a non negative number 'a' & $n \geq 2$, $n \in N$ $\sqrt[n]{a} = a^{1/n}$.

(ix) Absolute Value Function /Modulus Function :

The symbol of modulus function is $f(x) = |x|$ and is defined as : $y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$



y = |x|

Properties of Modulus :

For any $a, b \in R$

(i) $|a| \geq 0$

(ii) $|a| = |-a|$

(iii) $|a| \geq a, \ |a| \geq -a$

(iv) $|ab| = |a| \ |b|$

(v) $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$

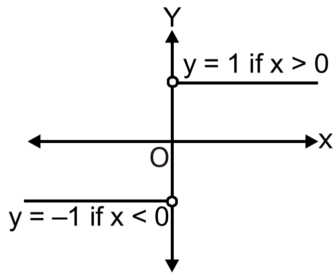
(vi) $|a + b| \leq |a| + |b|$

(vii) $|a - b| \geq ||a| - |b||$

(x) Signum Function :

A function f (x) = sgn (x) is defined as follows :

$$f(x) = \text{sgn}(x) = \begin{cases} 1 & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x = 0 \\ -1 & \text{for} \quad x < 0 \end{cases}$$



y = sgn x

It is also written as $\text{sgn}\, x = \begin{cases} \frac{|(x)|}{(x)}; & x \neq 0 \\ 0; & x = 0 \end{cases}$

☞ $\text{sgn}\, f(x) = \begin{cases} \frac{|f(x)|}{f(x)}; & f(x) \neq 0 \\ 0; & f(x) = 0 \end{cases}$

(xi) Greatest Integer Function or Step Up Function :

The function y = f(x) = [x] is called the greatest integer function where [x] equals to the greatest integer less than or equal to x. For example :

for $1 \leq x < 0$; [x] = -1 ; for $0 \leq x < 1$; [x] = 0

for $1 \leq x < 2$; [x] = 1 ; for $2 \leq x < 3$; [x] = 2 and so on.

Properties of greatest integer function :

(a) $x - 1 < [x] \leq x$

(b) $[x \pm m] = [x] \pm m$ iff m is an integer.

(c) $[x] + [y] \leq [x + y] \leq [x] + [y] + 1$

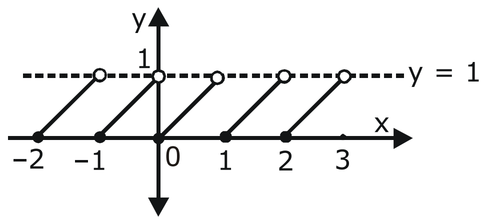
(d) $[x] + [-x] = \begin{cases} 0 ; & \text{if } x \text{ is an integer} \\ -1 ; & \text{otherwise} \end{cases}$

(xii) Fractional Part Function :

It is defined as, y = {x} = x – [x].

e.g. the fractional part of the number 2.1 is 2.1 – 2 = 0.1 and the fractional part of –3.7 is 0.3.

The period of this function is 1 and graph of this function is as shown.



y = (x)

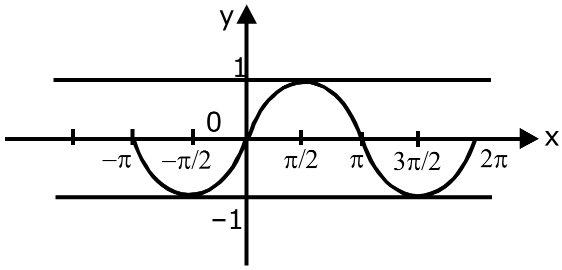
Properties of fractional part function

(a) $\{x \pm m\} = \{x\}$ iff m is an integer

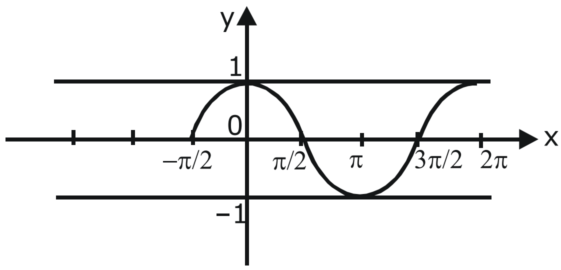
(b) $\{x\} + \{-x\} = \begin{cases} 0 ; & \text{if } x \text{ is an integer} \\ 1 ; & \text{otherwise} \end{cases}$

Graphs of Trigonometric functions :

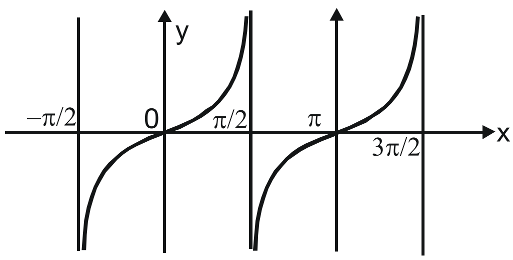
(a) y = sin x $x \in R$; $y \in [-1,1]$



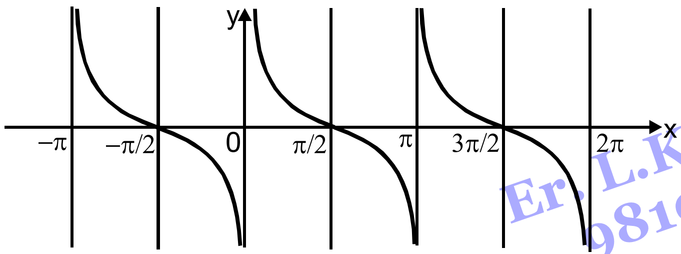
(b) y = cos x $x \in R$; $y \in [-1,1]$



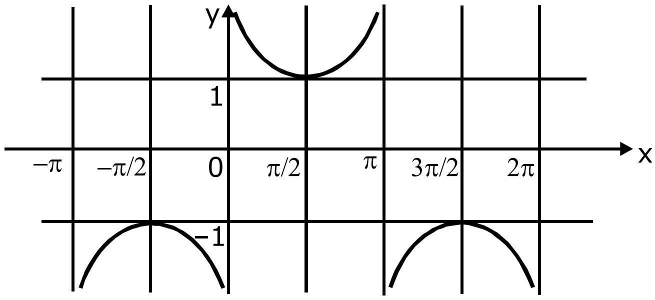
(c) y = tan x $x \in R-(2n+1)\pi/2$; $n \in I$; $y \in R$



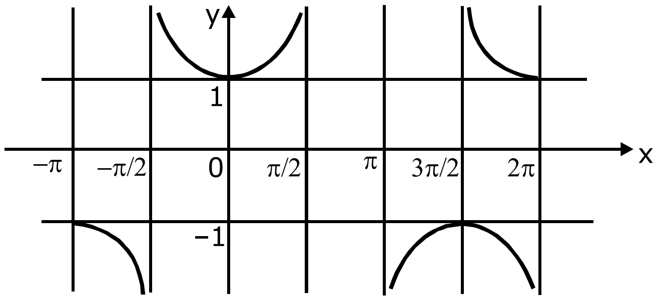
(d) y = cot x $x \in R-n\pi$; $n \in I$; $y \in R$



(e) y = cosec x $x \in R - n\pi;\ n \in I;\ y \in (-\infty, -1,] \cup [1, \infty)$



(f) y = sec x $x \in R - (2n+1)\pi/2;\ n \in I;$ $y \in (-\infty, -1,] \cup [1, \infty)$



Trigonometric Functions of sum or Difference of Two Angles :

(a) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

(b) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

(c) $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A+B).\sin(A-B)$

(d) $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A+B).\cos(A-B)$

(e) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

(f) $\cot(A \pm B) = \dfrac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$

(g) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

Factorisation of the Sum or Difference of Two sines or cosines :

(a) $\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$ (b) $\sin C - \sin D = 2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

(c) $\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$ (d) $\cos C - \cos D = -2\sin\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

Transformation of Products into Sum or Differences of Sines & Cosines :

(a) $2\sin A \cos B = \sin(A+B) + \sin(A-B)$ (b) $2\cos A \sin B = \sin(A+B) - \sin(A-B)$

(c) $2\cos A \cos B = \cos(A+B) + \cos(A-B)$ (d) $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

Multiple and Sub-multiple Angles :

(a) $\sin 2A = 2 \sin A \cos A$; $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

(b) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$; $2\cos^2\frac{\theta}{2} = 1 + \cos\theta$, $2\sin^2\frac{\theta}{2} = 1 - \cos\theta$.

(c) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$; $\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

(d) $\sin 2A = \frac{2\tan A}{1+\tan^2 A}$; $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$

(e) $\sin 3A = 3\sin A - 4\sin^3 A$

(f) $\cos 3A = 4\cos^3 A - 3\cos A$

(g) $\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

Important Trigonometric Ratios :

(a) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$, where $n \in I$

(b) $\sin 15^\circ$ or $\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75^\circ$ or $\cos\frac{5\pi}{12}$

$\cos 15^\circ$ or $\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75^\circ$ or $\sin\frac{5\pi}{12}$

$\tan 15^\circ = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3} = \cot 75^\circ$; $\tan 75^\circ = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3}$ or $\cot 15^\circ$

(c) $\sin\frac{\pi}{10}$ or $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$ & $\cos 36^\circ$ or $\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$

Conditional Identities :

If $A + B + C = \pi$ then :

(i) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(ii) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(iii) $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$

(iv) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2} . \sin\frac{B}{2} . \sin\frac{C}{2}$

(v) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(vi) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

(vii) $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2} . \cot\frac{B}{2} . \cot\frac{C}{2}$

(viii) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

(ix) $A + B + C = \frac{\pi}{2}$ then $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$

Range of Trigonometric Expression :

$E = a \sin\theta + b \cos\theta$

$E = \sqrt{a^2+b^2}\,\sin(\theta+\alpha)$, where $\tan\alpha = \frac{b}{a}$

$= \sqrt{a^2+b^2}\,\cos(\theta-\beta)$, where $\tan\beta = \frac{a}{b}$

Hence for any real value of θ, $-\sqrt{a^2+b^2} \le E \le \sqrt{a^2+b^2}$

Sine and Cosine Series :

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots\ldots + \sin\left(\alpha+\overline{n-1}\beta\right) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\,\sin\left(\alpha + \frac{n-1}{2}\beta\right)$$

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots. + \cos\left(\alpha+\overline{n-1}\beta\right) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\,\cos\left(\alpha + \frac{n-1}{2}\beta\right)$$

2. QUADRATIC EQUATION

If $a_0 \neq 0$, polynomial equation of 'n' degree is represented by $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ...a_n = 0$, where $n \in N$. If n = 1, equation is termed as 'linear' and if n=2, equation is termed as quadratic equation

Note:

(i) Values of 'x' which satisfy the polynomial equation is termed as its roots or zeros.

(ii) If general, polynomial equation of 'n' degree is having n roots, but if it is having more than n roots, then it represents an **identity**.

for example: $\frac{(x-a)(x-b)}{(c-a)(c-b)} + \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} = 1$ is quadratic equation but have more than two roots (viz: x = a,b,c, ... etc.)

(iii) If $ax^2 + bx + c = 0$ is an identity, then a=b=c=0.

(iv) An identity is satisfied for all real values of the variable.

for example: $\sin^2x + \cos^2x = 1 \ \forall \ x \in R$.

2. Relation Between Roots and Coefficients :

(i) The solutions of quadratic equation $ax^2 + bx + c = 0$, $(a \neq 0)$ is given by

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, where $b^2-4ac=D$ is called discriminant of quadratic equation.

(ii) If α, β are the roots of quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, then:

$ax^2 + bx + c \equiv a(x-\alpha)(x-\beta) \Rightarrow$

- $\alpha + \beta = -\frac{b}{a}$
- $\alpha\beta = \frac{c}{a}$
- $|\alpha - \beta| = \frac{\sqrt{D}}{|a|}$

(iii) A quadratic equation whose roots are α and β is $(x-\alpha)(x-\beta) = 0$

$\Rightarrow$ $x^2 -$ {sum of roots}x + {product of roots} = 0

Note:

- If α, β, γ are the roots of $ax^3 + bx^2 + cx + d = 0$, then

$S_1 = \alpha + \beta + \gamma = -\frac{b}{a}$, $S_2 = \beta\gamma + \gamma\alpha + \alpha\beta = \frac{c}{a}$ and $S_3 = \alpha\beta\gamma = -\frac{d}{a}$; (where S_k represents sum of roots taking k roots at a time)

- If $\alpha, \beta, \gamma, \delta$ are the roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, then

$\alpha + \beta + \gamma + \delta = -b/a$, $(\alpha+\beta)(\gamma+\delta) + \alpha\beta + \gamma\delta = c/a$, $\alpha\beta(\gamma+\delta) + \gamma\delta(\alpha+\beta) = -d/a$, $\alpha\beta\gamma\delta = e/a$.

- If $\alpha_1, \alpha_2, \alpha_3, ...\alpha_n$ are roots of the equation $f(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$ ....(1)

then, $f(x) = a_n(x-\alpha_1)(x-\alpha_2)......(x-\alpha_n)$

$\therefore \quad a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0 \equiv a_n(x-\alpha_1)(x-\alpha_2)........(x-\alpha_n)$

Comparing the coefficients of like powers of x on both sides, we get

$S_1 = \sum_{i=1}^{n} \alpha_i = \alpha_1 + \alpha_2 + \alpha_3 + ... + \alpha_n = -\frac{a_{n-1}}{a_n} = -\frac{\text{coefficient of } x^{n-1}}{\text{coefficient of } x^n}$

$$S_2 = \sum_{i \neq j} \alpha_i \alpha_j = \alpha_1\alpha_2 + \ldots\ldots = (-1)^2 . \frac{a_{n-2}}{a_n} = -\frac{\text{coefficient of } x^{n-2}}{\text{coefficient of } x^n}$$

$$S_3 = \sum_{i \neq j \neq k} \alpha_i \alpha_j \alpha_k = \alpha_1\alpha_2\alpha_3 + \alpha_2\alpha_3\alpha_4 + \ldots\ldots = (-1)^3 . \frac{a_{n-3}}{a_n} = (-1)^3 \frac{\text{coefficient of } x^{n-3}}{\text{coefficient of } x^n}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$S_n = a_1 a_2 a_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots a_n = (-1)^n \frac{a_0}{a_n} = (-1)^n \frac{\text{constant term}}{\text{coeficient of } x^n}.$$

3. Nature of Roots :

Consider the quadratic equation $ax^2 + bx + c = 0$ having α, β as its roots, $D = b^2 - 4ac$

Nature of Roots

$D = 0$
(Roots are equal, $\alpha = \beta = -b/2a$)

$D \neq 0$
(Roots are unequal)

$a, b, c \in R$ and $D > 0$
Roots are real and unequal

$a, b, c \in R$ and $D < 0$
Roots are imaginary

$a, b, c \in Q$ and
D is a perfect square
$\Rightarrow$ Roots are rational

$a, b, c \in Q$ and
D is not a perfect square
$\Rightarrow$ Roots are irrational

$\downarrow$

$a = 1$, $b, c \in I$ and D is a perfect square
$\Rightarrow$ Roots are integral.

Note:

(i) If the coefficients of the equation $ax^2 + bx + c = 0$ are all real and $\alpha + i\beta$ is its root, then $\alpha - i\beta$ is also a root. i.e. **imaginary roots occur in conjugate pairs.**

(ii) If the coefficients in the equation are all rational and $\alpha + \sqrt{\beta}$ is one of its roots, then $\alpha - \sqrt{\beta}$ is also a root where $\alpha, \beta \in Q$ and β is not a perfect square.

(iii) If a quadratic expression $f(x) = ax^2 + bx + c$ is a perfect square of a linear expression then $D = b^2 - 4ac = 0$.

4. Common Roots :

Consider two quadratic equations, $a_1x^2 + b_1x + c_1=0$ and $a_2x^2 + b_2x + c_2=0$.

(i) If two quadratic equations have both roots common, then the equation are identical and their co-efficient are in proportion. i.e.

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

(ii) If only one root is common, then the common root 'α' will be :

$$\alpha = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} = \frac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1}$$

Hence the condition for one common root is $\mathbf{(a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1) = (c_1a_2 - c_2a_1)^2}$

Note:

(i) If f(x) = 0 and g(x) = 0 are two polynomial equation having some common root(s) then these common root(s) is/are also the root(s) of h(x)=af(x)+bg(x)=0.

(ii) To obtain the common root, make coefficients of x^2 in both the equations same and subtract one equation from the other to obain a linear equation in x and then solve it for x to get the common root.

5. Factorisation of Quadratic Expressions :

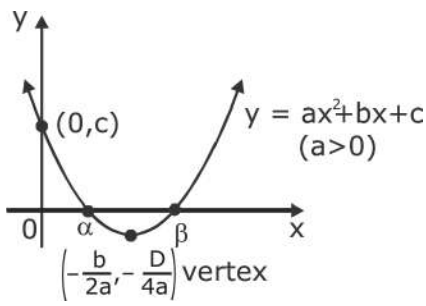
If a quadratic expression $f(x,y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ may be resolved into two linear factors, then

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \text{ OR } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

6. Graph of Quadratic Expression :

$$y = f(x) = ax^2 + bx + c; \quad a \neq 0$$

$$\Rightarrow \quad \left(y + \frac{D}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2$$



Note:

(i) $f(x) = ax^2+bx+c$ represents a parabola.

(ii) the co-ordinate of vertex are $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

(iii) If a>0 then the shape of the parabola is concave upwards and if a < 0 then the shape of the parabola is concave downwards.

(iv) the parabola intersect the y-axis at point (0,c).

(v) the x-co-ordinate of point of intersection of parabola with x-axis are the real roots of the quadratic equation f(x) = 0.

7. Range of Quadratic Expression f(x) = ax² + bx + c.

(1) **Absolute Range :**

If a > 0 $\Rightarrow$ $f(x) \in \left[-\frac{D}{4a}, \infty\right)$ and if a < 0 $\Rightarrow$ $f(x) \in \left(-\infty, -\frac{D}{4a}\right]$

Hence maximum and minimum values of the expression f(x) is $-\frac{D}{4a}$ in respective cases

and it occurs at x = $-\frac{b}{2a}$ (at vertex).

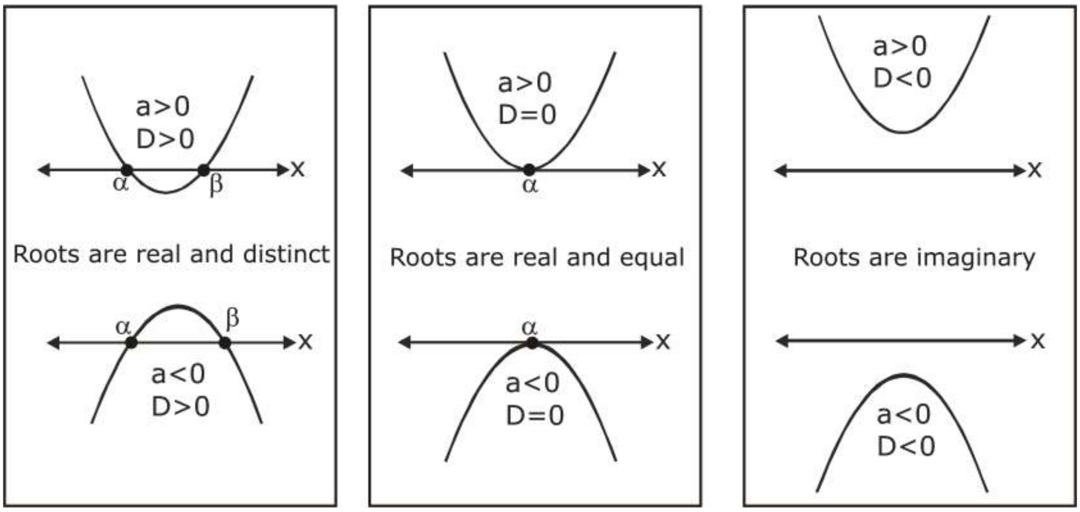
(2) **Range in restricted domain :**

(a) If $-\frac{b}{2a} \notin [x_1, x_2]$, then $f(x) \in \left[\min\{f(x_1), f(x_2)\}\ ,\ \max\{f(x_1), f(x_2)\}\right]$

(b) If $-\frac{b}{2a} \in [x_1, x_2]$, then $f(x) \in \left[\min\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\}\ ,\ \max\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\}\right]$

8. Sign of quadratic Expressions :

The value of expression f(x) = ax² + bx + c at x = x_0 is equal to y-co-ordinate of a point on parabola y = ax² + bx + c whose x-co-ordinate is x_0. Hence if the point lies above the x-axis for some x = x_0 then $f(x_0) > 0$ as illustrated in following graphs:



Note:

(i) $ax^2 + bx + c \geq 0\ \forall\ x \in R \Rightarrow a > 0$ and $D \leq 0$.

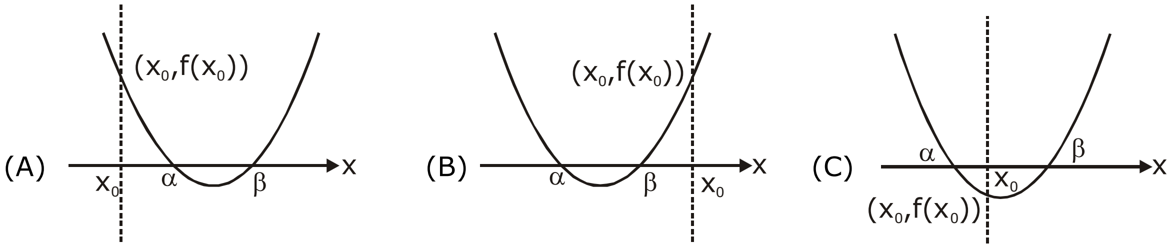
(ii) $ax^2 + bx + c \leq 0\ \forall\ x \in R \Rightarrow a < 0$ and $D \leq 0$.

(iii) $ax^2 + bx + c > 0\ \forall\ x \in R \Rightarrow a > 0$ and $D < 0$.

(iv) $ax^2 + bx + c < 0\ \forall\ x \in R \Rightarrow a < 0$ and $D < 0$.

10. Location of Roots :

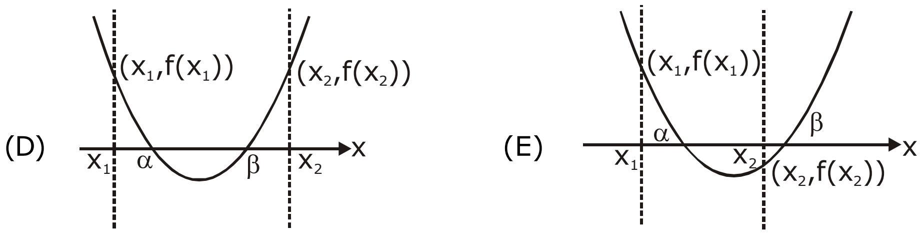
Let $f(x) = ax^2 + bx + c$, where $a\ b\ c \in R$.



(i) Conditions for both the roots of $f(x) = 0$ to be greater than a specified number 'x_0' are $b^2 - 4ac \geq 0$; $af(x_0) > 0$ and $(-b/2a) > x_0$. **(refer figure no. (A))**

(ii) Conditions for both the roots of $f(x) = 0$ to be smaller than a specified number 'x_0' are $b^2 - 4ac \geq 0$; $af(x_0) > 0$ and $(-b/2a) < x_0$. **(refer figure no. (B))**

(iii) Conditions for both roots of $f(x) = 0$ to lie on either side of the number 'x_0' (in other words the number 'x_0' lies between the roots of $f(x) = 0$) is $b^2 - 4ac > 0$ and $af(x_0) < 0$. **(refer figure no. (C))**



(iv) Condition that both roots of $f(x) = 0$ to be confined between the numbers x_1 and x_2, $(x_1 < x_2)$ are $b^2 - 4ac \geq 0$; $af(x_1) > 0$; $af(x_2) > 0$ and $x_1 < (-b/2a) < x_2$. **(refer figure no. (D))**

(v) Conditions for exactly one root of $f(x) = 0$ to lie in the interval (x_1, x_2) i.e. $x_1 < \alpha < x_2$ is $f(x_1).f(x_2) < 0$. **(refer figure no. (E))**

Important Results

(i) If α is a root of the equation $f(x) = 0$, then the polynomial $f(x)$ is divisible by $(x - \alpha)$ or $(x - \alpha)$ is a factor of $f(x)$

(ii) An equation of odd degree will have odd number of real roots and an equation of even degree will have even numbers of real roots.

(iii) Every equation $f(x) = 0$ of odd degree has at least one real root of a sign opposite to that of its last constant term. (If coefficient of highest degree term is positive).

(iv) The quadratic equation $f(x) = ax^2 + bx + c = 0$, $a \neq 0$ has α as a repeated root if and only if $f(\alpha) = 0$ and $f'(\alpha) = 0$. In this case $f(x) = a(x - \alpha)^2 \Rightarrow \alpha = -b/2a$.

(v) If polynomial equation $f(x) = 0$ has a root α of multiplicity r (where r > 1), then f(x) can be written as $f(x) = (x - \alpha)^r g(x)$, where $g(\alpha) \neq 0$, also, $f'(x) = 0$ has α as a root with multiplicity r–1.

(vi) If polynomial equation f(x) = 0 has n distinct real roots, then

$f(x) = a_0(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$, where $\alpha_1, \alpha_2, \ldots \alpha_n$ are n distinct real roots and

$$\frac{f'(x)}{f(x)} = \sum_{k=1}^{n} \frac{1}{(x-\alpha_k)}$$

(vii) If there be any two real numbers 'a' and 'b' such that f(a) and f(b) are of opposite signs, then f(x) = 0 must have odd number of real roots (also at least one real root) between 'a' and 'b'

(viii) If there be any two real numbers 'a' and 'b' such that f(a) and f(b) are of same signs, then f(x) = 0 must have even number of real roots between 'a' and 'b'

(ix) If polynomial equation f(x) = 0 has n real roots, then f'(x) = 0 has atleast (n – 1) real roots.

(x) **Descartes rule** of signs for the roots of a polynomial

- The maximum number of positive real roots of a polynomial equation f(x) = 0 is the number of changes of the signs of coefficients of f(x) from positive to negative or negative to positive.
- The maximum number of negative real roots of the polynomial equation f(x) = 0 is the number of changes from positive to negative or negative to positive in the signs of coefficients of f(–x) = 0.

(xi) To form an equation whose roots are reciprocals of the roots in equation

$a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$, x is replaced by 1/x and then both sides are multiplied by x^n.

3. COMPLEX NUMBER

(1) Complex number system :

A number of the form a + ib where a , b $\in$ R . and $i = \sqrt{-1}$ is called a complex number. If z = a + ib , then the real part of z is denoted by Re(z) and the imaginary part of z is denoted by Im(z). A complex number z is said to by purely real if Im(z) = 0 and is said to be purely imaginary if Re(z) = 0.

Note:

- Set of all complex numbers is denoted by **C** , where $C = \{a + ib : b \in R \wedge i = \sqrt{-1}\}$
- Zero is the only number which is purely real as well as purely imaginary.
- $i = \sqrt{-1}$ is the imaginary unit and is termed as 'iota'.
- If $n \in I$, then $(i)^{4n} = 1$, $(i)^{4n+1} = i$, $(i)^{4n+2} = -1$ and $(i)^{4n+3} = -i$
- Sum of four consecutive powers of 'i' is always zero (i.e. $(i)^n + (i)^{n+1} + (i)^{n+2} + (i)^{n+3} = 0$), where $n \in I$.
- The property of multiplication $(\sqrt{a})(\sqrt{b}) = \sqrt{ab} \in$ is valid only is atleast one of a or b is non-negative.

(2) Algebraic Operations :

1. Addition : (a + bi) + (c + di) = (a + c) + (b + d)i
2. Subtraction : (a + bi) − (c + di) = (a − c) + (b − d)i
3. Multiplication : $(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$
4. Division : $\dfrac{a+bi}{c+di} = \dfrac{a+bi}{c+di} \cdot \left(\dfrac{c-di}{c-di}\right) = \dfrac{ac+bd+(bc-ad)i}{c^2+d^2} = \left(\dfrac{ac+bd}{c^2+d^2}\right) + \left(\dfrac{bc-ad}{c^2+d^2}\right)i$

Note:

(i) In real numbers if $a^2 + b^2 = 0$ then a = b = 0 but in complex numbers, $z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$.

(ii) Inequalities in complex numbers are not defined (Law of order is not applicable for complex numbers).

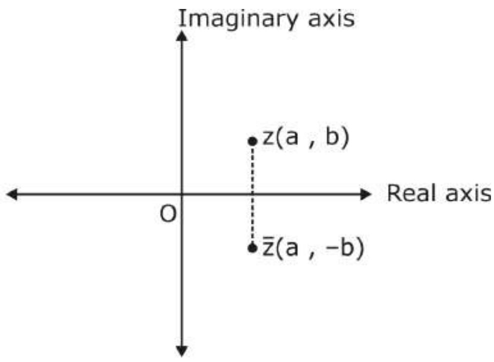
(3) Equality In Complex Number :

Two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are equal if and only if their corresponding real and imaginary parts are equal respectively

$\therefore \quad z_1 = z_2 \Rightarrow Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.

(4) Conjugate of Complex Number :

Conjugate of a complex number z = a + ib is denoted by $\bar{z}$ and and is defined as $\bar{z}$ = a−ib. Geometrically a complex number z = a + ib is represented by ordered pair (a , b) on the complex plane (or Argand plane) and the mirror image of z about real axis represents the conjugate of z.



" $\bar{z}$ is mirror image of z about Real axis "

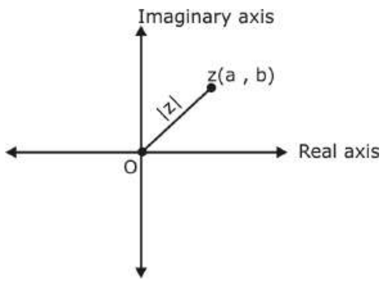
Properties of Conjugate of a Complex Number :

- $z_1 = z_2 \Leftrightarrow \bar{z}_1 = \bar{z}_2$
- $\overline{(\bar{z})} = z$
- $z + \bar{z} = 2\,\mathrm{Re}(z)$
- $z - \bar{z} = 2i\ \mathrm{Im}(z)$
- $z = \bar{z} \Leftrightarrow z$ is purely real
- $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary.
- $z\bar{z} = [\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2 = |z|^2$
- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
- $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$
- $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
- $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$ if $z_2 \neq 0$
- $(\bar{z})^n = \overline{z^n}$
- If $f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$ where a_0 , a_1 ,.... a_n and z are complex number , then $\overline{f(z)} = \bar{a}_0 + \bar{a}_1(\bar{z}) + \bar{a}_2(\bar{z})^2 + \ldots + \bar{a}_n(\bar{z})^n$

For example : $\overline{\left(\dfrac{2z + 3z^3}{4z + 1}\right)} = \dfrac{2\bar{z} + 3(\bar{z})^3}{4\bar{z} + 1}$

(5) Modulus of a Complex Number :

If z = a + ib , then modulus of complex number z is denoted by |z| and defined as $|z| = \sqrt{a^2 + b^2}$. Geometrically |z| is the distance of z from origin on the complex plane.



Properties of modulus :

- $|z| = 0 \Leftrightarrow z = 0$
- $|z| = |\bar{z}| = |-z| = |-\bar{z}|$
- $-|z| \le \mathrm{Re}(z) \le |z|$
- $-|z| \le \mathrm{Im}(z) \le |z|$
- $z\bar{z} = |z|^2$
- $|z_1 z_2| = |z_1|\,|z_2|$

- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, if $z_2 \neq 0$
- $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + \bar{z}_1 z_2 + z_1\bar{z}_2$
  $= |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z}_2)$
- $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - \bar{z}_1 z_2 - z_1\bar{z}_2$
  $= |z_1|^2 + |z_2|^2 - 2\text{Re}(z_1\bar{z}_2)$
- $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$
- If $z_1, z_2 \neq 0$, then $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \frac{z_1}{z_2}$ is purely imaginary.
- Geometrically $|z_1 - z_2|$ represents the distance between complex points z_1 and z_2 on argand plane.

Triangle Inequality :

If z_1 and z_2 are two complex numbers , then

- $|z_1 + z_2| \leq |z_1| + |z_2|$.
- $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$

The sign of equality holds iff z_1 , z_2 and origin are collinear and z_1 , z_2 lies on same side of origin.

(6) Argument of a Complex Number :

If non-zero complex number z is represented on the argand plane by complex point 'P' , then argument (or amplitude) of z is the angle which OP makes with positive direction of real axis.

Let complex number z = a + ib and $\alpha = \tan^{-1}\left|\frac{b}{a}\right|$, then principal argument of z is given as

- $\arg(z) = \alpha$, if z lies in I quadrant.
- $\arg(z) = (\pi - \alpha)$, if z lies in II quadrant.
- $\arg(z) = (-\pi + \alpha)$, if z lies in III quadrant.
- $\arg(z) = -\alpha$, if z lies in IV quadrant.

Note:

- Argument of a complex number is not unique , according to definition if θ is a value of argument , then $2n\pi + \theta \ \forall \ n \in I$ is also the argument of complex number.
- If θ is argument of complex number and $\theta \in (-\pi , \pi]$, then the argument θ is termed as principal argument.
- Unless otherwise mentioned , arg(z) implies the principal argument.

Properties of Argument of Complex Number :

- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi$ for some integer k.
- $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2) + 2k\pi$ for some integer k.
- $\arg(z^n) = n\arg(z) + 2k\pi$ for some integer k.
- $\arg(z) = 0 \Leftrightarrow$ z is real, for any complex number $z \neq 0$
- $\arg(z) = \pm\pi/2 \Leftrightarrow$ z is purely imaginary, for any complex number $z \neq 0$
- $\arg(\bar{z}) = -\arg(z)$

(7) Representation Of A Complex Number :

(a) Cartesian Form (Geometric Representation) :

Complex number z = x + i y is represented by a point on the complex plane (Argand plane/ Gaussian plane) by the ordered pair (x , y).

(b) Trigonometric/Polar Representation :

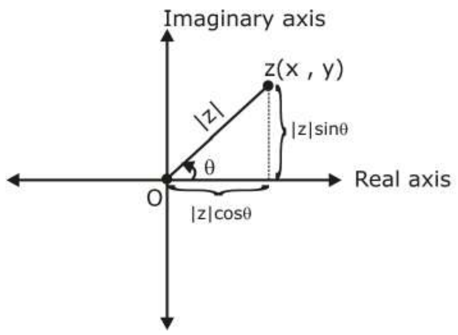
$z = r(\cos\theta + i\sin\theta)$, where $|z| = r$ and $\arg z = \theta$, represents the polar form of complex number

Note:

- $\cos\theta + i\sin\theta$ is also written as $\text{cis}\,\theta$ or $e^{i\theta}$.
- $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ and $\sin x = \dfrac{e^{ix} - e^{-ix}}{2}$ are known as Euler's Identities

(c) Euler's Representation :

$z = re^{i\theta}$, where $|z| = r$ and $\arg z = \theta$, represents the Euler's form of complex number



$\Rightarrow$ $z = x + iy$

$z = |z|\cos\theta + i\,|z|\sin\theta$

$\therefore$ $z = |z|\text{cis}\theta$ or $|z|e^{i\theta}$

(d) Vectorial Representation :

Every complex number can be considered as if it is the position vector of a point. If the point P represents the complex number z then , $\overrightarrow{OP} = z$ and $|\overrightarrow{OP}| = |z|$.

(8) Demoivre's Theorem :

- If n is any rational number , then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.
- If $z = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)....(\cos\theta_n + i\sin\theta_n)$ then $z = \cos(\theta_1 + \theta_2 + \theta_3 + ... + \theta_n) + i\sin(\theta_1 + \theta_2 + ... + \theta_n)$, where $\theta_1 , \theta_2 , \theta_3 ..... \theta_n \in R$
- If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer , then

  $$z^{1/n} = r^{1/n}\left[\cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)\right]$$, where k = 0 , 1 , 2 , 3 .... (n−1)
- If p , q $\in$ z and q $\neq$ 0 , then $(\cos\theta + i\sin\theta)^{p/q} = \cos\left(\frac{2k\pi + p\theta}{q}\right) + i\sin\left(\frac{2k\pi + p\theta}{q}\right)$,

  where k = 0 , 1 , 2 , 3 .... (q−1).

Note:

This theorem is not valid when n is not a rational number or the complex number is not in the form of $\cos\theta + i\sin\theta$.

(9) Cube Root Of Unity :

Let $x = (1)^{1/3}$

$\Rightarrow x^3 - 1 = 0 \Rightarrow (x-1)(x^2+x+1) = 0.$

$\Rightarrow x = 1 , \frac{-1+i\sqrt{3}}{2} , \frac{-1-i\sqrt{3}}{2}$

If $\omega = \frac{-1+i\sqrt{3}}{2}$, then $\omega^2 = \frac{-1-i\sqrt{3}}{2}$

$\therefore$ Cube Roots of unity is given by : $1 , \omega , \omega^2$.

Properties of cube roots of unity :

- $\omega^{3n} = 1$; $\omega^{3n+1} = \omega$; $\omega^{3n+2} = \omega^2 \ \forall \ n \in I$
- $1 + \omega + \omega^2 = 0$
- $\omega^3 = 1$
- $\bar{\omega} = \omega^2$ and $\overline{\omega^2} = \omega$
- $\omega^2 = \frac{1}{\omega}$ and $\frac{1}{\omega^2} = \omega$
- $1 + \omega^k + \omega^{2k} = \begin{cases} 0 \text{ , if } k \text{ is not multiple of } 3 \\ 1 \text{ , if } k \text{ is multiple of } 3 \end{cases}$
- In polar form the cube roots of unity are :

$\cos 0 + i \sin 0 , \cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3} , \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$

- The cube roots of unity , when represented on complex plane , lie on vertices of an equilateral triangle inscribed in a unit circle having centre at origin , one vertex being on positive real axis.
- A complex number a + ib , for which |a : b| = 1 : $\sqrt{3}$ or $\sqrt{3}$: 1 , can always be expressed in terms of i , ω , ω^2.

Note:

If $x , y , z \in R$ and ω is non-real cube root of unity , then

- $x^2 + x + 1 = (x-\omega)(x-\omega^2)$
- $x^2 - x + 1 = (x+\omega)(x+\omega^2)$
- $x^2 + xy + y^2 = (x - y\omega)(x - y\omega^2)$
- $x^2 - xy + y^2 = (x + y\omega)(x + y\omega^2)$
- $x^2 + y^2 = (x+iy)(x-iy)$
- $x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$

(10) n^{th} Roots of Unity :

The n^{th} roots of unity are given by the solution set of the equation

$x^n = 1 = \cos 0 + i \sin 0 = \cos 2k\pi + i \sin 2k\pi$

$x = [\cos 2k\pi + i \sin 2k\pi]^{1/n}$

$x = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$, where $k = 0 , 1 , 2 ,...., (n-1)$.

Properties of n^{th} roots of unity

(i) Let $\alpha = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} = e^{i(2\pi/n)}$, then n^{th} roots of unity can be expressed in the form of a G.P. with common ratio α. (i.e. $1, \alpha, \alpha^2 ,.....\alpha^{n-1}$.)

(ii) The sum of all n roots of unity is zero i.e., $1+\alpha+\alpha^2+.....+\alpha^{n-1}=0$
(iii) Product of all n roots of unity is $(-1)^{n-1}$.
(iv) Sum of p^{th} power of n^{th} roots of unity

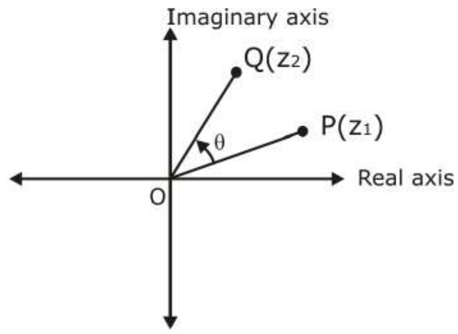
$$1+\alpha^p+\alpha^{2p}+....+\alpha^{(n-1)p}=\begin{cases}0, \text{ when p is not multiple of n}\\ n, \text{ when p is a multiple of n}\end{cases}$$

(v) The n, n^{th} roots of unity if represented on a complex plane locate their positions at the vertices of a regular plane polygon of n sides inscribed in a unit circle having centre at origin, one vertex on vertex on positive real axis.

(11) Rotation Theorem :

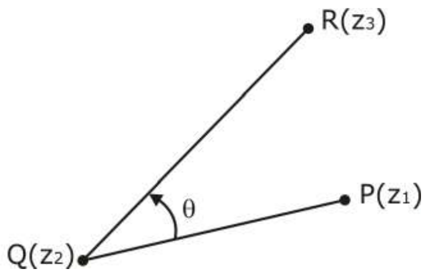
(i) If $P(z_1)$ and $Q(z_2)$ are two complex numbers such that $|z_1| = |z_2|$, then :

$$\frac{z_2}{z_1} = e^{i\theta}, \text{ where } \angle POQ = \theta$$



(ii) If $P(z_1)$, $Q(z_2)$ and $R(z_3)$ are three complex numbers and $\angle PQR = \theta$, then:

$$\frac{z_3 - z_2}{z_1 - z_2} = \left(\frac{RQ}{PQ}\right)e^{i\theta} = \frac{|z_3 - z_2|}{|z_1 - z_2|}e^{i\theta}$$



(12) Logarithm Of A Complex Number :

Let z = x + iy , then $z=|z|(e)^{i\theta}$

$\therefore \quad \log_e^{Z} = \log_e^{|Z|} + \log_e^{e^{i\theta}}$

$\Rightarrow \quad \log_e^{Z} = \frac{1}{2}\log_e^{(x^2+y^2)} + i\arg(z)$

(13) Standard Loci in the Argand Plane :

- If z is a variable point in the argand plane such that $\arg(z) = \theta$, then locus of z is a straight line (excluding origin) through the origin inclined at an angle θ with x-axis.

- If z is a variable point and z_1 is a fixed point in the argand plane such that $\arg(z - z_1) = \theta$, then locus of z is a straight line passing through the point representing z_1

and inclined at an angle θ with x-axis. Note that line point z_1 is excluded from the locus.

- If z is a variable point and z_1, z_2 are two fixed points in the argand plane, then

(i) $|z - z_1| = |z - z_2|$ $\Rightarrow$ Locus of z is the perpendicular bisector of the line segment joining z_1 and z_2

(ii) $|z - z_1| + |z - z_2| = \text{constant } (\neq |z_1 - z_2|)$ $\Rightarrow$ Locus of z is an ellipse

(iii) $|z - z_1| + |z - z_2| = |z_1 - z_2|$ $\Rightarrow$ Locus of z is the line segment joining z_1 and z_2

(iv) $|z - z_1| - |z - z_2| = |z_1 - z_2|$ $\Rightarrow$ Locus of z is a straight line joining z_1 and z_2 but z does not lie between z_1 and z_2.

(v) $|z - z_1| - |z - z_2| = \text{constant } (\neq |z_1 - z_2|)$ $\Rightarrow$ Locus of z is hyperbola.

(vi) $|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$ $\Rightarrow$ Locus of z is a circle with z_1 and z_2 extremities of diameter

(vii) $|z - z_1| = k\,|z - z_2|$ $k \neq 1$ $\Rightarrow$ Locus of z is a circle

(viii) $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = \alpha$ $\Rightarrow$ Locus of z is a segment of circle.

(ix) $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = \pm\pi/2$ $\Rightarrow$ Locus of z is a circle with z_1 and z_2 as the vertices of diameter.

(x) $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = 0$ or π $\Rightarrow$ Locus z is a straight line passing through z_1 and z_2.

(xi) The equation of the line joining complex numbers z_1 and z_2 is given by

$$\frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1} \text{ or } \begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

(13) Geometrical Properties :

(i) Distance formula :

If $P(z_1)$ and $Q(z_2)$ are two complex points on complex plane then distance between P and Q is given by : $PQ = |z_1 - z_2|$

(ii) Section formula :

If $P(z_1)$ and $Q(z_2)$ are two complex points on complex plane and R(z) divides the line segment PQ in ratio m : n , then :

$$z = \frac{mz_2 + nz_1}{m + n} \quad \text{(for internal division)}$$

$$z = \frac{mz_2 - nz_1}{m - n} \quad \text{(for external division)}$$

Note:

If a, b, c are three real numbers such that $az_1 + bz_2 + cz_3 = 0$, where $a + b + c = 0$ and a, b, c are not all simultaneously zero, then the complex numbers z_1, z_2 and z_3 are collinear.

(iii) Triangular Properties :

If the vertices A, B and C of a triangle is represented by the complex numbers z_1, z_2 and z_3 respectively and a, b, c are the length of sides , then :

- Centroid (Z_G) of the $\Delta ABC = \dfrac{z_1 + z_2 + z_3}{3}$

- Orthocentre (Z_o) of the $\Delta ABC = \dfrac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\tan A + \tan B + \tan C}$

- Incentre (Z_I) of the $\Delta ABC = \dfrac{az_1 + bz_2 + cz_3}{a+b+c}$

- Circumcentre of the $\Delta ABC = \dfrac{z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}$

Note:

Triangle ABC with vertices $A(z_1)$, $B(z_2)$ and $C(z_3)$ is equilateral if and only if

$$\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$$

$$\Leftrightarrow \; z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

(iv) Equation of a Straight Line :

- An equation of a straight line joining the two points $A(z_1)$ and $B(z_2)$ is $\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$

- An equation of the straight line joining the point $A(z_1)$ and $B(z_2)$ is $z = tz_1 + (1 - t)z_2$ where t is is a real parameter.

- The general equation of a straight line is $\bar{a}z + a\bar{z} + b = 0$ where a is a non-zero complex number and b is a real number

(v) Complex Slope of a Line :

If $A(z_1)$ and $B(z_2)$ are two points is the complex plane , then complex slope of AB is defined to be $\mu = \dfrac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$ two lines with complex slopes μ_1 and μ_2 are

- parallel , if $\mu_1 = \mu_2$

- perpendicular , if $\mu_1 + \mu_2 = 0$ the complex slope of the line $\bar{a}z + a\bar{z} + b = 0$ is given by $-(a/\bar{a})$

(vi) Length of Perpendicular from a Point to a Line :

Length of perpendicular of point $A(\omega)$ from the line $\bar{a}z + a\bar{z} + b = 0$, where $a \in C - \{0\}$, $b \in \mathbf{R}$, is given by : $p = \dfrac{|\bar{a}\omega + a\bar{\omega} + b|}{2|a|}$

(vii) Equation of Circle :

- An equation of the circle with centre at z_0 and radius r is

$$|r - z_0| = r$$

or $z = z_0 + re^{i\theta}$, $0 \le \theta < 2\pi$ (parametric from)

or $z\bar{z} - z_0\bar{z}_0 z + z_0\bar{z}_0 - r^3 = 0$

- General equatio of a circle is

$$z\bar{z} + a\bar{z} + \bar{a}z + b = 0 \qquad (1)$$

where a is a complex number and b is a real number such that $a\bar{a} - b \ge 0$

Centre of (1) is –a and its radius is $\sqrt{a\bar{a} - b}$.

- Diameter Form of a Circle

An equation of the circle one of whose diameter is the segment joining $A(z_1)$ and $B(z_2)$ is

$$(z-z_1)(\bar{z}-\bar{z}_2)+(\bar{z}-\bar{z}_1)(z-z_2)=0$$

- An equation of the circle passing through two points $A(z_1)$ and $B(z_2)$ is

$$(z-z_1)(\bar{z}-\bar{z}_2)+(\bar{z}-\bar{z}_1)(z-z_2)+k\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix}=0 \text{ where k is a parameter}$$

- Equation of a circle passing through three non-collinear points.

Let three non-collinear points be $A(z_1)$, $B(z_2)$ and $C(z_3)$. Let $P(z)$ be any point on the circle.
Then either $\angle ACB = \angle APB$ [when angles are in the same segment]

$\angle ACB + \angle APB = \pi$ [when angles are in the opposite segment]

$$\Rightarrow \quad \arg\left(\frac{z_3-z_2}{z_3-z_1}\right)-\arg\left(\frac{z-z_2}{z-z_1}\right)=0$$

$$\arg\left(\frac{z_3-z_2}{z_3-z_1}\right)+\arg\left(\frac{z-z_1}{z-z_2}\right)=\pi$$

$$\Rightarrow \quad \arg\left[\left(\frac{z_3-z_2}{z_3-z_1}\right)\left(\frac{z-z_1}{z-z_2}\right)\right]=0$$

$$\text{or} \quad \arg\left[\left(\frac{z-z_1}{z-z_2}\right)\left(\frac{z_3-z_2}{z_3-z_1}\right)\right]=\pi$$

[using $\arg\left(\frac{z_1}{z_2}\right)=\arg(z_1)-\arg(z_2)$ and $\arg(z_1z_2)=\arg(z_1)+\arg(z_2)$]

In any case , we get $\dfrac{(z-z_1)(z_3-z_2)}{(z-z_2)(z_3-z_1)}$ is purely real.

$$\Leftrightarrow \quad \frac{(z-z_1)(z_3-z_2)}{(z-z_2)(z_3-z_1)}=\frac{(\bar{z}-\bar{z}_1)(\bar{z}_3-\bar{z}_2)}{(\bar{z}-\bar{z}_2)(\bar{z}_3-\bar{z}_1)}$$

- Condition for four points to be concyclic. Four points z_1 , z_2 , z_3 are z_4 will lie on the same circle if and only if $\dfrac{(z_4-z_1)(z_3-z_2)}{(z_4-z_2)(z_3-z_1)}$ is purely real.

$\Leftrightarrow \quad \dfrac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$ is purely real.

4. BINOMIAL THEOREM

An algebraic expression which contains two distinct terms is called a binomial expression.

For example : $(2x+y)$, $\left(x+\frac{3}{y}\right)$, $\left(\frac{3}{x}+\sqrt{x+1}\right)$ etc.

General form of binomial expression is (a+b) and the expansion of $(a+b)^n$, $n \in N$ is called the binomial theorem.
For example :
$(a + b)^1 = a + b$
$(a + b)^2 = a^2 + 2ab + b^2$
$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Note:

Coefficients in the binomial expansion is termed as binomial coefficients and these binomial coefficients have a fixed pattern which can be seen through pascal triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

(2) Statement of Binomial theorem :

If a, b $\in$ **C** and $n \in N$, then :
$(a + b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + \ldots + {}^nC_r\, a^{n-r} b^r + \ldots + {}^nC_n\, a^0 b^n$

or $\Rightarrow$ $(a+b)^n = \sum_{r=0}^{n} {}^nC_r a^{n-r} b^r$

General term in binomial expansion $(a + b)^n$ is given by T_{r+1}, where $T_{r+1} = {}^nC_r a^{n-r} b^r$

Now, putting a = 1 and b = x in the binomial theorem

$$(1 + x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2\, x^2 + \ldots + {}^nC_r\, x^r + \ldots + {}^nC_n\, x^n$$

$$\Rightarrow (1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$$

General term in binomial expansion $(1 + x)^n$ is given by T_{r+1}, where $T_{r+1} = {}^nC_r x^r$

(3) Properties of Binomial Theorem :

(i) The number of terms in the expansion is n + 1.

(ii) The sum of the indices of 'a' and 'b' in each term is n.

(iii) The binomial coefficients (i.e. ${}^nC_0, {}^nC_1 \ldots\ldots\ {}^nC_n$) of the terms equidistant from the beginning and the end are equal, i.e. ${}^nC_0 = {}^nC_n$, ${}^nC_1 = {}^nC_{n-1}$ etc. $\{\therefore {}^nC_r = {}^nC_{n-r}\}$

(iv) Middle term (s) in expansion of $(a + b)^n$, $n \in N$:

If n is even, then number of terms are odd and in this case only one middle term exists which is $\left(\frac{n+2}{2}\right)^{th}$ therm if n is odd, then number of terms are even and in this case two middle terms exist which are $\left(\frac{n+2}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ terms.

7. Properties of Binomial coefficients :

$(1 + x)^n = C_0 + C_1x + C_2x^2 + \dots\dots\dots + C_rx^r + \dots\dots\dots + C_nx^n$ .....(1)

(1) The sum of the binomial coefficients in the expansion of $(1 + x)^n$ is 2^n

Putting x = 1 in (1)

${}^nC_0 + {}^nC_1 + {}^nC_2 + \dots\dots + {}^nC_n = 2^n$ ......(2)

or $\sum_{r=0}^{n} {}^nC_r = 2^n$

(2) Again putting x = –1 in (1), we get

${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \dots\dots\dots\dots + (-1)^n\,{}^nC_n = 0$ ......(3)

or $\sum_{r=0}^{n}(-1)^r\,{}^nC_r = 0$

(3) The sum of the binomial coefficients at odd position is equal to the sum of the binomial coefficients at even position and each is equal to 2^{n-1}.

from (2) and (3)

${}^nC_0 + {}^nC_2 + {}^nC_4 + \dots\dots\dots\dots = 2^{n-1}$

${}^nC_1 + {}^nC_3 + {}^nC_5 + \dots\dots\dots\dots = 2^{n-1}$

(4) Sum of two consecutive binomial coefficients

${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

$$\text{L.H.S.} = {}^nC_r + {}^nC_{r-1} = \frac{n!}{(n-r)!r!} + \frac{n!}{(n-r+1)!(r-1)!}$$

$$= \frac{n!}{(n-r)!(r-1)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!}{(n-r)!(r-1)!}\,\frac{(n+1)}{r(n-r+1)}$$

$$= \frac{(n+1)!}{(n-r+1)!r!} = {}^{n+1}C_r \qquad = \text{R.H.S.}$$

(5) Ratio of two consecutive binomial coefficients

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$$

(6) $${}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1} = \frac{n(n-1)}{r(r-1)}\,{}^{n-2}C_{r-2} = \dots\dots\dots = \frac{n(n-1)(n-2)\dots\dots\dots(n-(r-1))}{r(r-1)(r-2)\dots\dots\dots2.1}$$

8. Binomial Theorem For Negative Integer Or Fractional Indices

If $n \in R$ then,

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots + \frac{n(n-1)(n-2)\ldots\ldots.(n-r+1)}{r!}x^r + \ldots\ldots\ldots\ldots\infty.$$

Remarks

(i) The above expansion is valid for any rational number other than a whole number if $|x| < 1$.

(ii) When the index is a negative integer or a fraction the number of terms in the expansion of $(1 + x)^n$ is infinite, and the symbol ${}^nC_r$ cannot be used to denote the coefficient of the general term.

(iii) The first terms must be unity in the expansion, when index 'n' is a negative integer or fraction

(iv) The general term in the expansion of $(1 + x)^n$ is T_{r+1}

$$= \frac{n(n-1)(n-2)\ldots\ldots.(n-r+1)}{r!}x^r$$

(v) When 'n' is any rational number other than whole number then approximate value of $(1 + x)^n$ is $1 + nx$(x^2 and higher powers of x can be neglected)

(vi) Expansions to be remembered ($|x| < 1$)

(a) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots. + (-1)^r x^r + \ldots\ldots.\infty$

(b) $(1 - x)^{-1} = 1 + x + x^2 + x^5 + \ldots\ldots\ldots\ldots. + x^r + \ldots\ldots.\infty$

(c) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots..(-1)^r (r + 1) x^r + \ldots\ldots.$

(d) $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots.. + (r + 1)x^r + \ldots\ldots.\infty$

5. PERMUTATION AND COMBINATION

Permutations are arrangements and combinations are selections. In this chapter we discuss the methods of counting of arrangements and selections. The basic results and formulas are as follows :

(1) Fundamental Principle of Counting :

(i) **Principle of Multiplication :**

If an event can occur in 'm' different ways, following which another event can occur in 'n' different ways, then total number of different ways of simultaneous occurrence of both the events in a definite order is m + n.

(ii) **Principle of Addition :**

If an event can occur in 'm' different ways, and another event can occur in 'n' different ways, then exactly one of the events can happen in m + n ways.

(2) The Factorial :

Let n be a positive integer. Then the continued product of first n natural numbers is called factorial n, be denoted by n! or $\underline{|n}$. Also , we define 0! = 1.

when n is negative or a fraction, n ! is not defined.

Thus , n ! = n (n−1) (n−2).......3.2.1.

Also , $1! = 1 \times (0!) \Rightarrow 0! = 1$.

(3) Arrangement or Permutation :

The number of permutations of n different things, taking r at a time without repetition is given by ${}^nP_r$

$${}^nP_r = n(n-1)(n-2) \ldots (n-r+1) = \frac{n!}{(n-r)!}$$

Note:

(i) The number of permutations of n different things, taking r at a time with repetition is given by n^r

(ii) The number of arrangements that can be formed using n objects out ow which p are identical (and of one kind) q are identical (and of another kind), r are identical (and of another kind) and the rest are distinct is $\dfrac{n!}{p!q!r!}$.

(4) Conditional Permutations :

(i) Number of permutation of n dissimilar things taken r at a time which p particular things always occur $= {}^{n-p}C_{r-p}\, r!$

(ii) Number of permutation of n dissimilar things taken r at a time which p particular things never occur $= {}^{n-p}C_{r-p}\, r!$

(iii) The total number of permutation of n dissimilar things taken not more than r at a time, when each thing may be repeated any number of times, is $\frac{n(n^r-1)}{n-1}$.

(iv) Number of permuutations of n different things, taken all at a time, when m specified things always come together is $m! \times (n-m+1)!$

(v) Number of permutation of n different things, taken all at a time when m specified things never come together is $n! - m! \times (n-m+1)!$

(5) Circular Permutation:

The number of circular permutations of n different things taken all at a time is given by $(n-1)!$, provided clockwise and anti-clockwise circular permutations are considered to be differnt

Note:

(i) The number of circular permutations of n different things taken all at a time , clockwise and anti-clockwise circular permutations are considered to be same , is given by $\frac{(n-1)!}{2}$.

(6) Selection or Combinations :

The number of combinations of n different things taken r at a time is given by ^nC_r, where

$$^nC_r = \frac{n!}{r!(n-r)!} = \frac{^nP_r}{r!} \text{ where } r \le n \text{ ; } n \in N \text{ are } r \in W.$$

Note:

(i) ^nC_r is a natural number.
(ii) $^nC_0 = {}^nC_n = 1$, $^nC_1 = n$
(iii) $^nC_r = {}^nC_{n-r}$
(iv) $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
(v) $^nC_x = {}^nC_y \Leftrightarrow x = y$ or $x + y = n$
(vi) If n is even then the greatest value of ^nC_r is $^nC_{n/2}$
(vii) If n is odd then the greatest value of ^nC_r is $^nC_{(n+1)/2}$ or $^nC_{(n-1)/2}$
(viii) $^nC_r = \frac{n}{r}.{}^{n-1}C_{r-1}$
(ix) $\frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}$
(x) $^nC_r = \frac{r+1}{n+1}.{}^{n+1}C_{r+1}$
(xi) $^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n$

(7) Selection of one or more objects

(a) Number of ways in which atleast one object be selected out of 'n' distinct objects is
$^nC_1 + {}^nC_2 + {}^nC_3 + \dots\dots + {}^nC_n = 2^n-1$

(b) Number of ways in which atleast one object may be selected out of 'p' alike objects of one type 'q' alike objects of second type and 'r' alike of third type is
$(p + 1)(q + 1)(r + 1)-1$

(c) Number of ways in which atleast one object may be selected from 'n' objects where 'p' alike of one type 'q' alike of second type and 'r' alike of third type and rest $n-(p + q + r)$ are different , is
$(p + 1)(q + 1)(r + 1)\, 2^{n-(p+q+r)}-1$

(8) Formation of Groups :

Number of ways in which (m + n +p) different things can be divided into three different groups contiaining m, n & p things respectively is $\frac{(m+n+p)!}{m!n!p!}$,

If m = n = p and the groups have identical qualitative characteristic then the number of groups = $\frac{(3n)!}{n!n!n!3!}$.

However , if 3n things are to be divided equally among three people then the number of ways = $\frac{(3n)!}{(n!)^3}$.

8. Multinomial Theorem :

Coefficient of x^r in expansion of $(1-x)^{-n} = {}^{n+r-1}C_r$ $(n \in N)$

Number of ways in which it is possible to make a selection from m + n + p = N things, where p are alike of one kind , m alike of second kind & n alike of third kind taken r at a time is given by coefficient of x^r in the expansion of

$$(1 + x + x^2 + ... + x^p)\ (1 + x + x^2 + ... + x^m)\ (1 + x + x^2 + ... + x^n).$$

(i) For example the number of ways in which a selection of four letters can be made from the letters of the word **PROPORTION** is given by coefficient of x^4 in $(1 + x + x^2 + x^3)\ (1 + x + x^2)\ (1 + x + x^2)\ (1 + x)\ (1 + x)$.

(ii) **Method of fictious partition :**
Number of ways in which n identical things may be distributed among p persons if each person may receive one, one or more things is; ${}^{n+p-1}C_n$

9. Let $N = p^a\ q^b\ r^c$ ... where p, q, r... are distinct primes & a, b, c... are natural numbers then :

(a) The total numbers of divisors of N including 1 & N is = $(a+1)(b+1)(c+1)$.......

(b) The sum of these divisors is =
$(p^0 + p^1 + p^2 + ... + p^a)\ (q^0 + q^1 + q^2 + ... + q^b)\ (r^0 + r^1 + r^2 + ... + r^c)$....

(c) Number of ways in which N can be resolved as a product of two factors is
$1/2\ (a+1)(b+1)(c+1)$.... if N is not a perfect square
$1/2\ [(a+1)(b+1)(c+1)...+1]$ if N is a perfect square

(d) Number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1} where n is the number of different prime factors in N.

10. Let there be 'n' types of objects, with each type containing atleast r objects. Then the number of ways of arranging r objects in a row is n^r.

11. Dearrangement :

Number of ways in which 'n' letters can be put in 'n' corresponding envelopes such that no letter goes to correct envelope is

$$n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\text{.........}\ \text{.......}+(-1)^n\frac{1}{n!}\right).$$

Exponent of Prime p in n!

If p is a prime number and n is a positive integer , then $E_p(n)$ denotes the exponent of the prime p in the positive integer n.

$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \dots \left[\frac{n}{p^s}\right]$ where S is the largest naturla number. Such that $p^s \le n < p^{s+1}$.

Some Important Results for Geometical problems

(i) Number of total different straight lines formed by joining the n points on a plane of which m(< n) are collinear is ${}^nC_2 - {}^mC_2 + 1$.
(ii) Number of total triangle formed by joining the n points on a plane of which m (< n) are collinear is ${}^nC_3 - {}^mC_3$.
(iii) Number of diagonals in a polygon of n sides is ${}^nC_2 - n$.
(iv) If m parallel lines in a plane are intersected by a family of other n parallel line. Then total number of parallelograms so formed is ${}^mC_2 \times {}^nC_2$ i.e. $\frac{mn(m-1)(n-1)}{4}$
(v) Given n points on the circumference of a circle, then
(a) Number of straight lines = ${}^nC_2$ (b) Number of traingles = ${}^nC_3$
(c) Number of quadrilaterals = ${}^nC_4$.
(vi) If n straight lines are drawn in the plane such that no two lines are parallel and no three lines are concurrent. then the number of part into which these lines divide the plane is $= 1 + \sum n$.

(vii) Number of rectangle of any size in a square of $n \times n$ is $\sum_{r=1}^{n} r^3$ and number of squares of any size is $\sum_{r=1}^{n} r^2$

(viii) In a rectangle of $n \times p (n < p)$ number of rectangle of any size is $\frac{np}{4}(n+1)(p+1)$ and number of squares of any size is $\sum_{r=1}^{n} (n+1-r)(p+1-r)$.

6. PROBABILITY

Theory of probability measures the degree of certainty or uncertainty of an event, if an event is represented by 'E', then its probability is given by notation P(E), where $0 \le P(E) \le 1$.

(1) Basic terminology:

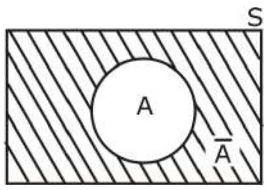
(i) Random Experiment :
It is a process which results in an outcome which is one of the various possible outcomes that are known to happen, for example : throwing of a dice is a random experiment as it results in one of the outcome from {1, 2, 3, 4, 5, 6}, similarly taking a card from a pack of 52 cards is also a random experiment.

(ii) Sample Space :
It is the set of all possible outcomes of a random experiment for example : {H, T} is the sample space associated with tossing of a coin.
In set notation it can be interpreted as the universal set.

(iii) Event :
It is subset of sample space. for example : getting a head in tossing a coin or getting a prime number is throwing a die. In general if a sample space consists 'n' elements, then a maximum of 2^n events can be associated with it.

Note:

- Each element of the sample space is termed as the sample point or an event point.
- The complement of an event 'A' with respect to a sample space S is the set of all elements of 'S' which are not in A . It is usually denoted by A' , $\overline{A}$ or A^c.
- $A \cup \overline{A} = S \quad \Rightarrow \quad P(A) + P(\overline{A}) = 1$



Complement of event

(iv) Simple Event or Elementary Event :
If an event covers only one point of sample space, then it is called a simple event, for example : getting a head followed by a tail in throwing of a coin 2 times is a simple event.

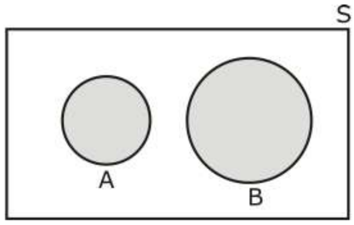
(v) Compound Event or mixed event (Composite Event) :
when two or more than two events occur simultaneously then event is said to be a compound event and it contains more than one element of sample space. for example : when a dice is thrown, the event of occurrence of an odd number is mixed event.

(vi) Equally likely Events :
If events have same chance of occurrence, then they are said to be equally likely events.
For example :

- In a single toss of a fair coin, the events {H} and {T} are equally likely.
- In a single throw of an unbiased dice the events {1}, {2} , {3}, {4}, {5} and {6} are equally likely.
- In tossing a biased coin the events {H} and {T} are not equally likely.

(vii) Mutually Exclusive / Disjoint / Incompatible Events :
Two events are said to be mutually exclusive if occurrence of one of them rejects the possibility of occurrence of the other (i.e. both cannot occur simultaneously).

Mathematically, $A \cap B = \phi \quad \Rightarrow \quad P(A \cap B) = 0$



A and B are Mutually Exclusive Events

For example:

- When a coin is tossed the event of occurrence of a head and the event of occurrence of a tail are mutually exclusive events.
- When a dice is thrown sample space S = (1, 2, 3, 4, 5, 6),

  Let A = the event of occurrence of a number greater than 4 = {5, 6}
  B = the event of occurrence of an odd number = {1, 3, 5}
  C = the event of occurrence of an even number = {2, 4, 6}

Now B and C are mutually exclusive events but A and B are not mutually exclusive because they can occur together (when the number 5 turns up).

(viii) Exhaustive System of Events :

If each outcome of an experiment is associated with at least one of the events E_1, E_2, E_3, ... E_n, then collectively the events are said to be exhaustive.

Mathematically, $E_1 \cup E_2 \cup E_3 .... E_n = S$.

For example.

In random experiment of rolling an unbiased dice, S = {1, 2, 3, 4, 5, 6}

Let E_1 = the event of occurence of prime number = {2, 3, 5}
E_2 = the event of occurence of an even number = {2, 4, 6}
E_3 = the event of occurence of number less than 3 = {1, 2}

Now, $E_1 \cup E_2 \cup E_3 = S$ and hence events E_1, E_2 and E_3 are mutually exhaustive events.

(2) Classical Definition of Probability :

If a random experiment results in a total of (m + n) outcomes which are equally likely and mutually exclusive and exhaustive and if 'm' outcomes are favourable to an event 'E' while 'n' are unfavourable, then the probability of occurrence of the event 'E', denoted by P(E), is given by :

$$P(E) = \frac{m}{m+n} = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}} \quad \Rightarrow \quad P(E) = \frac{n(E)}{n(S)}$$

Note:

(i) In above definition, odds in favour of 'E' are m : n, while odds against 'E' are n : m .

(ii) $P(\bar{E})$ or $P(E')$ or $P(E^c)$ denotes the probability of non-occurrence of E.

$$P(\bar{E}) = 1 - P(E) = \frac{n}{m+n}$$

(3) Notation of an event in set theory:

In dealing with problems of probability, it is important to convert the verbal description of an event into its equivalent set theoretic notation. Following table illustrates the verbal description and its corresponding notation in set theory.

Verbal description of Event	Notation is Set theory.
Both A and B occurs.	$A \cap B$
Atleast one of A or B occurs	$A \cup B$
A occurs but not B.	$A \cap \overline{B}$
Neither A nor B occurs.	$\overline{A} \cap \overline{B}$
Exactly one of A and B occurs	$(A \cap \overline{B}) \cup (\overline{A} \cap B)$
All three events occur simultaneously	$A \cap B \cap C$
Atleast one of the events occur.	$A \cup B \cup C$
Only A occurs and B and C don't occur.	$A \cap \overline{B} \cap \overline{C}$
Both A and B occurs but C don't occur.	$A \cap B \cap \overline{C}$
Exactly two of the event A, B, C occur.	$(A \cap B \cap \overline{C}) \cup (A \cap \overline{B} \cap C) \cup (\overline{A} \cap B \cap C)$
Atleast two of the events occur.	$(A \cap B) \cup (B \cap C) \cup (A \cap C)$
exactly one of the events occur.	$(A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C)$
None of the events A, B, C occur.	$\overline{A} \cap \overline{B} \cap \overline{C} = \overline{A \cup B \cup C}$.

Note:

(i) De Morgan's Law :

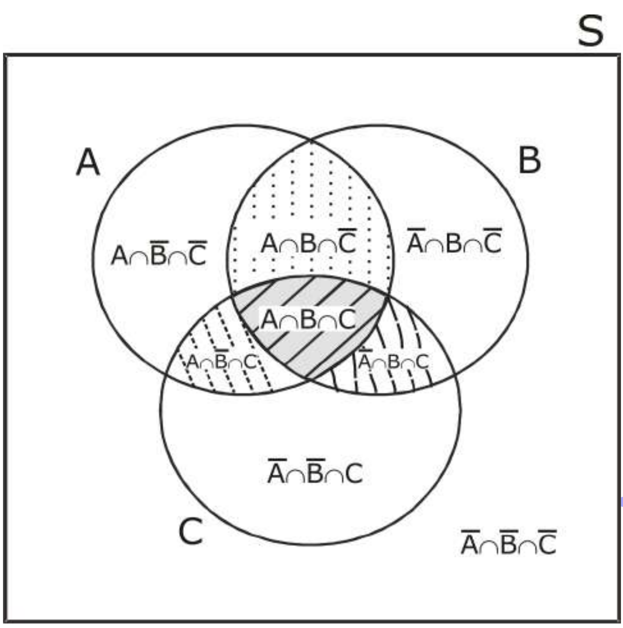
If A and B are two subsets of a universal set, then

$(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$.

(ii) Distributive Law :

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(iii) Venn diagram for events A , B , C :



(4) Addition theorem of probability :

If 'A' and 'B' are any two events associated with a random experiment, then

(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ or $P(A + B) = P(A) + P(B) - P(AB)$.

(ii) $P(\overline{A} \cap B) = P(B) - P(A \cap B)$

(iii) $P(A \cap \overline{B}) = P(A) - P(A \cap B)$

(iv) $P\left((A \cap \overline{B}) \cup (\overline{A} \cap B)\right) = P(A) + P(B) - 2P(A \cap B) = P(A \cup B) - P(A \cap B)$.

If A , B , C are three events associated with a random experiment, then

(v) $P(A \cup B \cup C) = P(A) + P(B) + P(c) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

(vi) P(at least two of A, B c occur) = $P(B \cap C) + P(C \cap A) + P(A \cap B) - 2P(A \cap B \cap C)$

(vii) P(exactly two of A, B, c occur) = $P(B \cap C) + P(C \cap A) + P(A \cap B) - 3P(A \cap B \cap C)$

(viii) P(exactly one of A, B , C occur) =

$$P(A) + P(B) + P(C) - 2P(B \cap C) - 2P(C \cap A) - 2P(A \cap B) + 3P(A \cap B \cap C)$$

(5) Independent and dependent events

If two events are such that occurence or non-occurence of one does not effect the chances of occurence or non-occurence of the other event, then the events are said to be independent. Mathematically : if $P(A \cap B) = P(A).P(B)$ then A and B are independent.

For example

- When a coin is tossed twice, the event of occurrence of head in the first throw and the event of occurrence of head in the second throw are independent events.
- When two cards are drawn out of a full pack of 52 playing cards with replacement (the first card drawn is put back in the pack and the second card is drawn), then the event of occurrence of a king in the first draw and the event of occurrence of a king in the second draw are independent events because the probability of drawing a king in the second draw is $\frac{4}{52}$ whether a king is drawn in the first draw or not. But if the two cards are drawn without replacement then the two events are not independent.

Note:

(i) If A and B are independent , then A' and B' are independent , A and B' are independent and A' and B are independent.

(ii) Three events A,B and C are independent if and only if all the three events are pairwise independent as well as mutually independent.

(iii) **Complementation Rule :**

If A and B are independent events, then

$$\begin{aligned} P(A \cup B) &= 1 - P(\overline{A \cup B}) \\ &= 1 - P(\overline{A} \cap \overline{B}) \\ &= 1 - P(\overline{A}).P(\overline{B}) \end{aligned}$$

Similarly, if A, B and C are three independent events, then

$P(A \cup B \cup C) = 1 - P(\overline{A}).P(\overline{B}).P(\overline{C})$.

(6) Conditional Probability

Let A and B be any two events, $B \neq \phi$, then P(A/B) denotes the conditional probability of occurrence of event A when B has already occurred.

For example :
When a die is thrown, sample space S = {1, 2, 3, 4, 5, 6}

Let A = the event of occurrence of a number greater than 4.
= {5, 6}
B = the event of occurrence of an odd number.
= {1, 3, 5}

Then P(A/B) = probability of occurrence of a number greater than 4, when an odd number has occurred

Note:

(i) If $B \neq \phi$, then $P\left(\frac{A}{B}\right) + P\left(\frac{\overline{A}}{B}\right) = 1$ and $P\left(\frac{A}{\overline{B}}\right) + P\left(\frac{\overline{A}}{\overline{B}}\right) = 1$.

(ii) If $A \neq \phi$, and B is dependent on A, then

$$P(B) = P(A).P\left(\frac{B}{A}\right) + P(\overline{A}).P\left(\frac{B}{\overline{A}}\right).$$

(7) Multiplication Theorem of Probability :

If A and B are any two events, then

$P(A \cap B) = P(B).\ P\left(\frac{A}{B}\right)$, where $B \neq \phi$ and $P\left(\frac{A}{B}\right)$ denotes the probability of occurrence of event A when B has already occurred.

If A and B are independent events, then probability of occurrence of event A is not affected by occurrence or nor occurrence of event B, therefore

$$P\left(\frac{A}{B}\right) = P(A)$$

(8) Total Probability Theorem

If an event A can occur with one of the n mutually exclusive and exhaustive events $B_1, B_2, \ldots, B_n$ and the probabilities $P(A/B_1), P(A/B_2), \ldots\ldots\ldots, P(A/B_n)$ are known , then

$$P(A) = \sum_{i=1}^{n} P(B_i).P(A/B_i).$$

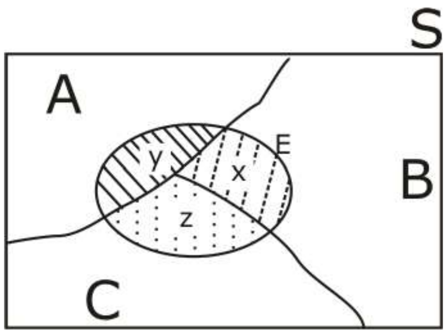
(9) Bayes' Theorem (Inverse Probability) :

Let B_1 , B_2 , B_3 , ... , B_n is a set of n mutually exclusive and exhaustive events and event A can occur with any of the event B_1 , B_2 , ... , B_n . If event 'A' has occured, then probability that event 'A' had occured with event B_i is given by $P\left(\frac{B_i}{A}\right)$, where

$$P\left(\frac{B_i}{A}\right) = \frac{P(B_i).P\left(\frac{A}{B_i}\right)}{\sum_{i=1}^{n} P(B_i).P\left(\frac{A}{B_i}\right)}$$

Particular Case :

Let A , B , C be three mutually exclusive and exhaustive events and event 'E' can occur with any of the events A , B and C as shown in diagram:



Now, if event 'E' had occurred, then

$$P\left(\frac{A}{E}\right)=\left(\frac{y}{x+y+z}\right)$$

$$\Rightarrow \qquad P\left(\frac{A}{E}\right)=\frac{P(A).P\left(\frac{E}{A}\right)}{P(A)P\left(\frac{E}{A}\right)+P(B)P\left(\frac{E}{B}\right)+P(c)P\left(\frac{E}{C}\right)}$$

(10) Binomial Probability Distribution :

Let a random experiment is conducted for n trials and in each trial an event 'E' is defined for which the probability of its occurence is 'p' and probability of non-occurence is 'q', where p+q = 1. Now if in n trials, event 'E' occurs for r times, then the probability of occurence of event 'E' for r times is given by:

$P(r)={}^nC_r(p)^r(q)^{n-r}$.

Note:

(i) $(q+p)^n={}^nC_0\,q^n+{}^nC_1(p)(q)^{n-1}+{}^nC_2(p)^2(q)^{n-2}+\ldots\ldots..{}^nC_n(p)^n$

$\Rightarrow \quad p(0)+p(1)+p(2)+\ldots\ldots..p(n)=1$

(ii) For Binomial Probability distribution, it mean and variance is given by np and npq respectively

(10) Geometrical Applications :

If the number of sample points in sample space is infinite, then classical definition of probability can't be applied. For uncountable uniform sample space, probability P is given by:

$$P=\frac{\text{favourable dimension}}{\text{total dimension}}$$

For example:

- If a point is taken at random on a given straight line segment AB, the chance that it falls on a particular segment PQ of the line segment is PQ/AB.

- If a point is taken at random on the area S which includes an area Δ, the chance that the point falls on Δ is Δ/S.

7. MATRICES

Any rectangular arrangement of numbers (real or complex) is called a **matrix**. If a matrix has 'm' rows and 'n' columns then the **order** of matrix is denoted by $m \times n$.

If $A = [a_{ij}]_{m\times n}$, where a_{ij} denotes the element of i^{th} row and j^{th} column, then it represents the following matrix :

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots\cdots & a_{ij} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots\cdots & a_{2j} & \cdots\cdots & a_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots\cdots & a_{ij} & \cdots\cdots & a_{in} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots\cdots & a_{mj} & \cdots\cdots & a_{mn} \end{bmatrix}$$

(1) Basic Definitions :

(i) Row matrix : A matrix having only one row is called as row matrix.
General form of row matrix is $A = [a_{11}\ \ a_{12}\ \ a_{13}\ \ \ldots\ldots\ \ a_{1n}]$

(ii) Column matrix : A matrix having only one column is called as column matrix.

General form of column matrix is $A = \begin{bmatrix} a_{11} \\ a_{21} \\ \cdots\cdots \\ a_{m1} \end{bmatrix}$

(iii) Singleton matrix : If in a matrix there is only one element then it is called singleton matrix. $A = [a_{ij}]_{m\times n}$ is a singleton matrix if $m = n = 1$. For example : [2] , [3] , [a] , [–3] are singleton matrices.

(iv) Square matrix : A matrix in which number of rows and columns are equal is called a square matrix. General form of a square matrix is $A = [a_{ij}]_n$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{bmatrix}$$

Note:

(i) In a square matrix , the elements of the form a_{ii} is termed as diagonal elements, where $i = 1, 2, 3, \ldots\ldots, n$ and line joining these elements is called the principal diagonal or leading diagonal or main diagonal.

(ii) Summation of the diagonal elements of a square matrix is termed as its trace.

$$\text{If } A = [a_{ij}]_{n\times n}, \text{ then } tr(A) = \sum_{i=1}^{n} a_{ii}$$

$$\Rightarrow \quad tr(A) = a_{11} + a_{22} + a_{33} + \dots\dots\dots + a_{nn}$$

(iii) If $A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$ and λ be a scalar, then :

- $tr(\lambda A) = \lambda tr(A)$
- $tr(A \pm B) = tr(A) \pm tr(B)$
- $tr(AB) = tr(BA)$
- $tr(A) = tr(A')$ or $tr(A^T)$
- $tr(I_n) = n$
- $tr(O) = 0$
- $tr(AB) \neq tr\,A.\,tr\,B$

(v) Rectangular Matrix : A matrix in which number of rows in not equal to number of columns is termed as rectangular matrix. $A = [a_{ij}]_{m\times n}$ is rectangular if $m \neq n$.
If $m > n$, matrix is termed as vertical matrix and if $m < n$, matrix is termed as horizontal matrix.

(vi) Zero matrix : $A = [a_{ij}]_{m\times n}$ is called a zero matrix if $a_{ij} = 0 \ \forall$ i and j.

For example : $O_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; $O_{3\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

(vii) Diagonal matrix : A square matrix $[a_{ij}]_n$ is said to be a diagonal matrix if $a_{ij} = 0$ for $i \neq j$. (i.e. all the elements of the square matrix other than diagonal elements are zero).
Diagonal matrix of order n is denoted as diag $(a_{11}, a_{22}, \dots\dots, a_{nn})$.

(viii) Scalar matrix : Scalar matrix is a diagonal matrix in which all the diagonal elements are same. $A = [a_{ij}]_n$ is a scalar matrix if $a_{ij} = 0$ for $i \neq j$ and $a_{ij} = k$ for $i = j$.

(ix) Unit matrix (Identity matrix) : Unit matrix is a diagonal matrix in which all the diagonal elements are unity. Unit matrix of order 'n' is denoted by I_n (or I). $A = [a_{ij}]_n$ is a unit matrix if $a_{ij} = 0$ for $i \neq j$ and $a_{ii} = 1$ for $i = j$

For example : $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(x) Triangular Matrix : A square matrix $[a_{ij}]_n$ is said to be triangular matrix if each element above or below the principal diagonal is zero. It is of two types :

- **Upper triangular matrix :**
  $A = [a_{ij}]_n$ is said to be upper triangular if $a_{ij} = 0$ for $i > j$ (i.e. all the elements below the diagonal elements are zero).

- **Lower triangular matrix :**
  $A = [a_{ij}]_n$ is said to be lower triangular matrix if $a_{ij} = 0$ for $i < j$. (i.e. all the elements above the diagonal elements are zero.)

Note:

- Minimum numbers of zero in a triangular matrix is given by $\frac{n(n-1)}{2}$, where n is order of triangular matrix.
- Diagonal matrix is both upper and lower triangular.

(xi) Singular and Non-singular matrix : Any square matrix A is said to be non-singular if $|A| \neq 0$, and a square matrix A is said to be singular if $|A| = 0$. det (A) or |A| represents determinant of square matrix A.

(xii) Comparable matrices : Two matrices A and B are said to be comparable if they have the same order.

(xiii) Equality of matrices : Two matrices A and B are said to be equal if they are compa rable and all the corresponding elements are equal.

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{p \times q}$, then

$A = B \Rightarrow m = p,\ n = q$ and $a_{ij} = b_{ij}\ \forall$ i and j.

(2) Scalar Multiplication of matrix :

let λ be a scalar (real or complex number) and $A = [a_{ij}]_{m \times n}$ be a matrix , then λA is defined as $[\lambda a_{ij}]_{m \times n}$

Note:

If A and B are matrices of the same order and λ, μ are any two scalars , then

(i) $\lambda(A+B) = \lambda A + \lambda B$ (ii) $(\lambda + \mu)A = \lambda A + \mu A$

(iii) $\lambda(\mu A) = (\lambda\mu)A = \mu(\lambda A)$ (iv) $(-\lambda A) = -(\lambda A) = \lambda(-A)$

(3) Addition of matrices :

Let A and B be two matrices of same order (i.e. comparable matrices) , then A + B is defined as:

$$A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n}$$

$$= [c_{ij}]_{m \times n}, \text{ where } c_{ij} = a_{ij} + b_{ij}\ \forall \text{ i and j.}$$

Note:

If A , B and C are matrices of same order , then

- $A + B = B + A$ (Commutative law)
- $(A + B) + C = A + (B + C)$ (Associative law)
- $A + O = O + A = A$, where O is zero matrix which is additive identity of the matrix.
- $A + (-A) = 0 = (-A) + A$, where (–A) is obtained by changing the sign of every element of A, which is additive inverse of the matrix.
- $\left.\begin{array}{l} A + B = A + C \\ B + A = C + A \end{array}\right\} \Rightarrow B = C$ (Cancellation law)

(4) Multiplication of Matrices :

Two matrices A and B are conformable for the product AB if the number of columns in A (per-multiplier) is same as the number of rows in B (post multiplier).

Thus, if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$ are two matrices of order $m \times n$ and $n \times p$ respectively,

then their product AB is of order $m \times p$ and is defined as $AB = [c_{ij}]_{m\times p}$, where $c_{ij} = \sum_{r=1}^{n} a_{ir}b_{rj}$

For example : $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}_{2\times 2} \times \begin{bmatrix} 1 & 0 & 2 \\ -2 & 1 & 1 \end{bmatrix}_{2\times 3} = \begin{bmatrix} 1-4 & 0+2 & 2+2 \\ -1-6 & 0+3 & -2+3 \end{bmatrix}_{2\times 3} = \begin{bmatrix} -3 & 2 & 4 \\ -7 & 3 & 1 \end{bmatrix}_{2\times 3}$

Note:

(i) If A, B and C are three matrices such that their product is defined, then

- $AB \neq BA$ (Generally not commutative)
- $(AB)C = A(BC)$ (Associative Law)
- $IA = A = AI$ (I is identity matrix for matrix multiplication)

- $A(B+C) = AB + AC$ (Distributive law)
- If $AB = AC \not\Rightarrow B = C$ (Cancellation law is not always applicable)
- If $AB = 0$ (It does not imply that $A = 0$ or $B = 0$, product of two non zero matrix may be a zero matrix.)

(ii) If A and B are two matrices of the same order, then

- $(A + B)^2 = A^2 + B^2 + AB + BA$
- $(A - B)^2 = A^2 + B^2 - AB - BA$
- $(A - B)(A + B) = A^2 - B^2 + AB - BA$
- $(A + B)(A - B) = A^2 - B^2 - AB + BA$
- $A(-B) = (-A)B = -(AB)$

(iii) The positive integral powers of a matrix A are defined only when A is a square matrix $A^2 = A.A$, $A^3 = A.A.A = A^2A$. For any positive integers m, n

- $A^mA^n = A^{m+n}$
- $(A^m)^n = A^{mn} = (A^n)^m$
- $I^n = I$, $I^m = I$
- $A^0 = I_n$ where A is a square matrix of order n.

(5) Transpose of a Matrix :

Let $A = [a_{ij}]_{m\times n}$, then the transpose of A is denoted by A' (or A^T) and is defined as $A^T = [b_{ij}]_{n\times m}$, where $b_{ij} = a_{ji}$ $\forall$ i and j.

For example : Transpose of matrix $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}_{2\times 3}$ is $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}_{3\times 2}$

Note:

Let A and B be two matrices then

- $(A^T)^T = A$

- $(A \pm B)^T = A^T \pm B^T$, A and B being comparable.
- $(kA)^T = kA^T$, k be any scalar (real or complex)
- $(AB)^T = B^T A^T$, A and B being conformable for the product AB
- $(A_1A_2A_3 .... A_{n-1} A_n)^T = A_n^T A_{n-1}^T .... A_3^T A_2^T A_1^T$ (Reversal law)
- $I^T = I$
- $(A^n)^T = (A^T)^n$
- $(A^{-1})^T = (A^T)^{-1}$
- $|A^T| = |A|$

(6) Symmetric and Skew-symmetric Matrix :

A square matrix $A = [a_{ij}]$ is called **symmetric matrix** if $a_{ij} = a_{ji}$ $\forall$ i, j (i.e. $A^T = A$)

For example: $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix.

A square matrix $A = [a_{ij}]$ is called skew-symmetric matrix if $a_{ij} = -a_{ji}$ $\forall$ i, j (i.e. $A^T = -A$)

For example: $B = \begin{bmatrix} 0 & x & y \\ -x & 0 & -z \\ -y & z & 0 \end{bmatrix}$ is a skew symmetric matrix.

Note:

- Every unit matrix, scalar matrix and square zero matrix are symmetric matrices.
- Maximum number of different elements in a symmetric matrix of order 'n' is $\frac{n(n+1)}{2}$.
- All principal diagonal elements of a skew-symmetric matrix are always zero because for any diagonal element. $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$
- Trace of a skew symmetric matrix is always 0.
- If A is skew symmetric of odd order, then det(A) = 0.

(7) Properties of Symmetric and skew-symmetric Matrices:

(i) If A is a square matrix, then $A + A^T$, AA^T, $A^T A$ are symmetric matrices, while $A - A^T$ is skew-symmetric matrix.

(ii) If A is a symmetric matrix, then $-A$, KA, A^T, A^n, A^{-1}, $B^T AB$ are also symmetric matrices, where $n \in N$, $K \in R$ and B is a square matrix of order as that of A.

(iii) If A is a skew-symmetric matrix, then

- A^{2n} is a symmetric matrix for $n \in N$
- A^{2n+1} is a skew-symmetric matrix for $n \in N$
- kA is skew-symmetric matrix, where $K \in R$
- B^TAB is skew-symmetric matrix, where B is a square matrix of order as that of A.

(iv) If A and B are two symmetric matrices, then

- $A \pm B$, $AB + BA$ are also symmetric matrix

- AB–BA is a skew-symmetric matrix
- AB is symmetric matrix when AB = BA.

(v) If A and B are two skew-symmetric matrices , then

- $A \pm B$, $AB - BA$ are skew-symmetric matrices
- AB + BA is a symmetric matrix.

(vi) If A a skew-symmetric matrix and C is a column matrix, then $C^T AC$ is a zero matrix.

(vii) Every square matrix A can uniquely be expressed as sum of symmetric and

skew-symmetric matrix $\left\{\text{i.e.}\quad A = \left[\frac{1}{2}(A + A^T)\right] + \left[\frac{1}{2}(A - A^T)\right]\right\}$

(8) Submatrix, Minors, Cofactors :

Submatrix : In a given matrix A, the matrix obtained by deleting some rows or columns (or both) of A is called as submatrix of A.

For example : If $A = \begin{bmatrix} a & b & c & d \\ p & q & r & s \\ x & y & z & w \end{bmatrix}$, then

$\begin{bmatrix} a & b \\ p & q \\ x & y \end{bmatrix}, \begin{bmatrix} a & b & c \\ p & q & r \end{bmatrix}, \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$ are all submatrices of A.

Minors and Cofactors : If $A = [a_{ij}]_n$ is a square matrix , then minor of element a_{ij}, denoted by M_{ij} , is defined as the determinant of the submatrix obtained by deleting i^{th} row and j^{th} column of A.

Cofactor of element a_{ij}, denoted by C_{ij} , is defined as $C_{ij} = (-1)^{i+j}.M_{ij}$

For example: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$M_{11} = d = C_{11}$
$M_{12} = c$, $C_{12} = -c$
$M_{21} = b$, $C_{21} = -b$
$M_{22} = a = C_{22}$

If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, then

$M_{11} = ei-hf$, $M_{32} = af-dc$
and $C_{22} = ai-cg$, $C_{31} = bf-ec$, $C_{23} = bg-ah$, etc.

(9) Determinant of A matrix :

If $A = [a_{ij}]_n$ is a square matrix of order $n > 1$, then determinant of A is defined as the summation of products of elements of any one row (or any one column) with the corresponding cofactors.

For example: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$|A| = a_{11}C_{11} + a_{12}\, C_{12} \; + a_{13}\, C_{13}\,.$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{33} \end{vmatrix} \text{, or}$$

$$|A| = a_{12}\, C_{12} + a_{22}\, C_{22} + a_{32}\, C_{32}\,.$$

$$= -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

Properties of determinant :

- If $A = [a_{ij}]_n$, then the summation of the products of elements of any row with corresponding cofactors of any other row is zero. (Similarly , the summation of the products of elements of any column with corresponding cofactors of any other column is zero).

For example: $a_{11}\, C_{21} + a_{12}\, C_{22} + a_{13}\, C_{23} = 0$, and $a_{12}\, C_{11} + a_{22}\, C_{21} + a_{32}\, C_{31} = 0.$

- $|A| = |A^T|$ for any square matrix A.

- $|AB| = |A||B| = |BA|$
- If λ be a scalar , then $\lambda\,|A|$ is obtained by multiplying any one row (or any one column) of $|A|$ by λ

$\mathbf{|\,\lambda A\,| = \lambda^n\,|\,A\,|}$, where $A = [a_{ij}]_n$

- If A is a skew symmetric matrix of odd order then $|A| = 0$

- If $A = \text{diag}\,(a_1 , a_2 , \ldots\ldots a_n)$ then $|A| = a_1a_2\ldots a_n$
- $|\,A\,|^n = |\,A^n\,|$, $n \in N$ and $|\,A^{-1}\,| = \dfrac{1}{|\,A\,|}$

- If two rows are identical (or two columns are identical) then $|A| = 0$.

- If any two rows (or columns) of a determinant be interchanged , the determinant is unaltered in numerical value but is changed in sign only.

- If each element of any row (or column) can be expressed as a sum of two terms , then the determinant can be expressed as the sum of two determinants.

- The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column).

(10) Cofactor matrix and adjoint matrix :

If $A = [a_{ij}]_n$ is a square matrix , then matrix obtained by replacing each element of A by corresponding cofactor is called as cofactor matrix of A, denoted as cofactor A.

The transpose of cofactor matrix of A is called as adjoint of A, denoted by **adj A**.

If $A = [a_{ij}]_n$, then cofactor $A = [C_{ij}]_n$ where C_{ij} is the cofactor of a_{ij} $\forall$ i and j.

$\Rightarrow$ $\mathbf{adj(A) = [C_{ji}]_n}$

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{ then } \mathbf{adj\ A} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix};$$

Where C_{ij} denotes the cofactor of a_{ij} in A.

Note:

If A and B are square matrices of order n and I_n is unit matrix, then

- $A(\text{adj } A) = |A|\, I_n = (\text{adj } A)A$
- $|\text{adj } A| = |A|^{n-1}$
- $\text{adj}(\text{adj } A) = |A|^{n-2} A$
- $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$
- $\text{adj}(A^T) = (\text{adj } A)^T$
- $\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$
- $\text{adj}(A^m) = (\text{adj } A)^m$, $m \in N$
- $\text{adj}(kA) = k^{n-1}(\text{adj } A)$, $k \in R$
- $\text{adj}(I_n) = I_n$
- $\text{adj}(O) = O$
- A is symmetric $\Rightarrow$ adj A is also symmetric.
- A is diagonal $\Rightarrow$ adj A is also diagonal
- A is triangular $\Rightarrow$ adj A is also triangular.
- A is singular $\Rightarrow$ $|\text{adj } A| = 0$

(11) Inverse of a matrix (reciprocal matrix) :

A non-singular square matrix of order n is invertible if there exists a square matrix B of the same order such that $AB = I_n = BA$.

In such a case, we say that the inverse of A is B and we write $A^{-1} = B$

$$\therefore \quad AA^{-1} = A^{-1}A = I.$$

The inverse of A is given by $A^{-1} = \frac{1}{|A|}.\text{adj } A$

The necessary and sufficient condition for the existence of the inverse of a square matrix A is that $|A| \neq 0$

Note:

If A and B are invertible matrices of the same order , then

- $(A^{-1})^{-1} = A$
- $(A^T)^{-1} = (A^{-1})^T$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^k)^{-1} = (A^{-1})^k$, $k \in N$
- $\operatorname{adj}(A^{-1}) = (\operatorname{adj} A)^{-1}$
- $|A^{-1}| = \dfrac{1}{|A|} = |A|^{-1}$
- $A = \operatorname{diag}(a_1 a_2 \dots a_n) \Rightarrow A^{-1} = \operatorname{diag}(a_1^{-1} a_2^{-1} \dots a_n^{-1})$

- A is symmetric $\Rightarrow A^{-1}$ is also symmetric.
- A is diagonal, $|A| \neq 0 \Rightarrow A^{-1}$ is also diagonal.
- A is scalar matrix $\Rightarrow A^{-1}$ is also scalar matrix.
- A is triangular, $|A| \neq 0 \Rightarrow A^{-1}$ is also triangular.

- Every invertible matrix possesses a unique inverse.

- If A is a non-singular matrix, then $AB = AC \Rightarrow B = C$ and $BA = CA \Rightarrow B = C$.

- In general $AB = O$ does not imply $A = O$ or $B = O$. But if A is non singular and $AB = O$, then $B=O$. Similarly B is non singular and $AB = O \Rightarrow A = O$. Therefore, $AB = O \Rightarrow$ either both are singular or one of them is O.

(12) Important Definitions on Matrices :

(i) Orthogonal matrix :

A square matrix A is said to be an orthogonal matrix if, $AA^T = I = A^TA$.

For example : $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

(ii) Involutory matrix :

A square matrix A is said to be involutory if $A^2 = I$, I being the identity matrix.

For example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an involutory matrix.

(iii) Idempotent matrix :

A square matrix A is said to be idempotent if $A^2 = A$.

For example: $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ is an idempotent matrix.

(iv) Nilpotent matrix :

A square matrix is said to be nilpotent of index p, if p is the least positive integer such that $A^p = O$.

For example : $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is nilpotent matrix of index 2.

(v) Periodic matrix :

A matrix A will be called a periodic matrix if $A^{k+1} = A$, where k is a positive integer. If however k is the least positive integer for which $A^{k+1} = A$, then k is said to be the period of A.

(vi) Differentiation of a matrix:

If $A = \begin{bmatrix} f(x) & g(x) \\ h(x) & l(x) \end{bmatrix}$ then $\frac{dA}{dx} = \begin{bmatrix} f'(x) & g'(x) \\ h'(x) & l'(x) \end{bmatrix}$ is the differentiation of matrix A.

For example: If $A = \begin{bmatrix} x^2 & \sin x \\ 2x & 2 \end{bmatrix}$ then $\frac{dA}{dx} = \begin{bmatrix} 2x & \cos x \\ 2 & 0 \end{bmatrix}$

(vii) Conjugate of a matrix :

If matrix $A = [a_{ij}]_{m\times n}$, then the conjugate of matrix A is given by $\overline{A}$, where $\overline{A} = [\overline{a}_{ij}]_{m\times n}$

For example : If $A = \begin{bmatrix} 2+3i & i \\ 4 & 3-i \end{bmatrix}$; then $\overline{A} = \begin{bmatrix} 2-3i & -i \\ 4 & 3+i \end{bmatrix}$

(viii) Hermitian and skew-Hermitian matrix:

A square matrix $A = [a_{ij}]$ is said to be Hermitian matrix if $a_{ij} = \overline{a}_{ji}\ \forall\ i, j \Rightarrow A = A^{\theta}$

For example: $\begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$

A square matrix, $A = [a_{ij}]$ is said to be a skew-Hermitian if $a_{ij} = -\overline{a}_{ji}\ \forall\ i, j \Rightarrow A = -A^{\theta}$

For example: $\begin{bmatrix} 0 & -2+i \\ 2-i & 0 \end{bmatrix}$

(ix) Unitary matrix :

A square matrix A is said to be unitary if $AA^{\theta} = I$, where A^{θ} is the transpose of complex conjugate of A.

(13) System of Linear Equations and Matrices

Consider the system

$a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n = b_1$
$a_{21}x_1 + a_{22}x_2 + \dots\dots\dots + a_{2n}x_n = b_2$
....
$a_{m1}x_1 + a_{m2}x_2 + \dots\dots\dots + a_{mn}x_n = b_n$

Let $A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots. & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots. & a_{2n} \\ \ldots. & \ldots. & \ldots\ldots. & \ldots. \\ a_{m1} & a_{m2} & \ldots\ldots. & a_{mn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \ldots. \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ \ldots \\ b_n \end{bmatrix}$

Then the above system can be expressed in the matrix form as AX = B.

The system is said to be consistent if it has atleast one solution.

(i) **System of linear equations and matrix inverse :**

If the above system consist of n equations in n unknowns, then we have AX = B where A is a square matrix. If A is non-singular, solution is given by $X = A^{-1}B$.

If A is singular, (adj A) B = 0 and all the columns of A are not proportional, then the system has infinite many solution.

If A is singular and (adj A) B $\neq$ 0, then the system has no solution (we say it is inconsistent).

(ii) **Homogeneous system and matrix inverse :**

If the above system is homogeneous, n equations in n unknowns, then in the matrix for m it is AX = O. ($\because$ in this case $b_1 = b_2 = \ldots.. b_n = 0$) , where A is a square matrix.

If A is non-singular, the system has only the trivial solution (zero solution) X = 0.

If A is singular, then the system has infinitely many solutions (including the trivial solution) and hence it has non trivial solutions.

(iii) **Rank of a matrix :**

Let $A = [a_{ij}]_{m\times n}$. A natural number ρ is said to be the rank of A if A has a non-singular submatrix of order ρ and it has no nonsingular submatrix of order more than ρ. Rank of zero matrix is regarded to be zero.

e.g. $A = \begin{bmatrix} 3 & -1 & 2 & 5 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix}$

we have $\begin{bmatrix} 3 & 2 \\ 0 & 2 \end{bmatrix}$ as a non singular submatrix.

The square matrices of order 3 are

$$\begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}, \begin{bmatrix} 3 & -1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 & 5 \\ 0 & 2 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} -1 & 2 & 5 \\ 0 & 2 & 0 \\ 0 & 5 & 0 \end{bmatrix}$$

and all these are singular. Hence **rank of A is 2**.

(iv) Elementary row transformation of matrix :

The following operations on a matrix are called as elementary row transformations.

(a) Interchanging two rows.

(b) Multiplications of all the elements of row by a nonzero scalar.

(b) Addition of constant multiple of a row to another row.

Note : Similar to above we have elementary column transformations also.

Remark :

1. Elementary transformation on a matrix does not affect its rank.

2. Two matrices A & B are said to be equivalent if one is obtained from other using elementary transformations. We write $A \approx B$.

(v) **Echelon form of a matrix :** A matrix is said to be in Echelon form if it satisfy the following :

(a) The first non-zero element in each row is 1 & all the other elements in the corresponding column (i.e. the column where 1 appears) are zeroes.

(b) The number of zeroes before the first non zero element in any non zero row is less than the number of such zeroes in succeeding non zero rows.

Result : Rank of a matrix in Echelon form is the number of non zero rows (i.e. number of rows with atleast one non zero element).

Remark :

1. To find the rank of a given matrix we may reduce it to Echelon form using elementary row transformations and then count the number of non zero rows.

(vi) System of linear equations & rank of matrix :

Let the system be $AX = B$ where A is an $m \times n$ matrix, X is the n-column vector & B is the m-column vector. Let [AB] denote the augmented matrix (i.e. matrix obtained by accepting elements of B as $(n + 1)^{th}$ column & first n columns are that of A).

ρ (A) denote rank of A and ρ ([AB]) denote rank of the augmented matrix.

Clearly $\rho(A) \leq \rho([AB])$

(a) If $\rho(A) < \rho([AB])$ then the system has no solution (i.e. system is inconsistent).

(b) If $\rho(A) = \rho([AB])$ = number of unknowns, then the system has unique solution. (and hence is consistent)

(c) If $\rho(A) = \rho([AB])$ < number of unknowns, then the systems has infinitely many solutions (and so is consistent).

(vii) Homogeneous system & rank of matrix :

let the homogenous system be $AX = 0$, m equations in 'n' unknowns. In this case $B = 0$ and so $\rho(A) = \rho([AB])$.

Hence if $\rho(A) = n$, then the system has only the trivial solution. If $\rho(A) < n$, then the system has infinitely many solutions.

Cayley-Hemilton Theorem :

Every matrix satisfies its characteristic equation e.g. let A be a square matrix then $|A-xI|=0$ is characteristics equation of A. If $x^3 - 4x^2 - 5x - 7 = 0$ is the characteristic equation for A, then $A^3 - 4A^2 + 5A - 7I = 0$

Roots of characteristic equation for A are called Eigen values of A or characteristic roots of A or latent roots of A. If λ is characteristic root of A, then λ^{-1} is characteristic root of A^{-1}.

8. DETERMINANTS

Consider the equations $a_1x + b_1y = 0$ and $a_2x + b_2y = 0$

$$\Rightarrow \quad \frac{y}{x} = -\frac{a_1}{b_1} \text{ and } \frac{y}{x} = -\frac{a_2}{b_2}$$

$$\Rightarrow \quad \frac{a_1}{b_1} = \frac{a_2}{b_2}$$

$$\therefore \quad a_1b_2 - a_2b_1 = 0$$

Now the obtained eliminant $a_1b_2 - a_2b_1$ is expressed as $\begin{vmatrix} a_1 & b_1 \\ b_2 & b_2 \end{vmatrix}$, which is called the determinant of order two.

$$\therefore \quad \begin{vmatrix} a_1 & b_1 \\ b_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

(1) Minors and Cofactors :

If $A = [a_{ij}]_n$ is a square matrix , then minor of element a_{ij}, denoted by M_{ij} , is defined as the determinant of the submatrix obtained by deleting i^{th} row and j^{th} column of A.

Cofactor of element a_{ij} , denoted by C_{ij} , is defined as $C_{ij} = (-1)^{i+j}.M_{ij}$

For example: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$M_{11} = d = C_{11}$
$M_{12} = c$, $C_{12} = -c$
$M_{21} = b$, $C_{21} = -b$
$M_{22} = a = C_{22}$

If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, then

$M_{11} = ei-hf$, $M_{32} = af-dc$
and $C_{22} = ai-cg$, $C_{31} = bf-ec$, $C_{23} = bg-ah$, etc.

(2) Determinant of A matrix :

If $A = [a_{ij}]_n$ be a square matrix of order $n > 1$, then determinant of A is defined as the summation of products of elements of any one row (or any one column) with the corresponding cofactors.

For example: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} .$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{33} \end{vmatrix} \text{ , or}$$

$$|A| = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$$

$$= -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

Note:

If $A = [a_{ij}]_n$, then the summation of the products of elements of any row with corresponding cofactors of any other row is zero. (Similarly , the summation of the products of elements of any column with corresponding cofactors of any other column is zero).

For example: $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = 0$, and

$a_{12}C_{11} + a_{22}C_{21} + a_{32}C_{31} = 0$.

(3) Properties of Determinants :

(i) The value of a determinant remains unaltered, if the rows and columns are interchanged.

i.e. $$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_1 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \Rightarrow |A| = |A^T|$$

(ii) If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only. For example:

$$\text{If } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},\ R_1 \leftrightarrow R_2 \Rightarrow D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}, \text{ then } D' = -D$$

(iii) If a determinant has all the elements zero in any row or column then its value is zero. For example:

$$D = \begin{vmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

(iv) If a determinant has any two rows(or columns) identical, then its value is zero.

For example: $$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0. \quad (R_1 \equiv R_2)$$

(v) If all the elements of any row (or column) is multiplied by the same number, then the determinant is multiplied by that number. For example:

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } D' = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \text{ then } D' = KD$$

Note:

If λ be a scalar , then $\lambda|A|$ is obtained by multiplying any one row(or any one column) of $|A|$ by λ.

$$|\lambda A| = \lambda^n |A|, \text{ where } A = [a_{ij}]_n$$

(vi) If each element of any row (or column) can be expressed as a sum of two terms then the determinant can be expressed as the sum of two determinants.
For example:

$$\begin{vmatrix} a_1+x & b_1+x & c_1+x \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(vii) The value of a determinant is not altered by adding to the elements of any row (or column) a constant multiple of the corresponding elements of any other row (or column). For example:

$$\text{If } \; D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \; R_1 \to R_1 + mR_2 \text{ and } R_3 \to R_3 + nR_1$$

$$\Rightarrow \; D' = \begin{vmatrix} a_1+ma_2 & b_1+mb_2 & c_1+mc_2 \\ a_2 & b_2 & c_2 \\ a_3+na_1 & b_3+nb_1 & c_3+nc_1 \end{vmatrix}, \text{ then } D' = D$$

(4) Summation of Determinant :

$$\text{Let } \; \Delta(r) = \begin{vmatrix} f(r) & g(r) & h(r) \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

where a_1, a_2, a_3, b_1, b_2, b_3 are constants (independent of r) , then

$$\sum_{r=1}^{n} \Delta(r) = \begin{vmatrix} \sum_{r=1}^{n} f(r) & \sum_{r=1}^{n} g(r) & \sum_{r=1}^{n} h(r) \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Note:
If more than one row or one column are function of r , then first the determinant is simplified and then summation is calculated.

(5) Integration of a Determinant :

$$\text{Let } \; \Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

where a_1, a_2, a_3, b_1, b_2, b_3 are constants (independent of x) , then

$$\int_a^b \Delta(x)dx = \begin{vmatrix} \int_a^b f(x)dx & \int_a^b g(x)dx & \int_a^b h(x)dx \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Note:
If more than one row or one column are function of x , then first the determinant is simplified and then it is integrated.

(6) Differentiation of Determinant :

Let $\Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$, then

$$\Delta'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1'(x) & g_2'(x) & g_3'(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1'(x) & h_2'(x) & h_3'(x) \end{vmatrix}$$

(7) Multiplication Of Two Determinants :

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \ell_1 & m_1 \\ \ell_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1\ell_1 + b_1\ell_2 & a_1m_1 + b_1m_2 \\ a_2\ell_1 + b_2\ell_2 & a_2m_1 + b_2m_2 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{vmatrix} = \begin{vmatrix} a_1\ell_1 + b_1\ell_2 + c_1\ell_3 & a_1m_1 + b_1m_2 + c_1m_3 & a_1n_1 + b_1n_2 + c_1n_3 \\ a_2\ell_1 + b_2\ell_2 + c_2\ell_3 & a_2m_1 + b_2m_2 + c_2m_3 & a_2n_1 + b_2n_2 + c_2n_3 \\ a_3\ell_1 + b_3\ell_2 + c_3\ell_3 & a_3m_1 + b_3m_2 + c_3m_3 & a_3n_1 + b_3n_2 + c_3n_3 \end{vmatrix}$$

Multiplication expressed in the calculation is row $\times$ column which also applicable in matrix multiplication , but in case of multiplication of determinants , row $\times$ column , row $\times$ row , column $\times$ row and column $\times$ column all are applicalble.

Note:

Power cofactor formula :

If $\Delta = |a_{ij}|$ is a determinant of order n , then the value of the determinant $|C_{ij}| = \Delta^{n-1}$, where C_{ij} is cofactor of a_{ij}.

(8) Standard Results :

- $\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = \begin{vmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{vmatrix} = abc$

- $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2.$

- $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc) = -(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac)$

  $= -\frac{1}{2}(a + b + c)\left((a - b)^2 + (b - c)^2 + (c - a)^2\right)$

- $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a - b)(b - c)(c - a)$

- $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a - b)(b - c)(c - a)(a + b + c)$

- $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a - b)(b - c)(c - a)(ab + bc + ca)$

(8) Cramer's Rule : System of Linear equations

(I) Consider the system of non-homogenous equations :

$a_1x + b_1y + c_1z = d_1$
$a_2x + b_2y + c_2z = d_2$
$a_3x + b_3y + c_3z = d_3$

Where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ and $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$,then

according to Cramer's rule : $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$, $z = \frac{D_3}{D}$

Non-homogenous system of equations may or may not be consistent ,following cases explains the consistency of equations :

- If $D \neq 0$, then the system of equations are consistent and have unique solution.
- If $D = D_1 = D_2 = D_3 = 0$, then the system of equations have either infinite solutions or no solution.
- If $D = 0$ and atleast one of D_1, D_2, D_3 is not zero then the equations are inconsistent and have no solution.

(II) Consider the system of homogenous equations :

$a_1x + b_1y + c_1z = 0$
$a_2x + b_2y + c_2z = 0$
$a_3x + b_3y + c_3z = 0$

In homogenous system , $D_1 = D_2 = D_3 = 0$, hence :

- If $D \neq 0$, then the system of equations are consistent and have unique solution (i.e. **Trivial solution**)
- If $D = 0$, then the system of equations consitent and have infinite solutions. (i.e. **Non-trivial solution**)

Note:

Homogenous system of equations are always consistent , either having Trivial solution or non-trivial solution

(III) System of equations , $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ represents two straight lines in 2-dimonseional plane , hence :

- If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$: System of equations are consistent and have uniqe solution

  (i.e. Intersecting lines)
- If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$: Wywtem of equation are consistent and have infinite solution

  (i.e. coincident lines)
- If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$: System of equation in inconsistent and have no solution

  (i.e. parellel lines).

Note:

Three equation in two variables :

If x and y are not zero, then condition for $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$ and

$a_3x + b_3y + c_3 = 0$ to be consistent in x and y is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

(10) Application of Determinants :

(i) Area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is :

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

If $\Delta = 0$, then the three points are collinear.

(ii) Equation of a straight line passing through (x_1, y_1) and (x_2, y_2) is :

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \qquad \text{(two point form of line)}$$

(iii) If lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are non-parallel and non-coincident, then lines are concurrent, if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

(iv) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if :

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \Rightarrow abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

9. SEQUENCE AND SERIES

(1) Basic Definitions :

Sequence :

A sequence is a function whose domain is the set of natural numbers.Since the domain for every sequence is the set N of natural numbers, therefore a sequence is represented by its range. If $f : N \to R$, then $f(n) = T_n$, $n \in N$ is called a sequence and is denoted by $\{T_n\}$
$\{f(1), f(2), f(3), ...\} = \{T_1, T_2, T_3, ...\}$

Note:

(i) A sequence whose range is a subset of R is called a real sequence.
(ii) Finite sequences : A sequence is said to be finite if it has finite number of terms.
(iii) Infinite sequences : A sequence is said to be infinite if it has infinite number of terms.
(iv) It is not necessary that the terms of a sequence always follow a certain pattern or they are described by some explicit formula for the nth term. Sequences whose terms follow certain patterns are called progressions.
(v) All progressions are sequences, but all sequences are not progressions. For example: Set of prime numbers is a sequence but not a progression

Series :

By adding or substracting the terms of a sequence, we get an expression which is called a series. If $a_1, a_2, a_3, ... a_n$ is a sequence, then the expression $a_1 + a_2 + a_3 + ... + a_n$ is a series.

For example : (i) $1 + 2 + 3 + 4 + ... + n$
(ii) $2 + 4 + 8 + 16 + ...$

n^{th} term of sequence:

If summation of n terms of a sequence is given by S_n , then its n^{th} term is given by :
$T_n = S_n - S_{n-1}$.

(2) Arithmetic progression (A.P.) :

A.P. is a sequence whose terms increase or decrease by a fixed number. This fixed number is called the common difference. If a is the first term and d is the common difference , then A.P. can be written as :

$a , a + d, a + 2d ,.......... a + (n-1) d , ...$

n^{th} term of an A.P. :

Let a be the first term and d be the common difference of an A.P., then
$T_n = a + (n-1)d$ and $T_n - T_{n-1} =$ Constant. (i.e. , common difference)

Sum of first n terms of an A.P. :

If a is first term and d is common difference then

$$S_n = \frac{n}{2}(2a + (n-1)d) \quad \Rightarrow \quad = \frac{n}{2}(a + \ell)$$

where ℓ is the last term

p^{th} term of an A.P. from the end :

Let 'a' be the first term and 'd' be the common difference of an A.P. having n terms. Then p^{th} term from the end is $(n - p + 1)^{th}$ term from the beginning.

p^{th} term from the end $= T_{(n-p+1)} = a + (n - p)d$

(3) Properties of A.P. :

(i) If a_1 , a_2 , a_3..... are in A.P. whose common difference is d, then for fixed non-zero number $K \in R$.

- $a_1 \pm K, a_2 \pm K, a_3 \pm K, ...$ will be in A.P. , whose common difference will be d.
- Ka_1 , Ka_2 , Ka_3 ..... will be in A.P. with common difference = Kd.
- $\frac{a_1}{K}, \frac{a_2}{K}, \frac{a_3}{K}$..... will be in A.P. with common difference = d/k.
- $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, ...$ will be in H.P.
- If $x \in R^+ - \{1\}$, then $x^{a_1}, x^{a_2}, x^{a_3}, ...$ will be in G.P.

(ii) The sum of terms of an A.P. which are equidistant from the beginning and the end is constant and is equal to sum of first and last term. i.e. $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = ...$

(iii) Any term (except the first term) of an A.P. is equal to half of the sum of terms equidistant from the term i.e. $a_n = \frac{1}{2}(a_{n-k} + a_{n+k}), k < n$.

(iv) If number of terms of any A.P. is odd, then sum of the terms is equal to product of middle term and number of terms.

(v) If number of terms of any A.P. is even then A.M. of middle two terms is A.M. of first and last term.

(vi) If the number of terms of an A.P. is odd then its middle term is A.M. of first and last term.

(vii) If a_1, a_2 , ... a_n and b_1 , ... b_n are the two A.P.'s. Then $a_1 \pm b_1, a_2 \pm b_2, ...a_n \pm b_n$ are also A.P.'s with common difference $d_1 \neq d_2$, where d_1 and d_2 are the common difference of the given A.P.'s .

(viii) Three numbers a , b , c are in A.P. iff 2b = a + c.

(ix) If the terms of an A.P. are chosen at regular intervals, then they form an A.P.

(x) Three numbers in A.P. can be taken as a–d, a, a + d; four numbers in A.P. can be taken as a–3d, a–d, a + d, a + 3d; five numbers in A.P. are a–2d, a–d, a , a + d, a + 2d and six terms in A.P. are a–5d, a–3d, a–d, a+d, a+3d, a+5d etc.

(4) Arithmetic Mean (A.M.) :

If three terms are in A.P. then the middle term is called the A.M. between the other two, so if a, b, c are in A.P. , b is A.M. of a and c.

$$b = \frac{a+c}{2}$$

n-Arithmetic Means Between Two Numbers :

If a, b are any two given numbers and a, A_1, A_2....., A_n, b are in A.P. then A_1, A_2, ... A_n are the n A.M.'s between a and b.

$$A_1 = a + \frac{b-a}{n+1},\ A_2 = a + \frac{2(b-a)}{n+1}\ ,....,\ A_n = a + \frac{n(b-a)}{n+1}$$

Note:

Sum of n A.M.'s inserted between a and b is equal to n times the single A.M. between a and b i.e. $\sum_{r=1}^{n} A_r = nA$ where A is the single A.M. between a and b.

(5) Geometric Progression (G.P.) :

G.P. is a sequence of numbers whose first term is non zero and each of the succeeding terms is equal to the preceding terms multiplied by a constant. Thus in a G.P. the ratio of n^{th} term and $(n-1)^{th}$ term is constant. This constant factor is called the **common ratio** of the series

a, ar, ar^2, ar^3, ar^4,....... is a G.P. with a as the first term and r as common ratio.

n^{th} term of G.P. :

$T_n = a(r)^{n-1}$

Sum of the first n terms : $S_n = \begin{cases} \dfrac{a(r^n - 1)}{r-1}, & r \neq 1 \\ na, & r = 1 \end{cases}$

Sum of an infinite G.P. , when |r| < 1.

$$S_\infty = \frac{a}{1-r}.$$

p^{th} term from the end of a finite G.P. :

If G.P. consists of 'n' terms, p^{th} term from the end = $(n - p + 1)^{th}$ term from the beginning = ar^{n-p}.

Also, the p^{th} term from the end of a G.P. with last term ℓ and common ratio r is $\ell\left(\frac{1}{r}\right)^{n-1}$

(5) Properties of G.P. :

(i) If each term of a G.P. be multiplied or divided or raised to power by the some non-zero quantity, the resulting sequence is also a G.P.

(ii) The reciprocal of the terms of a given G.P. form a G.P. with common ratio as reciprocal of the common ratio of the original G.P.

(iii) In a finite G.P. , the product of terms equidistant from the beginning and the end is always the same and is equal to the product of the first and last term.
i.e. , if a_1 , a_2 , a_3 , ... a_n be in G.P. Then $a_1 a_n = a_2 a_{n-1} = a_3 a_{n-2} = a_4 a_{n-3} = ... = a_r . a_{n-r+1}$

(iv) If the terms of a given G.P. are chosen at regular intervals, then the new sequence so formed also forms a G.P.

(v) Three non-zero numbers a , b , c are in G.P. iff $b^2 = ac$.

(vi) Every term (except first term) of a G.P. is the square root of terms equidistant from it.
i.e. $T_r = \sqrt{T_{r-p}.T_{r+p}}$; [r > p]

(vii) If first term of a G.P. of n terms is a and last term is ℓ, then the product of all terms of the G.P. is $(a\ell)^{n/2}$.

(viii) If there be n quantities in G.P. whose common ratio is r and S_m denotes the sum of the first m terms , then the sum of their product taken two by two is $\frac{r}{r+1} S_n\ S_{n-1}$.

(ix) Any three consecutive terms of a G.P. can be taken as $\frac{a}{r}$ a, ar, in general we take

$\frac{a}{r^k}, \frac{a}{r^{k-1}}, \frac{a}{r^{k-2}}$,....a, ar, ar^2, ...ar^k in case we have to take 2k + 1 terms in a G.P..

(x) Any four consecutive terms of a G.P. can be taken as $\frac{a}{r^3}, \frac{a}{r}$, ar, ar^3, in general we take

$\frac{a}{r^{2k-1}}, \frac{a}{r^{2k-3}}, ...\frac{a}{r}$, ar, ... ar^{2k-1} in case we have to take 2k terms in a G.P.

(xi) If a_1, a_2, a_3, ... and b_1, b_2, b_3......., are two G.P.'s with common ratio r_1 and r_2 respectively then the sequence a_1b_1, a_2b_2, a_3b_3,.. is also a G.P. with common ratio r_1r_2.

(xii) If a_1, a_2, a_3,....... are in G.P. where each $a_i > 0$, then log a_1, log a_2, log a_3............. are in A.P. and its converse is also true.

(6) Geometric Means (G.M.) :

If a, b, c are in G.P., b is the G.M. between a and c.
$b^2 = ac$

n-Geometric Means Between a, b:

If a, b are two given numbers and a, G_1, G_2, ..., G_n, b are in G.P.. Then
G_1, G_2, G_3,...., G_n are n G.M.s between a and b.

$G_1 = a(b/a)^{1/n+1}$, $G_2 = a(b/a)^{2/n+1}$,........., $G_n = a(b/a)^{n/n+1}$

Note:

The product of n G.M.s between a and b is equal to the nth power of the single G.M. between a and b

$\prod_{r=1}^{n} G_r = (G)^n$, where G is the single G.M. between a and b.

(7) Harmonic Progression :

The sequence a_1 , a_2 , ... , a_n , ... where $a_i \neq 0$ for each i is said to be in harmonic progression (H.P.) if the sequence $1/a_1$, $1/a_2$, ... , $1/a_n$, ... is in A.P. Note that a_n , the n^{th} term of the H.P. , is given by

$$a_n = \frac{1}{a + (n-1)d} \quad \text{where } a = \frac{1}{a_1} \text{ and } d = \frac{1}{a_2} - \frac{1}{a_1}.$$

Note:

(i) If a and b are two non-zero numbers, then the harmonic mean of a and b is number H such that the sequence a , H , b is an H.P. We have

$$\frac{1}{H} = \frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right)$$

(ii) The n numbers H_1 , H_2 , ... , H_n are said to be harmonic means between a and b if a, H_1 , H_2 , ... , H_n , b are in H.P., that is, if 1/a, $1/H_1$, $1/H_2$, ... , 1/b are in A.P. Let d be the common difference of this A.P. Then

$$\frac{1}{b} = \frac{1}{a} + (n + 2 - 1)d$$

...

$$\Rightarrow \qquad d = \frac{1}{n+1}\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{a-b}{(n+1)ab}$$

Thus

$$\frac{1}{H_1} = \frac{1}{a} + \frac{a-b}{(n+1)ab},$$

$$\frac{1}{H_2} = \frac{1}{a} + \frac{2(a-b)}{(n+1)ab}, \ldots, \frac{1}{H_n} = \frac{1}{a} + \frac{n(a-b)}{(n+1)ab}$$

From here , we can get the values of H_1 , H_2 , ... , H_n.

(8) Relation between means :

If A, G, H are respectively A.M., G.M., H.M. between a and b both being unequal and positive then,

$G^2 = AH \Rightarrow$ A, G, H are in G.P.

Let $a_1, a_2, a_3, \ldots\ldots\ldots\, a_n$ be n positive real numbers, then **A.M. $\geq$ G.M. $\geq$ H.M.** , where

$$A.M. = \frac{a_1 + a_2 + a_3 + \ldots\ldots + a_n}{n}, \; G.M. = (a_1.a_2.a_3.a_4 \ldots\ldots a_n)^{1/n} \text{ and}$$

$$H.M. = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots\ldots + \frac{1}{a_n}}.$$

Note:

(i) The inequality of A.M. , G.M. and H.M. is only applicable for positive real numbers and the sign of equality holds for equal number.

$$\therefore \qquad A.M. = G.M. = H.M.$$

$$\Rightarrow \qquad a_1 = a_2 = a_3 = \ldots\ldots = a_n.$$

(ii) If $a, b, c \in R^+$, then $\dfrac{a+b+c}{3} \geq (abc)^{1/3} \geq \dfrac{3abc}{ab+bc+ac}$

(9) Arithmetico-Geometric Series :

A series each term of which is formed by multiplying the corresponding term of an A.P. and G.P. is called the Arithmetico-Geometric Series. e.g. $1 + 3x + 5x^2 + 7x^3 + \ldots$

Here 1, 3, 5, ... are in A.P. and $1, x, x^2, x^3 \ldots\ldots$ are in G.P...

Sum of n terms of an Arithmetico-Geometric Series :

Let $S_n = a + (a + d)r + (a + 2d)\, r^2 + \ldots\ldots + [a + (n-1)d]r^{n-1}$, then

$$S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{1-r}, \quad r \neq 1.$$

Sum To Infinity : If $|r| < 1$ & $n \to \infty$, then $S_\infty = \dfrac{a}{1-r} + \dfrac{dr}{(1-r)^2}$

(10) Important Results :

(i) $\sum_{r=1}^{n}(a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$ (ii) $\sum_{r=1}^{n} ka_r = k\sum_{r=1}^{n} a_r$

(iii) $\sum_{r=1}^{n} k = k + k + k \ldots\ldots\ldots\ldots\ldots n \text{ times} = nk$; where k is a constant

(iv) $\sum_{r=1}^{n} r = 1 + 2 + 3 + \ldots\ldots\ldots + n = \frac{n(n+1)}{2}$

(v) $\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(vi) $\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + \ldots\ldots\ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

(vii) $2.\sum_{1 \le i < j \le n} a_i.a_j = (a_1 + a_2 + \ldots\ldots\ldots + a_n)^2 - (a_1^2 + a_2^2 + \ldots\ldots + a_n^2)$

(viii) $\prod_{r=1}^{n} a_r = a_1.a_2.a_3 \ldots\ldots\ldots a_n$ (ix) $\prod_{r=1}^{n} ka_r = k^n.\prod_{r=1}^{n} a^r$

(x) $\prod_{r=1}^{n} r = n!$ (xi) $\prod_{r=1}^{n}(a_r.b_r) = \left(\prod_{r=1}^{n} a_r\right)\left(\prod_{r=1}^{n} b_r\right)$

(11) Method of Difference :

If the differences of the successive terms of a series are in A.P. of G.P. , we can find n^{th} term of the series by the following steps :

Step 1: Denote the n^{th} term by T_n and the sum of the series upto n terms by S_n.

Step 2: Rewite the given series with each term shifted by one place to the right.

Step 3: By subtracting the later series from the former, find T_n.

Step 4: From T_n , S_n can be found by appropriate summation.

10. FUNCTIONS AND GRAPH

(1) Definition :

Function is a particular case of relation, from a non empty set A to a non empty set B, that associates each and every member of A to a unique member of B. Symbolically, we write $f : A \to B$. We read it as "f is a function from A to B".

Note:

(i) Let $f : A \to B$, then the set A is known as the domain of f & the set B is known as co-domain of f.

(ii) If a member 'a' of A is associated to the member 'b' of B, then 'b' is called **the f-image** of 'a' and we write $b = f(a)$. Further 'a' is called **a pre-image** of 'b'.

(iii) The set $\{f(a) : \forall a \in A\}$ is called **the range** of f and is denoted by f(A). Clearly $f(A) \subseteq B$.

(iv) Sometimes if only definition of f(x) is given (domain and s are not mentioned), then domain is set of those values of 'x' for which f(x) is defined, while codomain is considered to be $(-\infty, \infty)$.

(v) A function whose domain and range both are sets of real numbers is called a **real function**. Conventionally the word **"FUNCTION"** is used only as the meaning of real function.

(2) Classification of Functions :

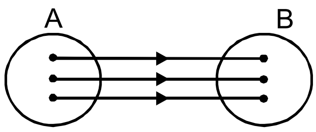
On the basis of mapping functions can be classified as follows:

(i) One-One Function (Injective Mapping)

A function $f : A \to B$ is said to be a one-one function or injective mapping if different elements of A have different f images in B.

Thus for $x_1, x_2 \in A$ & $f(x_1), f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.

Diagrammatically an injective mapping can be shown as



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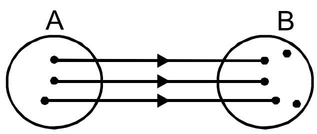
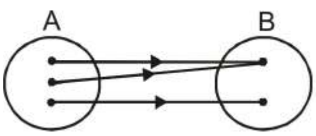


(ii) Many - One function :

A function $f : A \to B$ is said to be a many one function if two or more elements of A have the same f image in B.

Thus $f : A \to B$ is many one iff there exists at least two elements $x_1, x_2 \in A$, such that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Diagrammatically a many one mapping can be shown as



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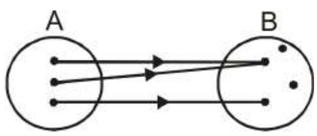


Note:

(i) If $x_1, x_2 \in A$ & $f(x_1), f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$, then function is ONE-ONE otherwise MANY-ONE.

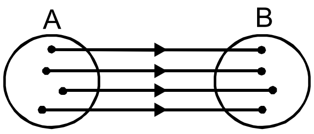
(ii) If there exists a straight line parallel to x-axis, which cuts the graph of the function atleast at two points, then the function is MANY-ONE, otherwise ONE-ONE.

(iii) If either $f'(x) \geq 0, \forall x \in$ complete domain or $f'(x) \leq 0$ $\forall x \in$ complete domain, where equality can hold at discrete point(s) only, then function is ONE-ONE, otherwise MANY-ONE.

(iii) Onto function (Surjective mapping)

If the function $f : A \to B$ is such that each element in B (co-domain) must have atleast one pre-image in A, then we say that f is a function of A 'onto' B. Thus $f : A \to B$ is surjective iff $\forall\ b \in B$, there exists some $a \in A$ such that $f(a) = b$.

Diagrammatically surjective mapping can be shown as



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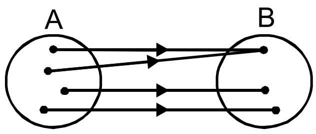
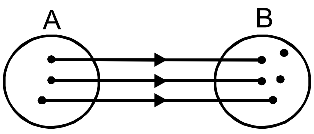


Note: If range $\equiv$ co-domain, then f(x) is onto, otherwise into

(iv) Into function :

If $f : A \to B$ is such that there exists atleast one element in co-domain which is not the image of any element in domain, then f(x) is into.

Diagrammatically into function can be shown as



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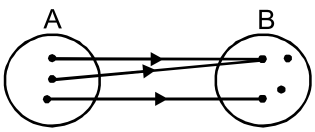
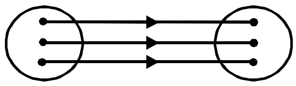


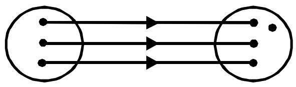
Note:

(i) function can be one of following four types :

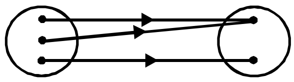
(a) one-one onto (injective & surjective)



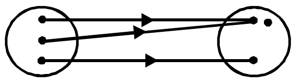
(b) one-one into (injective but not surjective)



(c) many-one onto (surjective but not injective)



(d) many-one into (neither surjective not injective)



(ii) If f is both injective & surjective, then it is called a bijective mapping. The bijective functions are also named as invertible, non-singular or biuniform functions.

(iii) If a set A contains 'n' distinct elements then the number of different functions defined from $A \to A$ is n^n and out of which n! are one one.

(3) Various Types of Functions :

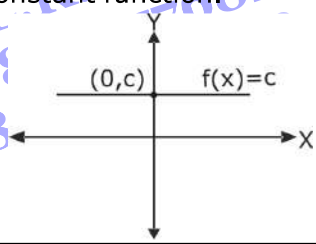
(i) Constant function :

A function $f : A \to B$ is said to be a constant function if every element of A has the same image in B. Thus $f : A \to B$; $f(x) = c, \forall x \in A, c \in B$ is a constant function.

Note:

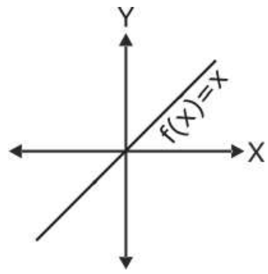
(i) Constant function is even and f(x) = 0 is the only function which is both even and odd

(ii) Constant function is periodic with indeterminate periodicity.



(ii) Identity function :

The function $f : A \to A$ defined by $f(x) = x \ \forall \ x \in A$ is called the identity function on A and is denoted by I_A. It is easy to observe that identity function is a bijection.



(iii) Polynomial function :

If a function f is defined by $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$ where n is a non **negative integer** and $a_0, a_1, a_2, \ldots\ldots\ldots, a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n.

for example: linear functions, quadratic functions, cubic functions etc.

(iv) Algebraic Function :

y is an algebraic function of x, if it is a function that satisfies an algebraic equation of the form, $P_0(x)\, y^n + P_1(x)\, y^{n-1} + \ldots.. + P_{n-1}(x)y + P_n(x) = 0$ where n is a positive integer and $P_0(x), P_1(x)$.....are polynomials in x. e.g. $y = |x|$ is an algebraic function-tion, since it satisfies the equation $y^2 - x^2 = 0$.

Note:

(i) All polynomial functions are algebraic but not the converse.

(ii) A function that is not algebraic is called **Transcendental Function**.

(v) Fractional/Rational Function :

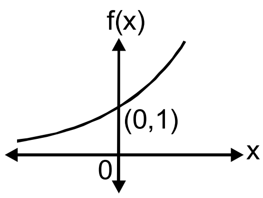
A rational function is a function of the form $y = f(x) = \dfrac{g(x)}{h(x)}$, where g(x) & h(x) are polynomials and $h(x) \neq 0$.

(vi) Exponential Function :

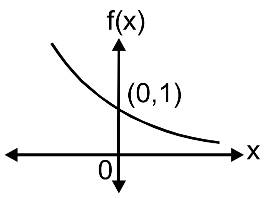
A function $f(x)=a^x=e^{x\ln a}\ (a > 0, a \neq 1, x \in R)$ is called an exponential function.

Graph of exponential function can be as follows :

For a > 1

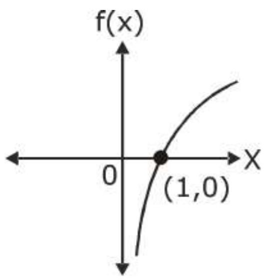


For 0 < a < 1

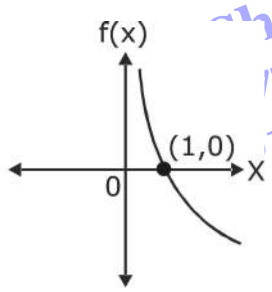


(vii) Logarithmic Function : $f(x) = \log_a x$ is called logarithmic function where $a > 0$ and $a \neq 1$ and $x > 0$. Its graph can be as follows

For a > 1

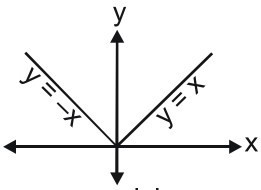


For 0 < a < 1



(viii) Absolute Value Function / Modulus Function:

The symbol of modulus function is f(x) = |x| and is defined as : $y = |x| = \begin{cases} x & \text{if} \quad x \ge 0 \\ -x & \text{if} \quad x < 0 \end{cases}$.



y = |x|

Note:

(i) $\sqrt{x^2} = |x|$ and $|x| = \max\{x, -x\}$

(ii) Absolute value function is piece-wise defined function and its domain and range are R and $[0, \infty)$ respectively

(iii) If 'a' and 'b' are non-negative real numbers, then

- $|x| \le a \Rightarrow x \in [-a, a]$; {i.e. $-a \le x \le a$}
- $|x| \ge a \Rightarrow x \in R / (-a, a)$; {i.e. $x \le -a$ or $x \ge a$}
- $a \le |x| \le b \Rightarrow x \in [-b, -a] \cup [a, b]$
- $|x| + |y| \ge |x \pm y|$
- $\left||x| - |y|\right| \le |x \pm y|$
- $|x + y| = |x| + |y| \Rightarrow xy \ge 0$.
- $|x - y| = \left||x| - |y|\right| \Rightarrow xy \ge 0$.

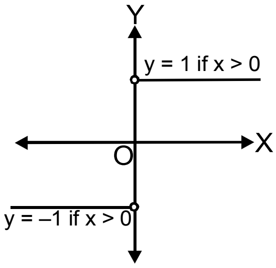
(iv) $\max\{f(x), g(x)\} = \dfrac{f(x) + g(x)}{2} + \left|\dfrac{f(x) - g(x)}{2}\right|$

(v) $\min\{f(x), g(x)\} = \dfrac{f(x) + g(x)}{2} - \left|\dfrac{f(x) - g(x)}{2}\right|$

(ix) Signum Function :

A function f(x) = sgn (x) is defined as follows :

$$f(x) = \text{sgn}\,(x) = \begin{cases} 1\,; & x > 0 \\ 0\,; & x = 0 \\ -1; & x < 0 \end{cases}$$



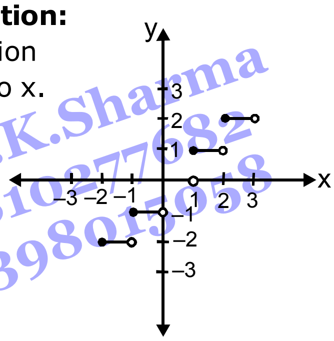
y = sgn x

Note:

(i) $\text{sgn}\,x = \begin{cases} \dfrac{|x|}{x} & x \ne 0 \\ 0; & x = 0 \end{cases}$ (ii) $\text{sgn}\,f(x) = \begin{cases} \dfrac{|f(x)|}{f(x)}; & f(x) \ne 0 \\ 0 \quad ; & f(x) = 0 \end{cases}$

(x) Greatest Integer Function or Step Up Function/Box function:

The function y = f(x)=[x] is called the greatest integer function where [x] equals to the greatest integer less than or equal to x.



Note:

(i) $[x] = x \ \forall\, x \in I$.

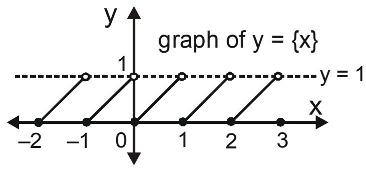
(ii) $[x \pm n] = [x] \pm n \ \forall\ n \in I$.

(iii) $x - 1 < [x] \le x$

(iv) $[x] + [-x] = \begin{cases} 0; & x \in I \\ -1; & x \notin I \end{cases}$ (v) $[x] + [y] \le [x + y]$

(xi) Fractional Part Function :

It is defined as, $y = \{x\} = x - [x]$ and $\{x\} \in [0, 1)$

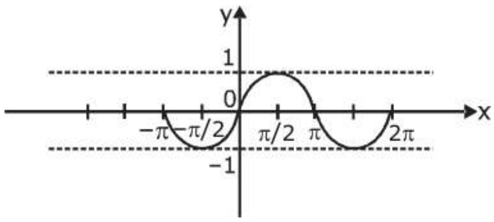


Note:

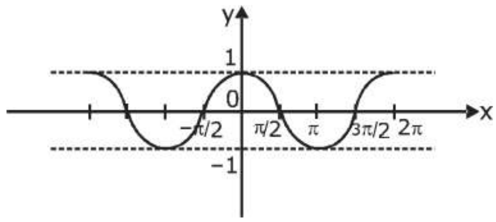
$f(x) = x - [x], \sqrt{x - [x]}$ or $(x - [x])^2$ are periodic with period 1.

(4) Trigonometric Function :

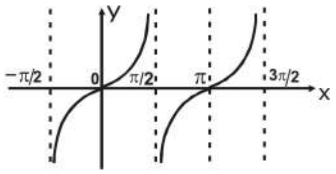
(i) $y = \sin x$; $x \in R$; $y \in [-1, 1]$



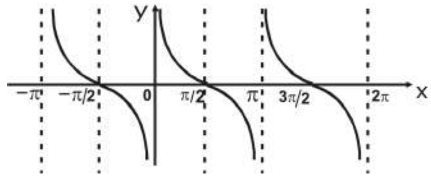
(ii) $y = \cos x$; $x \in R$; $y \in [-1, 1]$



(iii) $y = \tan x$; $x \in R - (2n+1)\pi/2,\ n \in I; y \in R$

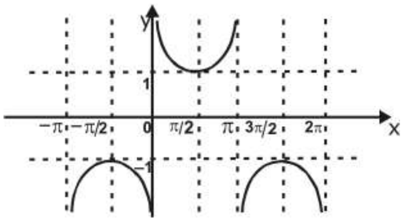


(iv) $y = \cot x$; $x \in R - n\pi,\ n \in I; y \in R$



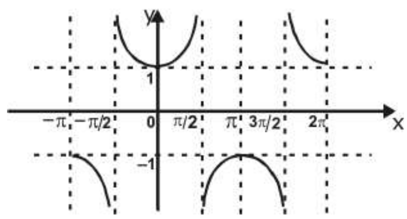
(v) $y = \text{cosec } x$;

$x \in R - n\pi,\ n \in I; y \in R/(-1, 1)$



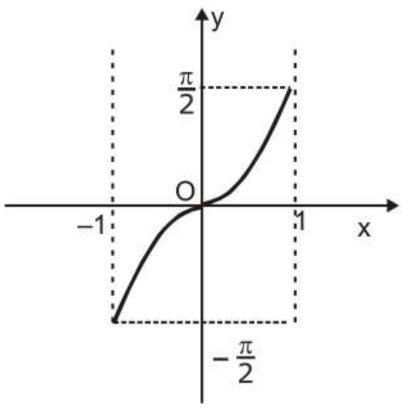
(vi) $y = \sec x$;

$x \in R - (2n+1)\pi/2,\ n \in I; y \in R/(-1, 1)$

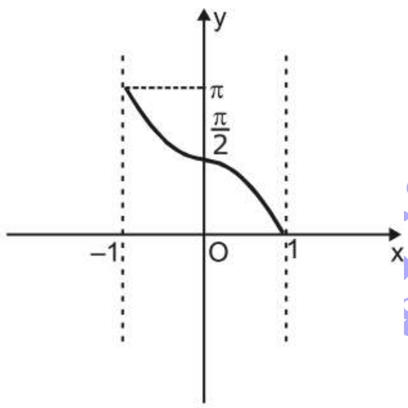


(5) Inverse Trigonometric Function :

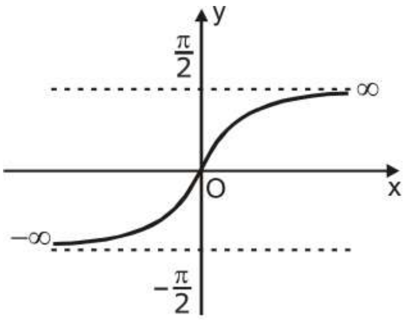
(i) $y = \sin^{-1} x,\ |x| \le 1,\ y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



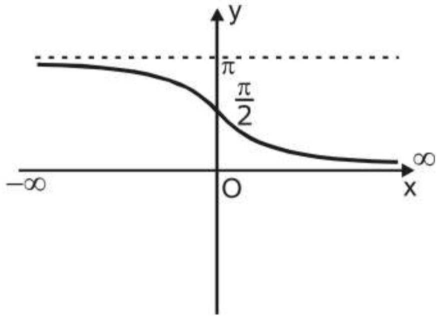
(ii) $y = \cos^{-1} x,\ |x| \le 1,\ y \in [0, \pi]$



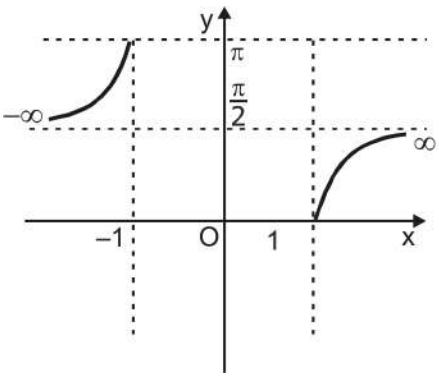
(iii) $y = \tan^{-1} x,\ x \in R,\ y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



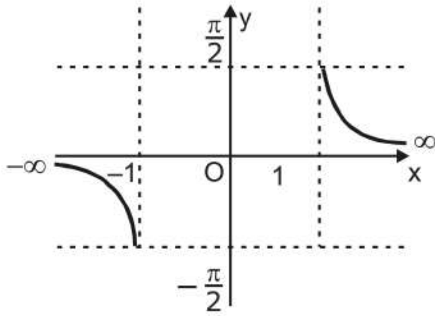
(iv) $y = \cot^{-1} x,\ x \in R,\ y \in (0, \pi)$



(v) $y = \sec^{-1} x,\ |x| \geq 1,\ y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$



(vi) $y = \text{cosec}^{-1} x,\ |x| \geq 1, y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$



(6) Equal or Identical Function :

Two functions f(x) and g(x) are said to be identical (or equal) iff :

(i) The domain of f $\equiv$ the domain of g.

(ii) The range of f $\equiv$ the range of g and

(iii) f(x) = g(x), for every x belonging to their common domain, e.g. $f(x) = \frac{1}{x}$ and $g(x) = \frac{x}{x^2}$ are identical functions. But f(x) = x and $g(x) = \frac{x^2}{x}$ are not identical functions.

(7) Bounded Function

The function f(x) is said to be bonded above if there exists M such that $y=f(x) \not> M$ (i.e. not greater that M) for all x of the domain and M is called upper bound. Similarly f(x) is said to be bounded below if there exists in such that $y=f(x) \not< m$(i.e. never less that m) for all x of the domain and m is called the lower bond.

If however, there does not exist M and m as stated above, the function is said to be unbounded.

(8) Even and odd Functions :

- If f (–x) = f(x) for all x in the domain of 'f' then f is said to be an even function .
  If f(x) – f(–x) = 0 $\Rightarrow$ f(x) is even.

- If $f(-x) = -f(x)$ for all x in the domain of 'f' then f is said to be an odd function.
  If $f(x) + f(-x) = 0 \Rightarrow f(x)$ is odd.

Note:

(i) Even functions are symmetrical about the y-axis and odd functions are symmetrical about origin

(ii) Any given function can be uniquely expressed as the sum of even function and odd function for example: $f(x) = \frac{1}{2}\{f(x)+f(-x)\} + \frac{1}{2}\{f(x)-f(-x)\}$

(iii) For any given function f(x), $g(x)=f(x)+f(-x)$ is even function and $h(x)=f(x)-f(-x)$ is odd function.

(iv) If an odd function is defined at $x = 0$, then $f(0) = 0$.

(v) Let the definition of the function f(x) is given only for $x \geq 0$. Even extension of this function implies to define the function for $x < 0$ assuming it to be even. In order to get even extension replace x by $-x$ in the given definition.Similarly, odd extension implies to define the function for $x < 0$ assuming it to be odd. In order to get odd extension, multiply the definition of even extension by -1.

(9) Periodic Function :

A function f(x) is called periodic with a period T if there exists a positive real number T such that for each x in the domain of f the numbers $x - T$ and $x + T$ are also in the domain of f and $f(x) = f(x + T)$ for all x in the domain of 'f'. Domain of a periodic function is always unbounded. Graph of a periodic function with period T is repeated after every interval of 'T'. e.g. The function sin x & cos x both are periodic over 2π and tan x is periodic over π.

The least positive period is called the **principal or fundamental period** of f or simply the period of f.

Note:

(i) $\sin^n$, $\cos^n x$, $\sec^n x$, and $\mathrm{cosec}^n x$ are periodic function with π period when n is even and 2π when n is odd or fraction. e.g. period of $\sin^2 x$ is π but period of $\sin^3 x$, $\sqrt{\sin x}$ is 2π.

(ii) $\tan^n x$ and $\cot^n x$ are periodic functions with period π irrespective of 'n'.

(iii) |sin x|, |cos x|, |tan x|, |cot x|, |sec x|, and |cosec x| are periodic functions with period π.

(iv) If f(x) is periodic with period T, then:
  (a) k. f(x) is periodic with period T.
  (b) f(x+b) is periodic with period T.
  (c) f(x)+c is periodic with period T.

(v) If f(x) has a period T, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ also have a period T.

(vi) If f(x) has a period T then $f(ax + b)$ has a period $\frac{T}{|a|}$.

(vii) If f(x) has a period T_1 and g(x) has a period T_2 then period of $f(x) \pm g(x)$ or f(x). g(x) or $\frac{f(x)}{g(x)}$ is L.C.M of T_1 & T_2 provided their L.C.M. exists. However that L.C.M. (if exists) need not to be fundamental period. If L.C.M. does not exists $f(x) \pm g(x)$ or f(x).g(x) is not periodic.
  e.g. |sin x| has the period π, |cos x| also has the period π.
  $\therefore$ |sin x| + |cosx| also has a period π. But the fundamental period of |sin x | + |cos x| is $\frac{\pi}{2}$.

(10) Composite Function :

Let $f : X \rightarrow Y_1$ and $g : Y_2 \rightarrow Z$ be two functions and the set $D = \{x \in X : f(x) \in Y_2\}$. If $D \neq \phi$ then the function h defined on D by $h(x) = g\{f(x)\}$ is called composite function of g and f and is denoted by gof. It is also called function of a function.

Note:

(i) Domain of gof is D which is a subset of X (the domain of f). Range of gof is a subset of the range of g . If D = X, then $f(x) \subseteq Y_2$.

(ii) In general gof $\neq$ fog (i.e. not commutative)

(iii) The composite of functions are associative i.e. if three functions f, g, h are such that fo(goh) and (fog)oh are defined, then fo(goh)=(fog)oh.

(iv) If f and g both are one-one, then gof and fog would also be one-one.

(v) If f and g both are onto, then gof or fog may or may not be onto.

(vi) The composite of two bijections is a bijection iff f & g are two bijections such that gof is defined, then gof is also a bijection only when co-domain of f is equal to the domain of g.

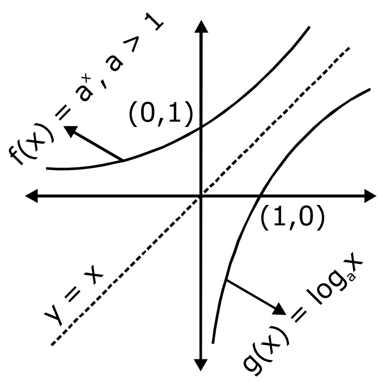
(vii) If g is a function such that gof is defined on the domain of f and f is periodic with T, then gof is also periodic with T as one of its periods.

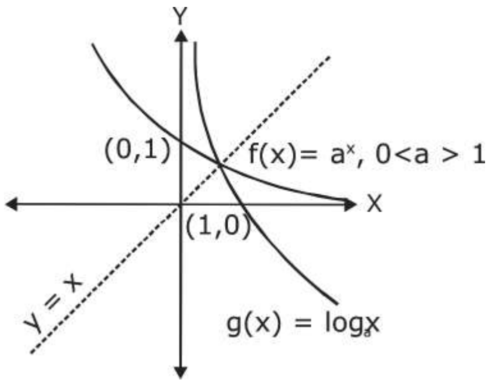
(11) Inverse of a Function :

Let $f : A \rightarrow B$ be a function, then f is invertible iff there is a function $g : B \rightarrow A$ such that gof(x) is an identity function on A and fog(x) is an identity function on B, g(x) is called inverse of f and is denoted by f^{-1}. For a function to be invertible it must be bijective.

Note:

(i) The graphs of f & g are the mirror images of each other in the line y = x. For example $f(x) = a^x$ and $g(x) = \log_a x$ are inverse of each other, and their graphs are mirror images of each other on the line y = x as shown below.





(ii) Normally points of intersection of f and f^{-1} lie on the straight line y = x. However it must be noted that f(x) and f^{-1} (x) may intersect otherwise also.

(iii) In general fog(x) and gof(x) are not equal but if they are equal then in majority of cases f and g are inverse of each other or atleast one of f and g is an identity function.

(iv) If f & g are two bijections $f : A \rightarrow B$, $g : B \rightarrow C$ then the inverse of gof exists and $(gof)^{-1} = f^{-1} \text{ o } g^{-1}$.

(v) If f(x) and g(x) are inverse function of each other then $f'(g(x)) = \dfrac{1}{g'(x)}$

(12) Functional relationships:

If x, y are independent variables, then :

(i) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \ln x$ or $f(x) = 0$.

(ii) $f(xy) = f(x).\ f(y) \Rightarrow f(x) = x^n,\ n \in R$

(iii) $f(x + y) = f(x).\ f(y) \Rightarrow f(x) = a^{kx}$.

(iv) $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant.

(v) $f(x).f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right) \Rightarrow f(x)=1 \pm x^n$ where $n \in N$.

(vi) $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \Rightarrow f(x)=ax+b$, a and b are constants.

11. LIMITS

(1) Definition :

Let f(x) be a function , if for every positive number ε there exists a positive number δ , such that $0 < |x - a| < \delta$ and $|f(x) - L| < \varepsilon$, then f(x) tends to limit L as x tends to a and the limiting value of f(x) at location x = a is represented by $\lim\limits_{x\to a} f(x)$

- In above definition of limit if x tends to a from the values of x greater than a , then limiting value is termed as right hand limit (RHL.) and represented by $\lim\limits_{x\to a^+} f(x)$ or $f(a^+)$

$\therefore$ $f(a^+) = \lim\limits_{h\to 0} f(a+h)$ where h is infinitely small positive number.

- In above definition of limit if x tends to a from the values of x less than a , then limiting value is termed as left hand limit (LHL.) and represented by $\lim\limits_{x\to a^-} f(x)$ or $f(a^-)$

$\therefore$ $f(a^-) = \lim\limits_{h\to 0} f(a-h)$ where h is infinitely small positive number.

Note:

(i) Finite limit of a function f(x) is said to exist as x approaches a , if :

$\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) =$ some finite value.

(ii) If LHL and RHL both approaches to ∞ or $-\infty$, then limit of function is said to be infinite limit.

(2) Indeterminate Forms :

Let f(x) be a function which is defined is same neighbourhood of location x = a , but functioning value f(a) is indeterminate because of the form of indeterminacy $\left(\text{i.e. } \frac{0}{0}\ ,\ \infty - \infty\ ,\ \frac{\infty}{\infty}\ ,\ 0\times\infty\ ,\ \infty^0\ ,\ 0^0 \text{ and } 1^\infty\right)$.

For example :

$f(x) = \dfrac{x^2-4}{x-2}$ is at $x = 2$, $f(x) = \dfrac{1}{x^2} - \dfrac{1}{\sin^2 x}$ at $x = 0$, $f(x) = (1-\cos x)^x$ at $x = 0$, etc.

To know the behavior of function at indeterminate locations , limiting values are calculated , which is represented by $\lim\limits_{x\to a} f(x)$, where x = a is the location of indeterminacy for function f(x).

Note:

If function f(x) is determinate at locations x = a , then $\lim\limits_{x\to a} f(x) = f(a)$, provided the limit exists (i.e. RHL = LHL).

For example : If $f(x) = \cos x$, then $\lim\limits_{x\to 0} f(x) = f(0)$

- If $f(x) = \dfrac{x^2-4}{x-2}$, then $\lim\limits_{x\to 2} f(x) \neq f(2)$.

(3) Fundamental Theorems on Limits :

Let $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$. If L and M exists , then :

(i) $\lim\limits_{x\to a}\{f(x)\pm g(x)\} = L+M \quad \{L\pm M \neq \infty-\infty\}$

(ii) $\lim\limits_{x\to a}\{f(x).g(x)\} = L.M \qquad \{L.M \neq 0\times\infty\}$

(iii) $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, provided $m \neq 0 \quad \left\{\dfrac{L}{M} \neq \dfrac{0}{0}, \dfrac{\infty}{\infty}\right\}$

(iv) $\lim\limits_{x\to a}(f(x))^{g(x)} = (L)^M \quad \left\{(L)^M \neq 0^o, \infty^o, 1^{\infty}\right\}$

(v) $\lim\limits_{x\to a} k\,f(x) = k\lim\limits_{x\to a} f(x)$; where k is a constant.

(vi) $\lim\limits_{x\to a} f(g(x)) = f\left(\lim\limits_{x\to a} g(x)\right) = f(M)$; provided f is continuous at $\lim\limits_{x\to a} g(x) = M$.

(vii) $\lim\limits_{x\to a}\log\{f(x)\} = \{\lim\limits_{x\to a} f(x)\}$

(viii) $\lim\limits_{x\to a}\dfrac{d}{dx}(f(x)) = \dfrac{d}{dx}\left(\lim\limits_{x\to a} f(x)\right)$

(ix) If $\lim\limits_{x\to a} f(x) = +\infty$ or $-\infty$, then $\lim\limits_{x\to a}\dfrac{1}{f(x)} = 0$

(4) Standard Limits :

(a) If 'x' is measured in radians , then $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$, $\lim\limits_{x\to 0}\dfrac{\tan x}{x} = 1$

(b) $\lim\limits_{x\to\infty}\dfrac{\cos x}{x} = 0$

(c) $\lim\limits_{x\to a}\dfrac{x^n - a^n}{x-a} = n(a)^{n-1}$

(d) $\lim\limits_{x\to 0}\dfrac{\sin^{-1}x}{x} = 1$

(e) $\lim\limits_{x\to 0}\dfrac{\tan^{-1}x}{x} = 1$

(f) $\lim\limits_{x\to 0}\dfrac{e^x - 1}{x} = 1$

(g) $\lim\limits_{x\to 0}\dfrac{a^x - 1}{x} = \log_e a$

(h) $\lim\limits_{x\to 0}\dfrac{\log_e(1+x)}{x} = 1$

(i) $\lim\limits_{x\to 0}\dfrac{\log_a(1+x)}{x} = \log_a e$

(j) If m , n $\in$ N and a_0 , $b_0 \neq 0$, then

$$\lim_{x\to\infty}\frac{a_0x^n + a_1x^{n-1} + a_2x^{n-2}\ldots\ldots + a_n}{b_0x^m + b_1x^{m-1} + b_2x^{m-2}\ldots + b_m} = \begin{cases} \dfrac{a_0}{b_0} & ; m = n \\ 0 & ; n < m \\ \infty & ; n > m \end{cases}$$

(k) $$\lim_{x\to\infty} a^x = \begin{cases} 1 & ; a = 1 \\ 0 & ; |a| < 1 \\ \infty & ; a > 1 \\ \text{oscillates finitely} & ; a = -1 \\ \text{oscillates infinitely} & ; a < -1 \end{cases}$$

(l) $\lim\limits_{x\to 0}(1+x)^{1/x} = e$

(m) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$ (n) $\lim_{x\to a}(f(x))^{g(x)} = e^{\lim_{x\to a} g(x).\log_e(f(x))}$

(o) If $(f(x))^{g(x)} \to 1^\infty$ for $x \to a$, then $\lim_{x\to a}(f(x))^{g(x)} = e^{\lim_{x\to a}(f(x)-1)g(x)}$.

(5) Limits Using Expansion

Let f(n) be a function which is differentiable for all orders throughout some interval containing location x = 0 as an interior point, then Taylor's series generated by f(x) at x = 0 (i.e. Maclaurin's series) is given by:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots\ldots\ldots\ldots\ldots$$

For example: If $f(x) = e^x$, then $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$

In solving limits sometimes standard series are more convenient than standard methods and hence following series should be remembered.

(i) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$

(ii) $a^x = 1 + \frac{x\ln a}{1!} + \frac{x^2\ln^2 a}{2!} + \frac{x^3\ln^3 a}{3!} + \ldots\ldots.$ $(a > 0, a \neq 1)$

(iii) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$ $(-1 < x < 1)$

(iv) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots$ (v) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

(vi) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots..$ (vii) $\tan^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots$

(viii) $\sin^{-1} x = x + \frac{1^2}{3!}x^3 + \frac{1^2.3^2}{5!}x^5 + \frac{1^2.3^2.5^2}{7!}x^7 + \ldots\ldots$

(ix) $\sec^{-1} x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!}$ ....

(x) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots\ldots\ldots\ldots$ ($|x| < 1, n \in R$)

(6) L' Hospital's Rule :

Let f(x) and g(x) are differentiable function of x at location x = a such that:

$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$

Now, according to L' Hospital Rule

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Note:

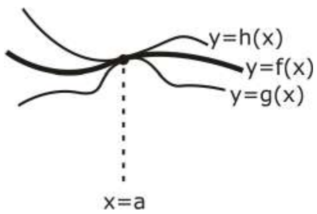
If $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ assumes the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and $f'(x)$, $g'(x)$ satisfy the conditions for L' Hospital rule then application the rule can be repeated

$\therefore \quad \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)}$.

(7) Sandwich Theorem (Squeeze Play Theorem):

Let $f(x)$, $g(x)$ and $h(x)$ be functions of x such that $g(x) \le f(x) \le h(x)\ \forall x \in (a-h\ ,\ a+h)$, where 'h' is infinitely small positive number , then $\lim_{x\to a} g(x) \le \lim_{x\to a} f(x) \le \lim_{x\to a} h(x)$.

Now , if $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L \ \Rightarrow\ \lim_{x\to a} f(x) = L$.



12. CONTINUITY AND DIFFERENTIABILITY

(1) Continuity of a function :

A function f(x) is said to be continuous at x = c , if :

$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$

Geometrically is not if a function f(x) is continous at x = c the graph of f(x) at the corresponding point (c , f(c)) will not be broken , but if f(x) is discontinuous at x = c the graph have break at the corresponding point.

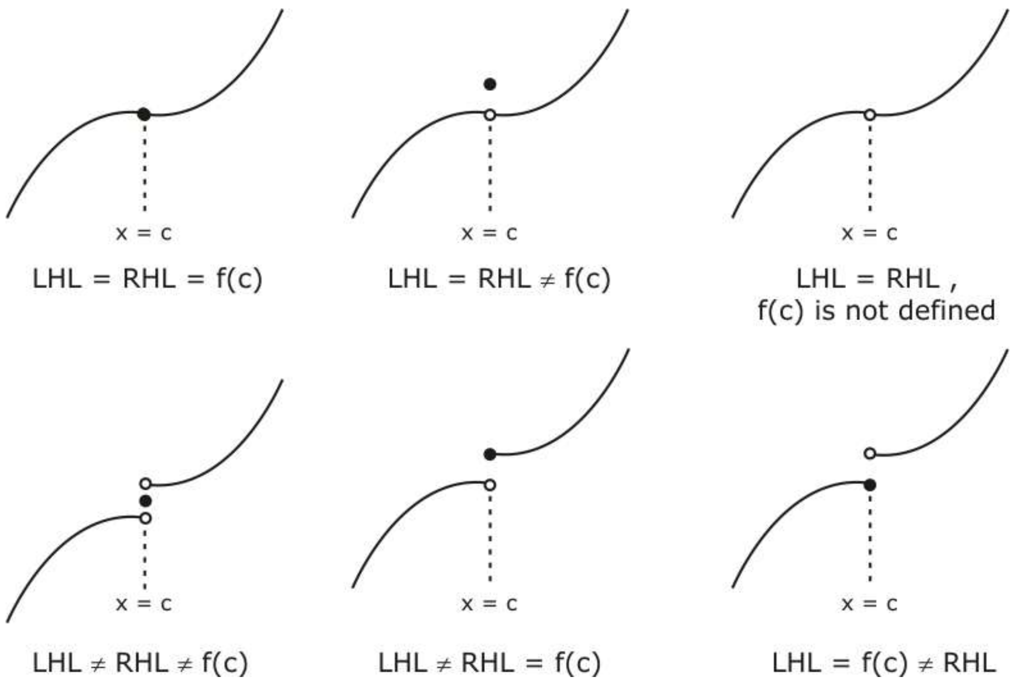
A function f(x) can be discontinuous at x = c because of any of the following three reasons:

(i) $\lim_{x \to c} f(x)$ does not exist (i.e. $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$)

(ii) f(x) is not defined at x = c

(iii) $\lim_{x \to c} f(x) \neq f(c)$

Following graph illustrates the different cases of discontinuity of f(x) at x = c.



(2) Types of Discontinuity :

(a) Discontinuity of First Kind:

Let f(x) be a function for which left hand limit and right hand limit at location x = a are finitely existing. Now if f(x) is discontinous at x = a is termed as discontinuity of first kind which may be removable or non-removable.

Removable Discontinuity of 1st kind:

If case $\lim_{x \to a} f(x)$ exists but is not equal to f(a) then the function is said to have a removable discontinuity at x = a. In this case function can be redefined such that $\lim_{x \to a} f(x) = f(a)$ and hence function can become continous at x = a.

Removable type of discontinuity of 1st kind can be further classified as :

Missing Point Discontinuity :

Where $\lim_{x\to a} f(x)$ exists finitely but f(a) is not defined.

For example: $f(x) = \dfrac{x^2 - 4}{x - 2}$ has a missing point discontinuity at x = 2.

$f(x) = \dfrac{\sin x}{x}$ has a missing point discontinuity at x = 0.

Isolated Point Discontinuity :

At locations x = a where $\lim_{x\to a} f(x)$ exists and f(a) also exists but $\lim_{x\to a} f(x) \neq f(a)$.

For example: $f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & ; \ x \neq 3 \\ 10 & ; \ x = 3 \end{cases}$ has isolated point discontinuity at x = 3

$f(x) = [x] + [-x] = \begin{cases} 0 & ; \ x \in I \\ -1 & ; \ x \notin I \end{cases}$ is discontinous at integral x.

Non-Removable discontinuity of Ist Kind:

If LHL and RHL exists for function f(x) at x = a but LHL $\neq$ RHL , then it is not possible to make the function continous by redefining it and this discontinous nature is termed as non-removable discontinuity of Ist kind.

For example: (i) f(x) = sgn (x) at x = 0

(ii) f(x) = x − [x] at all integral x.

Note:

• In case of non-removable discontinuity of the first kind the non-negative difference between the value of the RHL at x = a and LHL at x = a is called the jump of discontinuity.

Jump of discontinuity = |RHL−LHL|

• A function having a finite number of jumps in a given interval is called a Piece Wise Continuous or Sectionally Continuous function in this interval.

For example: f(x) = {x} , f(x) = [x].

(b) Discontinuity of Second Kind:

Let f(x) be a function for which atleast one of the LHL or RHL is non-existent or infinite at location x = a , then nature of discontinuity at x = a is termed as discontinuity of second kind and this is non-removable discontinuity.

Infinite discontinuity:

At location x = a , either LHL or RHL or both approaches to ∞ or $-\infty$.

For example: $f(x) = \dfrac{1}{x - 3}$ or $f(x) = \dfrac{1}{(x - 3)^2}$ at $x = 3$.

Oscillatory discontinuity:

At location x = a , function oscillates rigorously and can not have a finite limiting

For example: $f(x) = \sin\left(\dfrac{1}{x}\right)$ at $x = 0$

Note:

Point functions which are defined at single point only are discontinous of second kind.

For example: $f(x) = \sqrt{x - 4} + \sqrt{x - 4}$ at $x = 4$.

(3) Continuity in an Interval :

(a) A function f(x) is said to be continuous in open interval (a,b) if f(x) is continuous at each and every interior point of (a,b).

For example: f(x) = tanx is continous in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

(b) A function f(x) is said to be continuous in a closed interval [a,b] if :

- f(x) is continuous in the open interval (a,b) and
- f(x) is left continuous at x = a $\lim_{x\to a^+} f(x) = f(a) =$ a finite quantity.
- f(x) is right continuous at x = b $\lim_{x\to b^-} f(x) = f(b) =$ a finite quantity.

(4) Continuity of Composition of Functions:

(a) If f and g are two functions which are continuous at x = c then the functions defined by G(x) = f(x) $\pm$ g(x) ; H(x) = Kf(x) ; F(x) = f(x).g(x) are also continuous at x = c , where K is any real number. If g(c) is not zero , then Q(x) = $\frac{f(x)}{g(x)}$ is also continuous at x = c.

(b) If f(x) is continuous and g(x) is discontinuous at x = a then the product function ϕ(x) = f(x).g(x) may be continuous but sum or difference function ψ(x) = f(x) $\pm$ g(x) will necessarily be discontinuous at x = a.

For example: f(x) = x and g(x) = $\begin{cases} \sin\frac{\pi}{x} & x\neq 0 \\ 0 & x=0 \end{cases}$.

(c) If f(x) and g(x) both are discontinuous at x = a then the product function ϕ(x) = f(x).g(x) is not necessarily be discontinuous at x= a.

For example: f(x) = g(x) = $\begin{cases} 1 & x\geq 0 \\ -1 & x<0 \end{cases}$.

(d) If f is continuous at x = c and g is continuous at x = f(c) then the composite g(f(x)) is continuous at x = c.

For example: f(x) = $\frac{x\sin x}{x^2+2}$ and g(x) = |x| are continuous at x = 0 , hence the composite (gof) (x) = $\left|\frac{x\sin x}{x^2+2}\right|$ will also be continuous at x = 0.

(5) Differentiability of a function at a point :

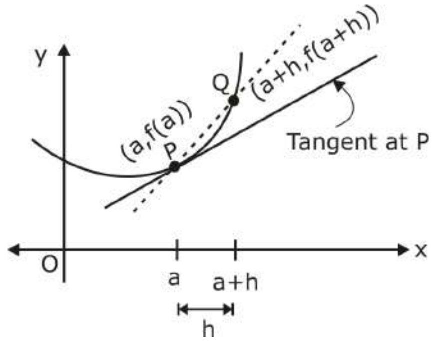
Let y = f(x) be a function and points P (a , f(a)) and Q (a + h , f(a + h)) lies on the curve of f(x).

Now , slope of secant passing through P and Q = $\frac{f(a+h)-f(a)}{h}$.

If positive scalar h tends to zero , then point Q approaches to point P from right hand side and the secant becomes tangent to curve f(x).

$\therefore$ Slope of tangent at $P = \lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h} = f'(a^+)$

f'(a⁺) represents the right hand differentiation of function f(x) at location x=a.

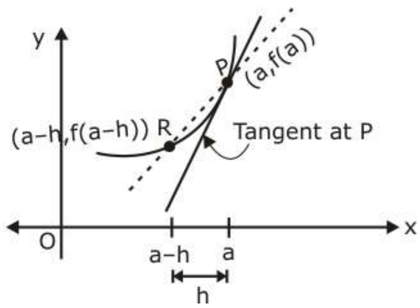


Similar to the above discussion , if P(a , f(a)) and R(a − h , f(a−h)) are two points on the curve y = f(x) , then slope of secant through P and R = $\dfrac{f(a-h)-f(a)}{-h}$.

Now , if positive scales h tends to zero , then point R approaches to point P from left hand side and the secant becomes tangent to curve f(x).

$\therefore$ Slope of tangent at $P = \lim\limits_{h\to 0}\dfrac{f(a-h)-f(a)}{-h} = f'(a^-)$

f'(a⁻) represents the left hand differentiation of function f(x) at location x=a.



Conditions for differentiability of function at location x = a:

(i) Necessary condition: function mast be continuous at x = a.

$\therefore \lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) = f(a)$ ...(1)

(ii) Sufficient condition: function must have finitely equal left hand and right hand derivative. (i.e. $f'(a^-) = f'(a^+)$)

$\therefore \lim\limits_{h\to 0}\dfrac{f(a-h)-f(a)}{-a} = \lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ = some finite quantity.

Differentiability of a function at x = a be confirmed if it satisfy both the conditions (i.e. necessary condition and sufficient condition).

Note:

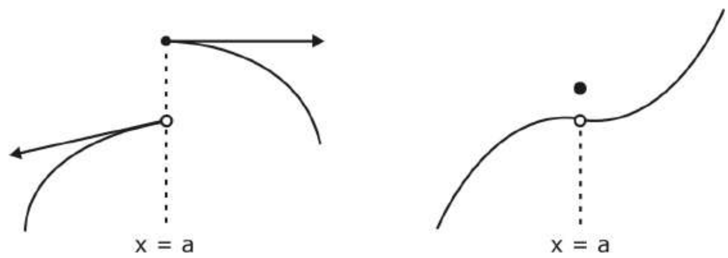
- If function is differentiable at a location than it is always continuous at that location but the converse is not always true. Hence all differentiable function are continous but all continous functions are not differentiable.

For example: $f(x) = e^{-|x|}$ is continous at x = 0 but not differentiable at x = 0

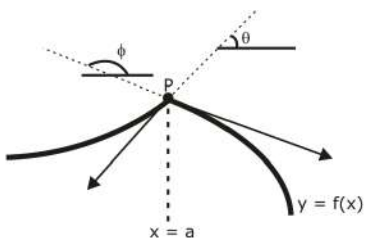
- If f(x) is differentiable at location x = c, then alternative formula for calculating differentiation is given by : $f'(c) = \lim\limits_{x\to c}\dfrac{f(x)-f(c)}{x-c}$

(6) Geometrical Interpretation of Differentiation:

Geometrically a function is differentiable at $x = a$ if there exists a unique tangent of finite slope at location $x = a$.
Non-differentiability of a function $y = f(x)$ at location $x = a$ can be visualized geometrically in the following cases:
(i) Discontinuity in the graph of $f(x)$ at location $x = a$.

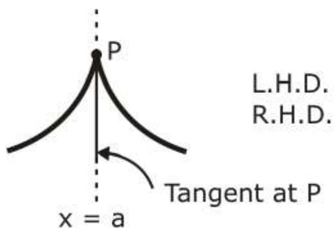


(ii) a corner or sharp change in the curvature of graph at $x = a$, where the left hand and right derivatives differ.



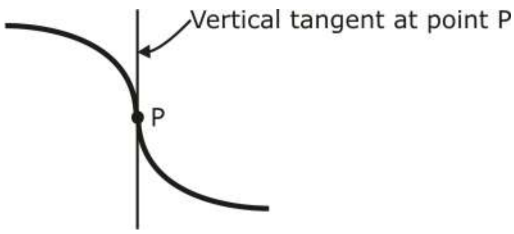
L.H.D. $= \tan\theta$
R.H.D. $= \tan\phi$
$\Rightarrow$ L.H.D. $\neq$ R.H.D

(iii) A cusp , where the slope of tangent at $x = a$ approaches ∞ from one side and $-\infty$ from the other side.



L.H.D. $\to \infty$
R.H.D. $\to -\infty$

(iv) Location of vertical tangent at $x = a$, where the slope of tangent approaches ∞ from both side or $-\infty$ from both sides.



Note:

• If the graph of a function is smooth curve or gradual curve without any sharp corner (or kink) then also it may be non-differentiable at some location.
For example: $f(x) = x^{1/3}$ represents a gradual curve without any sharp corner but it is non-differentiable at $x = 0$.
• If a function is non-differentiable at some location then also it may have a unique tangent.

For example: $y = \text{sgn}(x)$ is discontinous at $x = 0$ and hence non-differentiable but have a unique vertical tangent at $x = 0$.

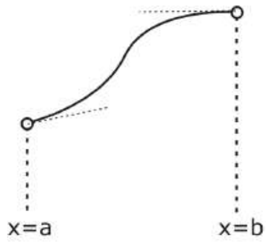
- Geometrically tangent is defined as a line which joins two closed points on the curve.

(7) Differentiable Over an Interval :

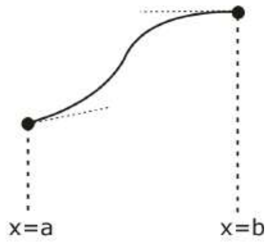
$f(x)$ is said to be differentiable over an open interval if it is differentiable at each and every point of the interval and $f(x)$ is said to be differentiable over a closed interval $[a,b]$ if $f(x)$ is continuous in $[a , b]$ and :

(i) for the boundary points $f'(a^+)$ and $f'(b^-)$ exist finitely , and

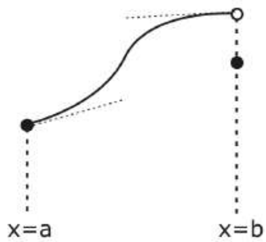
(ii) for any point c such that $a < c < b$, $f'(c^+)$ and $f'(c^-)$ exist finitely and are equal.



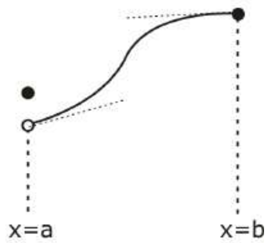
Differentiable in (a,b)



Differentiable in [a,b]



Differentiable in [a,b)



Differentiable in (a,b]

(8) Differentiability of Composition of Functions

(i) If $f(x)$ and $g(x)$ are differentiable at $x = a$ then the functions $\pm f(x)g(x))$, $f(x).g(x)$ will also be differentiable at $x = a$ and if $g(a) \neq 0$ then the function $f(x)/g(x)$ will also be differentiable at $x = a$.

(ii) If $f(x)$ is not differentiable at $x = a$ and $g(x)$ is differentiable at $x = a$, then the product function $f(x).g(x)$ may be differentiable at $x = a$

For example: $f(x) = |x|$ and $g(x) = x^2$.

(iii) If $f(x)$ and $g(x)$ both are not differentiable at $x = a$ then the product function $f(x).g(x)$ may be differentiable at $x = a$

For example: $f(x) = |x|$ and $g(x) = |x|$.

(iv) If $f(x)$ and $g(x)$ both are non-differentiable at $x = a$ then the sum function $f(x) \pm g(x)$ may be a differentiable function.

For example: $f(x) = |x|$ & $g(x) = -|x|$.

13. DIFFERENTIAL COEFFICIENTS

(1) First Principle of Differentiation (ab-initio Method):

- The derivative of a given function y = f(x) at a point x = a on its domain is represented by $f'(a)$ or $\left.\frac{dy}{dx}\right|_{x=a}$ and $f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$, provided the limit exists , alternatively the derivative of y = f(x) at x = a is given by $f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$, provided the limit exists.
- If x and x + h belong to the domain of function y = f(x), then derivative of function is represented by $f'(x)$ or $\frac{dy}{dx}$ and $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, provided the limit exists.

(2) Differentiation of Some elementary functions

	f(x)	f'(x)	
1.	x^n	nx^{n-1}	$(x \in R, n \in R)$
2.	a^x	$a^x \ell na$	$(a > 0 , a \neq 1)$
3.	$\ell n\|x\|$	$\frac{1}{x}$	$(x \neq 0)$
4.	$\log_a x$	$\frac{1}{x}\log_a e$	
5.	sin x	cos x	
6.	cos x	$-\sin x$	
7.	sec x	sec x tan x	
8.	cosec x	$-$cosec x cot x	
9.	tan x	$\sec^2 x$	
10.	cot x	$-\text{cosec}^2\, x$	
11.	$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$	; $\|x\| < 1$
12.	$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$	; $\|x\| < 1$
13.	$\tan^{-1} x$	$\frac{1}{1+x^2}$	; $x \in R$
14.	$\cot^{-1} x$	$\frac{-1}{1+x^2}$	; $x \in R$
15.	$\sec^{-1} x$	$\frac{1}{\|x\|\sqrt{x^2-1}}$	; $\|x\| > 1$
16.	$\text{cosec}^{-1} x$	$\frac{-1}{\|x\|\sqrt{x^2-1}}$	; $\|x\| > 1$

(3) Basic Theorems of Differentiation:

1. $\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$

2. $\frac{d}{dx}(k(f(x)) = k\frac{d}{dx}f(x)$

3. $\frac{d}{dx}(f(x).\,g(x)) = f(x)\,g'(x) + g(x)\,f'(x)$

4. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$

(4) Different methods of differentiation :

(4.1) Differentiation of function of a function:

If f(x) and g(x) are differentiable functions , then fog(x) is also differentiable and $\frac{d}{dx}(f(g(x))) = f'(g(x)).g'(x)$ is known as chain rule can be extended for two on move functions.

For example : $\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{du}.\frac{du}{dx}$.

Note:

- $\frac{dy}{dy} = 1 = \frac{dy}{dx}.\frac{dx}{dy} \Rightarrow \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$.

- $\frac{d^2y}{dx^2} \neq \frac{1}{\left(\frac{d^2x}{dy^2}\right)}$

- If $y = f(x)$ and $x = g(y)$ are inverse functions of each other , then g(f(x)) = x and

$g'(f(x)).f'(x) = 1 \quad \Rightarrow \quad g'(f(x)) = \frac{1}{g'(x)}$

$\therefore \quad g'(y).f'(x) = 1$.

(4.2) Implicit differentiation :

If f(x,y) = 0 is an implicit function , then

$\frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}$, where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial x}$ are partial differential coefficients of f(x,y) with respect to x and y respectively.

Note:

Partial differential coefficient of f(x , y) with respect to x means the ordinary differential coefficient of f(x , y) with respect to x keeping y constant.

(4.3) Logarithmic Differentiation :

If function is of the form $y = (f(x))^{g(x)}$ or $y = \dfrac{f_1(x).f_2(x).......}{g_1(x).g_2(x)......}$, where $g_i(x) \neq 0$ and $f_i(x)$, $g_i(x)$ are differentiable functions $\forall\ i = \{1, 2, 3, ..., n\}$, then differentiations after taking log of LHS and RHS is convenient and this method of differentiation is termed as logarithmic differentiation.

$$y = (f(x))^{g(x)} \Rightarrow \log_e y = g(x).\log_e f(x)$$

$$\Rightarrow y = e^{g(x).\log_e f(x)}$$

$$\therefore \frac{dy}{dx} = e^{g(x).\log_e f(x)} \left\{\frac{f'(x)}{f(x)}.g(x) + g'(x).\log_e f(x)\right\}$$

(4.4) Parametric Differentiation :

If $x = f(\theta)$ and $y = g(\theta)$, then $\dfrac{dy}{dx} = \dfrac{dg}{df}$

$$\Rightarrow \frac{dy}{dx} = \frac{g'(\theta)}{f'(\theta)} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}.$$

Note:

$$\frac{d^2y}{dx^2} \neq \frac{g''(\theta)}{f''(\theta)}.$$

(4.5) Derivative of one function with respect to another :

Let $u = f(x)$ and $v = g(x)$ be two functions of x, then derivative of $f(x)$ w.r.t. $g(x)$ is given by:

$$= \frac{\left(\frac{du}{dx}\right)}{\left(\frac{dv}{dx}\right)} = \frac{df(x)}{dg(x)} = \frac{f'(x)}{g'(x)}$$

(4.6) Differentiation of infinite series:

If y is given in the form of infinite series of x and we have to find out $\dfrac{dy}{dx}$ then we remove one or more terms, it does not affect the series

(i) If $y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + ...\infty}}}$, then $y = \sqrt{f(x) + y} \Rightarrow y^2 = f(x) + y$

$$2y\frac{dy}{dx} = f'(x) + \frac{dy}{dx}, \quad \therefore \frac{dy}{dx} = \frac{f'(x)}{2y - 1}$$

(ii) If $y = f(x)^{f(x)^{f(x)^{f(x)....\infty}}}$ then $y = f(x)^y$

$$\frac{1}{y}\frac{dy}{dx} = \frac{y.f'(x)}{f(x)} + \log f(x), \quad \therefore \frac{dy}{dx} = \frac{y^2 f'(x)}{f(x)[1 - y\log f(x)]}$$

(iii) If $y = f(x) + \cfrac{1}{f(x) + \cfrac{1}{f(x) + ...\infty}}$ then $\frac{dy}{dx} = \frac{yf'(x)}{2y - f(x)}$

(4.7) Derivative Of Inverse Trigonometric Functions :

Inverse trigonometric functions can be differentiation directly by use of standard results and the chain rule , but is is always easier by use of the proper substitution however the conditions of substitution must be applied carefully.

For example:

(4.8) Differentiation using substitution :

In case of differentiation of irrational function , method of substitution makes the calculations simpler and there the following substitution must be remember.

Some suitable substitutions

S. N.	**Functions**	**Substitutions**
(i)	$\sqrt{a^2 - x^2}$	$x = 0\sin\theta$ or $a\cos\theta$
(ii)	$\sqrt{x^2 + a^2}$	$x = a\tan\theta$ or $a\cot\theta$
(iii)	$\sqrt{x^2 - a^2}$	$x = a\ \sec\theta$ or $a\ \text{cosec}\ \theta$
(iv)	$\sqrt{\frac{a-x}{a+x}}$	$x = a\ \cos 2\theta$
(v)	$\sqrt{\frac{a^2-x^2}{a^2+x^2}}$	$x^2 = a^2\cos 2\theta$
(vi)	$\sqrt{ax - x^2}$	$x = a\sin^2\theta$
(vii)	$\sqrt{\frac{x}{a+x}}$	$x = a\tan^2\theta$
(viii)	$\sqrt{\frac{x}{a-x}}$	$x = a\sin^2\theta$
(ix)	$\sqrt{(x-a)(x-b)}$	$x = a\sec^2\theta - b\tan^2\theta$
(x)	$\sqrt{(x-a)(b-x)}$	$x = a\cos^2\theta + b\sin^2\theta$

(4.9) Differentiation of determinant :

If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where f, g, h, l, m, n, u, v, w are differentiable functions of x then $F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$

(4.10) Differentiation of definite integral :

If $g_1(x)$ and $g_2(x)$ both functions are defined on [a , b] and differentiable at point $x \in (a,b)$ and f(t) is continuous for $g_1(a) \le f(t) \le g_2(b)$

Then $\frac{d}{dx}\int_{g_1}^{g_2} f(t)\,dt = f[g_2(x)]g'_2(x) - f[g_1(x)]g'_1(x) = f[g_2(x)]\frac{d}{dx}g_2(x) - f[g_1(x)]\frac{d}{dx}g_1(x)$.

(5) Derivatives of Higher Order:

Let a function y = f(x) be defined on an open interval (a,b). It's derivative, if it exists on (a,b) is a certain function f'(x) [or (dy/dx) or y'] and is called the first derivative of y w.r.t. x.

If it happens that the first derivative has a derivative on (a,b) then this derivative is called the second derivative of y w.r.t. x & is denoted by f''(x) or $\left(\frac{d^2y}{dx^2}\right)$ or y''.

Similarly, the 3rd order derivative of y w.r.t. x, if it exists, is defined by $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$.

It is also denoted by f'''(x) or y'''.

14. TANGENT AND NORMAL

(1) Equation of Tangent and Normal :

Let y = f(x) be the equation of a curve and poit $P(x_1, y_1)$ lies on the curve. For the curve y = f(x) , geometrically $f'(x_1)$ represents the slope to tongent to curve at location $x = x_1$, hence the equations of tangent can be obtained by point slope form of line.

$\therefore \quad \dfrac{y-y_1}{x-x_1} = f'(x_1)$ is the equation of tangent at point $P(x_1, y_1)$ to curve y = f(x).

Slope of noraml to cuver y = f(x) at point $P(x_1, y_1)$ is given by $\left\{\dfrac{-1}{f'(x_1)}\right\}$

$\therefore \quad \dfrac{y-y_1}{x-x_1} = \dfrac{-1}{f'(x_1)}$ is equation of normal at point $P(x_1, y_1)$ to curve y = f(x).

Note:

If slope of tangent is zero , then tangent is termed as 'horizontal tangent' and if slope of targent approaches infinite , then tangent is termed as 'vertical tangent'.

(2) Length of Tangent and Normal :

Let $P(x_1, y_1)$ be any point on curve y = f(x). If tangent drawn at point P meets x-axis at T and normal at point P meets x-axis at N , then the length PT is called the length of tangent and PN is called length of normal.

Projection of segment PT on x-axis (i.e. QT) is called the subtangent and similarly projection of line segment PN on x-axis (i.e. QN) is called subnormal. (refer figure (1)).

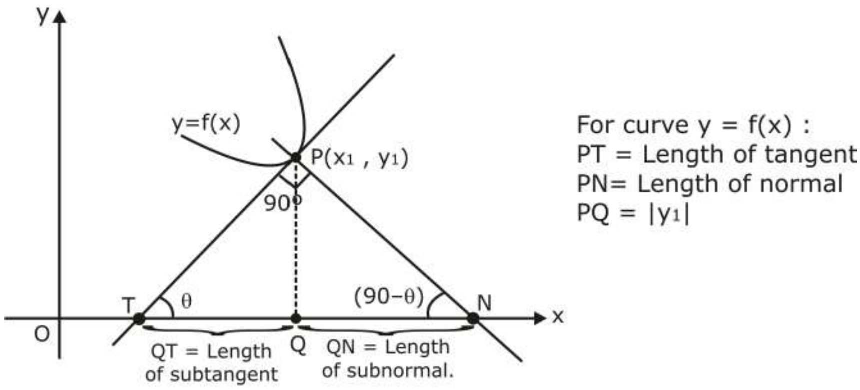


Figure (1)

Let $m = \left(\dfrac{dy}{dx}\right)_P$ = slope of tangent at $P(x_1, y_1)$ to curve y = f(x).

$\therefore \quad m = \tan\theta = \left(\dfrac{dy}{dx}\right)_P$

- In ΔPTQ , $\sin\theta = \dfrac{y_1}{PT} \Rightarrow PT = |\, y_1 \operatorname{cosec}\theta\, |$

$\therefore$ Length of tangent $= PT = \left| y_1\sqrt{1+\left(\dfrac{dx}{dy}\right)_P^2}\right|$

- In ΔPQN , $\sin(90-\theta)=\dfrac{y_1}{PN} \Rightarrow PN = |y_1 \sec\theta|$

$\therefore$ Length of normal $= PN = \left| y_1\sqrt{1+\left(\dfrac{dy}{dx}\right)_P^2} \right|$

- In ΔPTQ , $\tan\theta=\dfrac{y_1}{QT} \Rightarrow QT = |y_1 \cot\theta|$

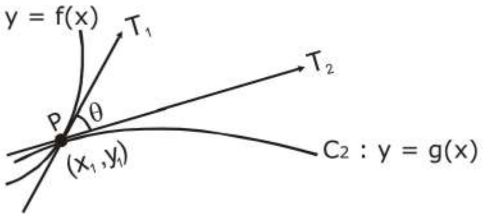
$\therefore$ Length of subtangent $= \left| \dfrac{y_1}{\left(\dfrac{dy}{dx}\right)_p} \right|$

- In ΔPQN , $\tan(90-\theta) = \dfrac{y_1}{QN} \Rightarrow QN = |y_1 \tan\theta|$

$\therefore$ Length of subnormal $= \left| y_1\left(\dfrac{dy}{dx}\right)_P \right|$

(3) Angle between the curves :

Angle between two intersecting curves is defined as the angle between their tangents or the normals at the point of intersection of two curves.



Slope of tangent T_1 : $m_1 = f'(x_1)$
Slope of tangent T_2 : $m_2 = g'(x_1)$

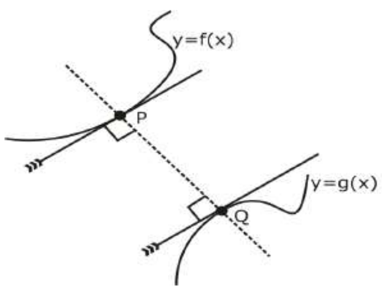
$\tan\theta = \left|\dfrac{m_1-m_2}{1+m_1m_2}\right|$, where θ is acute angle of intersection and $(\pi-\theta)$ is obtuse angle of intersection.

Note:

- The curves must intersect for the angle between them to be defined.
- If the curves intersect at more than one point then angle between the curves is mentioned with references to the point of intersection.
- Two curves are said to be orthogonal if angle between them at each point of intersection is right angle (i.e. $m_1m_2 = -1$).

(4) Shortest distance between two curves :

If shortest distance is defined between two non-intersecting curves , then it lies along the common normal.



Shortest distance between curves $y = f(x)$ and $y = g(x)$ is equal to PQ.

15. ROLLE'S THEOREM & MEAN VALUE THEOREM

(1) Rolle's Theorem:

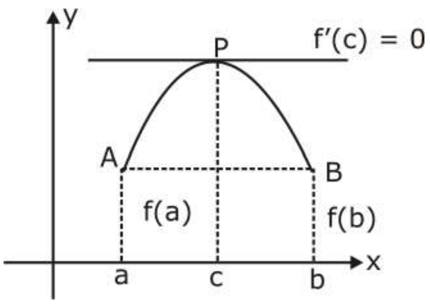
If a function f(x) is

(i) continuous in a closed interval [a, b]

(ii) derivable in the open interval (a, b)

(iii) f(a) = f(b),

then there exists at least one real number c is in (a, b) such that f '(c) = 0



(2) Langrange's Mean Value Theorem:

If a function f(x) is

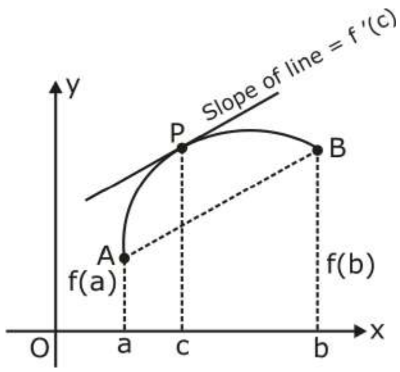
(i) continuous in the closed interval [a,b] and

(ii) derivable in the open interval (a, b),

then there exists at least one real number c in (a,b) such that

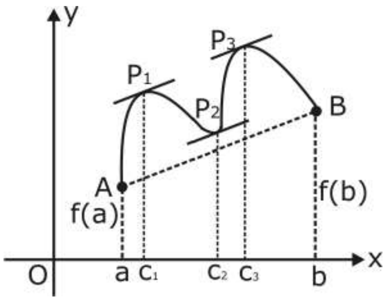
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Geometrical Illustration:



Let P be a point on the curve f(x) such that $\frac{f(b) - f(a)}{b - a} = f'(c)$ ...(i)

The slope of chord AB is $\frac{f(b) - f(a)}{b - a}$ and that of the tangent of point P is f'(c). These being equal, by (i), it follows that there exists a point P on the curve, the tangent at which is parallel to the chord AB.



There may exist more than one point between A and B, the tangents at which are parallel to the chord AB. Lagrange's mean value theorem ensures the existence of at least one real number c in (a,b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

(3) Alternative Form of Langrange's Theorem:

If a function f(x) is

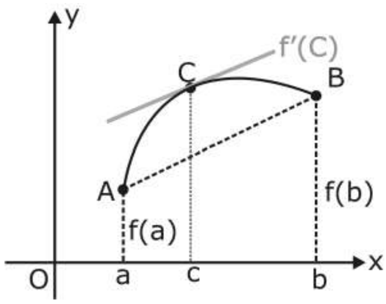
(i) continuous is a closed interval [a, a + h]

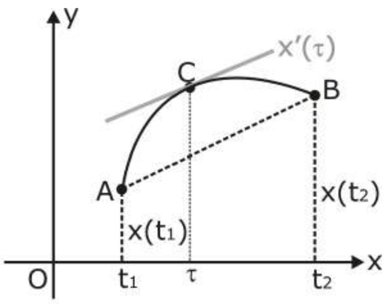
(ii) differentiable in a open interval (a, a + h), then their exists at least one number θ lying between 0 and 1 such that $f'(a+\theta h) = \frac{f(a+h) - f(a)}{h}$

i.e. $f(a+h) = f(a) + hf'(a+\theta h)$ $\quad (0 < \theta < 1)$

(4) Physical Illustration:

[f(b)–f(a)] is the change in the function f as x changes from a to b so that $\frac{[f(b) - f(a)]}{(b - a)}$ is the average rate of change of the function over the interval [a,b]. Also f'(c) is the actual rate of change of the function for x = c. Thus, the theorem states that the average rate of change of a function over an interval is also the actual rate of change of the function at some point of the interval. In particular, for instance, the average velocity of a particle over an interval of time is equal to the velocity at some instant belonging to the interval. i.e. their exists a tiem $\tau \in (t_1, t_2)$ for which $x'(\tau) = \frac{x(t_2) - x(t_1)}{t_2 - t_1}$ i.e. instantaneous velocity = average velocity.





(5) Intermediate Value Theorem (IVT)

If a function f is continuous in a closed interval [a,b] then there exists at least one $c \in (a,b)$ such that $f(c) = \frac{f(a) + \lambda f(b)}{1 + \lambda}$ for all $\lambda \in R^+$.

16. MONOTONOCITY

Basic Definitions

(i) **Strictly increasing function:**
A function f(x) is said to be strictly increasing in an interval 'D', if for any two locations $x = x_1$ and $x = x_2$, where $x_1 > x_2 \Leftrightarrow f(x_1) > f(x_2)\ \forall x_1, x_2 \in D$.
If function f(x) is differentiable in interval 'D', then for strictly increasing function $f'(x) > 0$. for example: $f(x) = e^x\ \forall x \in R$

(ii) **Strictly decreasing function:**
A function f(x) is said to be strictly decreasing in an interval 'D', if for any two locatious $x = x_1$ and $x = x_2$, where $x_1 > x_2 \Leftrightarrow f(x_1) < f(x_2)\ \forall x_1, x_2 \in D$.
If function f(x) is differentiable in interval 'D', then for strictly decreasing function $f'(x) < 0$. for example: $f(x) = e^{-x}\ \forall x \in R$

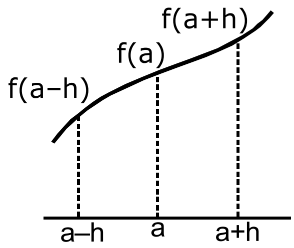
Note:

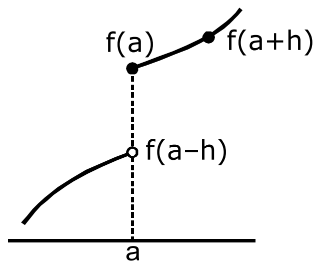
For strictly increasing or strictly decreasing functions, $f'(x) \neq 0$. For any point location or sub interval

A. Monotonocity about a point

1. A function f(x) is called an increasing function at point x = a. If in a sufficiently small neighbourhood around x = a.

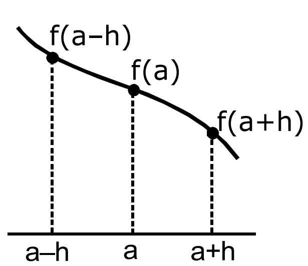
$$f(a-h) < f(a) < f(a+h)$$

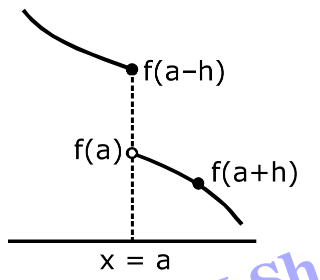




2. A function f(x) is called a decreasing function at point x = a if in a sufficiently small neighbourhood around x = a.

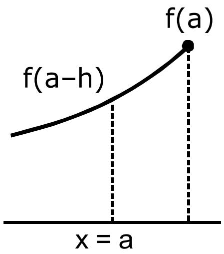
$$f(a-h) > f(a) > f(a+h)$$



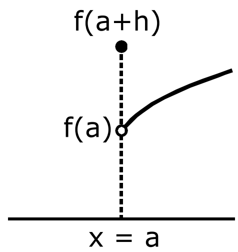


Note:

If x = a is a boundary point then use the appropriate one sided inequality to test monotonocity of f(x).



f(a) > f(a–h)
increasing at x = a



f(a+h) < f(a)
decreasing at x = a

3. **Test for increasing and decreasing functions at a point**

(i) If $f'(a) > 0$ then $f(x)$ is increasing at $x = a$.

(ii) If $f'(a) < 0$ then $f(x)$ is decreasing at $x = a$.

(iii) If $f'(a) = 0$ then examine the sign of $f'(a^+)$ and $f'(a^-)$

(a) If $f'(a^+) > 0$ and $f'(a^-) > 0$ then increasing

(b) If $f'(a^+) < 0$ and $f'(a^-) < 0$ then decreasing

(c) otherwise neither increasing nor decreasing.

B. Monotonocity over an interval

1. A function $f(x)$ is said to be monotonically increasing for all such interval (a, b) where $f'(x) \geq 0$ and equality may hold only for discreet values of x. i.e. $f'(x)$ does not identically become zero for $x \in (a,b)$ or any sub interval.

2. $f(x)$ is said to be monotonically decreasing for all such interval (a,b) where $f'(x) \leq 0$ and equality may hold only for discrete values of x.{By discrete points, it mean that points where $f'(x) = 0$ don't form an interval}

Note:

(i) A function is said to be monotonic if it's either increasing or decreasing.

(ii) The points for which $f'(x)$ is equal to zero or doesn't exist are called **critical points.** Here it should also be noted that critical points are the interior points of an interval.

(iii) The stationary points are the points where $f'(x) = 0$ in the domain.

C. Classification of functions

Depending on the monotonic behaviour, functions can be classified into following cases.

1. Increasing functions
2. Non decreasing functions
3. Decreasing functions
4. Non-increasing functions

This classification is not complete and there may be function which cannot be classified into any of the above cases for some interval (a, b).

12. MAXIMA & MINIMA

(1) Basic definitions:

(i) **Stationary points:** For a given function f(x), stationary points are those locations of x (let it be 'x_0') for which $f'(x_0) = 0$, and functioning value $f(x_0)$ is termed as the stationary value of function.

(ii) **Critical points:** For a given function f(x), critical points are those locations of x(let it be 'x') in the domain of f(x) for which $f'(x_0) = 0$ or $f'(x_0)$ doesn't exist.

Note: At critical points for a function, local maxima, local minima or point of inflection may exist

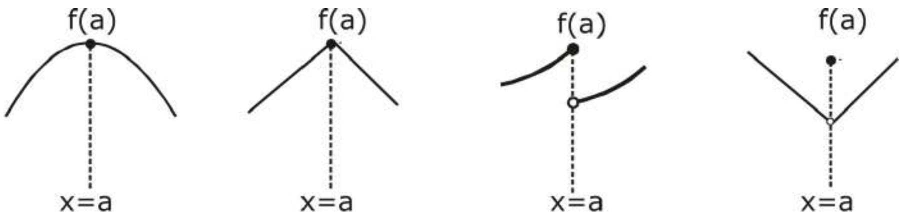
(iii) **Points of extremum:** critical points of a function at which either maxima or minima exists is termed as points of extremum.

(iv) **Point of inflection:** critical locations of function at which neither maxima nor minima exists is termed as point of in flection, it is also known as point of no extremum.

(v) **Local maxima/regional maxima:** A function f(x) is said to have a local maximum at critical point x= a if $f(a) \geq f(x)\ \forall\ x \in (a-h, a+h)$, where h is a very small positive arbitrary number.

If x = a is the point of local maxima, then $f(a) \geq \lim\limits_{x \to a^-} f(x)$ and $f(a) \geq \lim\limits_{x \to a^+} f(x)$.

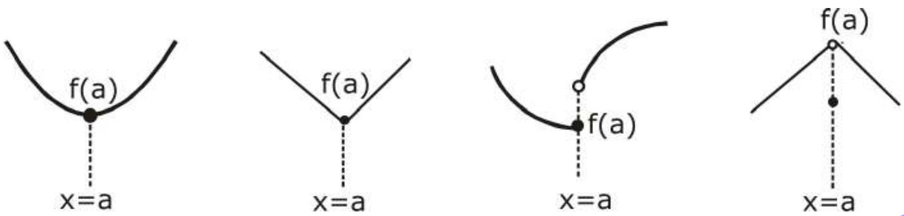
following figure illustrates the point of local maxima in different situations:



(vi) **Local minima/regional minima:** A function f(x) is said to have local minimum at critical point x = a if $f(a) \leq f(x)\ \forall\ x \in (a-h, a+h)$, where h is a very small positive arbitrary number.

If x = a is the point of local minima, if then $f(a) \leq \lim\limits_{x \to a^-} f(x)$ and $f(a) \leq \lim\limits_{x \to a^+} f(x)$.

following figure illustrates the point of local minima in different situations:



Note:

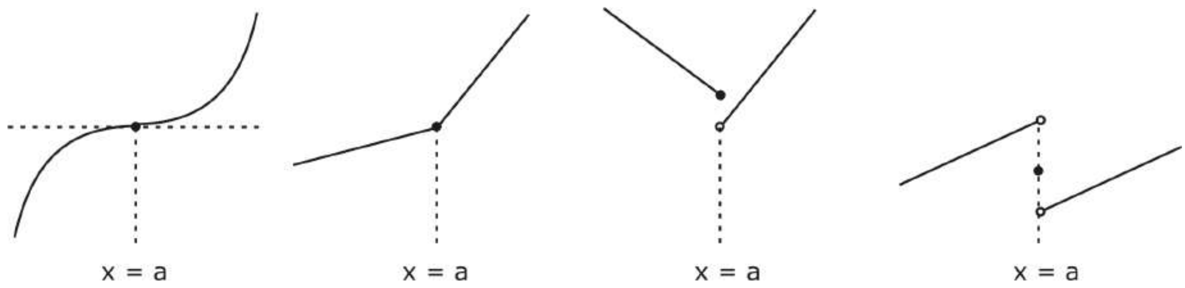
(i) local maxima of a function at x = a is the largest value of function only in the nearby neighbourhood of critical point x = a.

(ii) local minima of a function at x = a is the smallest value of function only in the nearby neighbourhood of critical point x = a.

(iii) If a function f(x) is defined on [a,b], then f(x) is having point of extremum at the boundary points.

(iv) for function f(x), if local maxima on local minima doesn't exist at critical point x = a, then point of inflection exists.

Following figure illustrates the point of inflection or point of no extremum



(2) Test for Maxima/Minima

1. I[st] derivative Test

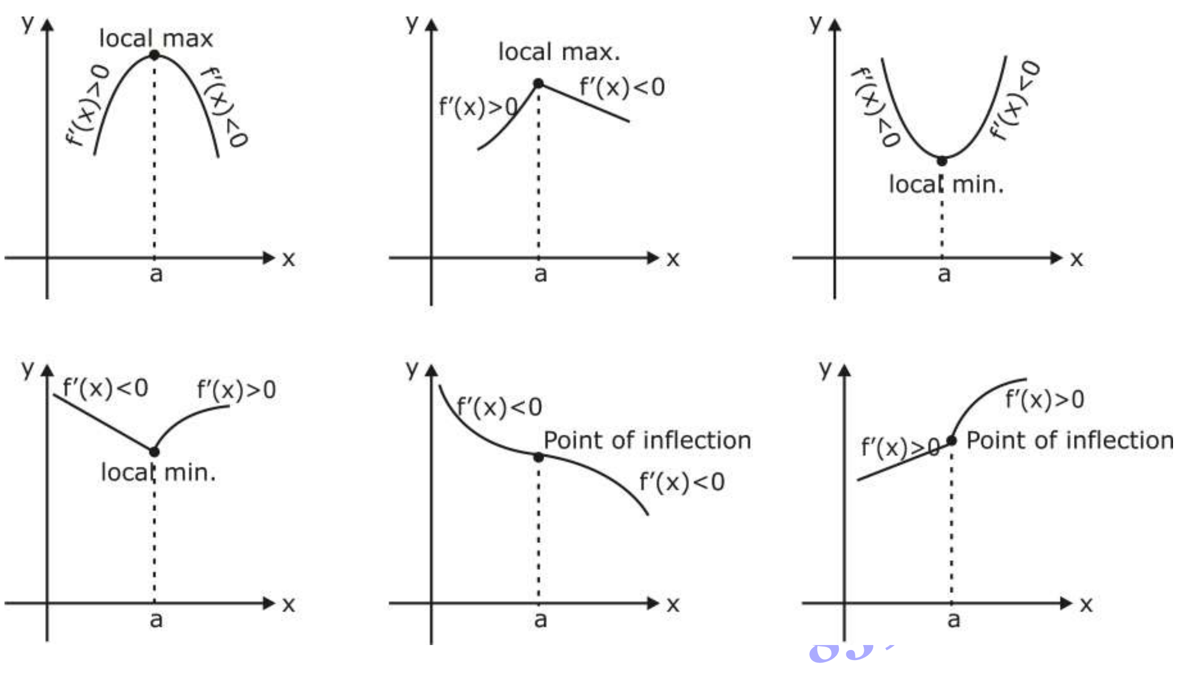
First derivative test is applicable only in those function for which the critical points are the points of continuity.

(i) If f '(x) changes sign from negative to positive while passing through x = a from left to right then x = a is a point of local minima.

(ii) If f '(x) changes sign from positive to negative while passing through x = a from left to right then x = a is a point of local maxima.

(iii) If f '(x) does not changes its sign about x = a then x = a is neither a point of maxima nor minima. (i.e. x=a is point of inflection)

following figure illustrates the I[st] derivative test



2. IInd derivative Test

If f(x) is continuous function in the neighborhood of x = 0 such that f'(x) = 0 and f''(a) exists then we can predict maxima or minima at x = 0 by examining the sign of f''(a)

(i) If f''(a) > 0 then x = a is a point of local minima.
(ii) If f''(a) < 0 then x = a is a point of local maxima.
(iii) If f''(a) = 0 then second derivative test does not give any use conclusive results.

3. nth derivative test

Let f(x) be function such that f'(a) = f''(a) = f''(a) = ... = $f^{n-1}(a) = 0$ and $f^n(a) = 0$, where f^n denotes the nth derivative, then

(i) If n is even and f^n (a) < 0, there is a local maximum at a, white if $f^n(a) > 0$, there is a local minimum at a.
(ii) If n is odd, there is no extremum at the point a.

(3) Maxima and Minima of Parametrically defined Functions

Let a function y = f(x) be defined parametrically as y = g(t) and x = h(t) where g and h both are twice differentiable functions for a certain interval of t and h'(t) $\neq 0$.

Now, $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{g'(t)}{h'(t)} = 0 \Rightarrow g'(t) = 0$

Let one of the root of g'(t) = 0 is say t_0.

Now $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{g'(t)}{g'(t)}\right) = \dfrac{d}{dt}\left(\dfrac{g'(t)}{h'(t)}\right).\dfrac{1}{dx/dt} = \dfrac{h'(t) - g'(t)h''(t)}{{}^3\{h'(t)\}}$

$\therefore \dfrac{d^2y}{dx^2} = \dfrac{g''(t)}{\{h'(t)\}^2}$ because g'(t) = 0. Here $\{h'(t)\}^2 > 0$,

Thus sign of $\dfrac{d^2y}{dx^2}$ is same as that of the sign of g"(t)

Now, if $(g''(t))_{t=t_0} < 0, f(x)$ has a max^m. at x = x_0 = h(t_0)

if $(g''(t))_{t=t_0} > 0, f(x)$ has a min^m. at x = x_0 = h(t_0) and so on.

(4) Global Maxima/Minima (or Absolute Maxima/Minima)

Global maximum or minimum value of f(x) $\forall\ x \in [a, b]$ refers to the greatest value and least value of f(x), mathematically

(i) If f(c) $\geq$ f(x) for $\forall\ x \in [a, b]$ then f(c) is called global maximum or absolute maximum value of f(x).

(ii) Similarly if f(d) $\leq$ f(x) $\forall\ x \in [a,b]$ then f(d) is called global minimum or absolute minimum value.

Note:

For function f(s) defined on [a,b], is c_1, c_2, c_3, ... c_n are the locations of critical point, then

Global maximum value = max{f(a), f(c_1), f(c_2), ... f(c_n), f(b)}
Global minimum value = min{f(a), f(c_1), f(c_2), ... f(c_n), f(b)}

(5) Useful Formulae of Mensuration to Remember :

1. Volume of a cuboid = ℓbh.

2. Surface area of cuboid = 2(ℓb + bh + hℓ).

3. Volume of cube = a^3

4. Surface area of cube = $6a^2$

5. Volume of a cone = $\frac{1}{3}\pi r^2 h$.

6. Curved surface area of cone = $\pi r \ell$ (ℓ = slant height)

7. Curved surface of a cylinder = $2\pi rh$.

8. Total surface of a cylinder = $2\pi rh + 2\pi r^2$.

9. Volume of a sphere = $\frac{4}{3}\pi r^3$.

10. Surface area of a sphere = $4\pi r^2$.

11. Area of a circular sector = $\frac{1}{2}r^2\theta$, when θ is in radians.

12. Volume of a prism = (area of the base) $\times$ (height).

13. Lateral surface of a prism = (perimeter of the base) $\times$ (height)

14. Total surface of a prism = (lateral surface) + 2(area of the base)

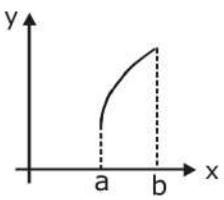
15. Volume of a pyramid = $\frac{1}{3}$(area of the base) $\times$ (height).

16. Curved surface of a pyramid = $\frac{1}{2}$(perimeter of the base) $\times$ (slant height)

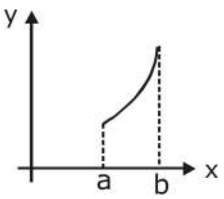
(6) Curvature of function

For continuous function f(x), If $f''(x_0) = 0$ or doesnot exist at points where $f'(x_0)$ exists and if $f''(x)$ changes sign when passing through $x = x_0$, then x_0 is called a point of inflection . At the point of inflection the curve changes its concavity i.e.

(i) If $f''(x) < 0$, $x \in (a,b)$ then the curve y = f(x) is convex in (a,b)



(ii) If $f''(x) > 0$, $x \in (a,b)$ then the curve y = f(x) is concave in (a,b).



18. INDEFINITE INTEGRAL

If f and g are functions of x such that g'(x) = f(x), then

$\int f(x)dx = g(x) + c \Leftrightarrow \frac{d}{dx}\{g(x) + c\} = f(x)$, where c is called the constant of integration.

Note:

(i) Integral of a function is also termed as primitive or antiderivative

(ii) The indefinite integral of a function geometrically represents a family of curves having parallel tangents at their points of intersection with the line orthogonal to the axis of variable of integration.

(1) Theorems on Integration

(i) $\int c\ f(x).dx = c\int f(x).dx$

(ii) $\int (f(x)\pm g(x))dx = \int f(x)dx \pm \int g(x)dx$

(iii) $\frac{d}{dx}\left(\int f(x)dx\right) = f(x)$

(iv) $\int \frac{d}{dx}(f(x)).dx = f(x) + c$.

(v) $\int f(x)dx = g(x)+c \Rightarrow \int f(ax+b)dx = \frac{g(ax+b)}{a}+c$

(2) Standard Formula :

(i) $\int x^2 dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$

(ii) $\int \frac{1}{x}dx = \ell n\ |x| + C$

(iii) $\int e^x dx = e^x + C$

(iv) $\int a^x dx = a^x/\ell n a + C$

(v) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c,\ n \neq -1$

(vi) $\int \frac{dx}{ax+b} = \frac{1}{a}\ \ell n\ |(ax + b)| + c$

(vii) $\int e^{ax+b}dx = \frac{1}{a}\ e^{ax+b} + c$

(viii) $\int a^{px+q}dx = \frac{1}{p}\ \frac{a^{px+q}}{\ell n a}\ a^{px+q}/\ell n a + c;\ a > 0$

(ix) $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax + b) + c$

(x) $\int \cos(ax+b)dx = \frac{1}{a} \sin(ax+b) + c$

(xi) $\int \tan(ax+b)dx = \frac{1}{a} \ln|\sec(ax+b)| + c$

(xii) $\int \cot(ax+b)dx = \frac{1}{a} \ln|\sin(ax+b)| + c$

(xiii) $\int \sec^2(ax+b)dx = \frac{1}{a} \tan(ax+b) + c$

(xiv) $\int \text{cosec}^2(ax+b)dx = -\frac{1}{a} \cot(ax+b) + c$

(xv) $\int \sec(ax+b).\tan(ax+b)dx = \frac{1}{a} \sec(ax+b) + c$

(xvi) $\int \text{cosec}(ax+b).\cot(ax+b)dx = -\frac{1}{a} \text{cosec}(ax+b) + c$

(xvii) $\int \sec x\, dx = \ln|(\sec x + \tan x)| + c$ or $\ln\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + c$

or $-\ln|(\sec x - \tan x)| + c$

(xviii) $\int \text{cosec}\, x\, dx = \ln|(\text{cosec}\, x - \cot x)| + c$

or $\ln\left|\tan\frac{x}{2}\right| + c$ or $-\ln|(\text{cosec}\, x + \cot x)| + c$

(xix) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$

(xx) $\int \frac{dx}{\sqrt{a^2+x^2}} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

(xxi) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

(xxii) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|\left[x+\sqrt{x^2+a^2}\right]\right| + C$

(xxiii) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|\left[x+\sqrt{x^2-a^2}\right]\right| + C$

(xxiv) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$

(xxv) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$

(xxvi) $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

(xxvii) $\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left(\frac{x+\sqrt{x^2+a^2}}{a}\right) + c$

(xxviii) $\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left(\frac{x+\sqrt{x^2-a^2}}{a}\right) + c$

(xxix) $\int e^{ax}.\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + c$ or $\frac{e^{ax}}{\sqrt{a^2+b^2}} \sin\left(bx - \tan^{-1}\left(\frac{b}{a}\right)\right) + C$

(xxx) $\int e^{ax}.\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) + c$ or $\frac{e^{ax}}{\sqrt{a^2+b^2}} \cos\left(bx - \tan^{-1}\left(\frac{b}{a}\right)\right) + C$

(3) Integration by Subsitutions

If $x = \phi(t)$ is substituted in a integral, then

- everywhere x is replaced in terms of t.
- dx gets converted in terms of dt.
- $\phi(t)$ should be able to take all possible value that x can take.

Note:

(i) $\int [f(x)^n f'(x)\, dx = \frac{(f(x))^{n+1}}{n+1} + C$

(ii) $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$

(iii) $\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + C$

(iv) $\int \sqrt{f(x)}.f'(x)dx = \frac{2}{3} f(x)\sqrt{f(x)} + C$

(4) Integration by Part :

If f(x) and g(x) are the first and second function respectively then

$\int f(x).g(x)dx = f(x).\int g(x)dx - \int \left\{f'(x).\int g(x)dx\right\} dx$

- The choice of f(x) and g(x) is decided by **ILATE rule**. The function which comes later is taken as second function (integral function).

  I $\rightarrow$ Inverse function
  L $\rightarrow$ Logarithmic function
  A $\rightarrow$ Algebraic function
  T $\rightarrow$ Trigonometric function
  E $\rightarrow$ Exponential function

Note:

(i) $\int e^x[f(x)+f'(x)]dx = e^x.f(x)+c$

(ii) $\int [f(x)+xf'(x)]dx = xf(x)+c$

(5) Integration of Rational Algebraic Functions by using Partial Fractions :

PARTIAL FRACTIONS :

If f(x) and g(x) are two polynomials, then $\frac{f(x)}{g(x)}$ defines a rational algebraic function of a rational function of x.

If degree of f(x) < degree of g(x), then $\frac{f(x)}{g(x)}$ is called a proper rational function.

If degree of f(x) $\geq$ degree of g(x then $\frac{f(x)}{g(x)}$ is called an improper rational function

If $\frac{f(x)}{g(x)}$ is an improper rational function, is divided f(x), by g(x) so that the rational function $\frac{f(x)}{g(x)}$ is expressed in the form $\phi(x) + \frac{\Psi(x)}{(g)x}$ where $\phi(x)$ and $\psi(x)$ are polynomials such that the degree of $\psi(x)$ is less than that of g(x). Thus, $\frac{f(x)}{g(x)}$ is expressible as the sum of a polynomial and a proper rational function.

Any proper rational function $\frac{f(x)}{g(x)}$ can be expressed as the sum of rational functions, each having a simple factor of g(x). Each such fraction is called a partial fraction and the process of obtaining them is called the resolution or decomposition of $\frac{f(x)}{g(x)}$ into partial fractions.

The resolution of $\frac{f(x)}{g(x)}$ into partial fractions depends mainly upon the nature of the factors of g(x) as discussed below.

CASE I When denominator is expressible as the product of non-repeating linear factors.
Let $g(x) = (x-a_1)(x-a_2)..........(x-a_n)$. Then, we assume that

$$\frac{f(x)}{g(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + ... + \frac{A_2}{x-a_n}$$

where $A_1, A_2, ... A_n$ are constants and can be determined by equating the numerator on R.H.S. to the numerator on L.H.S. and then substituting $x = a_1, a_2, ..., a_n$.
(Refer section 14,15,16,17 in **A List of Evaluation Techniques**)

Note:

In order to determine the value of constants in the numerator of the partial fraction corresponding to the non-repeated linear factor px + q in the denominator of a rational expression, we may proceed as follows :

Replace $x = -\frac{q}{p}$ (obtained by putting px + q = 0) everywhere in the given rational expression except in the factor px + q itself. For example, in the above illustration the value of A is obtained by replacing x by 1 in all factors of $\frac{3x+2}{(x-1)(x-2)(x-3)}$ except (x−1) i..e

$$A = \frac{3\times1+2}{(1-2)(1-3)} = \frac{5}{2}$$

Similarly, we have

$$B = \frac{3\times2+1}{(1-2)(2-3)} = -8 \text{ and, } C = \frac{3\times3+2}{(3-1)(3-2)} = \frac{11}{2}$$

A List of Evaluation Techniques

S.NO.	Form of Integrate	Value/Evaluation Techique
1.	$\int \sin^m x \cos^n x \, dx$ where where m, n $\in$ N	If m is odd put cos x = t If n is odd put sin x = t If both m and n are odd, put sin x = t if m $\geq$ n and cos x = t otherwise. If both m and n are even, use power reducting formulae $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ If m + n is a negative even integer , put tan x = t
2.	$\int \frac{a \sin x + b \cos x + c}{p \sin x + q \cos x + r} dx$ $\int \frac{ae^x + be^{-x}}{pe^x + qe^{-x}} dx$ (p, q, $\neq$ 0 and at least one of a, b 0)	Write Num = α (DEN) $\beta \frac{d}{dx}$ (DEN) + γ Find α, β and by comparing the eoefficients of cos x, sin x and constant term (or e^x, e^{-x}) write the integral as $\alpha \int dx + \beta \int \frac{\frac{d}{dx}(\text{DEN})}{\text{DEN}} dx + \gamma \int \frac{dx}{\text{DEN}}$
3.	$\int \frac{a \tan x + b}{p \tan x + q} dx$ $\int \frac{ae^{2x} + b}{pe^{2x} + q} dx$ (p , q $\neq$ 0, at least one of a , b $\neq$ 0)	Change to $\int \frac{a \sin x + b \cos x}{p \sin x + q \cos x} dx$ or $\int \frac{ae^x + be^{-x}}{be^x + qe^{-x}} dx$ and use (8) above
4.	$\int \frac{dx}{a \sin x \pm b \cos x}$ (Both a, b $\neq$ 0)	Use a = r cos α, b = r sin α to put the interal in the form $\frac{1}{r} \int \frac{dx}{\sin(x \pm \alpha)}$. Now, use formula for $\int \text{cosec } x \, dx$
5.	$\int \frac{dx}{\sqrt{Q}}$ or $\int \frac{L}{Q} dx$ or $\int \frac{dx}{Q}$ or $\int \frac{L}{Q} dx$ or $\int \sqrt{Q}$ or $\int L\sqrt{Q}$ where $L = px + q$, $p \neq 0$ and $Q = ax^2 + bx + c$, $a \neq 0$	Write $Q = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$ and put $t = x + \frac{b}{2a}$

S.NO.	Form of Integrate	Value/Evaluation Techique
6.	$\int \sqrt{\frac{L_1}{L_2}}dx$ where $L_1 = ax+b, a \neq 0$, $L_2 = px + q$, $p \neq 0$	Write as $\int \sqrt{\frac{L_1}{L_1L_2}}dx$ to reduce to form $\int \sqrt{\frac{L}{Q}}dx$
7.	$\int \frac{\sqrt{Q}}{L}dx$ where $L = px + q, p \neq 0$ and $Q=ax^2 + bx + c, a \neq 0$	Put $L = \frac{1}{t}$ to reduce to form $\int \frac{dx}{\sqrt{Q}}$
8.	$\int \frac{\sqrt{Q}}{L}dx$ where $L = px + q, p \neq 0$, $Q = ax^2 + bx + c$, $a \neq 0$	Rewrite the integrand as $\frac{Q}{L\sqrt{Q}}$ Divide Q by L to obtain $Q = L_1(L) + \alpha$ where L_1 is a linear expression in x and α is a constant. The integral reduces to $\int \frac{L_1}{\sqrt{Q}}dx + \alpha\int \frac{dx}{L\sqrt{Q}}$.
9.	$\int \frac{Q_1}{\sqrt{Q_2}}dx$ where $Q_1 = ax_2+bx+c$, $a \neq 0$, $Q_2 = px^2+qx+r$, $p \neq 0$	Write $Q_1 \equiv \alpha\ Q_2 + \beta\frac{d}{dx}[Q_2] + \gamma$ Find α, β, γ and write the integral as $\alpha\int \sqrt{Q_2}dx + \beta\int \frac{\frac{d}{dx}[Q_2]}{\sqrt{Q}}dx + y\int \frac{dx}{\sqrt{Q_2}}$
10.	$\int \frac{dx}{L_1\sqrt{L_2}}$ where $L_1 = ax + b$ and $L_2 = px + q$, with $a , p \neq 0, a \neq p$.	Put $L_2 = t^2$
11.	$\int \frac{dx}{Q_1\sqrt{Q_2}}$ where $Q_1=ax^2+b, Q_2=px^2+q$ with $a, p \neq 0$, $p \neq 0$	First put $x=1/t$ and simplify then put $p + qt^2 = u^2$
12.	$\int R(x^{1/p}, x^{1/q}, x^{1/t}\ldots)dx$ where R is a rational function.	Let l = lcm (p, q, r, ...) and put $x = t^l$

S.NO.	Form of Integrate	Value/Evaluation Techique
13.	$\int \frac{P(x)}{Q(x)}dx$ where P(x) and Q(x) are polynomials in x	If deg P(x) $\geq$ deg Q(x) first isolate the quotient A(x) when P(x) is divided by Q(x). Now, $\frac{P(x)}{Q(x)} = A(x) + \frac{B(x)}{Q(x)}$ where deg B(x) < deg Q(x) Now use techniques 22 to 29
14.	$\int \frac{P(x)}{Q(x)}dx$ where deg P(x)<deg Q(x) and Q(x) is a product of distinct linear factors, that is $Q(x)=A(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_r)$ where a_i's are distinct and A $\neq$ 0	Write $\frac{P(x)}{Q(x)} = \frac{A_1}{x-\alpha_1} + \frac{A_2}{x-\alpha_2} + \ldots + \frac{A_n}{x-\alpha_n}$ Evaluate A_i's and use $\int \frac{dx}{x-a} = \log\|x-a\|$
15.	$\int \frac{P(x)}{(x-a)^n}dx$ where $n \geq 1$ and P(x) is polynomial in x.	Put x–a = t, express P(x) in terms of t_s and then integrate.
16.	$\int \frac{P(x)dx}{(x-a)^m(x-b)^n}$ where $m, n \geq 1, a \neq b$, and deg P(x) < m + n.	Write the integrand as $\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \ldots \frac{A_m}{(x-a)^m} + \frac{B_1}{x-b} + \frac{B_2}{(x-b)^2} + \ldots + \frac{B_n}{(x-b)^n}$ Evaluate A_i's and B_j' and the integral.
17.	$\int \frac{P(x)dx}{(x-a)(x^2+bx+c)}$ where $b^2 - 4c < 0$ and deg P(x) < 3	Write the integrand as $\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$
18.	$\int \frac{x^2+1}{x^4+kx^2+1}dx$	Divide the numerator by x^2 to obtain $\int \frac{(1+1/x^2)}{x^2+1/x^2+k}$ Now, put x–1/x = t
19.	$\int \frac{x^2-1}{x^4+kx^2+1}dx$	Divide the numerator by x^2 to obtain $\int \frac{(1-1/x^2)dx}{x^2+1/x^2+k}$ Now , put x + 1/x = t

S.NO.	Form of Integrate	Value/Evaluation Techique
20.	$\int \frac{xP(x^2)}{Q(x^2)}dx$ where P and Q are polynomials in x	Put $x^2 = t$
21.	$\int \frac{P(x^2)}{Q(x^2)}dx$ where P and Q are polynomials in x	Put $x^2 = t$ for partial fraction not for integration
22.	$\int \frac{1}{(x^2+a^2)^n}dx = A_n$ where n > 1	Begin with A_{n-1} $A_{n-1} = \int 1.\frac{1}{(x^2+a^2)^{n-1}}dx$
23.	$\int R(\sin x, \cos x)dx$ where R is a rational function Special Cases	Universal substitution $\tan(x/2) = t$.
	(a) If $R(-\sin x, \cos x) = -R(\sin x, \cos x)$	Put $\cos x = t$
	(b) If $R(\sin x, -\cos x) = -R(\sin x, \cos x)$	Put $\sin x = t$
	(c) If $R(-\sin x, -\cos x) = R(\sin x, \cos x)$	Put $\tan x = t$
24.	$\int x^m(a+bx^n)^p\,dx$ where m, n, p are rational numbers,	
	(a) p is a positive integer	Expand $(a + bx^n)^n$ by using binomial theorem
	(b) p is a negative integer,	Put $x = t^k$
	$m = \frac{a}{b}$, $n = \frac{c}{d}$,	where k = L CM (b,d)
	a, b, c, d $\in$ I, b > 0, d > 0	where k = L CM (b,d)
	(c) $\frac{m+1}{n}$ + p is an integer	Put $a + bx^n = t^s$ where $p = r/s$, r, s $\in$ I, s > 0
	(d) $\frac{m+1}{n} + p$ is an integer	Put $a + bx^{n} = x^n t^s$ where $p = r/s$, r , s $\in$ I, s > 0
25.	$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}dx$ where $P_n(x)$ is a polynomial of degree n in x	Write $\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}dx = Q_{n-1}(x)\sqrt{ax^2+bx+c} + k\int \frac{dx}{\sqrt{ax^2+bx+c}}$ where $Q_{n-1}(x)$ is polynomial of degree (n–1) in x and k is a constant. Differentiate both the sides w.r.t. x and multiplying by $\sqrt{ax^2+bx+c}$ to get the identiy $P_n(x) = Q'_{n-1}(x)(ax^2 + bx+c) + 1/2\, Q_{n-1}(x)(2ax+b) + k$ Compare coefficients to obtain $Q_{n-1}(x)$ and k.

19. DEFINITE INTEGRAL

(1) Newton-Leibnitz formula

Let f(x) be a continuous function defined on [a,b] and $\int f(x)dx = F(x) + c$, then

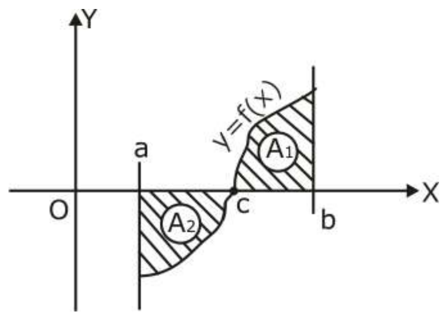
$\int_a^b \mathbf{f(x)dx = F(b) - F(a)}$ is called definite integral and this formula is known as Newton-Leibnitz formula.

Note:

(i) Newton-Leibnitz formula is only applicable iff integrand f(x) is continous in [a,b].

(ii) Geometrically definite integral $\int_a^b f(x)dx$ represents the algebraic sum of the area bounded by integrand f(x) with the x-axis, lines x = a and x = b, bounded area above the x-axis is treated as positive and bounded area below the x-axis is treated as negative.

$\int_a^b f(x)dx = A_1 - A_2.$



(2) Method of Substitution in a Definite Integral

If evaluation of definite integral $\int_a^b f(x)dx$, if variable x is substituted as $\phi(t)$, where $\phi(\alpha) = a$ and $\phi(\beta) = b$, then

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\phi(t)).\phi'(t)dt.$$

Note:

Substitution $x = \phi(t)$ must satisfy the following conditions.

- $\phi(t)$ is a continuous single-valued function defined in $[\alpha, \beta]$ and has in this interval a continuous derivative $\phi'(t)$;
- with t varying on $[\alpha, \beta]$ the values of the function $x = \phi(t)$ do not leave the limits of [a, b];
- $\phi(\alpha) = a$ and $\phi(\beta) = b$.

(3) Properties of Definite Integral

Property:1 $\int_a^b f(x)dx = \int_a^b f(t)dt$

definite integral is independent of variable of integration.

Property:2 $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Property:3 $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,$

where c may lie inside or outside the interval [a,b].

Property:4 $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

Note:

(i) If $I = \int_a^b f(x)dx$, than by property –4,

$$I = \int_a^b f(a+b-x)dx \Rightarrow 2I = \int_a^b \{f(x)+f(a+b-x)\}dx$$

(ii) If $I = \int_a^b \frac{f(x)}{f(x)+f(a+b-x)}dx$, then $I = \frac{b-a}{2}$.

Property:5 $\int_{-a}^{a} f(x)dx = \int_0^a (f(x)+f(-x))dx$

$$\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & ;\ \text{if } f(-x)=f(x)\ \text{(i.e. } f(x) \text{ is even function)} \\ 0 & ;\ \text{if } f(-x)=-f(x)\ \text{(i.e. } f(x) \text{ is odd function)} \end{cases}$$

Property:6 $\int_0^{2a} f(x)dx = \int_0^a (f(x)+f(2a-x))dx$

$$\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & ;\ \text{if } f(2a-x)=f(x) \\ 0 & ;\ \text{if } f(2a-x)=-f(x) \end{cases}$$

Note:

(i) If function f satisfy the condition f(2a–x) = f(x), then graph of f(x) is symmetrical about the line x = a.

(ii) If function f satisfy the condition f(2a–x) = –f(x), then graph of f(x) is symmetrical about point (a, 0)

(iii) $\int_0^{\pi/2} \log_e^{(\cos x)} dx = \int_0^{\pi/2} \log_e^{(\sin x)} dx = \frac{\pi}{2} \log_e^{1/2}$

Property:7 If f(x) is a periodic function with period T and m, n $\in$ I, then

(i) $\int_0^{nT} f(x)dx = n\int_0^{T} f(x)dx, \quad n \in z$

(ii) $\int_a^{a+nT} f(x)\,dx = n\int_0^{T} f(x)\,dx, \quad n \in z, a \in R$

(iii) $\int_{mT}^{nT} f(x)\,dx = (n-m)\int_0^{T} f(x)\,dx, \; m, n \in z$

(iv) $\int_{nT}^{a+nT} f(x)\,dx = \int_0^{a} f(x)\,dx, \; n \in z, a \in R$

(v) $\int_{a+nT}^{b+nT} f(x)\,dx = \int_a^{b} f(x)dx \; n \in z, a, b \in R$

(4) Standard Results for Definite Integral

(i) If $f(x) \ge 0 \;\forall\; x \in [a,b]$, then $\int_a^b f(x)dx \ge 0$

(ii) If $f(x) \le 0 \;\forall\; x \in [a,b]$, then $\int_a^b f(x)dx \le 0$

(iii) If $\psi(x) \le f(x) \le \phi(x)$ for $a \le x \le b$, then

$$\int_a^b \Psi(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b \phi(x)\,dx$$

(iv) If $m \le f(x) \le M$ for $a \le x \le b$, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

Further if f(x) is monotonically decreasing in (a,b) then

$f(b)\,(b-a) < \int_a^b f(x)\,dx < f(a)(b-a)$ and if f(x) is monotonically increasing in (a,b) then

$f(a)(b-a) < \int_a^b f(x)\,dx < f(b)\,(b-a)$.

(v) If f(x) is continous in [a,b], then $\int_a^b |f(x)|\,dx \ge \left|\int_a^b f(x)dx\right|$; the sign of equality holds if f(x) is non-negative or non-positive for all $x \in [a,b]$.

(vi) If f(x) is continous and increasing function in [a,b] and have concave graph, then

$$(b-a)f(a) < \int_a^b f(x)dx < (b-a)\left(\frac{f(a)+f(b)}{2}\right)$$

(vii) If f(x) is continous and increasing function in [a,b] and have convex graph, then

$$(b-a)\left(\frac{f(a)+f(b)}{2}\right) < \int_a^b f(x)dx < (b-a)f(b).$$

(viii) If f(x) and g(x) are continous function for all $x \in [a,b]$ and $f^2(x)$, $g^2(x)$ are integrable functions, then $\left|\int_a^b f(x).\ g(x)\,dx\right| \le \sqrt{\left(\int_a^b f^2(x)dx\right)\left(\int_a^b g^2(x)dx\right)}$

(ix) If f(x) is continuous on the interval [a,b], then $\int_a^b f(x)dx = f(c)(b-a)$; where a < c < b.

(x) **Leibnitz Rule:**

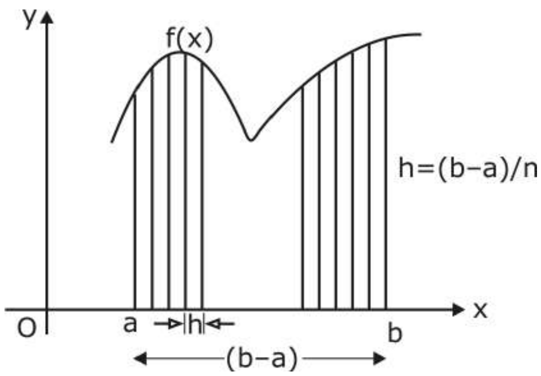
If $F(x) = \int_{\phi(x)}^{\psi(x)} f(t)dt$, then **$F'(x) = f(\psi(x)).\psi'(x) - f(\varphi(x)).\ \varphi'(x).$**

(5) Definite Integral as a Limit of Sum.

If f(x) is the continuous function in the interval [a,b] where a and b are finite and b > a and if the interval [a,b] be divided into n equal parts, each of width h so that n h = b – a , Then $\lim_{h\to 0}[h\{f(a)+f(a+h)+f(a+2h)+\dots\dots\dots+f(a+(n-1)h\}]$ is called the definite integral of f(x) between limits a and b,

$$\therefore \quad \int_a^b f(x)dx = \underset{h\to 0}{\text{lt}}\, h[\{f(a)+f(a+h)+f(a+2h)+\dots\dots+f(a+(n-1)h)]$$

$$= \lim_{h\to 0}\left[h.\sum_{r=0}^{n-1} f(a+rh)\right]$$



If should be noted that as $h \to 0$, $0 \to \infty$

nh = b – a

Putting a = 0 b = 1, so that h = 1/n.

We get $\int_0^1 f(x)\,dx = \lim_{n\to\infty} \frac{1}{n}\ .\ \sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)$

Working Rule

(i) Express the given series in the form of $\sum \frac{1}{n}\ f\left(\frac{r}{n}\right)$

(ii) The limit when $n \to \infty$ is its sum $\lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{1}{n}.\ f\left(\frac{r}{n}\right)$ Replace r/n by x, 1/n by dx and $\lim_{n\to\infty}\Sigma$ by the sign of integration $\int$.

(iii) The lower and upper limits of integration will be the value of r/n for the first and last term (or the limits of these values respectively).

Note:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r=pn+a}^{qn+b} f\left(\frac{r}{n}\right)=\int_p^q f(x)dx.$$

(6) Important Results:

If $I_n = \int_0^{\pi/2}\sin^n x dx = \int_0^{\pi/2}\cos^n x dx$, then

$$I_n = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right) .... I_0 \text{ or } I_1$$

according as n is even or odd , $I_0 = \frac{\pi}{2}$, $I_1 = 1$

$$\text{Hence } I_n = \begin{cases}\left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)...\left(\frac{1}{2}\right).\frac{\pi}{2} & ; \quad n \text{ is even}\\ \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)...\left(\frac{2}{3}\right).1 & ; \quad n \text{ is odd}\end{cases}$$

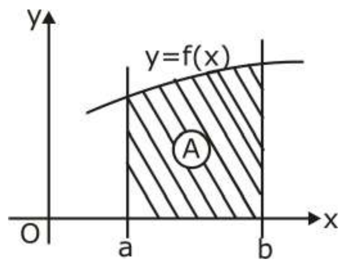
If $I_{m,n} = \int_0^{\pi/2}\sin^n x\cos^n x dx$ m, $n\in N$ then,

$$I_{m,n} = \begin{cases}\frac{(n-1)(n-3)(m-5)......(n-1)(n-3)(n-5).....}{(m+1)(m+n-2)(m+n-4)......}\frac{\pi}{2} & ; \quad \text{when both m,n are even}\\ \frac{(m-1)(m-3)(m-5)......(n-1)(n-3)(n-5).....}{(m+n)(m+n-2)(m+n-4)......} & ; \quad \text{when both m,n are not even}\end{cases}$$

20. AREA BOUNDED BY CURVES

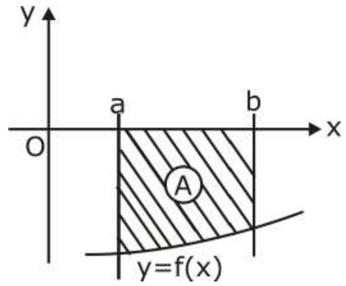
Following cases illustrates the method of computation of bounded area under different conditions:

(i)



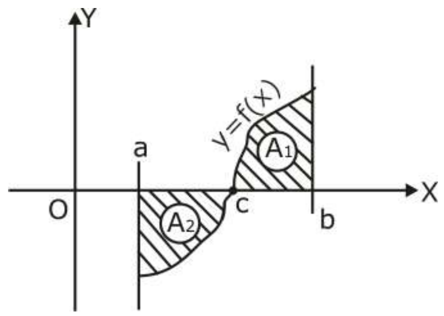
$$A = \int_a^b f(x)dx.$$

(ii)



$$A = -\int_a^b f(x)dx.$$

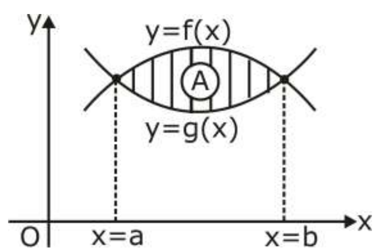
(iii)



$$A = A_1 + A_2 \Rightarrow$$

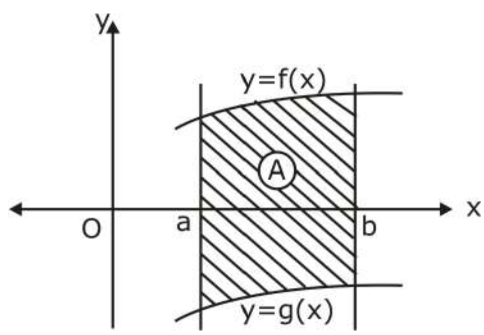
$$A = -\int_a^c f(x)dx + \int_c^b f(x)dx$$

(iv)



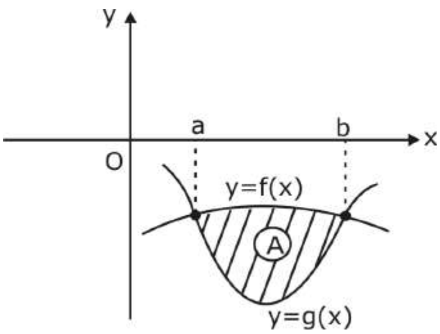
$$A = \int_a^b \left(f(x) - g(x)\right)dx.$$

(v)



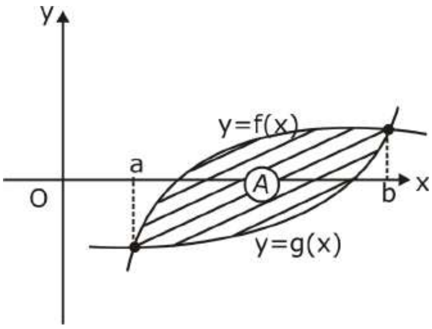
$$A = \int_a^b \left(f(x) - g(x)\right)dx.$$

(vi)



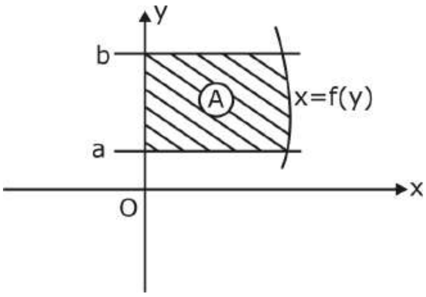
$$A = \int_a^b \left(f(x) - g(x)\right) dx.$$

(vii)



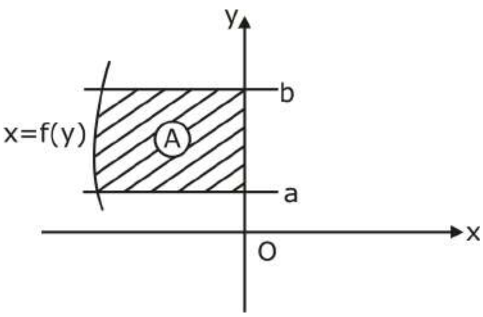
$$A = \int_a^b \left(f(x) - g(x)\right) dx.$$

(viii)



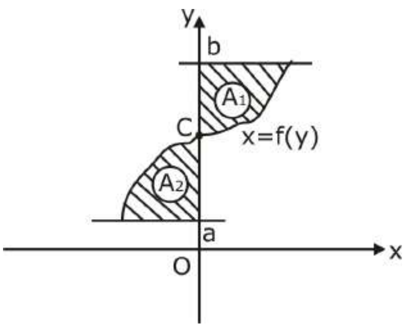
$$A = \int_a^b f(y)dy$$

(ix)



$$A = -\int_a^b f(y)dy$$

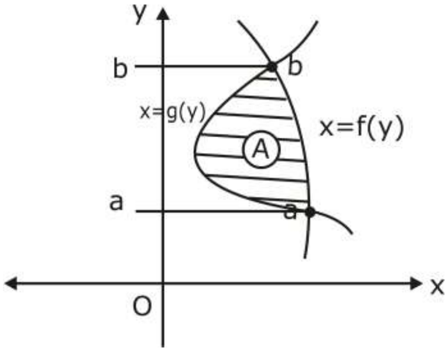
(x)



$$A = A_1 + A_2 \Rightarrow$$

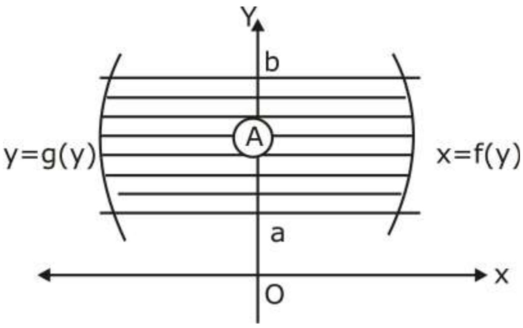
$$A = -\int_a^c f(y)dy + \int_c^b f(y)dy.$$

(xi)



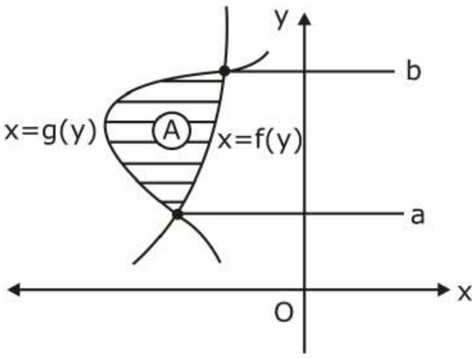
$$A = \int_a^b \left(f(y) - g(y)\right) dy$$

(xii)



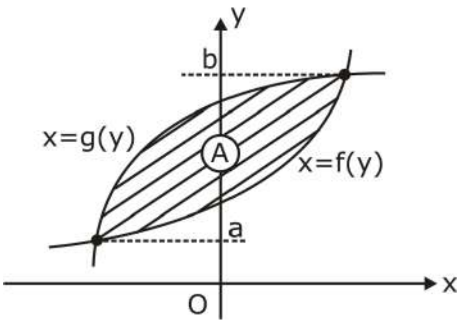
$$A = \int_a^b \left(f(y) - g(y)\right) dy$$

(xiii)



$$A = \int_a^b \left(f(y) - g(y)\right) dy$$

(xiv)



$$A = \int_a^b \left(f(y) - g(y)\right) dy$$

Curve Tracing :

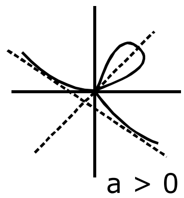
To find the approximate shape of a non-standard curve, the following steps must be checked .

(1) Symmetry about x-axis :
If all the powers of 'y' in the equation are even then the curve is symmetrical about the x-axis.

(2) Symmetry about y-axis :
If all the powers of 'x' in the equation are even then the curve is symmetrical about the y-axis.

(3) Symmetry about both axis :
If all the powers of 'x' and 'y; in the equation are even, the curve is symmetrical about the axis of 'x' as well as 'y'.

(4) Symmetry about the line y = x :
If the equation of the curve remains unchanged on interchanging 'x' and 'y', then the curve is symmetrical about the line y = x.



$a > 0$

E.g.: $x^2 + y^3 = 3axy$.

(5) Symmetry in opposite quadrants :
If the equation of the curve remains unaltered when 'x' and 'y' are replaced by $-x$ and $-y$ respectively then there is symmetry in opposite quadrants.

(6) Intersection points of the curve with the x-axis and the y-axis.

(7) Locations of maxima and minima of the curve.

(8) Increasing or decreasing behaviour of the curve.

(9) Nature of the curve when $x \to \infty$ or $x \to -\infty$.

(10) Asymptotes of the curve.

Asymptoto(s) is (are) line(s) whose distance from the curve tends to zero as point on curve moves towards infinity along branch of curve.

(i) If $\underset{x\to a}{\text{Lt}} f(x) = \infty$ or $\underset{x\to a}{\text{Lt}} f(x) = -\infty$ then x= a is asymptote of y = f(x)

(ii) If $\underset{x\to+\infty}{\text{Lt}} f(x) = k$ or $\underset{x\to-\infty}{\text{Lt}} f(x) = k$, then y = k is asymptote of y = f(x)

(iii) If $\underset{x\to\infty}{\text{Lt}} \dfrac{f(x)}{x} = m_1$, $\underset{x\to\infty}{\text{Lt}} f(x) = m_1(x) = c$, then $y = m_1 x + c_1$ is an asymptote.
(inclined to right)

(iv) If $\underset{x\to\infty}{\text{Lt}} \dfrac{f(x)}{x} = m_2$, $\underset{x\to\infty}{\text{Lt}} (f(x - m_2 x) = c_2$, then $y = m_2 x + c_2$ is an asymptote
(inclined to left).

21. DIFFERENTIAL EQUATION

(1) Introduction :

An equation involving independent and dependent variables and the derivatives of the dependent variables is called a **differential equation**. There are two kinds of differential equation .

Ordinary Differential Equation : If the dependent variables depend on one independent variable x, then the differential equation is said to be ordinary. For example:

$$\frac{dy}{dx} + xy = \sin x, \qquad \frac{d^3y}{dx^3} + 2\frac{dy}{dx} + y = e^x,$$

$$k\frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}, \; y = x\frac{dy}{dx} + k\sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}$$

Partial differential equation : If the dependent variables depend on two or more independent variables, then it is known as partial differential equation. For example

$$y^2\frac{\partial z}{\partial x} + y\frac{\partial^2 z}{\partial y} = ax, \; \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = xy.$$

(2) Linear and Non-Linear Differential Equations:

A differential equation is a linear differential equation if it is expressible in the form

$$P_0\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + P_{n-1}\frac{dy}{dx} + P_n y = Q$$

where P_0, P_1, P_2, ... , P_{n-1} , P_n and Q are either constants of function of independent variable x.

Thus, if a differential equation when expressed in the form of a polynomial involves the derivatives and dependent variable in the first power and there are no product of these, and also the coefficient of the various terms are either constants or function of the independent variable, then it is said to be linear differential equation. Otherwise, it is a non linear differential equation.

If follows from the above definition that a differential equation will be non-linear differential equation if

(i) its degree is more than one.

(ii) any of the differential coefficient has exponent more than one

(iii) exponent of the dependent variable is more than one.

(iv) products containing dependent variable and its differential coefficients are present.

(3) Order and Degree of a Differential Equation :

Order: Order of differential equation is the order of highest order derivative which is present in the equation.

Degree: degree of differential equation is the highest exponent with which the highest order derivative is raised in the differential equation, provided the differential equation is expressed in the polynomial form.

Note:

If differential equation can't be expressed in the polynomial form, then its degree is not defined.

Example: $\frac{d^2y}{dx^2} + x\left(\frac{d^3y}{dx^3}\right)^2 + y = 0$ is differential equation of 3 order and 2 degree.

(4) Geometrical Significance of Differential Equation:

Geometrically, differential equation represents a family of curves which is having as many number of parameters as that of the order of differential equation. For example

(i) $\frac{dy}{dx} = e^x$ represents $y = e^x + c$.

(ii) $y\frac{dy}{dx} + x = 0$ represents $x^2 + y^2 = c$.

(iii) $\frac{d^2y}{dx^2} = e^x$ represents $y = e^x + c_1x + c_2$.

(5) Formation of Differential Equation:

In order to formulate a differential equation for a given family of curves of n parameter (independent arbitrary constants), following steps is used.

Step (1): differentiate the family of curves 'n' times to obtain n set of equations.

Step (2): from the set of equations of step(1) and the equation of curve, all the arbitrary constants are eliminated to get the differential equation.

Note:

Differential equation for a family of curves should not have any arbitrary constant and its order must be same as that of the number of parameters in family of curves.

(6) Solution of a Differential Equation :

Finding the dependent variable from the differential equation is called solving or integrating it. The solution or the integral of a differential equation is , therefore, a relation between dependent and independent variables (free from derivatives) such that it satisfies the given differential equation

Note:

The solution of the differential equation is also called its primitive, because the differential equation can be regarded as a relation derived from it.

There can be three types of solution of a differential equation :

(i) General solution (or complete integral or complete primitive) :A relation in x and y satisfying the given differential equation and involving exactly same number of arbitrary constants as that of the order of differential equation.

(ii) Particular solution : A solution obtained by assigning values to one or more than one arbitrary constant of general solution of a differential equation.

(7) Elementary Types of First Order and First Degree Differential Equations :

Variables separable : If the differential equation can be put in the form, $f(x)\,dx = \phi(y)\,dy$ we say that variables are separable and solution can be obtained by integrating each side separately.

A general solution of this will be $\int f(x)\,dx = \int \phi(y)\,dy + c$,

where c is an arbitrary constant.

Equations Reducible to the Variables Separable form :

If a differential equation can be reduced into a variables separable form by a proper substitution, then it is said to be "Reducible to the variables separable type". Its general form is

$$\frac{dy}{dx} = f(ax + by + c) \quad \text{where } a, b \neq 0.$$

substitution, $ax + by + c = t$. reduces the general form in to variable separable form

Homogeneous differential Equations :

A differential equation of the from $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ where f and g are homogeneous function of x and y and of the same degree, is called homogeneous differential equation and can be solved by putting $y = vx$.

Equations Reducible to the Homogeneous form

Equations of the form $\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}$ .... (1)

can be made homogeneous (in new variables X and Y) by substituting $x = X + h$ and $y = Y + k$, where h and k are constants, we get $\frac{dY}{dX} = \frac{aX+bY+(ah+bk+c)}{AX+BY+(Ah+Bk+C)}$ ....(2)

Now, h and k are chosen such that $ah + bk + c = 0$, and $Ah + Bk + C = 0$; the differential equation can now be solved by putting $Y = vX$.

Note:

If the homogeneous equation is of the form :
$yf(xy)\,dx + xg(xy)\,dy = 0$, the variables can be separated by the substitution $xy = v$.

Exact Differential Equation :

The differential equation $M(x,y)dx + N(x,y)dy = 0$ ....(1)
where M and N are functions of x and y is said to be exact if it can be derived by direct differentiation of an equation of the form $f(x,y) = c$

Note:

(i) The necessary condition for exact differential equation is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

(ii) For finding the solution of exact differential equation, following exact differentials must be remembered:

(a) $xdy + ydx = d(xy)$ (b) $\frac{xdy-ydx}{x^2} = d\left(\frac{y}{x}\right)$ (c) $2(xdx + ydy) = d(x^2 + y^2)$

(d) $\frac{xdy-ydx}{xy} = d\left(\ln\frac{y}{x}\right)$ (e) $\frac{xdy-ydx}{x^2+y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$ (f) $\frac{xdy+ydx}{xy} = d(\ln xy)$

(g) $\frac{xdy+ydx}{x^2y^2} = d\left(-\frac{1}{xy}\right)$ (h) $\frac{xdx+ydy}{x^2+y^2} = d\left(\frac{1}{2}\ln(x^2+y^2)\right)$

Linear differential equations of first order and first degree:

The differential equation $\frac{dy}{dx} + Py = Q$, is linear in y.
where P and Q are functions of x.

Integrating Factor **(I.F.) :** It is an expression which when multiplied to a differential equation converts it into an exact form.

I.F for linear differential equation = $e^{\int Pdx}$
(constant of integration need not to be considered)
$\therefore$ after multiplying above equation by I. F it becomes ;

$$\frac{dy}{dx}.\ e^{\int Pdx} + Py.e^{\int Pdx} = Q.e^{\int Pdx}$$

$$\Rightarrow \frac{d}{dx}(y.e^{\int Pdx}) = Q.e^{\int Pdx} \quad \Rightarrow \quad y.e^{\int Pdx} = \int Q.e^{\int Pdx} + C \quad \text{(General solution).}$$

Bernoulli's equation :

Equations of the form $\frac{dy}{dx} + Py = Q.y^n$, $n \neq 0$ and $n \neq 1$
where P and Q are functions of x, is called Bernoulli's equation

To convert Bernoulli's equation to linear form $\frac{1}{y^{n-1}}$ is substituted as t.

Clairaut's Equation: Differential equation of the form $y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right)$ is termed as clairaut's equation. To get the solution of this form , the equation is differentiated as , explained in the next step:

$$\cancel{\frac{dy}{dx}} = x\frac{d^2y}{dx^2} + \cancel{\frac{dy}{dx}} + f'\left(\frac{dy}{dx}\right).\frac{d^2y}{dx^2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}\left(x + f'\left(\frac{dy}{dx}\right)\right) = 0.$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 0 \qquad \text{or} \qquad x + f'\left(\frac{dy}{dx}\right) = 0.$$

$$\Rightarrow \quad \frac{dy}{dx} = \text{Constant} = c \qquad \Big| \Rightarrow \quad f'\left(\frac{dy}{dx}\right) = -x.\ \text{gives}$$

$\therefore$ general solution is given by $y = cx + f(c)$. | another solution, which is termed as **singular solution**

Differential equation reducible to the linear differential equation of first order and first degree.

(i) $\frac{dx}{dy} + P(y)x = Q(y).$

I.F. $= e^{\int P(y)dy}$ and general solution is given by:

$$x(\text{I.F.}) = \int Q(y)\text{I.F.}\,dy + c$$

(ii) $f'(y)\frac{dy}{dx} + P(x).f(y) = Q(x)$

Put $f(y) = t \quad \therefore \quad f'(y)\frac{dy}{dx} = \frac{dt}{dx} \quad \Rightarrow \quad \frac{dt}{dx} + P(x)t = Q(x).$

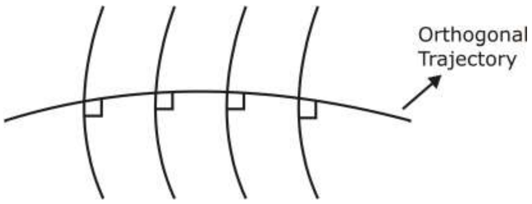
(iii) $f'(x)\frac{dx}{dy} + P(y)f(x) = Q(y)$.

Put $f(x) = t \quad \therefore \quad f'(x)\frac{dx}{dy} = \frac{dt}{dy}$

$\Rightarrow \qquad \frac{dt}{dy} + P(y)t = Q(y)$.

(8) Orthogonal Trajectory :

An orthogonal trajectory of a given system of curves is defined to be a curve which cuts every member of the given family of curve at right angle.



Steps to find orthogonal trajectory :

(i) Let f(x,y,c) = 0 be the equation of the given family of curves, where 'c' is an arbitrary constant.

(ii) Differentate the given equation with respect to x and then eliminate c.

(iii) Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in the equation obtained in (ii).

(iv) Solving the differenetial equation obtained in step (iii) gives the required orthogonal trajectory.

22. BASICS OF 2 Dimensional GEOMETRY

In 2-dimensional coordinate system any point P is represented by (x,y), where |x| and |y| are the distances of 'P' from the y-axis and x-axis respectively

Note:

(i) If both x and $y \in I$, point 'P' is termed as **integral point**.

(ii) If both x and $y \in Q$, point 'P' is termed as **rational point**.

(iii) If atleast one of $x, y \notin Q$, point 'P' is termed as **irrational point**.

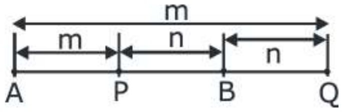
(1) Distance Formula :

The distance between the points $A(x_1,y_1)$ and $B(x_2,y_2)$ is $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

(2) Section Formula :

If P(x,y) divides the line joining $A(x_1,y_1)$ and $B(x_2,y_2)$ internally in the ratio m : n, then :

$\left(x = \frac{mx_2+nx_1}{m+n}\ ;\ y = \frac{my_2+ny_1}{m+n}\right)$ and for external division P(x, y) is $\left(x = \frac{mx_2-nx_1}{m-n}\ ;\ y = \frac{my_2-ny_1}{m-n}\right)$



Note:

(i) The mid-point of AB is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

(ii) If P divides AB internally in the ratio m : n and Q divides AB externally in the ratio m : n then P and Q are said to be *harmonic conjugates* to each other with respect to A and B. Mathematically,

$$\frac{2}{AB} = \frac{1}{AP} + \frac{1}{AQ} \quad (\because \text{ AP, AB \& AQ are in H.P.})$$

(3) Standard points of triangle.

If $A(x_1,y_1)$, $B(x_2,y_2)$, $C(x_3,y_3)$ are the vertices of triangle ABC, whose sides BC, CA, AB are of lengths a, b, c respectively, then the coordinates of the special points of triangle ABC are as follows :

(i) **Centroid** $G \equiv \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$ (ii) **Incentre** $I \equiv \left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$

(iii) **Excentre** $I_1 \equiv \left(\frac{-ax_1+bx_2+cx_3}{-a+b+c}, \frac{-ay_1+by_2+cy_3}{-a+b+c}\right)$

(iv) **Orthocentre** $H \equiv \left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right)$

(v) **Circumcentre** $C \equiv \left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$

Note:

(i) Orthocentre, Centroid and Circumcentre are collinear and centroid divides the line joining orthocentre and circumcentre in the ratio 2 : 1.

(ii) In an isosceles triangle G, H, I and C lie on the same line and in an equilateral triangle all these four points coincide with each other.

(iii) Incentre and excentre are harmonic conjugate of each other with respect to the angle bisector on which they lie.

(iv) In right angle triangle, orthocentre lies at the vertex of right angle and circumcentre lies at midpoint of hypotenuse.

(4) Area of a Triangle :

If $A(x_1,y_1)$, $B(x_2,y_2)$, $C(x_3,y_3)$ are the vertices of triangle ABC, then its area is equal to

$$\Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Note: Area of n-sided polygon formed by points (x_1,y_1); (x_2,y_2) ; .... (x_n,y_n) is given by

$$\frac{1}{2}\left(\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \ldots\ldots\ldots \begin{vmatrix} x_{n-1} & x_n \\ y_{n-1} & y_n \end{vmatrix} + \begin{vmatrix} x_n & x_1 \\ y_n & y_2 \end{vmatrix}\right)$$

(5) Change of Axes:

(i) **Rotation of Axes:** If the axes are rotated through an angle θ in the anticlockwise direction keeping the origin fixed, then the coordinates (X,Y) of point P(x,y) with respect to the new system of coordinate are given by $X = x\cos\theta + y\sin\theta$ and $Y = y\cos\theta - x\sin\theta$.

(ii) **Translation of Axes:** The shifting of origin of axes without rotation of axes is called *Translation of axes.* If the origin (0, 0) is shifted to the point (h,k) without rotation of the axes then the coordinates (X,Y) of a point P(x,y) with respect to the new system of coordinates are given $X = x - h$, $Y = y - k$.

(6) Slope Formula :

If θ is the angle at which a straight line is inclined to the positive direction of x-axis and $0 \le \theta < \pi, \theta \ne \pi/2$, then the slope of the line denoted by m is defined by $m = \tan\theta$.

If $A(x_1,y_1)$ and $B(x_2,y_2)$, $x_1 \ne x_2$, are points on a straight line , then the slope m of the line is given by $\left(\frac{y_2-y_1}{x_2-x_1}\right)$.

(7) Condition of collinearity of three points :

Points $A(x_1,y_1)$, $B(x_2,y_2)$, $C(x_3,y_3)$ are collinear if

(i) $m_{AB} = m_{BC} = m_{CA}$

(ii) $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(iii) AC = AB + BC or |AB − BC|

(8) Locus:

The locus of a moving point is the path traced out by that point under one or more given condition.

Approach to find the locus of a point: Let (h,k) be the co-ordinates of the moving points say P. Now apply the geometrical condition on h, k. This gives a relation between h and k. Now replace h by x and k by y in th eleminant and resulting equation would be the equation of the locus.

23.STRAIGHT LINE

(1) Equation of a straight Line in various forms :

(i) **Point-Slope form :** $y - y_1 = m(x - x_1)$ is the equation of a straight line whose slope is m and which passes through the point (x_1,y_1).

(ii) – **Slope-intercept form :** y = mx + c is the equation of a straight line whose slope is m and which makes an intercept c on the y-axis.

(iii) **Two point form :** $y - y_1 = \left\{\frac{y_2 - y_1}{x_2 - x_1}\right\}(x - x_1)$ is the equation of a straight line which passes through the points (x_1,y_1) and (x_2,y_2).

(iv) **Determinant form :** Equation of line passing through (x_1,y_1) and (x_2,y_2) is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(v) **Intercept form :** $\frac{x}{a} + \frac{y}{b} = 1$ is the equation of a straight line which makes intercepts a and b on X-axis and Y-axis respectively.

(vi) **Perpendicular/Normal form :** $x\cos\alpha + y\sin\alpha = p$ (where $p > 0, 0 \le \alpha < 2\pi$) is the equation of the straight line where the length of the perpendicular from the origin on the line is p and this perpendicular makes an angle α with positive x -axis.

(vii) **Parametric form :** $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$ or $P(r) = (x,y) \equiv (x_1 + r\cos\theta, y_1 + r\sin\theta)$ is the equation of the line in parametric form , where 'r' is the parameter whose absolute value is the distance of any point (x, y) on the line from the fixed point (x_1,y_1) on the line. 'r' is take as positive for co-ordinate in increasing direction of 'y' and negative for decreasing direction of 'y'. θ is angle of inclination of the line and $\theta \in [0, \pi)$

(viii) **General Form :** ax + by + c = 0 is the equation of a straight line in the general form in this case, slope of line $= -\frac{a}{b}$

(2) Angle between two straight lines:

If m_1 and m_2 are the slopes of two intersecting straight lines $(m_1m_2 \neq -1)$ and θ is the acute angle between them, then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$.

Note:

(i) Let m_1, m_2, m_3 are the slopes of three lines $L_1 = 0$; $L_2 = 0$; $L_3 = 0$ where $m_1 > m_2 > m_3$ then the interior angles of the ΔABC formed by these lines are given by,

$\tan A = \frac{m_1 - m_2}{1 + m_1m_2}$; $\tan B = \frac{m_2 - m_3}{1 + m_2m_3}$ & $\tan C = \frac{m_3 - m_1}{1 + m_3m_1}$

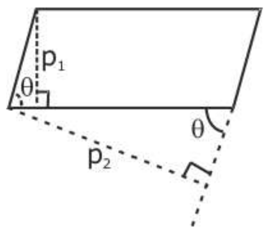
(ii) The equation of lines passing through point (x_1, y_1) and making angle α with the line y = mx + c are given by :
$(y - y_1) = \tan(\theta - \alpha)(x - x_1)$ and $(y - y_1) = \tan(\theta + \alpha)(x - x_1)$, where $\tan\theta = m$.

(iii) When two straight lines are parallel their slopes are equal. Thus any line parallel to y = mx + c is of the type y = mx + k, where k is a parameter.

(iv) Two lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are parallel if $\frac{a}{a'} = \frac{b}{b'} \neq \frac{c}{c'}$.

Thus any line parallel to $ax + by + c = 0$ is of the type $ax + by + k = 0$, where k is a parameter.

(v) The distance between two parallel lines with equations $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $\left|\frac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right|$.



(vi) The area of the parallelogram $= \frac{p_1 p_2}{\sin\theta}$, where p_1 & p_2 are distances between two pairs of opposite sides & θ is the angle between any two adjacent sides.

(vii) Area of the parallelogram bounded by the lines $y = m_1x + c_1$, $y = m_1x + c_2$ and $y = m_2x + d_1$, $y = m_2x + d_2$ is given by $\left|\frac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2}\right|$.

(viii) When two lines of slopes m_1 and m_2 are at right angles, the product of their slopes is -1, i.e. $m_1 m_2 = -1$. Thus any line perpendicular to $y = mx + c$ is of the form $y = -\frac{1}{m}x + d$ where d is any parameter.

(ix) Two lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are perpendicular if $aa' + bb' = 0$. Thus any line perpendicular to $ax + by + c = 0$ is of the from $bx - ay + k = 0$, where k is any parameter.

(3) Position of the point (x_1, y_1) with respect to the line $ax + by + c = 0$:

If $ax_1 + by_1 + c$ is of the same sign as c, then the point (x_1, y_1) lie on the origin side of $ax + by + c = 0$. But if the sign of $ax_1 + by_1 + c$ is opposite to that of c, the point (x_1, y_1) will lie on the non-origin side of $ax + by + c = 0$.

In general two points (x_1, y_1) and (x_2, y_2) will lie on same side or opposite side of $ax + by + c = 0$ according as $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of same or opposite sign respectively.

Note: If b is positive in line $ax+by+c = 0$, then point $P(x_1, y_1)$ lies above the line if $ax_1 + by_1 + c > 0$ and point P lies below the line if $ax_1 + by_1 + c < 0$.

(4) Length of perpendicular from a point on a line :

The length of perpendicular from $P(x_1, y_1)$ on $ax + by + c = 0$ is $\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

(5) Reflection of a point about a line :

(i) Foot of the perpendicular from a point (x_1, y_1) on the line $ax+by+c = 0$ is given by:

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -\left(\frac{ax_1 + by_1 + c}{a^2 + b^2}\right)$$

(ii) The image of a point (x_1, y_1) about the line $ax + by + c = 0$ is given by:

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -2\left(\frac{ax_1 + by_1 + c}{a^2 + b^2}\right)$$

(6) Bisectors of the angles between two lines :

Equations of the bisectors of angles between the lines ax + by + c = 0 and

a'x + b'y + c' = 0, ($ab' \neq a'b$) are : $\dfrac{ax+by+c}{\sqrt{a^2+b^2}} = \pm \dfrac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$

Note:

(i) Equation of straight lines passing through $P(x_1, y_1)$ and equally inclined with the lines $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ are those which are parallel to the bisectors between these two lines & passing through the point P.

(ii) If θ be the angle between one of the lines and one of the bisectors and $|\tan\theta| < 1$, then this bisector is the acute angle bisector, if $|\tan\theta| > 1$, then the bisector is obtuse angle bisector.

(iii) If aa' + bb' < 0, then the equation of the bisector of this acute angle is

$\dfrac{ax+by+c}{\sqrt{a^2+b^2}} = +\dfrac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$ (c and c´ constants are positive)

If aa' + bb' > 0, the equation of the bisector of the obtuse angle is :

$\dfrac{ax+by+c}{\sqrt{a^2+b^2}} = +\dfrac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$ (c and c´ constants are positive)

(iv) To discriminate between the bisector of the angle containing the origin and that of the angle not containing the origin. the equations are written as ax + by + c = 0 and a'x + b'y + c' =0 such that the constant terms c, c' are positive. Then

$\dfrac{ax+by+c}{\sqrt{a^2+b^2}} = +\dfrac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$ gives the equation of the bisector of the angle containing the origin & $\dfrac{ax+by+c}{\sqrt{a^2+b^2}} = -\dfrac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$ gives the equation of the bisector of the angle not containing the origin. In general equation of the bisector which contains the point (α, β) is $\dfrac{ax+by+c}{\sqrt{a^2+b^2}} = +\dfrac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$ or $\dfrac{ax+by+c}{\sqrt{a^2+b^2}} = -\dfrac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$ according as $a\alpha + b\beta + c$ and $a'\alpha + b'\beta + c'$ having same sign or otherwise.

(7) Condition of Concurrency :

Three non parallel lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ & $a_3x + b_3y + c_3 = 0$ are concurrent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$ $\left(-\dfrac{a_1}{b_1} \neq -\dfrac{a_2}{b_2} \neq -\dfrac{a_3}{b_3}\right)$

Alternatively : If three constants A, B and C (not all zero) can be found such that $A(a_1x + b_1y + c_1) + B(a_2x + b_2y + c_2) + C(a_3x + b_3y + c_3) = 0$, then the three straight lines are concurrent.

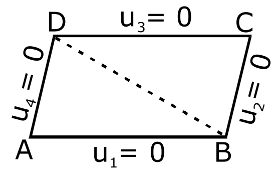
(8) Family of Straight Lines :

The equation of a family of straight lines passing through the point of intersection of the lines. $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$ is given by $L_1 + \lambda L_2 = 0$ i.e. $(a_1x + b_1y + c_1) + \lambda(a_2x + b_2y + c_2) = 0$, where λ is an arbitrary real number.

Note:

(i) If $u_1 = ax + by + c$, $u_2 = a'x + b'y + d$, $u_3 = ax + by + c'$, $u_4 = a'x + b'y + d'$

then $u_1 = 0$; $u_2 = 0$; $u_3 = 0$; $u_4 = 0$ form a parallelogram.
The diagonal BD can be given by $u_2u_3 - u_1u_4 = 0$.



(ii) The diagonal AC is given by $u_1 + \lambda u_4 = 0$ and
$u_2 + \mu u_3 = 0$, if the two equations are identical
for some real λ and μ.[For getting the values
of λ & μ compare the coefficients of x, y & the constant terms].

24. PAIR OF STRAIGHT LINES

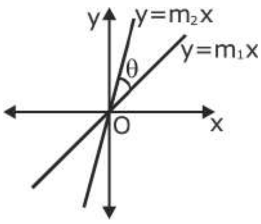
(1) A Pair of Straight lines through origin :

(i) A homogeneous equation of two degree $ax^2 + 2hxy + by^2 = 0$ always represents a pair of straight lines passing through the origin if :

(a) $h^2 > ab \Rightarrow$ lines are real and distinct.

(b) $h^2 = ab \Rightarrow$ lines are coincident.

(c) $h^2 < ab \Rightarrow$ lines are imaginary with real point of intersection i.e. (0,0)



(ii) If $y = m_1x$ and $y = m_2x$ be the two equations represented by $ax^2 + 2hxy + by^2 = 0$, then;

$$m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1 m_2 = \frac{a}{b}.$$

(iii) If θ is the acute angle between the pair of straight lines represented by

$$ax^2 + 2hxy + by^2 = 0, \text{ then; } \tan\theta = \left|\frac{2\sqrt{h^2-ab}}{a+b}\right|.$$

(iv) The condition that these lines are :

(a) At right angles to each other is $a + b = 0$. i.e. coefficient of x^2 + co-efficient of $y^2 = 0$.

(b) Coincident is $h^2 = ab$.

(c) Equally inclined to the axis of x is $h = 0$. i.e. coeff. of $xy = 0$.

Note: A homogeneous equation of degree n represents n straight lines passing through origin.

(v) The equation to the pair of straight lines bisecting the angle between the straight lines, $ax^2 + 2hxy + by^2 = 0$ is $\frac{x^2-y^2}{a-b} = \frac{xy}{h}$.

(2) General equation of second degree representing a pairs of straight lines:

(i) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if :

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0, \text{ i.e. if } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

(ii) The angle θ between the two lines representing by a general equation is the same as that between the two lines represented by its homogeneous part only.

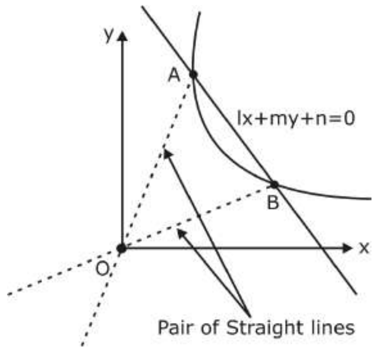
(3) Homogenization :

The equation of a pair of straight lines joining origin to the points of intersection of the line $L \equiv \ell x + my + n = 0$ and a second degree curve.

$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

is $ax^2 + 2hxy + by^2 + 2gx\left(\frac{\ell x - my}{-n}\right) + 2fy\left(\frac{\ell x + my}{-n}\right) + c\left(\frac{\ell x + my}{-n}\right)^2 = 0.$

The equation is obtained by homogenizing the equation of curve with the help of equation of line.



NOTE: Equation of any curve passing through the points of intersection of two curves $C_1 = 0$ and $C_2 = 0$ is given by $\lambda C_1 + \mu C_2 = 0$ where λ & μ are parameters.

(4) Some Important Results

(i) If $ax^2 + 2hxy + 2gx + 2fy + c = 0$ represent a pair of parallel straight lines, then the distance between them is given by $2\sqrt{\frac{g^2 - ac}{a(a+b)}}$ or $2\sqrt{\frac{f^2 - bc}{b(a+b)}}$

(ii) The lines joining the origin to the points of intersection of the curves $ax^2 + 2hxy + by^2 + 2gx = 0$ and $a'x^2 + 2h'xy + b'y^2 + 2'gx = 0$ will be mutually perpendicular, if $g'(a + b)$.

(iii) If the equation $hxy + gx + fy + c = 0$ represents a pair of straight lines, then $fg = ch$.

(iv) The pair of lines $(a^2 - 3b^2)x^2 + 8abxy + (b^2 - 3a^2)y^2 = 0$ with the line $ax+by+c = 0$ form an equilateral triangle and its area $= \frac{c^2}{\sqrt{3}(a^2 + b^2)}$.

(v) The area of a triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$ is given by $\frac{n^2\sqrt{h^2 - ab}}{am^2 - 2hlm + bl^2}$

(vi) The lines joining the origin to the points of intersection of line $y = mx + c$ and the circle $x^2 + y^2 = a^2$ will be mutually perpendicular, if $a^2(m^2 + 1) = 2c^2$.

(vii) If the distance of two lines passing through origin from the point (x_1,y_1) is d, then the equation of lines is $(xy_1 - yx_1)^2 = d^2(x^2 + y^2)$

(viii) The lines represented by the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ will be equidistant from the origin, if $f^4 - g^4 = c(bf^2 - ag^2)$

(ix) The product of the perpendiculars drawn from (x_1,y_1) on the lines $ax^2 + 2hxy + by^2 = 0$ is given by $\frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a-b)^2 + 4h^2}}$

(x) The product of the perpendiculars drawn from origin on the lines

$ax^2 + 2hxy + by^2 + 2gx + 2fy = 0$ is $\dfrac{c}{\sqrt{(a-b)^2 + 4h^2}}$

(xi) If the lines represented by the general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are perpendicular, then the square of distance between the point of intersection and origin is $\dfrac{f^2 + g^2}{h^2 + b^2}$ or $\dfrac{f^2 + g^2}{h^2 + a^2}$.

(xii) The square of distance between the point of intersection of the lines represented by the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and origin is $\dfrac{c(a+b) - f^2 - g^2}{ab - h^2}$.

(xiii) If one of the line given by the equation $ax^2 + 2hxy + by^2 = 0$ coincide with one of these given by $a'x^2 + 2h'xy + b'y^2 = 0$ and the other lines represented by them be perpendicular, then $\dfrac{ha'b'}{b'-a'} = \dfrac{h'ab}{b-a} = \dfrac{1}{2}\sqrt{(-aa'bb')}$.

(xiv) The straight lines joining the origin to the points of intersection of the straight line $kx + hy = 2hk$ with the curve $(x - h)^2 + (y - k)^2 = c^2$ are at right angles, if $h^2 + k^2 = c^2$.

25. CIRCLES

(1) Equations of a Circle

(i) An equation of a circle with centre (h, k) and radius r is $(x-h)^2 + (y-k)^2 = r^2$
(ii) An equation of a circle with centre (0, 0) and radius r is $x^2+y^2 = r^2$
(iii) An equation of the circle on the line segment joining (x_1, x_1) and (x_2, y_2) as diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$
(iv) General equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ where g, f and c are constants.

- Centre of this circle is $(-g, -f)$
- Its radius is $\sqrt{g^2 + f^2 - c}$, $(g^2 + f^2 \geq c)$.

(v) The intercepts made by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ on the x-axis and y-axis is given by $2\sqrt{g^2-c}$ and $2\sqrt{f^2-c}$ respectively.
(vi) General equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ in x and y, represents a circle if and only if

- coefficient of x^2 equals coefficient of y^2, i.e., $a = b \neq 0$.
- coefficient of x y is zero, i.e., $h = 0$.
- $g^2 + f^2 - ac \geq 0$

Note:

(i) Equation of circle circumscribing a triangle whose sides are given by $L_1 = 0$; $L_2 = 0$ and $L_3 = 0$ is given by ; $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$, provided co-efficient of xy = 0 and coefficient of x^2 = co-efficient of y^2.

(ii) Equation of circle circumscribing a quadrilateral whose side in order are represented by the lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ and $L_4 = 0$ are $u L_1 L_3 + \lambda L_2L_4 = 0$ where values of u and λ can be found out by using condition that co-efficient of x^2 = co-efficient of y^2 and co-efficient of xy = 0.

(2) Parametric Equations of a circle :

Parametric equation of circle $(x - h)^2 + (y - k)^2 = r^2$ is given by $\frac{x-h}{\cos\theta} = \frac{y-k}{\sin\theta} = r$, where (h, k) is the centre, r is the radius and θ is a parameter $(0 \leq \theta < 2\pi)$

Any point $P(\theta)$ on the circle $(x - h)^2 + (y - k)^2 = r^2$ can be assumed in parametric form as $P \equiv (h + r\cos\theta\ ,\ k + r\sin\theta)$

Note:

The equation of chord PQ to the circle $x^2 + y^2 = a^2$ joining two points $P(\alpha)$ and $Q(\beta)$ on it is given by $x \cos\frac{\alpha+\beta}{2} + y \sin\frac{\alpha+\beta}{2} = a \cos\frac{\alpha-\beta}{2}$.

(3) Position of a point with respect to a circle :

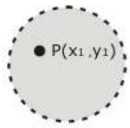
Let equation of the circle is given by $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then following condition illustrates the relative position of point P.

(i) Point 'P' lies on the circle if:

$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$.

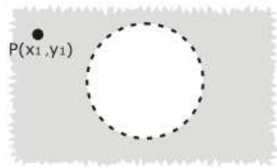
(ii) Point 'P' lies inside the circle if:

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c < 0$$



(iii) Point 'P' lies outride the circle if:

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c > 0$$



Note:

The greatest and the least distance of a point P from a circle with centre C and radius r is PC + r and PC − r respectively.

(4) Line and circle

If line $y = mx + c$ intersects the circle $x^2 + y^2 = a^2$ at point A and B, then the x-co-ordinate of the points of intersection is given by $x^2 + (mx + c)^2 - a^2 = 0$.

$\Rightarrow \quad (1 + m^2)\,x^2 + 2cm\,x + c^2 - a^2 = 0 \qquad$ ....(1)

In equation (1), if discriminant 'D' is positive, then two distinct real values of 'x' is obtained which confirms the intersection of line and circle at two distinct points

(refer figure no. (1)). $\therefore \quad D > 0 \Rightarrow c^2 < a^2(1 + m^2) \Rightarrow p < a$.

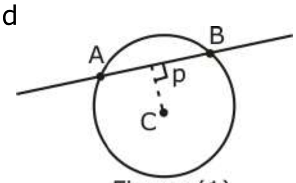


Figure (1)

If equation (1), if dircriminant 'D' is negative, then line is outside the circle (refer figure (2))

$\Rightarrow c^2 < a^2(1 + m^2) \Rightarrow p > a$

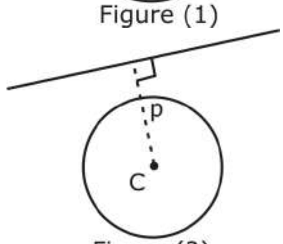


Figure (2)

If equation (1), if discriminant 'D' is zero, then line is tangential to the circle (refer figure(3)).

$\Rightarrow c^2 = a^2(1 + m^2) \Rightarrow p = a$

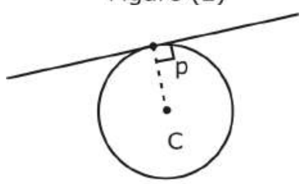


Figure (3)

(5) Tangent to a Circle:

(a) Slope Form :

Straight line $y = mx + c$ is tangent to the circle $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$. Hence,

equation of tangent is $y = mx \pm a\sqrt{1+m^2}$ and the point of contact is $\left(-\frac{a^2m}{c}, \frac{a^2}{c}\right)$.

Note:

Equation of tangent of slope 'm' to the circle $(x - h)^2 + (y - k)^2 = r^2$ is given by:

$(y - k) = m(x - h) \pm r\sqrt{1 + m^2}$

(b) Point form :

(i) The equation of the tangent to the circle $x^2 + y^2 = a^2$ at its point (x_1, y_1) is given by $x\,x_1 + y\,y_1 = a^2$.

(ii) The equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at its point

(x_1, y_1) is given by $x x_1 + y y_1 + g(x + x_1) + f(y + y_1) + c = 0$.

Note: In general the equation of tangent to any second degree curve at point (x_1, y_1) on it can be obtained by replacing x^2 by xx_1, y^2 by yy_1, x by $\frac{x+x_1}{2}$, y by $\frac{y+y_1}{2}$, xy by $\frac{x_1y+xy_1}{2}$ and c remains as c.

(c) Parametric form :

The equation of a tangent to circle $x^2 + y^2 = a^2$ at $(a\cos\alpha, a\sin\alpha)$ is $x\cos\alpha + y\sin\alpha = a$.

Note:

The point of intersection of the tangents at the points $P(\alpha)$ & $Q(\beta)$ is $\left(\frac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \frac{a\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$.

(6) Normal to Circle

If a line is normal/orthogonal to a circle then it must pass through the centre of the circle. Using this fact normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at $(x_1\ y_1)$ is given by:

$$y - y_1 = \frac{y_1+f}{x_1+g}(x - x_1).$$

(7) Pair of Tangents from a Point :

The equation of a pair of tangents drawn from the point $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is given by $SS_1 = T^2$.

Where $S \equiv x^2 + y^2 + 2gx + 2fy + c$; $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$
$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$.

Note:

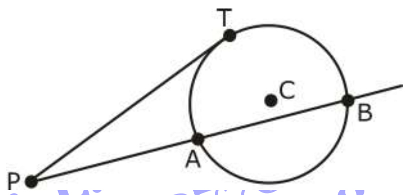
The locus of the point of intersection of two perpendicular tangents is called the director circle of the given circle. The director circle of a circle is the concentric circle having radius equal to $\sqrt{2}$ times the original circle.

(8) Length of a Tangent and Power of a Point :

The length of a tangent from an external point (x_1,y_1) to the circle

$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is given by $L = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2f_1y + c} = \sqrt{S_1}$

Note:



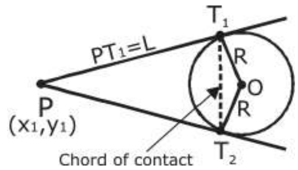
(i) Square of length of the tangent from the point P is called the power of point with respect to a circle.
(ii) Power of a point with respect to a circle remains constant i.e. $PA.PB = PT^2$.

(9) Chord of Contact :

If two tangents PT_1 and PT_2 are drawn from the point $P(x_1,y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of contact T_1T_2 is given by: $T \equiv 0$.

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$

Note:



(i) Chord of contact exists only if the point 'P' is outside the circle.

(ii) Length of chord of contact $T_1T_2 = \dfrac{2LR}{\sqrt{R^2 + L^2}}$

(iii) Area of the triangle formed by the pair of the tangents and its chord of contact is given by $\dfrac{RL^3}{R^2+L^2}$.

(iv) If $\angle T_1PT_2 = \theta$, then $\tan\theta = \dfrac{2RL}{L^2 - R^2}$

(v) Quadrilateral PT_1OT_2 is cyclic and the equation of the circle circumscribing the triangle PT_1T_2 is given by $(x - x_1)(x + g) + (y - y_1)(y + f) = 0$.

(10) Equation of the Chord with a given Middle Point :

The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its mid point $M(x_1, y_1)$ is given by $T = S_1$.

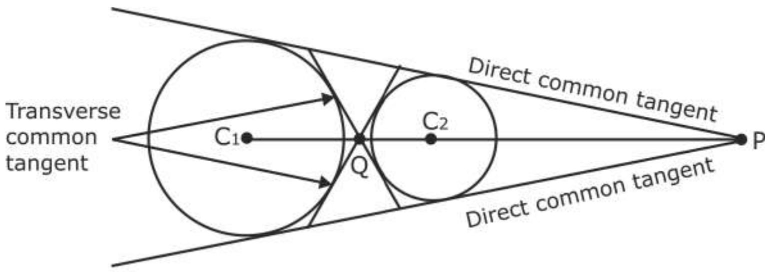
$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

(11) Common Tangents to two Circles :

Direct (or external) common tangents meet at a point which divides the line joining centre of circles externally in the ratio of their radii.

Transverse (or internal) common tangents meet at a point which divides the line joining centre of circles internally in the ratio of their radii.

If $(x - g_1)^2 + (y - f_1)^2 = r_1^2$ and $(x - g_2)^2 + (y - f_2)^2 = r_2^2$ are two circles with centres $C_1(g_1, f_1)$ and $C_2(g_2, f_2)$ and radii r_1 and r_2 respectively, then 'P' and 'Q' are the respective points from which direct and Transverse common tangents can be drawn (as shown in figure).



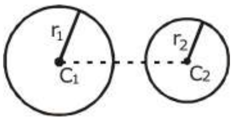
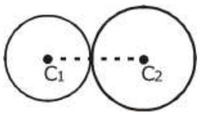
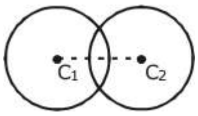
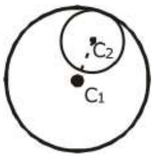
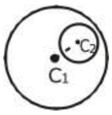
$$\frac{C_1P}{PC_2} = \frac{r_1}{r_2} \text{ (externally)}, \quad \frac{C_1Q}{QC_2} = \frac{r_1}{r_2} \text{ (internally)}$$

Note:

Length of an external (or direct) common tangent & internal (or transverse) common tangent to the circles are given by : $L_{ext} = \sqrt{d^2-(r_1-r_2)^2}$ & $L_{int} = \sqrt{d^2-(r_1+r_2)^2}$,

where d = distance between the centres of the two circles and r_1, r_2 are the radii of the two circles.

(12) Relative Position of two Circles

	Case	Number of Common Tangents	Condition
(i)		4 common tangents	$r_1 + r_2 < c_1 c_2$.
(ii)		3 common tangents.	$r_1 + r_2 = c_1 c_2$.
(iii)		2 common tangents.	$\lvert r_1 - r_2 \rvert < c_1 c_2 < r_1 + r_2$
(iv)		1 common tangent	$\lvert r_1 - r_2 \rvert = c_1 c_2$
(v)		No common tangent	$c_1 c_2 < \lvert r_1 - r_2 \rvert$

(13) Orthogonality Of Two Circles :

Two circles $S_1 = 0$ and $S_2 = 0$ are said to be orthogonal or said to intersect orthogonally if the tangents at their point of intersection include a right angle. The condition for two circles to be orthogonal is : $2g_1g_2 + 2f_1f_2 = c_1 + c_2$.

Note:

(i) The centre of a variable circle orthogonal to two fixed circles lies on the radical axis of two circles.

(ii) The centre of a circle which is orthogonal to three given circles is the radical centre provided the radical centre lies outside all the three circles.

(14) Radical Axis and Radical Centre :

The radical axis of two circles is the locus of a point whose powers with respect to the two circles are equal. The equation of radical axis of the two circles $S_1 = 0$ and $S_2 = 0$ is given by $S_1 - S_2 = 0$ i.e: $2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2) = 0$.

The common point of intersection of the radical axes of three circles taken two at a time is called the **radical centre** of three circles.

Note:

(i) The length of tangents from radical centre to the three circles are equal.

(ii) If two circles intersect, then the radical axis is the common chord of the two circles.

(iii) If two circles touch each other then the radical axis is the common tangent of the two circles at the common point of contact.

(iv) Radical axis is always perpendicular to the line joining the centres of the two circles.

(v) Radical axis will pass through the mid point of the line joining the centres of the two circles only if the two circles have equal radii.

(vi) Radical axis bisects a common tangent between the two circles.

(vii) A system of circles, every two of which have the same radical axis, is called a coaxial system.

(viii) Pairs of circles which do not have radical axis are concentric circles.

(15) Family of Circles :

(i) The equation of the family of circles passing through the points of intersection of two circles $S_1 = 0$ and $S_2 = 0$ is : $S_1 + \lambda S_2 = 0$; $\lambda \in R - \{-1\}$

($\lambda \neq -1$ if the co-efficient of x^2 and y^2 in S_1 and S_2 are same)

Note: If Coefficients of x^2 and y^2 are same in two intersecting circles $S_1 = 0$ and $S_2 = 0$, then the common chord of circles is given by: $S_1 - S_2 = 0$

(ii) The equation of the family of circles passing through the point of intersection of a circle $S = 0$ and a line $L = 0$ is given by $S + \lambda L = 0$; $\lambda \in R$

(iii) The equation of a family of circles passing through two given points (x_1,y_1) and (x_2,y_2) can be written in the form :

$$\{(x-x_1)(x-x_2) + (y-y_1)(y-y_2)\} + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0 \text{ where } \lambda \in R.$$

(iv) The equation of a family of circles touching a fixed line $ax+by+c = 0$ at the fixed point (x_1, y_1) is $\{(x-x_1)^2 + (y-y_1)^2\} + \lambda(ax+by+c) = 0$ where λ is a parameter.

(v) The equation of family of circles touching a fixed circle $S = 0$ at fixed point (x_1,y_1) is given by: $\{(x-x_1)^2+(y-y_1)^2\}+\lambda S=0$; $\lambda \in R$.

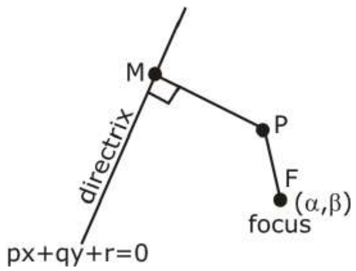
26. PARABOLA

(1) Conic Sections :

A conic section is the locus of a point which moves in a plane so that its distance from a fixed point (i.e. focus) is in a constant ratio (i.e. eccentricity) to its perpendicular distance from a fixed straight line (i.e. directrix). Let point P be (x,y) then equation of conic is given

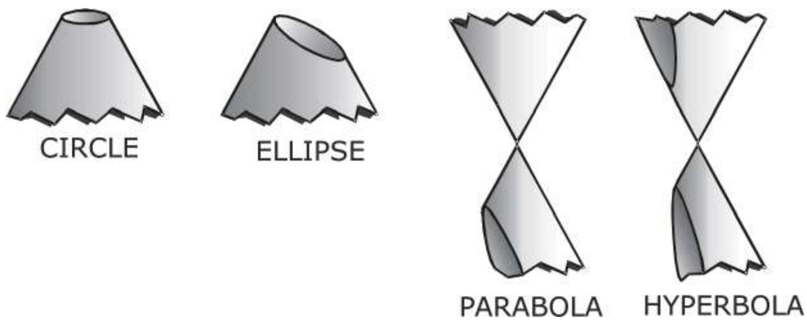
by: $\frac{PF}{PM} = e =$ constant ratio $\Rightarrow \sqrt{(x-\alpha)^2 + (y-\beta)^2} = e\left|\frac{px+qy+r}{\sqrt{p^2+q^2}}\right|$

$\Rightarrow \; ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$



Note:

All conic sections can be derived from a right circular cone, as shown in the figure



(2) Recognisation of Conics:

The equation of conics is represented by the general equation of second degree

$\mathbf{ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0}$..............(1)

where Δ $\mathbf{= abc + 2fgh - af^2 - bg^2 - ch^2}$

Different conics can be recognized by the conditions given in the tabular form.

Case I: when $\Delta = 0$

In this case equation (1) represents the **Degenerate conic** whose nature is given in the following table:

Condition	Nature of Conic
$\Delta = 0$ and $h^2 - ab = 0$	A pair of coincident lines
$\Delta = 0$ and $h^2 - ab \neq 0$	Real or imaginary pair of straight lines
$\Delta = 0$ and $h^2 - ab > 0$	A pair of intersecting straight line.
$\Delta = 0$ and $h^2 - ab < 0$	Point.

Case II: When $\Delta \neq 0$,

In this case equation (1) represents the **Non-degenerate conic** whose nature is given in the following table:

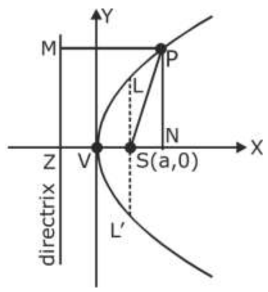
Condition	Nature of Conic
$\Delta \neq 0$, $h = 0$, $a = b$	a circle
$\Delta \neq 0$, $h^2 - ab = 0$	a parabola
$\Delta \neq 0$, $h^2 - ab < 0$	an Ellipse or empty set
$\Delta \neq 0$, $h^2 - ab > 0$	a Hyperbola
$\Delta \neq 0$, $h^2 - ab > 0$ and $a + b = 0$	a Rectangular hyperbola

PARABOLA

(3) Basic Definitions

A parabola is the locus of a point, whose distance from a fixed point (focus) is equal to perpendicular distance from a fixed straight line (directrix). Standard forms of the parabola are $y^2 = 4ax$; $y^2 = -4ax$; $x^2 = 4ay$; $x^2 = -4ay$

For parabola $y^2 = 4ax$:
(i) Vertex is (0,0)
(ii) focus is (a,0)
(iii) Axis is $y = 0$
(iv) Directrix is $x + a = 0$



Focal Distance : The distance of a point on the parabola from the focus.
Focal Chord : A chord of the parabola, which passes through the focus. $y^2 = 4ax$.
Double Ordinate : A chord of the parabola perpendicular to the axis of the symmetry
Latus Rectum : A double ordinate passing through the focus or a focal chord perpendicular to the axis of parabola is called the latus Rectum (L.R.).

For $y^2 = 4ax$. $\Rightarrow$ Length of the latus rectum = 4a.
$\Rightarrow$ ends of the latus rectum are L(a, 2a) & L' (a, $-2a$).

Note:

(i) Perpendicular distance from focus on directrix = half the latus rectum.
(ii) Vertex is middle point of the focus & the point of intersection of directrix & axis.
(iii) Two parabolas are said to be equal if they have the same latus rectum.

(4) Parametric Representation :

Parametric co-ordinates of a point on the parabola is $(at^2, 2at)$ i.e. the equations $x = at^2$ and $y = 2at$ together represents the parabola $y^2 = 4ax$, t being the parameter and $t \in R$

Note:

(i) The equation of a chord joining t_1 and t_2 is $2x - (t_1 + t_2)\,y + 2\,at_1t_2 = 0$.
(ii) If t_1 and t_2 are the ends of a focal chord of the parabola $y^2 = 4ax$ then $t_1t_2 = -1$. Hence the co-ordinates at the extremities of a focal chord are $(at^2, 2at)$ and $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$.
(iii) Length of the focal chord making an angle α with the x-axis is $4a\,\text{cosec}^2\,\alpha$.

(5) Position of a point Relative to a Parabola :

The point (x_1,y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as the expression $y_1^2 - 4ax_1$ is positive, zero or negative respectively.

(6) Line and a Parabola :

The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $c < \frac{a}{m}, c = \frac{a}{m}$ or $c > \frac{a}{m}$ respectively. Condition of tangency is $c = a/m$.

(7) Tangents to the Parabola $y^2 = 4ax$:

(i) $yy_1 = 2a(x + x_1)$ at the point (x_1, y_1);

(ii) $y = mx + \frac{a}{m} (m \neq 0)$ at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

(iii) $yt = x + at^2$ at $(at^2, 2at)$.

Note: Point of intersection of the tangents at the point t_1 and t_2 is $(at_1t_2, a(t_1 + t_2))$

(8) Normal to the parabola $y^2 = 4ax$:

(i) $y - y_1 = \frac{y_1}{2a}(x - x_1)$ at (x_1, y_1) ;

(ii) $y = mx - 2am - am^3$ at $(am^2, -2am)$

(iii) $y + tx = 2at^2 + at^3$ at $(at^2\ 2at)$.

Note:

(i) Point of intersection of normal at t_1 and t_2 are, $(a(t_1^2 + t_2^2 + t_1t_2 + 2), -at_1t_2(t_1 + t_2))$.

(ii) If the normals to the parabola $y^2 = 4ax$ at the point t_1 meets the parabola again at the point t_2 then $t_2 = -\left(t_1 + \frac{2}{t_1}\right)$.

(iii) If the normals to the parabola $y^2 = 4ax$ at the points t_1 and t_2 intersect again on the parabola at the point 't_3' then $t_1t_2 = 2$; $t_3 = -(t_1 + t_2)$ and the line joining t_1 and t_2 passes through a fixed point $(-2a,0)$.

(9) Pair of Tangents :

The equation to the pair of tangents which can be drawn from any point (x_1,y_1) to the parabola $y^2 = 4ax$ is given by : $SS_1 = T^2$ where :
$S \equiv y^2 - 4ax$; $S_1 = y_1^2 - 4ax_1$; $T \equiv yy_1 - 2a(x + x_1)$.

Note:

(i) From any point on the directrix perpendicular tangents can be drawn to the parabola and their chord of contact is the focal chord.

(ii) If tangents are drawn at the extremity of a focal chord of parabola, then these tangents are perpendicular and meet on the line of directrix.

(iii) Circle drawn with focal chord as the diameter always touches the directrix of parabola.

(10) Chord of Contact :

Equation to the chord of contact of tangents drawn from a point $P(x_1,y_1)$ is
$yy_1 = 2a(x + x_1)$.

Note:

The area of the triangle formed by the tangents from the point (x_1,y_1) and the chord of contact is $(y_1^2 - 4ax_1)^{3/2}/2a$.

(11) Chord with a given middle point :

Equation of the chord of the parabola $y^2=4ax$ whose middle point is (x_1,y_1) is $T = S_1$, where, $S_1 = y_1^2 - 4ax_1$; $T \equiv yy_1 - 2a(x + x_1)$.

(12) Important Results :

(i) If the tangent and normal at any point 'P' of the parabola intersect the axis at T and G then ST=SG=SP where 'S' is the focus. In other words the tangent and the normal at a point P on the parabola are the bisectors of the angle between the focal radius SP and the perpendicular from P on the directrix. From this we conclude that all rays emanating from S will become parallel to the axis of the parabola after reflection.

(ii) The portion of a tangent to a parabola cut off between the directrix and the curVe subtends a right angle at the focus.

(iii) The tangents at the extremities of a focal chord intersect at right angles on the directrix, and hence a circle on any focal chord as diameter touches the directrix. Also a circle on any focal radii of a point $P(at^2,2at)$ as diameter touches the tangent at the vertex and intercepts a chord of length $a\sqrt{1+t^2}$ on a normal at the point P.

(iv) Any tangent to a parabola & the perpendicular on it from the focus meet on the tangent at the vertex.

(v) If the tangents at P and Q meet in T, then :

- TP and TQ subtend equal angles at the focus S.
- $ST^2 = SP.\,SQ$ and
- The triangles SPT and STQ are similar.

(vi) Semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean between segments of any focal chord of the parabola.

(vii) The area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

(viii) If normal are drawn from a point P(h,k) to the parabola $y^2 = 4ax$ then

$k = mh - 2am - am^3$ i.e. $am^3 + m(2a - h) + k = 0$.

$$m_1 + m_2 + m_3 = 0;\ m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a-h}{a};\ m_1m_2m_3 = -\frac{k}{a}.$$

where m_1 m_2 & m_3 are the slopes of the three concurrent normals.

Note:

- algebraic sum of the slopes of the three concurrent normals is zero.
- algebraic sum of the ordinates of the three co-normal points on the parabola is zero.
- Centroid of the Δ formed by three co-normal points lies on the x-axis.
- Condition for three real and distinct normals to be drawn from a point P(h,k) is

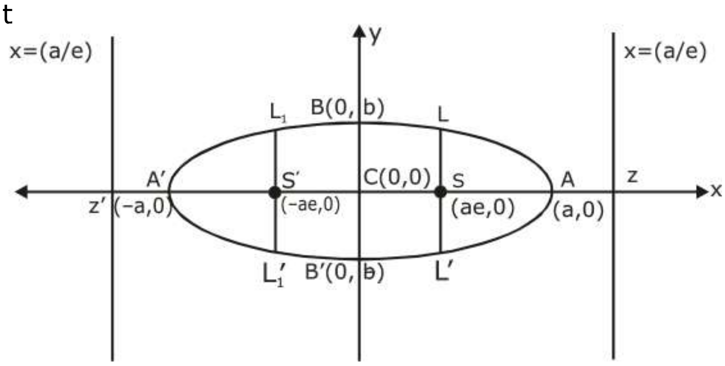
$$h > 2a \text{ and } k^2 < \frac{4}{27a}(h - 2a)^3$$

(ix) Length of subtangent at any point P(x,y) on the parabola $y^2 = 4ax$ equals twice the abscissa of the point P and the subtangent is bisected at the vertex.

(x) Length of subnormal is constant for all points on the parabola and is equal to the semi latus rectum.

27. ELLIPSE

Ellipse is the locus of a point which moves in such a way so that sum of its distances from two fixed points is always constant



Note:

(i) PS + PS' = AA'

(ii) Eccentricity (e) < 1

(1) Basic definitions:

Standard equation of an ellipse referred to its principal axes along the co-ordinate

axes is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$ & $b^2 = a^2 (1 - e^2)$

Eccentricity : $e = \sqrt{1-\frac{b^2}{a^2}}$, $(0 < e < 1)$

Foci : $S \equiv (ae,0)$ and $S' \equiv (-ae, 0)$.

Equations of Directrix : $x = \frac{a}{e}$ and $x = -\frac{a}{e}$.

Major Axis : The line segment AA' in which the foci S and S' lie is of length 2a and is called the major axis (a > b) of the ellipse. Point of intersection of major axis with directrix is called the foot of the directrix (Z and Z').

Minor Axis : The y-axis intersects the ellipse in the points $B' \equiv (0, -b)$ & $B \equiv (0,b)$. The line segment BB' is of length 2b (b<a) is called the minor axis of the ellipse.

Principal Axis : The major & minor axes together are called principal axis of the ellipse.

Vertices : Point of intersection of ellipse with major axis. $A' \equiv (-a,0)$ & $A \equiv (a,0)$.

Focal distances: The focal distance of the point (x,y) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are a+ex and a−ex

Focal Chord : A chord which passes through a focus is called a focal chord.

Double Ordinate : A chord perpendicular to the major axis is called a double ordinate.

Latus Rectum : The focal chord perpendicular to the major axis is called the latus rectum.

Length of latus rectum **(LL')** $= \frac{2b^2}{a} = 2\frac{(\text{minor axis})^2}{\text{major axis}} = 2a(1-e^2)$

Centre : The point which bisects every chord of the conic drawn through it, is called the centre of the conic. C = (0,0) the origin is the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

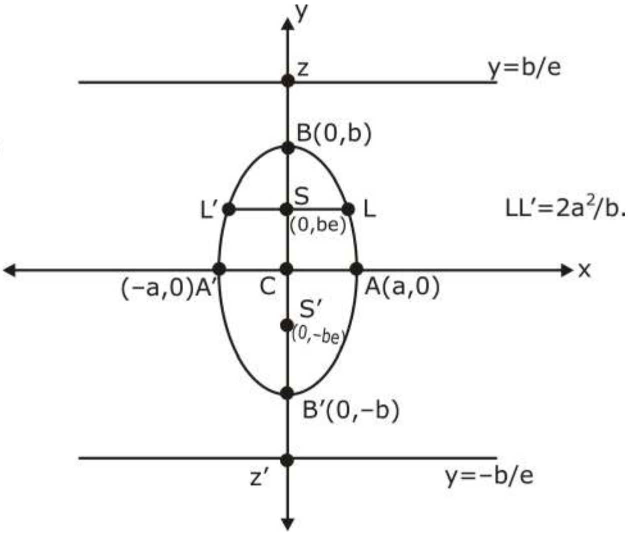
Note:

(i) If the equation of the ellipse is given as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and nothing is mentioned then the standard form (a > b) is assumed.

(ii) If b > a is given, then the y-axis become major axis and x-axis become the minor axis and all other points and lines change accordingly. (as shown if figure)

vertical ellipse is given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; (a < b)

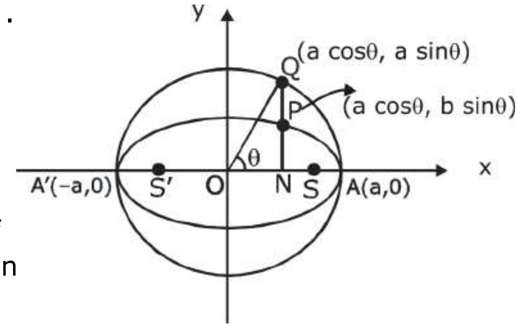
$$\left\{e = \sqrt{1+\frac{a^2}{b^2}}\right\}$$



Vertical Ellipse:

(2) Auxiliary Circle:

A circle described on major axis of ellipse as diameter is called the **auxiliary circle**.
Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ such that line through Q perpendicular to the x-axis on the way intersects the ellipse at P, then P & Q are called as the **Corresponding Points** on the ellipse and the auxiliary circle respectively. 'θ' is called the **Eccentric Angle** of the point P on the ellipse $(0 \le \theta < 2\pi)$.



Note:

(i) $\dfrac{\ell(PN)}{\ell(QN)} = \dfrac{b}{a} = \dfrac{\text{Semi minor axis}}{\text{Semi major axis}}$

(ii) If from each point of a circle perpendiculars are drawn upon a fixed diameter then the locus of the points dividing these perpendiculars in a given ratio is an ellipse of which the given circle is the auxiliary circle.

(3) Parametric Representation :

The equations $x = a\cos\theta$ & $y = b\sin\theta$ together represent the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Where θ is a parameter, if $P(\theta) \equiv (a\cos\theta, b\sin\theta)$ is on the ellipse then,
$P(\theta) \equiv (a\cos\theta, a\sin\theta)$ is on the auxiliary circle.

The equation to the chord of the ellipse joining two points with eccentric angles α & β is given by $\dfrac{x}{a}\cos\left(\dfrac{\alpha+\beta}{2}\right) + \dfrac{y}{b}\sin\left(\dfrac{\alpha+\beta}{2}\right) = \cos\left(\dfrac{\alpha-\beta}{2}\right)$

(4) Position of a Point with respect to an Ellipse :

The point $P(x_1,y_1)$ lies outside, inside or on the ellipse according as $\left\{\dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} - 1\right\}$ is positive, negative or zero respectively.

(5) Line and an Ellipse :

The line y = mx + c meets the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in two points real, coincident or imaginary according as c^2 is < = or > $a^2m^2 + b^2$.

Hence y = mx + c is tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $c^2 = a^2m^2 + b^2$.

(6) Tangents to Ellipse:

(a) **Slope form :** $y = mx \pm \sqrt{a^2m^2+b^2}$ is tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(b) **Point form :** $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ is tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at (x_1, y_1).

(c) **Parametric form :** $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$ is tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $(a\cos\theta, b\sin\theta)$.

Note:

(i) There are two tangents to the ellipse having the same slope m, i.e. there are two tangents parallel to any given direction. These tangents touches the ellipse at extremities of a diameter.

(ii) Point of intersection of the tangents to the ellipse at the points

'θ' and 'ϕ' is $\left(\frac{a\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}, \frac{b\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right)$

(iii) The eccentric angles of the points of contact of two parallel tangents differ by π.

(7) Normals of Ellipse:

(i) Equation of the normal at (x_1, y_1) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$.

(ii) Equation of the normal at the point $(a\cos\theta, b\sin\theta)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$ax\sec\theta - by\operatorname{cosec}\theta = (a^2-b^2)$.

Note:

(i) Atmost four normals can be drawn to an ellipse from a point in its plane.

(ii) In general, there are four points A, B, C and D on the ellipse the normals at which pass through a given point. These four points A, B, C, D are called the co-normal points.

(8) Pair of Tangents :

The equation to the pair of tangents which can be drawn from any point (x_1, y_1) to the ellipse is given by : $SS_1 = T^2$ where :

$S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1$; $S_1 = \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$; $T \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$.

Note:

Locus of the point of intersection of the tangents which meet at right angles is called the **Director Circle**. The equation to this locus is $x^2 + y^2 = a^2 + b^2$ i.e. a circle whose centre is the centre of the ellipse & whose radius is the length of the line joining the ends of the major and minor axes.

(9) Chord of Contact :

Equation to the chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is T = 0, where $T = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$

(10) Chord with a given middle point :

Equation of the chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose middle point is (x_1, y_1) is $T = S_1$,

where $S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$; $T \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$.

(11) Important results:

Refering to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

- The tangent & normal at a point P on the ellipse bisect the external & internal angles between the focal distances of P. This refers to the reflection property of the ellipse which states that rays from one focus are reflected through other focus & vice-versa. Hence we can deduce that the straight lines joining each focus to the foot of the perpendicular from the other focus upon the tangent at any point P meet on the normal PG and bisects it where G is the point where normal at P meets the major axis.
- The product of the lengths of the perpendicular segments from the foci on any tangent to the ellipse is b^2 and the feet of these perpendiculars lie on auxiliary circle and the tangents at these feet to the auxiliary circle meet on the ordinate of P and that the locus of their point of intersection is a similar ellipse as that of the original one.
- The portion of the tangent to an ellipse between the point of contact and the directrix subtends a right angle at the corresponding focus.
- If the normal at any point P on the ellipse with centre C meet the major and minor axes in G and g respectively and if CF be perpendicular upon this normal then

  (i) PF. PG = b^2 (ii) PF. Pg = a^2 (iii) PG. Pg = SP. S'P (iv) CG. CT = CS^2

  (v) locus of the mid point of Gg is another ellipse having the same eccentricity as that of the original ellipse.

  [S and S' are the focii of the ellipse and T is the point where tangent at P meet the major axis]
- The circle on any focal distance as diameter touches the auxiliary circle. Perpendiculars from the centre upon all chords which join the ends of any perpendicular diameters of the ellipse are of constant length.
- If the tangent at the point P of a standard ellipse meets the axes in Q and R and CT is the perpendicular on it from the centre then,

  (i) QR. PT = $a^2 - b^2$ (ii) least value of QR is a + b.

28. HYPERBOLA

Hyperbola is the locus of a point which moves is such a ways so that difference of its distances from two fixed points is always constant.

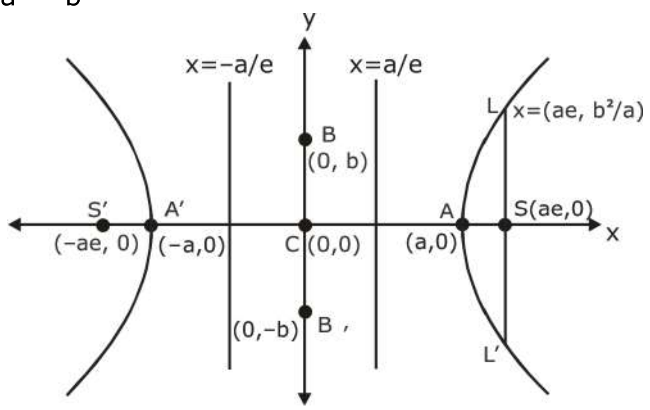
Note:

(i) $|PS-PS'|=AA'$; AA' is the length of transverse axis.

(ii) eccentricity (e) > 1.

(1) Basic Definitions:

Standard equation of the hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where $b^2 = a^2(e^2-1)$



Eccentricity (e) :

$$e=\sqrt{1+\frac{b^2}{a^2}}$$

Foci : $S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.

Equations Of Directrix :

$$x=\frac{a}{e} \text{ and } x=-\frac{a}{e}$$

Transverse Axis :
The line segment AA' of length 2a in which the foci S and S' both lie is called the transverse axis of the hyperbola.

Conjugate Axis :
The line segment BB' between the two points $B' \equiv (0,-b)$ and $B \equiv (0, b)$ is called as the conjugate axis of the hyperbola.

Principal Axes :
The transverse and conjugate axis together are called Principal Axes of the hyperbola.

Vertices :
$A \equiv (a, 0)$ and $A' \equiv (-a, 0)$

Focal distance: the focal distance of any point (x,y) on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are ex–a and ex+a.

Focal Chord :
A chord which passes through a focus is called a focal chord.

Double Ordinate :
A chord perpendicular to the transverse axis is called a double ordinate.

Latus Rectum (ℓ) :
The focal chord perpendicular to the transverse axis is called the latus rectum.

$$\ell=\frac{2b^2}{a}=2a(e^2-1)=2e \text{ (distance from focus to directrix)}$$

Centre :
The point which bisects every chord of the conic drawn through it is called the centre of the conic.

$C \equiv (0,0)$ the origin is the centre of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Note:

Since the fundamental equation to the hyperbola only differs from that to the ellipse in having $-b^2$ instead of b^2 it is found that many propositions for the hyperbola are derived from those for the ellipse by simply changing the sign of b^2.

(2) Conjugate Hyperbola :

Two hyperbolas such that transverse and conjugate axes of one hyperbola are respectively the conjugate & the transverse axes of the other are called Conjugate Hyperbolas of each other.

for example: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are conjugate hyperbolas of each other.

Note:

(i) If e and e' are the eccentricities of the hyperbola & its conjugate then $1/e^2 + 1/e'^2 = 1$.

(ii) The foci of a hyperbola and its conjugate are concyclic and form the vertices of a square.

(iii) Two hyperbolas are said to be similar if they have the same eccentricity.

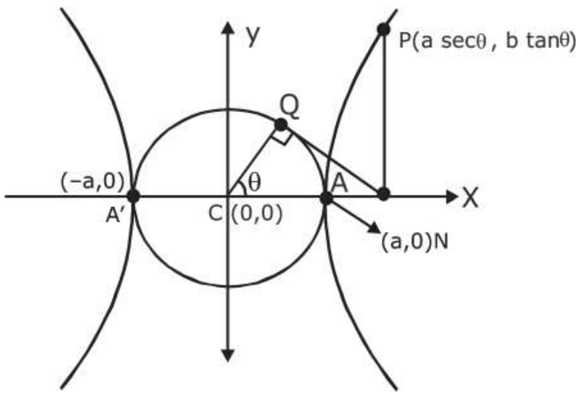
(iv) Two similar hyperbolas are said to be equal if they have same latus rectum.

(v) If a hyperbola is equilateral then the conjugate hyperbola is also equilateral.

(3) Auxiliary Circle :

A circle drawn with centre C and transverse axis as a diameter is called the **Auxiliary Circle** of the hyperbola. Equation of the auxiliary circle is $x^2 + y^2 = a^2$.

In the figure P and Q are called the **"Corresponding Points"** on the hyperbola and the auxiliary circle.



(4) Parametric Representation :

The equations $x = a \sec\theta$ and $y = b \tan\theta$ together represents the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

where θ is a parameter, which is termed as eccentric angle, $\theta \in [0, 2\pi) - \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$

If $P(\theta) \equiv (a\sec\theta, b\tan\theta)$ is on the hyperbola then corresponding point $Q(\theta) \equiv (a\cos\theta, a\sin\theta)$ is on the auxiliary circle.

The equation to the chord of the hyperbola joining two points with eccentric angles α & β

is given by $\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right) - \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)$.

(5) Position of Point 'P' with respect to Hyperbola :

The quantity $S_1 \equiv \left\{\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right\}$ is positive, zero or negative according as the point (x_1,y_1) lies inside, on or outside the curve respectively.

(6) Line And A Hyperbola :

The straight line y=mx+c is a secant, a tangent or passes outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ according as $c^2 >$ or $=$ or $< a^2m^2 - b^2$ respectively.

(7) Tangent of Hyperbola

(i) **Slope Form :** $y = mx \pm \sqrt{a^2m^2 - b^2}$ can be taken as the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ having slope 'm'.

(ii) **Point Form :** Equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

(iii) **Parametric Form :** Equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point. $(a\sec\theta, b\tan\theta)$ is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$.

Note:

(i) Point of intersection of the tangents at 'θ' and 'ϕ' is $\left\{\frac{a\cos\left(\frac{\theta-\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}, \frac{b\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}\right\}$

(ii) If $|\theta+\phi| = \pi$, then tangents at these points 'θ' and 'ϕ' are parallel.

(8) Normal of Hyperbola:

(i) The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $P(x_1,y_1)$ on it is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$.

(ii) The equation of the normal at the point $P(a\cos\theta, b\tan\theta)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$ or

(9) Pair of Tangents :

The equation to the pair of tangents which can be drawn from any point (x_1,y_1) to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is given by : $SS_1 = T^2$ where :

$$S \equiv \frac{x^2}{a^2}-\frac{y^2}{b^2}-1 \quad ; \quad S_1 = \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1 \quad ; \quad T \equiv \frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1.$$

Note:

The locus of the intersection point of tangents which are at right angles is known as the **Director Circle** of the hyperbola. The equation to the director circle is : $x^2 + y^2 = a^2 - b^2$.

If $b^2 < a^2$ this circle is real.

If $b^2 = a^2$ (rectangular hyperbola) the radius of the circle is zero & it reduces to a point circle at the origin. In this case the centre is the only point from which the tangents at right angles can be drawn to the curve.

If $b^2 > a^2$, the radius of the circle is imaginary, so that there is no such circle & so no pair of tangents at right angle can be drawn to the curve.

(10) Chord of Contact :

Equation to the chord of contact of tangents drawn from a point $P(x_1,y_1)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $T = 0$, where $T \equiv \frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$

(11) Chord with a given middle point :

Equation of the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ whose middle point is (x_1, y_1) is $T = S_1$, where $S_1 \equiv \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$; $T \equiv \frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$.

(12) Diameter :

The locus of the middle points of a system of parallel chords with slope 'm' of a hyperbola is called its diameter. It is a straight line passing through the centre of the hyperbola and has the equation $y = -\frac{b^2x}{a^2m}$, all diameters of the hyperbola passes through its centre.

(13) Asymptotes :

Definition : If the length of the perpendicular let fall from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called the Asymptote of the hyperbola.

Equations of Asymptote :

$$\frac{x}{a}+\frac{y}{b}=0 \text{ and } \frac{x}{a}-\frac{y}{b}=0 \Rightarrow \frac{x^2}{a^2}-\frac{b^2}{a^2}=0$$

Note:

(i) A hyperbola and its conjugate have the same asymptote.

(ii) The equation of the pair of asymptotes differs from the equation of hyperbola (or conjugate hyperbola) by the constant term only.

(iii) The asymptotes pass through the centre of the hyperbola & are equally inclined to the transverse axis of the hyperbola. Hence the bisector of the angles between the asymptotes are the principle axes of the hyperbola.

(iv) The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis.

(14) Rectangular Or Equilateral Hyperbola :

The hyperbola in which the lengths of the transverse & conjugate axis are equal is called an Equilateral Hyperbola, the eccentricity of the rectangular hyperbola is $\sqrt{2}$.

Rectangular Hyperbola ($xy = c^2$) :

It is referred to its asymptotes as axis of co-ordinates.

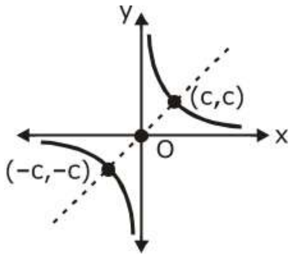
Vertices: (c, c) and $(-c, -c)$;

Foci: $(\sqrt{2}c, \sqrt{2}c)$ and $(-\sqrt{2}c, -\sqrt{2}c)$

Directrix: $x + y = \pm\sqrt{2}c$

Latus Rectum: $\ell = 2\sqrt{2}c$

Parametric equation $x = ct$, $y = c/t$, $t \in R - \{0\}$



Note:

(i) Equation of a chord joining the points $P(t_1)$ and $Q(t_2)$ is $x + t_1t_2y = c(t_1+t_2)$.

(ii) Equation of the tangent at $P(x_1,y_1)$ is $\frac{x}{x_1} + \frac{y}{y_1} = 2$ and at $P(t)$ is $\frac{x}{t} + ty = 2c$.

(iii) Equation of the normal at $P(t)$ is $xt^3 - yt = c(t^4 - 1)$.

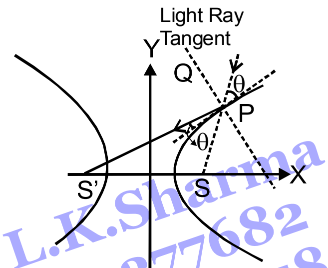
(iv) Chord with a given middle point as (h,k) is $kx + hy = 2hk$.

(15) Important Results :

- Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ upon any tangent is its auxiliary circle i.e. $x^2+y^2=a^2$ and the product of these perpendiculars is b^2.
- The portion of the tangent between the point of contact and the directrix subtends a right angle at the corresponding focus.
- The tangent & normal at any point of a hyperbola bisect the angle between the focal radii. This spells the reflection property of the hyperbola as "An incoming light ray" aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. It follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common point.

Note:

The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{a^2-k^2} - \frac{y^2}{k^2-b^2} = 1$ $(a > k > b > 0)$ are confocal and therefore orthogonal.



- The foci of the hyperbola and the points P and Q in which any tangent meets the tangents at the vertices are concyclic with PQ as diameter of the circle.
- If from any point on the asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point & the curve is

always equal to the square of the semi conjugate axis.

- Perpendicular from the foci on either asymptote meet it in the same points as the corresponding directrix & the common points of intersection lie on the auxiliary circle.
- The tangent at any point P on a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with centre C, meets the asymptotes in Q and R and cuts off a ΔCQR of constant area equal to ab from the asymptotes & the portion of the tangent intercepted between the asymptote is bisected at the point of contact. This implies that locus of the centre of the circle circumscribing the ΔCQR in case of a rectangular hyperbola is the hyperbola itself & for a standard hyperbola the locus would be the curve: $4(a^2x^2 + b^2y^2) = (a^2 + b^2)^2$.
- If the angle between the asymptote of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is 2θ then the eccentricity of the hyperbola is sec θ.
- A rectangular hyperbola circumscribing a triangle also passes through the orthocentre of this triangle. If $\left(ct_1, \frac{c}{t_1}\right)$ i = 1, 2, 3 be the angular points P, Q, R then orthocentre is $\left(\frac{-c}{t_1t_2t_3}, -ct_1t_2t_3\right)$.
- If a circle and the rectangular hyperbola $xy = c^2$ meet in the four points t_1, t_2, t_3 & t_4,then
  (i) $t_1\, t_2\, t_3\, t_4 = 1$
  (ii) the centre of the mean position of the four points bisects the distance between the centres of the two curves.
  (iii) the centre of the circle through the points t_1, t_2 & t_3 is:

$$\left\{\frac{c}{2}\left(t_1 + t_2 + t_3 + \frac{1}{t_1t_2t_3}\right), \frac{c}{2}\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + t_1 + t_2 + t_3\right)\right\}$$

29. VECTORS

(1) Basic Definitions :

Vector is a physical quantity having magnitude and definite direction. A directed line segment $\overrightarrow{AB}$ represents a vector having initial point 'A' and the terminal point 'B'. Magnitude of vector $\overrightarrow{AB}$ is represented by $|\overrightarrow{AB}|$.

In 3-dimensional space if a point P is represented by (x_1, y_1, z_1), then position vector of 'P' is given by $\overrightarrow{OP} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$. If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points in space, then :

$\overrightarrow{OP} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\overrightarrow{OQ} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

$\overrightarrow{PQ}$ = (position vector of Q)−(position vector of P)

$\therefore \quad \overrightarrow{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

$\Rightarrow |\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

(i) Zero Vector/Null Vector :

A vector having zero magnitude and indeterminate direction is termed as zero vector.

For example : $\overrightarrow{AA} = \vec{0}$ (i.e. vector having same initial and terminal point).

(ii) Unit Vector :

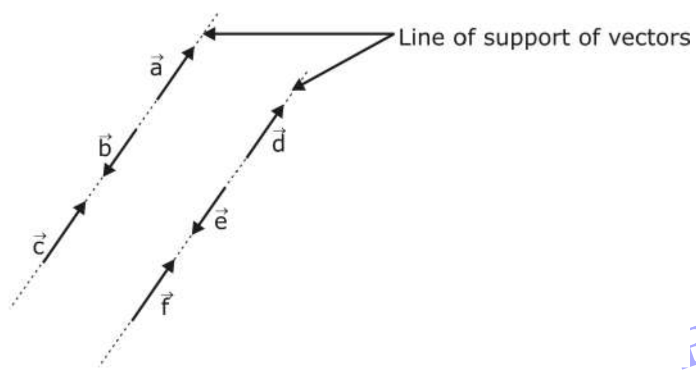
A vector having unit magnitude is termed as unit vector. Unit vector in the direction of $\vec{a}$ is represented by $\hat{a}$.

$\therefore \quad \hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$ or $\vec{a} = |\vec{a}|\,\hat{a}$.

(iii) Parallel Vector or Collinear Vectors :

A set of vectors is termed as parallel (or collinear) vectors if they have common line of support or parallel line of support.

- Parallel vectors are termed as like vectors if they have same direction and further if like vectors have same magnitude, then vectors are termed as equal vectors.
- Parallel vectors are termed as unlike vectors if they have opposite sense of direction and negative of vector $\vec{a}$ is represented by $-\vec{a}$



- Vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, $\vec{e}$ and $\vec{f}$ are all collinear (or parallel) vectors
- Vectors $\vec{a}$, $\vec{e}$, $\vec{d}$ and $\vec{f}$ are like vectors.

- Vectors $\vec{a}$ and $\vec{b}$ are unlike vectors , similarly $\vec{b}$ and $\vec{d}$ are unlike vectors.

Let $\vec{a}$ and $\vec{b}$ are two non-zero collinear vectors , then $\vec{a}=\lambda\vec{b}$, where $\lambda \in R$.

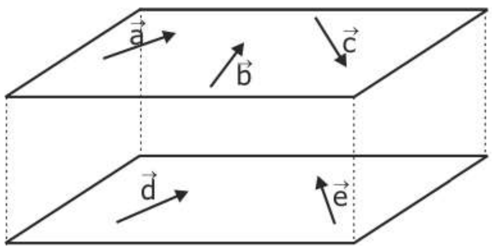
If $\vec{a}(a_1 , a_2 , a_3)$ and $\vec{b}(b_1 , b_2 , b_3)$ are collinear or parallel vectors , then.

$(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) = \lambda(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \Rightarrow \dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}$.

Note:

Three points $A(\vec{a})$, $B(\vec{b})$ and $c(\vec{c})$ are collinear , if there exists scalars x , y and z such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$, where x + y + z = 0 and x , y , z are not simultaneously zero.

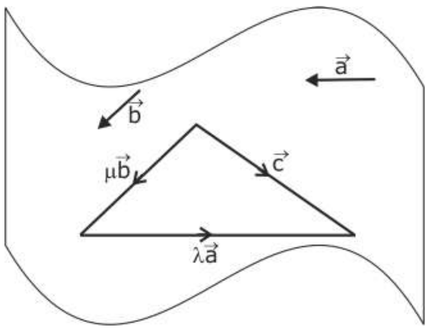
(iv) Coplanar Vectors :
Vectors which lie one the same plane of support or parallel plane of support are termed as coplanar vectors.



Vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ and $\vec{e}$ are all coplanar vectors.

Note:

- Two vectors are always coplanar and if three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar then any one of the vector can be expressed as a linear combination of other tow vectors.



$\vec{c} = \lambda\vec{a}+\mu\vec{b}$ for some scalars λ and μ

- If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar vectors , then $(\vec{a} \times \vec{b}).\vec{c} = 0$

- If $\vec{a}(a_1 , a_2 , a_3)$, $\vec{b}(b_1 , b_2 , b_3)$ and $\vec{c}(c_1 , c_2 , c_3)$ are coplanar , then

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

- Three points in space are always coplanar and if $A(\vec{a})$, $B(\vec{b})$, $C(\vec{c})$ and $D(\vec{d})$ are four coplanar points , then $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = 0$, where x + y + z + w = 0 and x , y , z , w are scalars which are not simultaneously zero.

- If A , B , C and D are four point lying on a plane , then $\left[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AC}\right] = 0$.

(v) Co-initial Vectors :
Vectors which are having same initial points are termed as co-initial vectors.

For example : $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ are coinitial vectors.

(vi) Coterminous Vectors :
Vectors which have same terminal point are termed as coterminous vectors.

For example : $\overrightarrow{PA}$, $\overrightarrow{QA}$, $\overrightarrow{RA}$ are coterminous vectors.

(2) Linear Combination of Vectors :

- **Linearly Independent Vectors :**

A set of vectors $\overrightarrow{a_1}$, $\overrightarrow{a_2}$, ..., $\overrightarrow{a_n}$ is said to be linearly independent iff

$$x_1\overrightarrow{a_1} + x_2\overrightarrow{a_2} + \ldots + x_n\overrightarrow{a_n} = 0 \Rightarrow x_1 = x_2 = \ldots = x_n = 0$$

- **Linearly Dependent Vectors:**

A set of vectors $\overrightarrow{a_1}$, $\overrightarrow{a_2}$, ..., $\overrightarrow{a_n}$ is said to be linearly dependent iff there exists scalars x_1 , x_2 , ... , x_n not all zero such that

$$x_1\overrightarrow{a_1} + x_2\overrightarrow{a_2} + \ldots + x_n\overrightarrow{a_n} = 0$$

Note:

- If two non-zero vectors $\vec{a}$ and $\vec{b}$ are linearly dependent , then $\vec{a}$ and $\vec{b}$ are collinear or parallel ($\therefore$ $\vec{a} \times \vec{b} = 0$)
- If two non-zero vectors $\vec{a}$ and $\vec{b}$ are linearly independent , then $\vec{a} \times \vec{b} \neq 0$ and $x\vec{a} + y\vec{b} = 0 \Rightarrow x = y = 0$.

For example : $x\hat{i} + y\hat{j} = 0 \Rightarrow x = y = 0$.

- If three non-zero vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are linearly dependent , then any one of the

vector can be other two vectors which further implies that $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar. ($\therefore$ $[\vec{a}\ \vec{b}\ \vec{c}] = 0$)

- If three non-zero vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are linearly independent , then $\left[\vec{a}\ \vec{b}\ \vec{c}\right] \neq \vec{c}$ and hence the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar.
- Any set of four or more non-orthogonal system of vectors is always linearly dependent

(3) Addition of Vectors :

If the vectors $\vec{a}$ and $\vec{b}$ are represented by $\overrightarrow{AB}$ and $\overrightarrow{AD}$, then $\vec{a}+\vec{b}$ is the vector addition which is represented by $\overrightarrow{AC}$, where AC is the diagonal of parallelogram ABCD.

Note:

- $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
- $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
- $\vec{a} + \vec{o} = \vec{o} + \vec{a} = \vec{a}$
- $\vec{a} + (-\vec{a}) = \vec{o}$

Triangle inequality:

- $|\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|$
- $|\vec{a}-\vec{b}| \ge \left||\vec{a}| - |\vec{b}|\right|$

Sign of equality holds if $\vec{a}$ and $\vec{b}$ are like vectors.

- $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}| \Rightarrow$ Angle between $\vec{a}$ and $\vec{b}$ is 90^o.
- $|\vec{a}\pm\vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 \pm 2|\vec{a}||\vec{b}|\cos\theta}$

(4) Multiplication of a Vector by Scalar :

Scalar multiplication of vector $\vec{a}$ and scalar quantity k , where $K \in R$, is represented by $K\vec{a}$. $K\vec{a}$ is vector which is parallel to $\vec{a}$ and having magnitude |K| times the magnitude of vector $\vec{a}$.

Note:

If P , $K \in R$, then

- $K(\vec{a}) = (\vec{a})K = K(\vec{a})$
- $K(P\vec{a}) = P(K\vec{a}) = PK(\vec{a})$
- $(K+P)\vec{a} = K\vec{a} + P\vec{a}$
- $K(\vec{a}+\vec{b}) = K\vec{a} + K\vec{b}$.

(5) Basic Application of Vectors to Geometry:

(i) If $\vec{a}$ and $\vec{b}$ are the position vectors of two points A and B , then position vector of points A and B , then position vector of point $C(\vec{r})$ which divides AB in the ratio m : n is given by

$$\vec{r} = \frac{m\vec{b}+b\vec{a}}{m+n} \quad \text{(for internal division).}$$

$$\vec{r} = \frac{m\vec{b}-b\vec{a}}{m-n} \quad \text{(for external division).}$$

(ii) If $A(\vec{a})$, $B(\vec{b})$ and $C(\vec{c})$ are the vertices of ΔABC , then :

- P.V. of Controid $= \dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$
- P.V. of incentre $= \dfrac{|\vec{b}-\vec{c}|\vec{a} + |\vec{c}-\vec{a}|\vec{b} + |\vec{a}-\vec{b}|\vec{c}}{|\vec{a}-\vec{b}| + |\vec{b}-\vec{c}| + |\vec{c}-\vec{a}|}$

(iii) If $A(\vec{a})$, $B(\vec{b})$, $C(\vec{c})$ and $D(\vec{d})$ are the vertices of a tetrahedron , then P.V. of centroid of tetrahedron is given by . $\vec{r} = \dfrac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$

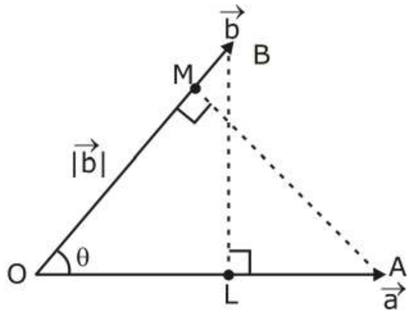
(6) Scalar or Dot Product :

Scalar product of two vector $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}.\vec{b}$ and is defined as :

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where θ is the angle $(0 \le \theta \le \pi)$ between vectors $\vec{a}$ and $\vec{b}$.

Projection of $\vec{b}$ on $\vec{a}$ = OL

Projection of $\vec{a}$ on $\vec{b}$ = OM.



In ΔOBL , $OL = |\vec{b}|\cos\theta$

$\therefore \quad OL = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|}$ (Projection of $\vec{b}$ on $\vec{a}$) $\Rightarrow \overrightarrow{OL} = \left(\dfrac{\vec{a}.\vec{b}}{|\vec{a}|^2}\right)\vec{a}$ and $\overrightarrow{LB} = \vec{b}$

Similarly , In ΔOAM , $OM = |\vec{a}|\cos\theta$

$\therefore \quad OM = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$ (Projection of $\vec{a}$ on $\vec{b}$)

$$\Rightarrow \overrightarrow{OM} = \left(\frac{\vec{a}.\vec{b}}{|\vec{b}|^2}\right)\vec{b} \text{ and } \overrightarrow{MA} = \vec{a} - \left(\frac{\vec{a}.\vec{b}}{|\vec{b}|^2}\right)\vec{b}.$$

Properties of dot Product:

- $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$
- $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$
- If $\hat{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\hat{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.
- Angle θ between vectors $\vec{a}$ and $\vec{b}$ is given by :

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \left\{\frac{a_1b_1 + a_2b_2 + a_3b_3}{\left(\sqrt{a_1^2 + a_2^2 + a_3^2}\right)\left(\sqrt{b_1^2 + b_2^2 + b_3^2}\right)}\right\}$$

If $a_1b_1 + a_2b_2 + a_3b_3 > 0$, then θ is Acute angle.
If $a_1b_1 + a_2b_2 + a_3b_3 < 0$, then θ is obtux angle.

If $a_1b_1 + a_2b_2 + a_3b_3 = 0$, then θ is 90^o.
If $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}$, then $\theta = 0$ or π.

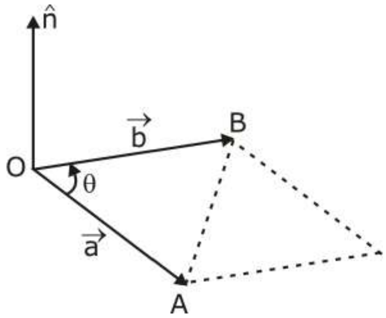
- $\vec{a}.\vec{a} = |\vec{b}|^2$
- $\vec{a}.\vec{b} = \vec{b}.\vec{a}$
- $\vec{a}.(\vec{b} + \vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$
- $(\vec{a}.\hat{i})\hat{i} + (\vec{b}.\hat{j})\hat{j} + (\vec{c}.\hat{k})\hat{k} = \vec{a}$
- $-|\vec{a}||\vec{b}| \le \vec{a}.\vec{b} \le |\vec{a}||\vec{b}|$
- $|\vec{a} \pm \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 \pm 2\,\vec{a}.\vec{b}$
- $|\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a})$

(7) Vector or Cross Product :

Vector product of two vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}\times\vec{b}$ and is defined as: $\vec{a}\times\vec{b}\left\{|\vec{a}||\vec{b}|\sin\theta\right\}\hat{n}$, where θ is the angle between vector $\vec{a}$ and $\vec{b}$, and $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. (direction of $\hat{n}$ is determined by right handed screw rule)

Area of $\Delta OAB = 1/2\ (OA)(OB)\sin\theta$

$\therefore\quad \Delta OAB = 1/2\ |\vec{a}\times\vec{b}|$



Geometrically $|\vec{a}\times\vec{b}|$ represents the area of parallelogram whose two adjacent sides are represented by $\vec{a}$ and $\vec{b}$

Properties of Cross Product :

- $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}.\hat{k}=0$
- $\hat{i}\times\hat{j}=\hat{k}$, $\hat{j}\times\hat{k}=\hat{i}$ and $\hat{k}\times\hat{i}=\hat{j}$
- $\vec{a}\times\vec{a}=0$
- If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$
- $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$
- $\vec{a}\times(\vec{b}+\vec{c})=(\vec{a}\times\vec{b})+\vec{a}\times\vec{c}$
- $\vec{a}\times(k\vec{b})=k(\vec{a}\times\vec{b})$
- $\vec{a}\times\vec{b}=0 \Leftrightarrow \vec{a}$ and $\vec{b}$ are parallel
- Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is given by : $\hat{n}=\pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$
- $|\vec{a}\times\vec{b}|^2+(\vec{a}.\vec{b})^2=|\vec{a}|^2|\vec{b}|^2$ (Lagrange's Idantity)
- If $A(\vec{a})$, $B(\vec{b})$ and $C(\vec{c})$ are the vertices of ΔABC , then area of

$\Delta ABC=\dfrac{1}{2}\left|\vec{a}\times\vec{b}+\vec{c}+\vec{c}\times\vec{a}\right|$ $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=0 \Rightarrow A$, B and C are collinear.

- If $A(\vec{a})$, $B(\vec{b})$, $C(\vec{c})$ and $D(\vec{d})$ are the vertices of a quadrileteral , then area of Q uad

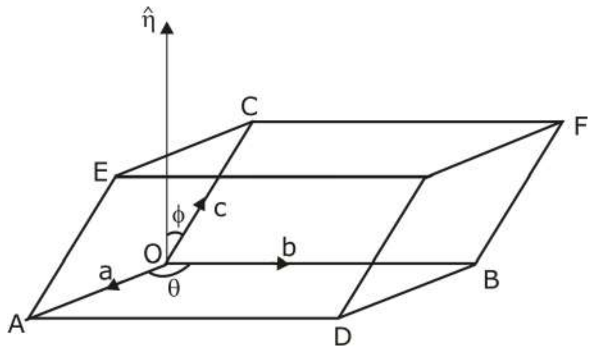
$$(ABCD)=\frac{1}{2}\left|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}\times\vec{c}\times\vec{d}+\vec{d}\times\vec{a}\right|$$

$$=\frac{1}{2}\left|\vec{AC}\times\vec{BD}\right|$$

8. Scalar Triple Product :

The scalar triple product of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is defined as : $\vec{a} \times \vec{b} . \vec{c} = |\vec{a}||\vec{b}||\vec{c}|$ $\sin\theta \cos\phi$ where θ is the angle between $\vec{a}$ and $\vec{b}$ and ϕ is the angle between $\vec{a} \times \vec{b}$ and $\vec{c}$.

It is also written as $[\vec{a}\ \vec{b}\ \vec{c}]$



Scalar triple product geometrically represents the volume of the parallelopiped whose three coterminous edges are represented by $\vec{a}$, $\vec{b}$ and $\vec{c}$ i.e. $v = [\vec{a}\ \vec{b}\ \vec{c}]$

Note:

(i) If $\vec{a} = a_1 i + a_2 j + a_3 k ; \vec{b} = b_1 i + b_2 j + b_3 k$ and $\vec{c} = c_1 i + c_2 j + c_3 k$ then $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

(ii) In a scalar triple product the position of dot and cross can be interchanged i.e. $\vec{a}\ .(\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}).\vec{c}$ or $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \left[\vec{b}\ \vec{c}\ \vec{a}\right] = \left[\vec{c}\ \vec{b}\ \vec{a}\right]$

(iii) $\vec{a}.(\vec{b} \times \vec{c}) = \vec{a},(\vec{c} \times \vec{b})$ i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{a}\ \vec{c}\ \vec{b}]$

(iv) If $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar $\Leftrightarrow$ $[\vec{a}\ \ \vec{b}\ \ \vec{c}] = 0$.

(v) Scalar product of three vectors, two of which are equal or parallel is 0.

(vi) If $\vec{a}$, $\vec{b}$, $\vec{c}$ are non-coplanar then $[\vec{a}\ \ \vec{b}\ \ \vec{c}] > 0$ for right handed system and $[\vec{a}\ \ \vec{b}\ \ \vec{c}] < 0$ for left handed system $[K\vec{a}\ \ \vec{b}\ \ \vec{c}] = [K\vec{a}\ \ \vec{b}\ \ \vec{c}]$

(vii) The volume of the tetrahedron OABC with O as origin and the pv's of A, B and C being $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively is given by $V = \frac{1}{6}[\vec{a}\ \ \vec{b}\ \ \vec{c}]$

The position vector of the centroid of a tetrahedron if the pv's of its vertices are $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are given by $\frac{1}{4}\ [\vec{a} + \vec{b} + \vec{c} + \vec{d}]$

Note that this is also the point of concurrency of the lines joining the vertices to the centroids of the opposite faces and is also called the centre of the tetrahedron. In case the tetrahedron is regular it is equidistant from the vertices and the four faces of the tetrahedron.

(9) Vector Triple Product :

Vector quantity $\vec{a} \times (\vec{b} \times \vec{c})$ or $(\vec{a} \times \vec{b}) \times \vec{c}$ is termed as the vector triple product for any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

- $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$
- $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

Let $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{R}$, then by definition of cross product , $\vec{R}$ is perpendicular to both $\vec{a}$ and $\vec{b} \times \vec{c}$.

$\Rightarrow$ $\vec{a}.\vec{R} = 0$ and $\vec{R}.(\vec{b} \times \vec{c}) = 0$

$\Rightarrow$ $\vec{R}$ is normal to $\vec{a}$ and $\vec{R}$, $\vec{b}$ and $\vec{c}$ are coplanar.

$\therefore$ $\vec{a} \times (\vec{b} \times \vec{c})$ represents a vector which is normal to $\vec{a}$ and lies in the plane containing $\vec{b}$ and $\vec{c}$.

Note:

- Unit vector normal to $\vec{a}$ and lying in plane of $\vec{b}$ and $\vec{c} = \pm \dfrac{\vec{a} \times (\vec{b} \times \vec{c})}{|\vec{a} \times (\vec{b} \times \vec{c})|}$
- In general , $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$.

(10) Scalar Product of four Vectors :

If $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ are four vectors , the products $(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d})$ is called scalar products of four vectors.

$$\text{i.e. , } (\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a}.\vec{b} & \vec{b}.\vec{c} \\ \vec{a}.\vec{d} & \vec{b}.\vec{d} \end{vmatrix}$$

(11) Vector product of four Vectors :

If $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ are four vectors , the products $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ is called scalar products of four vectors.

i.e. , $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\,\vec{b}\,\vec{d}]\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\vec{d}$

$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\,\vec{c}\,\vec{d}]\vec{b} - [\vec{a}\,\vec{c}\,\vec{c}]\vec{a}$.

(12) Reciprocal System of Vector :

If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three non coplanar vector such that $[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$, then the system of vectors $\vec{a}'$, $\vec{b}'$, $\vec{c}$ which satisfy the condition that $a.a' = b.b' = c.c' = 1$ and $\vec{a}.\vec{b}' = \vec{a}.\vec{c}' = \vec{b}.\vec{a}' = \vec{b}.\vec{c}' = \vec{c}.\vec{a}' = \vec{c}.\vec{b}' = 0$.

$$\vec{a}\,' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]} \qquad \vec{b}\,' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]} \quad \text{and} \quad \vec{c}\,' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Properties of reciprocal system

(a) $[\vec{a}\,\vec{b}\,\vec{c}]\,[\vec{a}\,'\,\vec{b}\,'\,\vec{c}\,'] = 1$

(b) $\vec{a} = \dfrac{\vec{b}\,'\times\vec{c}\,'}{[\vec{a}\,'\,\vec{b}\,'\,\vec{c}\,']} \quad \vec{b} = \dfrac{\vec{c}\,'\times\vec{a}\,'}{[\vec{a}\,'\,\vec{b}\,'\,\vec{c}\,']} \quad \vec{c} = \dfrac{\vec{a}\,'\times\vec{b}\,'}{[\vec{a}\,'\,\vec{b}\,'\,\vec{c}\,']}$

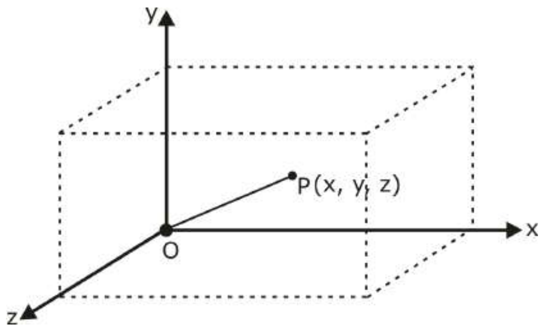
(c) The system of unit vectors $\hat{i}\,,\,\hat{j}\,,\hat{k}$ is its own reciprocal

i.e. $\hat{i}\,' = \hat{i} \quad \hat{j}\,' = \hat{j}$ and $\hat{k}\,' = \hat{k}$

Not that any vector can be expressed in terms of $\vec{a'}$, $\vec{b'}$ and $\vec{c'}$ as they also constitute a system of non coplanar vectors.

30. THREE DIMENSIONAL GEOMETRY

Coordinates of any point 'P' in the space is given by (x, y, z) and the point is located with reference to three mutually perpendiculer corordinates axes ox, oy and oz.



Position vector of point P is given by $\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$.

(1) Distance formula :

Distance between any two points $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$ is given by:

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Note:

If position vector of two points A and B are given as $\overrightarrow{OA}$ and $\overrightarrow{OB}$, then

$$AB = |\overrightarrow{OB} - \overrightarrow{OA}|$$

$$\Rightarrow \quad AB = |(x_2 i + y_2 j + z_2 k) - (x_1 i + y_1 j + z_1 k)|$$

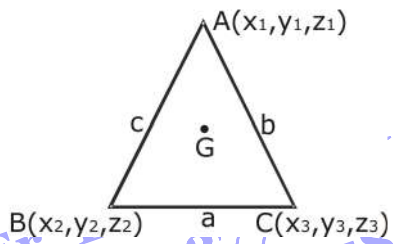
$$\Rightarrow \quad AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

(2) Section Formula :

If point P divides the distance between the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ internally in the ratio of m : n , then coordinates of P are given by :

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$

(3) Centroid of a Triangle ABC :



$$G \equiv \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

(4) Incentre of Triangle ABC :

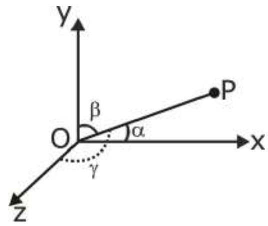
$$I \equiv \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}, \frac{az_1 + bz_2 + cz_3}{a+b+c}\right)$$

(5) Centroid of a tetrahedron :

A (x_1, y_1, z_1) , B(x_2, y_2, z_2) , C (x_3, y_3, z_3) and D(x_4, y_4, z_4) are the vertices of a tetrahedron then coordinate of its centroid (G) is given by :

$$G \equiv \left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4} \right)$$

(6) Direction Cosines And Direction Ratios :



Direction cosines: Let α , β , γ be the angles which a directed line makes with the positive directions of the axes of x, y and z respectively, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the direction cosines of the line. The direction cosines are usually denoted by (ℓ , m, n).

$\therefore \quad \ell = \cos\alpha, \quad m = \cos\beta, \quad n = \cos\gamma.$

Note:

(i) If ℓ , m, n be the direction cosines of a line, then $\ell^2 + m^2 + n^2 = 1$.

(ii) If ℓ , m, n are direction cosines of line L then $\ell\,\hat{i} + m\hat{j} + n\hat{k}$ is a unit vector parallel to the line L.

Direction ratios : Let a, b, c be proportional to the direction cosines ℓ , m , n then a, b, c are called the direction ratios.

$$\therefore \quad \frac{\ell}{a} = \frac{m}{b} = \frac{m}{c}$$

Note:

(i) If ℓ , m, n be the direction cosines and a, b , c be the direction ratios of a vector, then

$$\left(\ell = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}, n = \frac{c}{\sqrt{a^2 + b^2 + c^2}} \right)$$

(ii) If the coordinates of P and Q are (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively, then the direction ratios of line PQ can be assumed as:
$a = x_2 - x_1$, $b = y_2 - y_1$ and $c = z_2 - z_1$, and the direction cosines of line PQ are

$$\ell = \pm \frac{x_2 - x_1}{|PQ|}, \quad m = \pm \frac{y_2 - y_1}{|PQ|} \quad \text{and} \quad n = \pm \frac{z_2 - z_1}{|PQ|}.$$

(iii) If a, b, c are the direction ratios of any line L then $a\hat{i} + b\hat{j} + c\hat{k}$ is a vector parallel to the line L.
(iv) Direction cosines of axes:
Direction cosines of x-axis are (1, 0,0)
Direction cosines of y-axis are (0,1,0)
Direction cosines of z-axis are (0,0,1)

(7) Angle Between Two Line Segments :

If two lines have direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 respectively then vectors parallel to the lines are $a_1i + b_1j + c_1k$ and $a_2i + b_2j + c_2k$ and angle between them is given as :

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Note:

(i) The line will be perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

(ii) The lines will be parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(iii) Two parallel lines have same direction cosines i.e. $\ell_1 = \ell_2$, $m_1 = m_2$, $n_1 = n_2$.

A LINE

(8) Equation Of A Line

(i) A straight line in space is characterised by the intersection of two planes which are not parallel and therefore, the equation of a straight line is a solution of the system constituted by the equations of the two planes, $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$.
This form is also known as non-symmetrical form.

(ii) The equation of a line passing through the point (x_1, y_1, z_1) and having direction ratios a, b, c is $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \lambda$. This form is called symmetric form. A general point on the line is given by $(x_1 + a\lambda,\ y_1 + b\lambda,\ z_1 + c\lambda)$, where λ is a scalar quantity

(iii) Vector equation: Vector equation of a straight line passing through a fixed point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$, where λ is a scalar quantity.

(iv) The equation of the line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = \lambda$$

(v) Vector equation of a straight line passing through two points with position vectors $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$.

(vi) Reduction of cartesion form of equation of a line to vector form & vice versa

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \quad \Leftrightarrow \quad \vec{r} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda(a\hat{i} + b\hat{j} + c\hat{k})$$

Note:

(a) If lines $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ and $\frac{x - x_2}{a'} = \frac{y - y_2}{b'} = \frac{z - z_2}{c'}$ intersect each others at unique point, then : $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a & b & c \\ a' & b' & c' \end{vmatrix} = 0$

(b) If lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \lambda\vec{b_2}$ are intersecting, then : $\left(\vec{b_1} \times \vec{b_2}\right).\left(\vec{a_2} - \lambda\vec{a_1}\right) = 0$.

(9) Foot, Length Of perpendicular From A point To A Line :

Let equation of the line be $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}=\lambda$ .....(i)

and A(x' , y' , z') be the point.

Any point on line (i) is $P(a\lambda+x_1,\ b\lambda+y_1,\ c\lambda+z_1)$ ....(ii)

If P is the foot of the perpendicular from A on the line, then AP is perpendicular to the line.

$$\therefore\quad \{(a\lambda+x_1-x')a+(b\lambda+y_1-y')b+(c\lambda+z_1-z')c\}=0$$

From above equation, value of 'λ' is obtained and putting this value of λ in (ii), we get the foot of perpendicular from point A on the given line , since foot of perpendicular P is known, then the length of perpendicular is given by AP.

(10) To find image of a point w.r.t a line :

Image of point P(x' , y' , z') w.r.t. line $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$.

Let image of P(x' , y' , z') be $Q(\alpha,\beta,\gamma)$, hence PQ is perpandiclor to the given line

$\Rightarrow\quad a(\alpha-x')+b(\beta-y')+c(r-z')=0$ ......(i)

Now mid-point of PQ lies on the line

$$\therefore\quad \frac{\frac{\alpha+x'}{2}-x_1}{a}=\frac{\frac{\beta+y'}{2}-y_1}{b}=\frac{\frac{r+z'}{2}-z_1}{c}=\lambda \quad ......(ii)$$

from above equation, point $Q(\alpha,\beta,\gamma)$ can be calculeted in terms of λ, which further satisfy the equation (i) and hence value of 'λ' can be obtained

After getting the value of 'λ' point $Q(\alpha,\beta,\gamma)$ cam ne calculeted from equation (ii).

(11) Skew Lines :

The straight lines which are non-parallel and non-intersecting are termed as skew lines.

If $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ and $\frac{x-x_2}{a'}=\frac{y-y_2}{b'}=\frac{z-z_2}{c'}$ are skew-lines , then

$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a & b & c \\ a' & b' & c' \end{vmatrix}\neq 0$ and the shortest distance (S.D.) between the skew-lines is

given by : $$S.D.=\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a & b & c \\ a' & b' & c' \end{vmatrix}}{\sqrt{(bc'-b'c)^2+(ac'-a'c)^2+(ab'-a'b)^2}}.$$

Note:

(i) If $\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ and $\vec{r}=\vec{a_2}+\mu\vec{b_2}$ are the skew-lines in vector form, then

$\left(\vec{b_1}\times\vec{b_2}\right).\left(\vec{a_2}-\vec{a_1}\right)\neq 0$ and the shortest distance between them is given by :

$$\text{S.D.} = \left| \frac{[\vec{b_1} \;\; \vec{b_2} \;\; \vec{a_2} - \vec{a_1}]}{|\vec{b_1} \times \vec{b_2}|} \right|$$

(ii) Shortest distance 'd' between the two parallel lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and

$\vec{r} = \vec{a}_2 + \mu\vec{b}$ is given by : $d = \left| \frac{(\vec{a}_2 - \vec{a}_1) \times \vec{b}}{|\vec{b}|} \right|$.

A PLANE

If line joining any two points on a surface is perpendicular to some fixed straight line , then this surface is called a plane and the fixed line is called the normal to the plane.

(12) Equation Of A Plane

(i) **Normal form** of the equation of a plane is $\ell x + my + nz = p$, where, ℓ, m, n are the direction cosines of the normal to the plane and p is the distance of the plane from the origin.

(ii) **General form :** $ax + by + cz + d = 0$ is the equation of a plane, where a, b , c are the direction ratios of the normal to the plane.

(iii) The equation of a plane passing through the point (x_1, y_1, z_1) is given by $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ where a, b, c are the direction ratios of the normal to the plane.

(iv) Plane through three points: The equation of the plane through three non-collinear points(x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is given by :

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

(iv) **Intercept Form :** The equation of a plane cutting intercept a, b, c on the axes is

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

(v) Vector form : The equation of a plane passing through a point having position vector $\vec{a}$ normal to vector $\vec{n}$ is $(\vec{r} - \vec{a}).\vec{n} = 0$ or $\vec{r}.\vec{n} = \vec{a}.\vec{n}$

(vi) Vector equation of a plane normal to unit vector $\vec{n}$ and at a distance d from the origin is $\vec{r}.\vec{n} = d$

Note:

(a) **Coordinate planes**

(i) Equation of yz-plane is x = 0
(ii) Equation of xz-plane is y = 0
(iii) Equation of xy-plane is z = 0.

(b) **Planes parallel to the axes :**
If a = 0, the plane is parallel to x-axis i.e. equation of the plane parallel to the x-axis is by + cz + d = 0.
Similarly, equation of planes parallel to y-axis and parallel to z-axis are ax + cz + d = 0 and ax + by + d = 0 respectively.

(c) **Plane through origin :** Equation of plane passing through origin is ax + by + cz = 0.

(d) Any plane parallel to the given plane ax + by + cz + d = 0 is ax + by + cz + k = 0. Distance between two parallel planes ax + by + cz + d_1 = 0 and ax + by + cz + d_2=0

is given as $\dfrac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$

(e) **Equation of a plane passing through a given point & parallel to the given vectors:**

The equation of a plane passing through a point having position vector $\vec{a}$ and parallel to $\vec{b}$ and $\vec{c}$ is $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$ (parametric form) where λ and μ are scalars.

or $\vec{r}.(\vec{b} \times \vec{c}) = \vec{a}.(\vec{b} \times \vec{c})$ (non parametric form)

(f) A plane ax + by + cz +d = 0 divides the line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio $\left(-\dfrac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d}\right)$

(g) The xy-plane divides the line segment joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio $-\dfrac{z_1}{z_2}$. Similarly yz-plane in $-\dfrac{x_1}{x_2}$ and zx-plane in $-\dfrac{y_1}{y_2}$

(h) Coplanarity of four points

The points A(x_1 , y_1 , z_1), B(x_2 , y_2 , z_2) C(x_3 , y_3 , z_3) and D(x_4 , y_4 , z_4) are coplanar , then

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_2 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} = 0$$

(13) Volume Of A Tetrahedron :

Volume of a tetrahedron with vertices A(x_1, y_1, z_1), B(x_2, y_2, z_2), C(x_3, y_3, z_3) and D(x_4, y_4, z_4) is given by $V = \dfrac{1}{6}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_2 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix}$

(14) Sides of a plane:

A plane divides the three dimensional space in two equal parts.Two points A (x_1 , y_1 , z_1) and B(x_2 , y_2 , z_2) are on the same side of the plane ax + by + cz + d = 0 if ax_1 + by_1 + cz_1 + d and ax_2 + by_2 + cz_2 + d are both positive or both negative and are opposite side of plane if both of these values are in opposite sign.

(15) A Plane and A Point :

(i) Distance of the point (x_1 , y_1 , z_1) from the plane ax + by + cz + d = 0 is given by :

$$p = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|.$$

(ii) The length of the perpendicular from a point having position vector $\vec{a}$ to plane

$\vec{r}.\vec{n} = d$ is given by: $p = \left|\frac{\vec{a}.\vec{n} - d}{|\vec{n}|}\right|$

(iii) The coordinates of the foot of perpendicular from the point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ are given by :

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = -\left(\frac{(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}\right)$$

(iv) Reflection of a point w.r.t. a plane.
The coordinate of the image of point (x_1, y_1, z_1) w.r.t. the plane $ax + by + cz + d = 0$ are given by :

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = -2\left(\frac{(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}\right)$$

(16) Angle Between Two Planes :

(i) If two planes are $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$, then angle between these planes is the angle between their normals. Since direction ratios of their normals are $\langle a, b, c\rangle$ and $\langle a', b', c'\rangle$ respectively, hence , 'θ' the angle betweeen them, is given by

$$\cos\theta = \frac{aa' + bb' + cc'}{\sqrt{a^2 + b^2 + c^2}\sqrt{a'^2 + b'^2 + c'^2}}$$

Planes are perpendicular if $aa' + bb' + cc' = 0$ and planes are parallel if $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$

(ii) The angle θ between the planes $\vec{r}.\vec{n_1} = d_1$ and $\vec{r}.\vec{n_2} = d_2$ is given by,

$$\cos\theta = \left(\frac{\vec{n_1}.\vec{n_2}}{|\vec{n_1}|.|\vec{n_2}|}\right)$$

Planes are perpendicular if $\vec{n_1}.\vec{n_2} = 0$ and planes are parallel if $\vec{n_1} = \lambda\vec{n_2}$.

(17) Angle Between A Plane And A Line :

(i) If θ is the angle between line $\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ and the plane

$ax + by + cz + d = 0$, then $\sin\theta = \left[\frac{a\ell + bm + cn}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{\ell^2 + m^2 + n^2}}\right]$.

(ii) Vector form : If θ is the angle between a line $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r}.\vec{n} = d$, then

$$\sin\theta = \left[\frac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|}\right]$$

(iii) Condition for perpendicularity : $\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}$ or $\vec{b} \times \vec{n} = 0$

(iv) Condition for parallel case : $a\ell + bm + cn = 0$ or $\vec{b}.\vec{n} = 0$

(18) Condition For A Line To Lie in A Plane :

(i) Cartesian form: Line $\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ will lie in plane ax + by + cz + d = 0, if $ax_1 + by_1 + cz_1 + d = 0$ and $a\ell + bm + cn = 0$. holds simultanously

(ii) Vector form :Line $\vec{r}=\vec{a}+\lambda\vec{b}$ will lie in plane $\vec{r}.\vec{n}=d$ if $\vec{b}.\vec{n}=0$ and $\vec{a}.\vec{n}=d$ holds simultanously.

(19) Angle Bisectors :

(i) The equations of the planes bisecting the angle between two given planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are :

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

(ii) Equation of bisector of the angle containing origin: If both the constant terms d_1 and d_2 are positive ,then the positive sign in

$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$ gives the bisector of the angle which contains the origin.

(iii) Bisector of acute/obtuse angle: First make both the constant terms positive , then

$a_1a_2 + b_1b_2 + c_1c_2 > 0 \quad \Rightarrow$ origin lies on obtuse angle

$a_1a_2 + b_1b_2 + c_1c_2 < 0 \quad \Rightarrow$ origin lies in acute angle

(20) Reduction Of Non-Symmetrical Form To Symmetrical Form :

Let equation of the line in non-symmetrical form be $a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$. To find the equation of the line in symmetrical form, we must know (i) its direction ratios (ii) coordinate of any point on it.

Direction ratios : Let ℓ, m, n be the direction ratios of the line. since the line lies in both the planes, it must be perpendicular to normals of both planes. So $a_1\ell + b_1m + c_1n = 0$, $a_2\ell + b_2m + c_2n = 0$. From these equations, proportional values of ℓ, m, n can be calculated by cross - multiplication as $\frac{\ell}{b_1c_2-b_2c_1}=\frac{m}{c_1a_2-c_2a_1}=\frac{n}{a_1b_2-a_2b_1}$

Alternative method

The vector $\begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = i(b_1c_2-b_2c_1) + j(c_1a_2-c_2a_1) + k(a_1b_2-a_2b_1)$ will be parallel to the line of intersection of the two given planes.

$\therefore \ \ell : m : n = (b_1c_2-b_2c_1) : (c_1a_2-c_2a_1) : (a_1b_2-a_2b_1)$

Point on the line : As ℓ, m, n cannot be zero simultaneously, so at least one coordinate amust be non-zero. Let $a_1b_2-a_2b_1 \neq 0$, then the line cannot be parallel to xy plane, so it intersect it. Let it intersect xy-plane in $(x_1, y_1, 0)$. Then $a_1x_1 + b_1y_1 + d_1 = 0$ and $a_2x_1 + b_2y_1 + d_2 = 0$. solving these, we get a point on the line. Then its equation becomes.

$$\frac{x-x_1}{b_1c_2-b_2c_1}=\frac{y-y_1}{c_1a_2-c_2a_1}=\frac{z-0}{a_1b_2-a_2b_1}$$

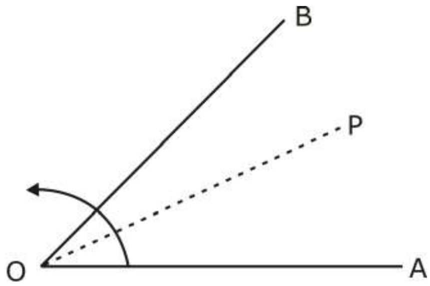
If $\ell \neq 0$, take a point on yz-plane as $(0, y_1, z_1)$ and if $m \neq 0$, take a point on xz-plane as $(x_1, 0, z_1)$.

(21) Family of Planes :

(i) Any plane passing through the line of intersection of non-parallel planes
$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by :
$(a_1x + b_1y + c_1z + d_1) + \lambda(a_2x + b_2y + c_2z + d_2) = 0$
(ii) The equation of plane passing through the intersection of the planes $\vec{r}.\overrightarrow{n_1} = d_1$ and $\vec{r}.\overrightarrow{n_2} = d_2$ is given by $\vec{r}.(\overrightarrow{n_1} + \lambda\overrightarrow{n_2}) = d_2 + \lambda d_1$ where λ is arbitrary scalar quantity.

31. TRIGONOMETRIC RATIO AND IDENTITIES

An angle is the amount of revolution which a line OP revolving about the point O has undergone in passing from its initial position OA into its final position OB.



If the rotation is in the clockwise sense, the angle measured is negative and it is positive if the rotation is in the anti-clockwise sense.
The two commonly used systems of measuring an angle are

1. Sexagesimal system is which
   1 right angle = 90 degrees (90º)
   1 degree = 60 minutes (60')
   1 minute = 60 seconds (60")
2. Circular systems in which the unit of measurement is the angle subtended at the centre of a circle by an are whose length is equal to the radius and is called a radian.

Relation between degree and radian

π radian = 180 degrees

1 radian = $\frac{180}{\pi}$ degress

= 57º 17' 45" (approximately)

The triagonometrical ratios of an angle are numberical quantities. Each one of them represent the ratio of the length of one side to another of a right angled triangle.

1. Basic Trigonometric Identities :

(a) $\sin^2\theta + \cos^2\theta = 1$; $-1 \le \sin\theta \le 1$; $-1 \le \cos\theta \le 1$ $\forall\ \theta \in R$

(b) $\sec^2\theta - \tan^2\theta = 1$; $|\sec\theta| \ge 1$ $\forall\ \theta \in R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$

(c) $\text{coses}^2\theta - \cot^2\theta = 1$; $|\text{cosec}\,\theta| \ge 1$ $\forall\ \theta \in R - \{n\pi, n \in I\}$

3. Trigonometric Functions Of Allied Angles :

If θ is any angle, then $-\theta$, $90 \pm \theta$, $180 \pm \theta$, $270 \pm \theta$, $360 \pm \theta$ etc. are called **ALLIED ANGLES**.

(a) $\sin(-\theta) = -\sin\theta$; $\cos(-\theta) = \cos\theta$

(b) $\sin(90^\circ - \theta) = \cos\theta$; $\cos(90^\circ - \theta) = \sin\theta$

(c) $\sin(90^\circ + \theta) = \cos\theta$; $\cos(90^\circ + \theta) = -\sin\theta$

(d) $\sin(180^\circ - \theta) = \sin\theta$; $\cos(180^\circ - \theta) = -\cos\theta$

(e) $\sin(180^\circ + \theta) = -\sin\theta$; $\cos(180^\circ + \theta) = -\cos\theta$

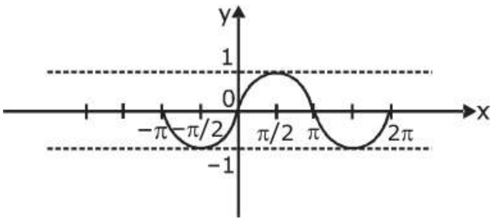
(f) $\sin(270^\circ - \theta) = -\cos\theta$; $\cos(270^\circ - \theta) = -\sin\theta$

(g) $\sin(270^\circ + \theta) = -\cos\theta$; $\cos(270^\circ + \theta) = \sin\theta$

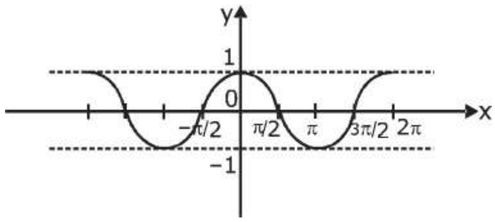
(h) $\tan(90^\circ - \theta) = \cot\theta$; $\cot(90^\circ - \theta) = \tan\theta$

(4) Graphs of Trigonometric Function :

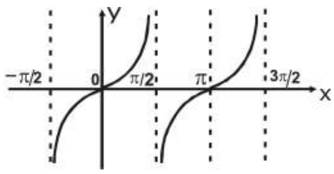
(i) $y = \sin x$; $x \in R$; $y \in [-1, 1]$



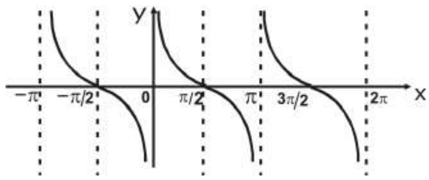
(ii) $y = \cos x$; $x \in R$; $y \in [-1, 1]$



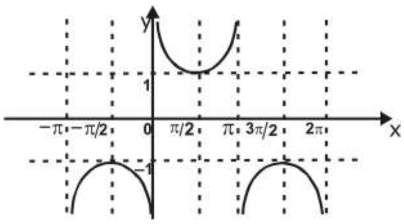
(iii) $y = \tan x$; $x \in R - (2n+1)\pi/2,\ n \in I ; y \in R$



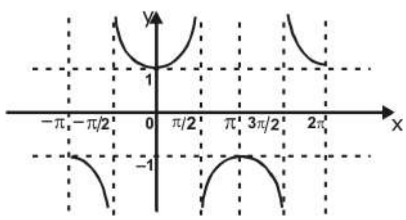
(iv) $y = \cot x$; $x \in R - n\pi,\ n \in I ; y \in R$



(v) $y = \text{cosec } x$;
$x \in R - n\pi,\ n \in I ; y \in R/(-1, 1)$



(vi) $y = \sec x$;
$x \in R - (2n+1)\pi/2,\ n \in I ; y \in R/(-1, 1)$



5. Trigonometric Functions of Sum or Difference of Two Angles :

(a) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

(b) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

(c) $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A+B).\sin(A-B)$

(d) $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A+B).\cos(A-B)$

(e) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

(f) $\cot(A \pm B) = \frac{\cot A \cot B \mp 1}{\cot B \mp \cot A}$

(g) $\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

6. Factorisation of the Sum of Difference of Two Sines or Cosines :

(a) $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

(b) $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

(c) $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

(d) $\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

7. Transformation of Products into Sum or Difference of Sines & Cosines:

(a) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

(b) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(c) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

(d) $2\sin A\cos B = \cos(A-B) - \cos(A+B)$

8. Multiple and Sub-multiple Angles :

(a) $\sin 2A = 2\sin A\cos A$; $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

(b) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$;

$$2\cos^2\frac{\theta}{2} = 1 + \cos\theta \quad 2\sin^2\frac{\theta}{2} = 1 - \cos\theta.$$

(c) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$; $\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

(d) $\sin 2A = \frac{2\tan A}{1+\tan^2 A}$, $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$

(e) $\sin 3A = 3\sin A - 4\sin^3 A$

(f) $\cos 3A = 4\cos^3 A - 3\cos A$

(g) $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

9. Important Trigonometric Ratios :

(a) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$, where $n \in I$

(b) $\sin 15^\circ$ or $\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75^\circ$ or $\cos\frac{5\pi}{12}$;

$\cos 15^\circ$ or $\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75^\circ$ or $\sin\frac{5\pi}{12}$

$\tan 15^\circ = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3} = \cot 75^\circ$; $\tan 75^\circ = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2-\sqrt{3} = \cot 15^\circ$

(c) $\sin\frac{\pi}{10}$ or $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$ & $\cos 36^\circ$ or $\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$

10. Conditional Identities :

If $A + B + C = \pi$ then :

(i) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(ii) $\sin A + \sin B + \sin C = 4 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$

(iii) $\cos 2A + \cos 2B + \cos 2C = -1-4 \cos A \cos B \cos C$

(iv) $\cos A + \cos B + \cos C = 1 + 4 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$

(v) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(vi) $\tan\frac{A}{2} \tan\frac{B}{2} + \tan\frac{B}{2} \tan\frac{C}{2} + \tan\frac{C}{2} \tan\frac{A}{2} = 1$

(vii) $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2} \cdot \cot\frac{B}{2} \cdot \cot\frac{C}{2}$

(viii) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

(ix) $A + B + C = \frac{\pi}{2}$ then $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$

11. Range of Trigonometric Expression:

$E = a \sin\theta + b \cos\theta$

$E = \sqrt{a^2+b^2} \sin(\theta+\alpha)$, where $\tan\alpha = \frac{b}{a}$

$= \sqrt{a^2+b^2} \cos(\theta-\beta)$, where $\tan\beta = \frac{a}{b}$

Hence for any real value of θ, $-\sqrt{a^2+b^2} \le E \le \sqrt{a^2+b^2}$

12. Sine and Cosine Series :

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + ...+ \sin(\alpha + \overline{n-1}\,\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \sin\left(\alpha+\frac{n-1}{2}\beta\right)$$

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) +...........+ \cos(\alpha + \overline{r-1}\,\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \cos\left(\alpha+\frac{n-1}{2}\beta\right)$$

32. TRIGONOMETRIC EQUATIONS

Trigonometric Equations

Principal Solutions :
The solutions of a trigonometric equation which lie in the interval $[0, 2\pi)$ are called **Principal solutions.**

General Solution :
The expression involving an integer 'n' which gives all solutions of a trigonometric equation is called **General solution.**

General solution of some standard trigonometric equations are given below :

(i) If $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $n \in I$.

(ii) If $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$ where $\alpha \in [0, \pi]$, $n \in I$.

(iii) If $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$ where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $n \in I$.

(iv) If $\sin^2\theta = \sin^2 a \Rightarrow \theta = n\pi \pm \alpha$

(v) $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.

(vi) $\tan^2\theta = \tan^2 n\alpha \Rightarrow \theta = n\pi \pm \alpha$

[Note : α is called the principal angle]

Types of Trigonometric Equations :

v

(i) Solutions of equations by factorising . Consider the equation :
$(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$.

(ii) Solutions of equations reducible to quadratic equations. Consider the equation ;
$3\cos^2 x - 10\cos x + 3 = 0$.

(iii) Solving equations by introducing an Auxiliary argument. Consider the equation :
$\sin x + \cos x = \sqrt{2}$ & $\sqrt{3}\cos x + \sin x = 2$.

(iv) Solving equations by Transforming a sum of Trigonometric functions into a product.
consider the example; $\cos 3x + \sin 2x - \sin 4x = 0$.

(v) Solving equations by transforming a product of trigonometric functions into a sum
Consider the equation, $\sin 5x . \cos 3x = \sin 6x . \cos 2x$.

(vi) Solving equations by change of variable :

(a) Equations of the form P(sin x $\pm$ cos x , sin x, cos x) = 0, where P(y z) is a polynomial, can be solved by the change.
$\cos x \pm \sin x = t \Rightarrow 1 \pm 2\sin x . \cos x = t^2$.
Consider the equation; $\sin x + \cos x = 1 + \sin x . \cos x$.

(b) Equations of the form of $a . \sin x + b . \cos x + d = 0$, where a , b & d are real number s and a, b $\neq 0$ can be solved by changing sin x & cos x into their corresponding tangent of half the angle.
consider the equation $3\cos x + 4\sin x = 5$.

(c) Many equations can be solved by introducing a new variable e.g. the equation, $\sin^4 2x + \cos^4 2x = \sin 2x . \cos 2x$ changes to
$$2(y + 1)\left(y - \frac{1}{2}\right) = 0 \text{ by substituting, } \sin 2x . \cos 2x = y.$$

(vii) Solving equations with the use of the Boundness of the functions sinx & cos x .

Trigonometric Inequalitities :

Solutions of elementary trigonometric inequalitities are obtained from graphs

Inequality	**Set of solutions of Inequality ($n \in z$)**
$\sin x > a$ ($\|a\| < 1$)	$x \in (\sin^{-1}a + 2\pi n, \pi - \sin^{-1}a + 2\pi n)$
$\sin x < a$ ($\|a\| < 1$)	$x \in (-\pi - \sin^{-1}a + 2\pi n, \sin^{-1}a + 2\pi n)$
$\cos x > a$ ($\|a\| < 1$)	$x \in (-\cos^{-1}a + 2\pi n, \cos^{-1}a + 2\pi n)$
$\cos x < a$ ($\|a\| < 1$)	$x \in (\cos^{-1}a + 2\pi n, 2\pi - \cos^{-1}a + 2\pi n)$
$\tan x > a$	$x \in (\tan^{-1}a + \pi n, \pi/2 + \pi n)$
$\tan x < a$	$x \in \left(-\frac{\pi}{2} + \pi n, \tan^{-1} a + \pi n\right)$

Inequalities of the form R(y) > 0, R(y)< 0, where R is a certain rational function and y is a trigonometric function (sine, cosine or tangent), are usually solved in two stages: first the rational inequality is solved for the unknown y and then follows the solution of an elementary trigonometric inequality.

33. PROPERTIES OF TRIANGLE

The angles of a triangle are denoted by capital letters A, B, C and the sides opposite to them are denoted by small letters a, b, c

In triangle $\quad A + B + C = \pi$

also $\quad a + b > c$ and $|a - b| < c$

$b + c > a$ and $|b - c| < a$

$c + a > b$ and $|c - a| < b$

(1) Sine Rule :

In any triangle ABC, the sines of the angles are proportional to the opposite sides

i.e. $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$; 'R' is circum-radius of $\triangle ABC$

(2) Cosine Formula :

(i) $\cos A = \frac{b^2+c^2-a^2}{2bc}$ or $a^2 = b^2 + c^2 - 2bc \cos A = b^2 + c^2 + 2bc\cos(B + C)$

(ii) $\cos B = \frac{c^2+a^2-b^2}{2ca}$ (iii) $\cos C = \frac{a^2+b^2-c^2}{2ab}$

(3) Projection Formula :

(i) $a = b\cos C + c\cos B$ (ii) $b = c\cos A + a\cos C$

(iii) $c = a\cos B + b\cos A$

(4) Napier's Analogy - tangent rule:

(i) $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$ (ii) $\tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2}$

(iii) $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$

(5) Trigonometric Functions of Half Angles :

(i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$; $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$; $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(ii) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$; $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$; $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(iii) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)}$ where $s = \frac{a+b+c}{2}$ is semi perimeter of triangle.

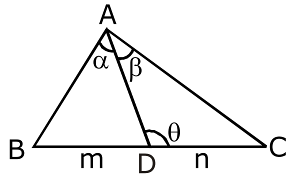
(iv) $\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{bc}$

(6) Area of Triangle (Δ)

$$\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \sqrt{s(s-a)(s-b)(s-c)}$$

(7) m - n Rule :

$$(m+n)\cot\theta = m\cot\alpha - n\cot\beta$$
$$= n\cot B - m\cot C$$



(8) Radius of Circumcircle :

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}$$

(9) Radius of The Incircle :

(i) $r = \dfrac{\Delta}{s}$

(ii) $r = (s-a)\tan\dfrac{A}{2} = (s-b)\tan\dfrac{B}{2} = (s-c)\tan\dfrac{C}{2}$

(iii) $r = \dfrac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$

(iv) $r = 4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

(10) Radius of The Ex-Circles :

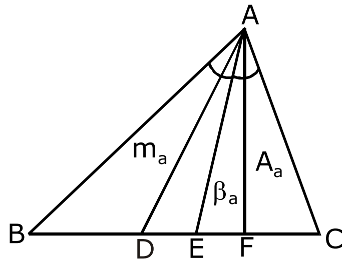
(i) $r_1 = \dfrac{\Delta}{s-a}$, $r_2 = \dfrac{\Delta}{s-b}$, $r_3 = \dfrac{\Delta}{s-c}$

(ii) $r_1 = s\tan\dfrac{A}{2}$; $r_2 = s\tan\dfrac{B}{2}$; $r_3 = s\tan\dfrac{C}{2}$

(iii) $r_1 \dfrac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$

(iv) $r_1 = 4R\sin\dfrac{A}{2}\cdot\cos\dfrac{B}{2}\cdot\cos\dfrac{C}{2}$

(11) Length of Angle Bisectors, Medians and Altitudes :



(i) Length of an angle bisector from the angle A = $\beta_a = \dfrac{2bc\cos\frac{A}{2}}{b+c}$

(ii) Length of median from the angle A = $m_a = \dfrac{1}{2}\sqrt{2b^2+2c^2-a^2}$

(iii) Length of altitude from the angle A = $A_a = \frac{2\Delta}{a}$

Note: $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$

(iii) Excentre (I_1) : $I_1 A = r_1 \operatorname{cosec}\frac{A}{2}$ and $I_{1a} = r_1$

(iv) Orthocentre(H) : HA = 2R cos A and H_a = 2R cos B cos C

(v) Centroid (G) : $GA = \frac{1}{3}\sqrt{2b^2+2c^2-a^2}$ and $G_a = \frac{2\Delta}{3a}$

(13) Orthocentre and Pedal Triangle :

The triangle which is formed by joining the feet of the altitudes is called the Pedal Triangle.

(i) Its angles are $\pi - 2a$, $\pi - 2B$ and $\pi - 2C$.

(ii) Its sides are a cosA = R sin 2A,

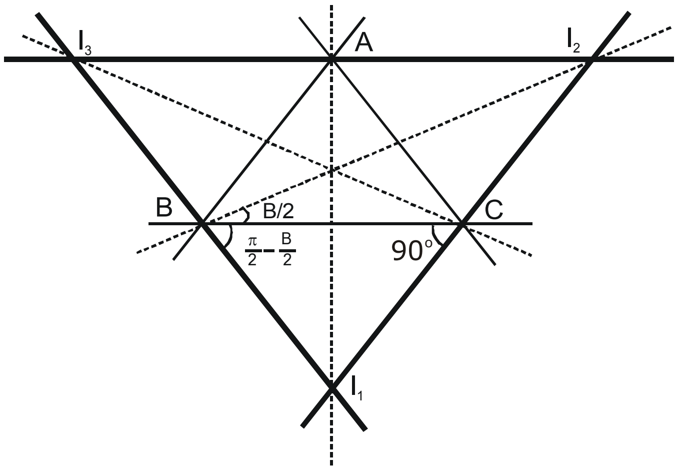
b cosB = R sin 2B and

c cosC = R sin 2C

(iii) Circumradii of the triangles PBC, PCA, PAB and ABC are equal.

(14) Excentral Triangle :

The triangle formed by joining the three excentres I_1, I_2 and I_3 of ΔABC is called the excentral or excentric triangle.



(i) ΔABC is the pedal triangle of the $\Delta\ I_1 I_2 I_3$,

(ii) Its angles are

$\frac{\pi}{2} - \frac{A}{2}, \frac{\pi}{2} - \frac{B}{2}$ and $\frac{\pi}{2} - \frac{C}{2}$.

(iii) Its sides are $4R\cos\frac{A}{2}$,

$4R\cos\frac{B}{2}$ and $4R\cos\frac{C}{2}$

(iv) $I I_1 = 4R\sin\frac{A}{2}$; $I I_2 = 4R\sin\frac{B}{2}$; $I I_3 = 4R\sin\frac{C}{2}$.

34. INVERSE TRIGONOMETRIC FUNCTION

(1) Domain and Range of Inverse Trigonometric/Circular Functions:

No.	Function	Domain	Range
(i)	$y = \sin^{-1}x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
(ii)	$y = \cos^{-1}x$	$-1 \le x \le 1$	$0 \le y \le \pi$
(iii)	$y = \tan^{-1}x$	$x \in R$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
(iv)	$y = \text{cosec}^{-1}x$	$x \le -1$ or $x \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \ne 0$
(v)	$y = \sec^{-1}x$	$x \le -1$ or $x \ge 1$	$0 \le y \le \pi$; $y \ne \frac{\pi}{2}$
(vi)	$y = \cot^{-1}x$	$x \in R$	$0 < y < \pi$

Note:

(a) 1st quadrant is common to the range of all the inverse functions.

(b) 3rd quadrant is not used in inverse functions.

(c) 4th quadrant is used in the clockwise direction i.e. $-\frac{\pi}{2} \le y \le 0$.

(d) No inverse function is periodic.

(2) Properties of Inverse Trigonometric Functions :

Property - (i)

(i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$, $-1 \le x \le 1$

(ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$, $x \in R$

(iii) $\text{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$, $|x| \ge 1$

Property - (ii)

(i) $\sin(\sin^{-1}x) = x$, $-1 \le x \le 1$

(ii) $\cos(\cos^{-1}x) = x$, $-1 \le x \le 1$

(iii) $\tan(\tan^{-1}x) = x$, $x \in R$

(iv) $\cot(\cot^{-1}x) = x$, $x \in R$

(v) $\sec(\sec^{-1}x) = x$, $x \le -1$, $x \ge 1$

(vi) $\text{cosec}(\text{cosec}^{-1}x) = x$, $x \le 1$, $x \ge 1$

Property - (iii)

(i) $\sin^{-1}(\sin x) = x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

(ii) $\cos^{-1}(\cos x) = x$, $0 \le x \le \pi$

(iii) $\tan^{-1}(\tan x) = x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(iv) $\cot^{-1}(\cot x) = x$, $0 < x < \pi$

(v) $\sec^{-1}(\sec x) = x$, $0 \le x \le \pi$, $x \ne \frac{\pi}{2}$

(vi) $\text{cosec}^{-1}(\text{cosec}\, x) = x$; $x \ne 0$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

Property - (iv)

(i) $\sin^{-1}(-x) = -\sin^{-1}x,\ -1 \le x \le 1$ (ii) $\tan^{-1}(-x) = -\tan^{-1}x,\quad x \in R$

(iii) $\cos^{-1}(-x) = \pi - \cos^{-1}x,\ -1 \le x \le 1$ (iv) $\cot^{-1}(-x) = \pi - \cot^{-1}x,\quad x \in R$

Note:

The functions $\sin^{-1}x$, $\tan^{-1}x$ and $\text{cosec}^{-1}x$ are odd functions and rest are neither even nor odd

Property-(v)

(i) $\text{cosec}^{-1}x = \sin^{-1}\frac{1}{x}$;x−1 , $x \ge 1$

(ii) $\sec^{-1}x = \cos^{-1}\frac{1}{x}$; $x \le -1,\ x \ge 1$

(iii) $\cot^{-1}x = \begin{cases} \tan^{-1}\frac{1}{x} & ;\quad x > 0 \\ \pi + \tan^{-1}\frac{1}{x}; & \quad x < 0 \end{cases}$

Property - (vi)

(i) $\sin(\cos^{-1}x) = \cos(\sin^{-1}x) = ,\ \sqrt{1-x^2}\ -1 \le x \le 1$

(ii) $\tan(\cot^{-1}x) = \cot(\tan^{-1}x) = \frac{1}{x},\ x \in R,\ x \ne 0$

(iii) $\text{cosec}(\sec^{-1}x) = \sec(\text{cosec}^{-1}x) = \frac{|x|}{\sqrt{x^2-1}},\ |x| > 1$

(3) Identities of Addition and Substraction:

Property-(i)

(i) $\sin^{-1}x + \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}$, $x \ge 0,\ y \ge 0$ and $(x^2 + y^2) \le 1$

$= \pi - \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}],\ x \ge 0,\ y \ge 0$ and $x^2 + y^2 > 1$

Note:

$x^2 + y^2 \le 1 \Rightarrow 0 \le \sin^{-1}x + \sin^{-1}y \le \frac{\pi}{2}$

$x^2 + y^2 > 1 \Rightarrow \frac{\pi}{2} < \sin^{-1}x + \sin^{-1}y < \pi$

(ii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}[xy - \sqrt{1-x^2}\ \sqrt{1-y^2}],\ x \ge 0,\ y \ge 0$

(iii) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$, $x > 0,\ y > 0\ \&\ xy < 1$

$= \pi + \tan^{-1}\frac{x+y}{1-xy}$, $x > 0,\ y > 0\ \&\ xy > 1$

$= \frac{\pi}{2}$, $x > 0,\ y > 0\ \&\ xy = 1$

Note:

$xy < 1 \Rightarrow 0 < \tan^{-1}x + \tan^{-1}y < \frac{\pi}{2}$; $xy > 1 \Rightarrow \frac{\pi}{2} < \tan^{-1}x + \tan^{-1}y < \pi$

Property-(ii)

(i) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} - y\sqrt{1-x^2}]$, $x \geq 0, y \geq 0$

(ii) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}[xy + \sqrt{1-x^2}\sqrt{1-y^2}]$, $x \geq 0, y \geq 0, x \leq y$

(iii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\dfrac{x-y}{1+xy}$, $x \geq 0, y \geq 0$

Note:
For x < 0 and y < 0 these identities can be used with the help of properties 2(C) i.e. change x and y to $-x$ and $-y$ which are positive.

Property-(iii)

(i) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \begin{bmatrix} 2\sin^{-1}x & \text{if } |x| \leq \frac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1}x & \text{if } x > \frac{1}{\sqrt{2}} \\ -(\pi + 2\sin^{-1}x) & \text{if } x < -\frac{1}{\sqrt{2}} \end{bmatrix}$

(ii) $\cos^{-1}(2x^2-1) = \begin{bmatrix} 2\cos^{-1}x & \text{if } 0 \leq x \leq 1 \\ 2\pi - 2\cos^{-1}x & \text{if } -1 \leq x < 0) \end{bmatrix}$

(iii) $\tan^{-1}\dfrac{2x}{1-x^2} = \begin{bmatrix} 2\tan^{-1}x & \text{if } |x| < 1 \\ \pi + 2\tan^{-1}x & \text{if } x < -1 \\ -(\pi - 2\tan^{-1}x) & \text{if } x > 1 \end{bmatrix}$

(iv) $\sin^{-1}\dfrac{2x}{1+x^2} = \begin{bmatrix} 2\tan^{-1}x & \text{if } |x| \leq 1 \\ \pi - 2\tan^{-1}x & \text{if } x > 1 \\ -(\pi + 2\tan^{-1}x) & \text{if } x < -1 \end{bmatrix}$

(v) $\cos^{-1}\dfrac{1-x^2}{1+x^2} = \begin{bmatrix} 2\tan^{-1}x & \text{if } x \geq 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{bmatrix}$

Property-(iv)

If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left[\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right]$ if, $x > 0, y > 0, z > 0$ & $(xy+yz+zx)<1$

Note:

(i) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$ then $x + y + z = xyz$

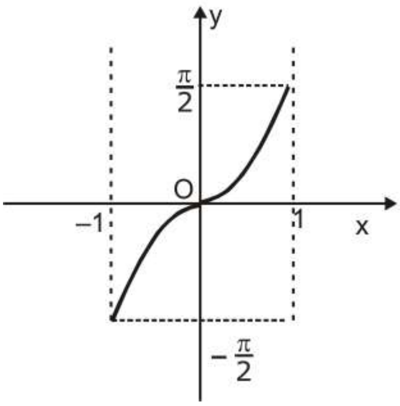
(ii) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \dfrac{\pi}{2}$ then $xy + yz + zx = 1$

(iii) $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$

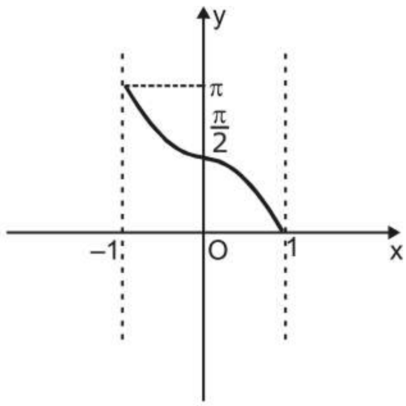
(iv) $\tan^{-1}1 + \tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{3} = \dfrac{\pi}{2}$

(4) Graphs of Inverse Trigonometric Function :

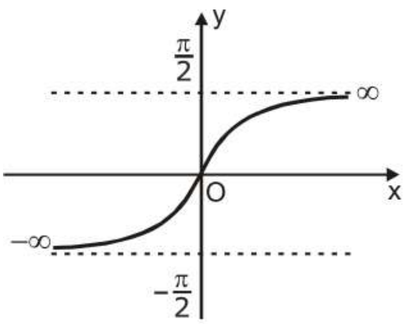
(i) $y = \sin^{-1} x,\ |x| \le 1,\ y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



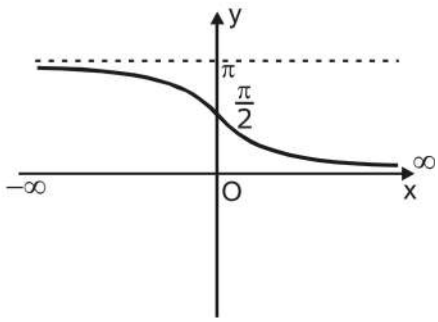
(ii) $y = \cos^{-1} x,\ |x| \le 1,\ y \in [0, \pi]$



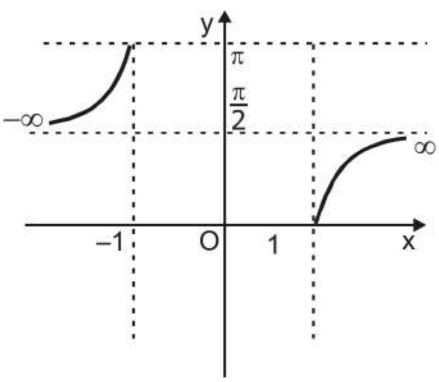
(iii) $y = \tan^{-1} x,\ x \in R,\ y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



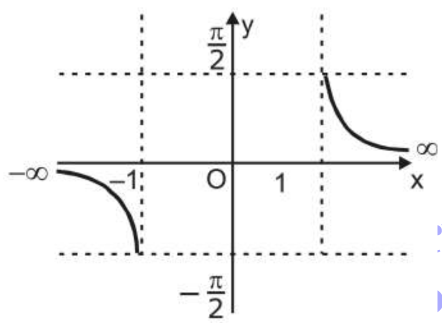
(iv) $y = \cot^{-1} x,\ x \in R,\ y \in (0, \pi)$



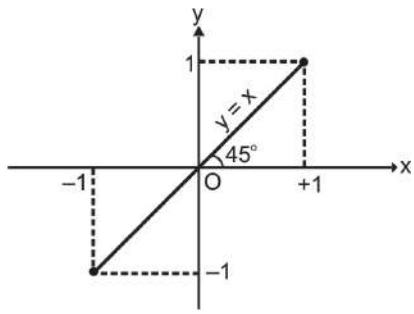
(v) $y = \sec^{-1} x,\ |x| \ge 1,\ y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$



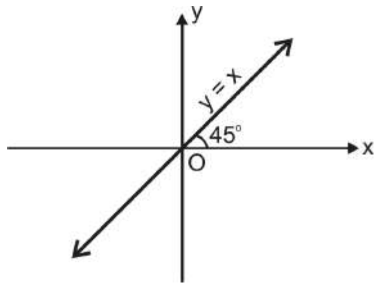
(vi) $y = \operatorname{cosec}^{-1} x,\ |x| \ge 1, y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$



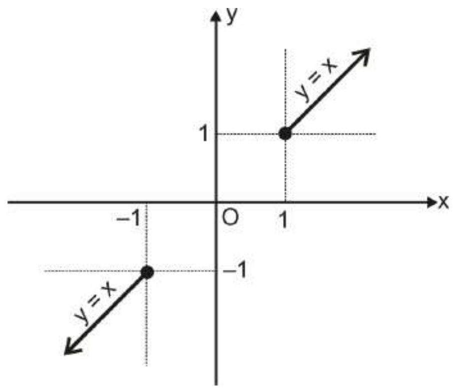
(vii) $y = \sin(\sin^{-1} x) = \cos(\cos^{-1} x) = x$, $x \in [-1, 1]$, $y \in [-1, 1]$; y is aperiodic



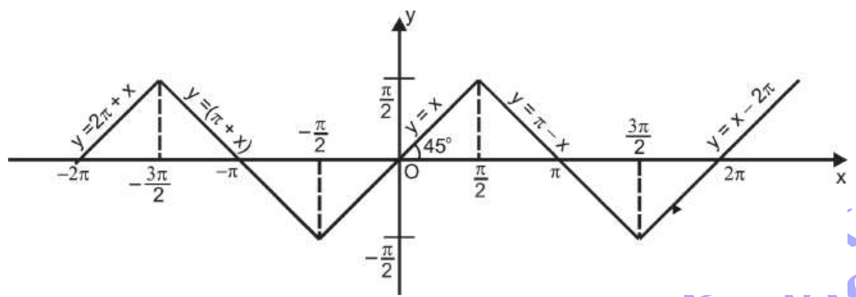
(viii) $y = \tan(\tan^{-1} x) = \cot(\cot^{-1} x) = x$, $x \in R$, $y \in R$; y is aperiodic



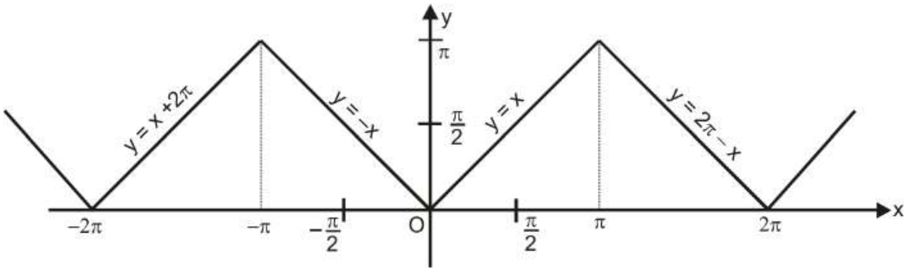
(ix) $y = \text{cosec}(\text{cosec}^{-1} x) = \sec(\sec^{-1} x) = x$, $|x| \geq 1$, $|y| \geq 1$; y is aperiodic



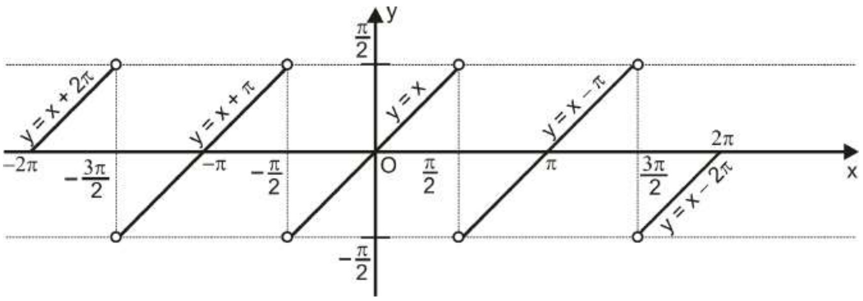
(x) $y = \sin^{-1}(\sin x)$, $x \in R$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, is periodic with period 2π



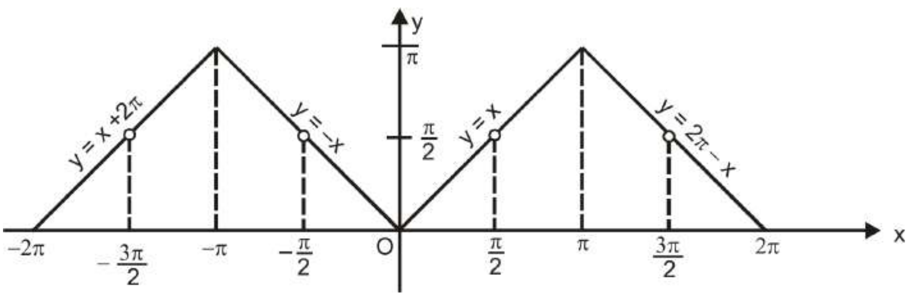
(xi) $y = \cos^{-1}(\cos x)$, $x \in R$, $y \in [0, \pi]$, is periodic with period 2π



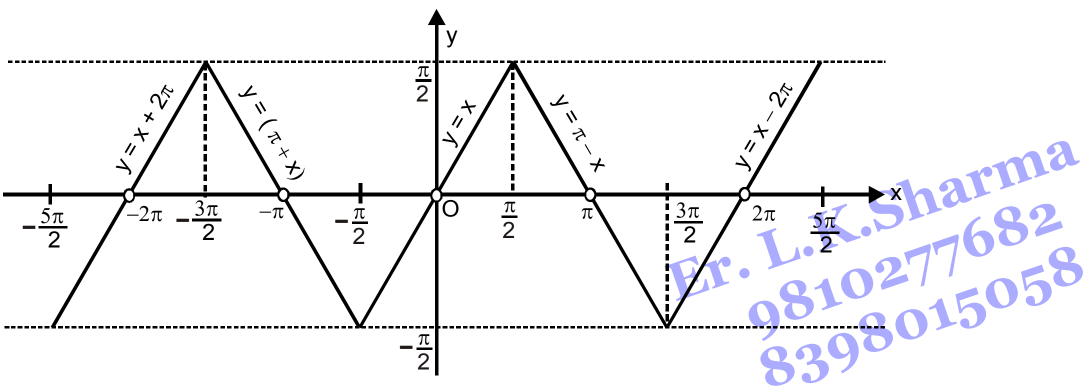
(xii) $y = \tan^{-1}(\tan x)$, $x \in R - \left\{(2n-1)\frac{\pi}{2}, n \in I\right\}$, $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is periodic with period π



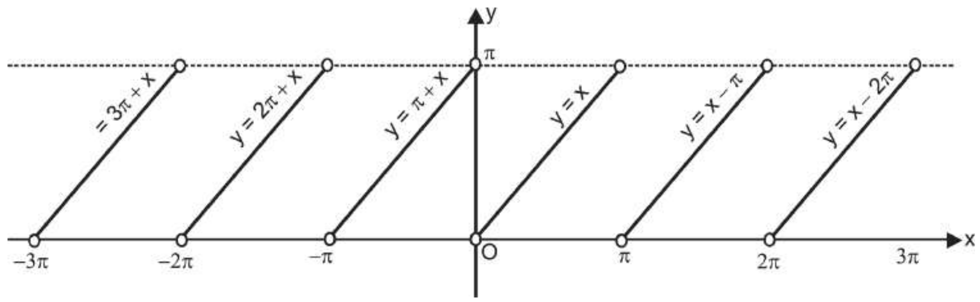
(xiii) $y = \sec^{-1}(\sec x)$, y is periodic with period 2π; $x \in R - \left\{(2n-1)\frac{\pi}{2}, n \in I\right\}$, $y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$



(xiv) $y = \text{cosec}^{-1}(\text{cosec}\, x)$, y is periodic with period 2π; $x \in R - \{n\pi, n \in I\}$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

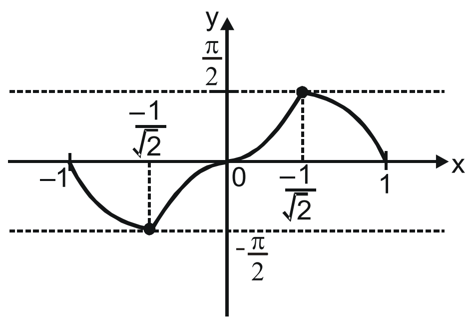


(xv) $y = \cot^{-1}(\cot x)$, y is periodic with period π ; $x \in R - \{n\pi,\ n \in I\}$, $y \in \left(0, \frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right)$

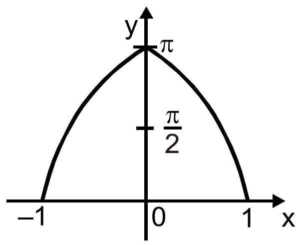


Part - 3(C)

(i) graph of $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$



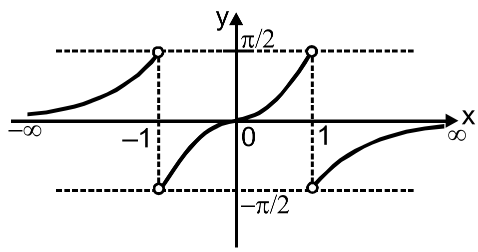
(ii) graph of $y = \cos^{-1}(2x^2 - 1)$



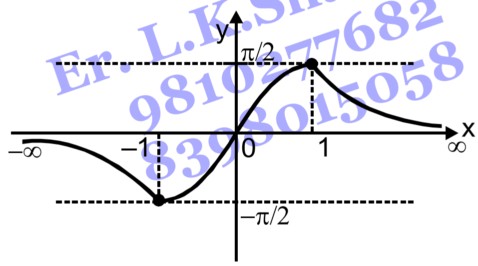
Note:

In this graph it is advisable not to check its derivability just by the inspection of the graph because it is difficult to judge from the graph that at x = 0 there is a shapr corner or not.

(iii) graph of $y = \tan^{-1}\frac{2x}{1-x^2}$



(iv) graph of $y = \sin^{-1}\frac{2x}{1+x^2}$



(v) graph of $y = \cos^{-1}\frac{1-x^2}{1+x^2}$