

Physics I

Oscillations and Waves

Somnath Bharadwaj and S. Pratik Khastgir
Department of Physics and Meteorology
IIT Kharagpur

Preface

The book “Oscillations and waves” is an account of one semester course, PHYSICS-I, given by the authors for the last three years at IIT, Kharagpur. The book is targeted at the first year undergraduate science and engineering students. Starting with oscillations in general, the book moves to interference and diffraction phenomena of waves and concludes with elementary applications of Schrödinger’s wave equation in quantum mechanics. Authors have attempted a simplified presentation of the essential topics rather than taking a comprehensive and detailed approach. Since the area of waves and oscillations is ubiquitous in science and engineering, authors hope that this book would be beneficial for the students of Indian colleges and universities.

The authors are indebted to Prof. G. P. Sastry for his pedagogy and unstinting encouragement. The authors are also indebted to Prof. Anushree Roy and Prof. Tapan K. Nath for providing us with data and figures for Raman effect and X-ray diffraction respectively.

Authors apologise in advance for the misprints.

IIT, Kharagpur

Somnath Bharadwaj
S. Pratik Khastgir

Contents

1	Oscillations	7
1.1	Simple Harmonic Oscillators SHO	7
1.2	Complex Representation.	9
1.3	Energy.	10
1.4	Why study the SHO?	11
2	The Damped Oscillator.	15
2.1	Underdamped Oscillations	16
2.2	Over-damped Oscillations.	17
2.3	Critical Damping.	18
2.4	Summary	18
3	Oscillator with external forcing.	21
3.1	Complementary function and particular integral	21
3.2	Effect of damping	23
4	Resonance.	31
4.1	Electrical Circuits	31
4.2	The Raman Effect	32
5	Coupled Oscillators	35
5.1	Normal modes	36
5.2	Resonance	37
6	Sinusoidal Waves.	43
6.1	What is $a(x, t)$?	43
6.2	Angular frequency and wave number	43
6.3	Phase velocity.	45
6.4	Waves in three dimensions.	46
6.5	Waves in an arbitrary direction.	46
7	Electromagnetic Waves.	49
7.1	Electromagnetic Radiation.	49
7.2	Electric dipole radiation.	52
7.3	Sinusoidal Oscillations.	53
7.4	Energy density, flux and power.	56

8	The vector nature of electromagnetic radiation.	61
8.1	Linear polarization	61
8.2	Circular polarization	62
8.3	Elliptical polarization	63
9	The Spectrum of Electromagnetic Radiation.	65
9.1	Radiowave and Microwave	65
9.1.1	21cm radiation.	66
9.1.2	Cosmic Microwave Background Radiation.	67
9.1.3	Molecular lines.	68
9.2	Infrared	69
9.3	Visible light	69
9.4	Ultraviolet (UV)	70
9.5	X-rays	70
9.6	Gamma Rays	71
10	Interference.	73
10.1	Young's Double Slit Experiment.	73
10.1.1	A different method of analysis.	76
10.2	Michelson Interferometer	78
11	Coherence	85
11.1	Spatial Coherence	85
11.2	Temporal Coherence	88
12	Diffraction	93
12.1	Single slit Diffraction Pattern.	94
12.1.1	Angular resolution	97
12.2	Chain of sources	99
12.2.1	Phased array	101
12.2.2	Diffraction grating	102
13	X-ray Diffraction	107
14	Beats	111
15	The wave equation.	117
15.1	Longitudinal elastic waves	117
15.2	Transverse waves in stretched strings	120
15.3	Solving the wave equation	122
15.3.1	Plane waves	122
15.3.2	Spherical waves	123
15.3.3	Standing Waves	124

16 Polarization	131
16.1 Natural Radiation	131
16.2 Producing polarized light	132
16.2.1 Dichroism or selective absorption	133
16.2.2 Scattering	134
16.2.3 Reflection	135
16.2.4 Birefringence or double refraction	136
16.2.5 Quarter wave plate	138
16.3 Partially polarized light	140
17 Wave-particle duality	143
17.1 The Compton effect	143
17.2 The wave nature of particles	145
18 Interpreting the electron wave	149
18.1 An experiment with bullets	149
18.2 An experiment with waves.	150
18.3 An experiment with electrons	151
18.4 Probability amplitude	151
19 Probability	153
20 Quantum Mechanics	159
20.1 The Laws of Quantum Mechanics	159
20.2 Particle in a potential.	164
20.2.1 In Classical Mechanics	165
20.2.2 Step potentials	165
20.2.3 Particle in a box	167
20.2.4 Tunnelling	169
20.2.5 Scanning Tunnelling Microscope	172

Chapter 1

Oscillations

Oscillations are ubiquitous. It would be difficult to find something which never exhibits oscillations. Atoms in solids, electromagnetic fields, multi-storeyed buildings and share prices all exhibit oscillations. In this course we shall restrict our attention to only the simplest possible situations, but it should be borne in mind that this elementary analysis provides insights into a diverse variety of apparently complex phenomena.

1.1 Simple Harmonic Oscillators SHO

We consider the spring-mass system shown in Figure 1.1. A massless spring, one of whose ends is fixed has its other attached to a particle of mass m which is free to move. We choose the origin $x = 0$ for the particle's motion at the position where the spring is unstretched. The particle is in stable equilibrium at this position and it will continue to remain there if left at rest. We are interested in a situation where the particle is disturbed from equilibrium. The particle experiences a restoring force from the spring if it is either stretched or compressed. The spring is assumed to be elastic which means that it follows Hooke's law where the force is proportional to the displacement $F = -kx$ with

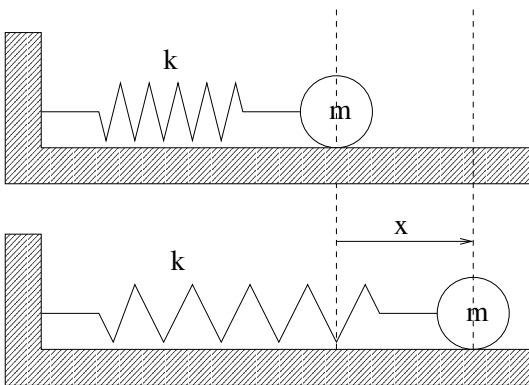


Figure 1.1:

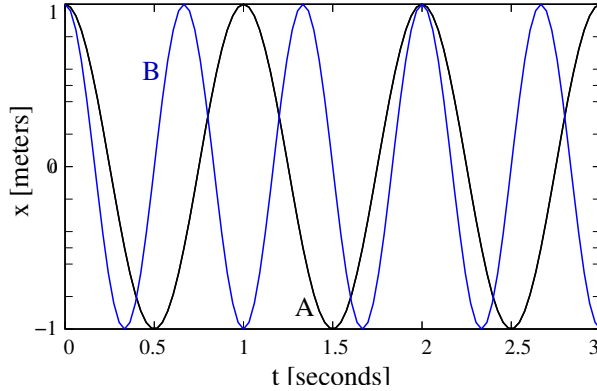


Figure 1.2:

spring constant k .

The particle's equation of motion is

$$m \frac{d^2 x}{dt^2} = -kx \quad (1.1)$$

which can be written as

$$\ddot{x} + \omega_0^2 x = 0 \quad (1.2)$$

where the dots denote time derivatives and

$$\omega_0 = \sqrt{\frac{k}{m}} \quad (1.3)$$

It is straightforward to check that

$$x(t) = A \cos(\omega_0 t + \phi) \quad (1.4)$$

is a solution to eq. (1.4).

We see that the particle performs sinusoidal oscillations around the equilibrium position when it is disturbed from equilibrium. The angular frequency ω_0 of the oscillation depends on the intrinsic properties of the oscillator. It determines the time period

$$T = \frac{2\pi}{\omega_0} \quad (1.5)$$

and the frequency $\nu = 1/T$ of the oscillation. Figure 1.2 shows oscillations for two different values of ω_0 .

Problem 1: What are the values of ω_0 for the oscillations shown in Figure 1.2? What are the corresponding spring constant k values if $m = 1 \text{ kg}$?

Solution: For A $\omega_0 = 2\pi \text{ s}^{-1}$ and $k = (2\pi)^2 \text{ Nm}^{-1}$; For B $\omega_0 = 3\pi \text{ s}^{-1}$ and $k = (3\pi)^2 \text{ Nm}^{-1}$

The amplitude A and phase ϕ are determined by the initial conditions. Two initial conditions are needed to completely specify a solution. This follows from the fact that the governing equation (1.2) is a second order differential

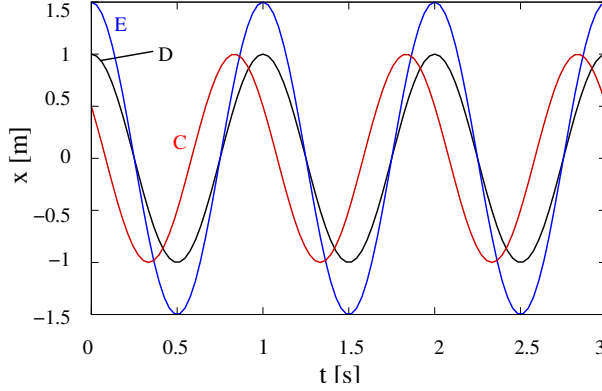


Figure 1.3:

equation. The initial conditions can be specified in a variety of ways, fixing the values of $x(t)$ and $\dot{x}(t)$ at $t = 0$ is a possibility. Figure 1.3 shows oscillations with different amplitudes and phases.

Problem 2: What are the amplitude and phase of the oscillations shown in Figure 1.3?

Solution: For C, $A=1$ and $\phi = \pi/3$; For D, $A=1$ and $\phi = 0$; For E, $A=1.5$ and $\phi = 0$;

1.2 Complex Representation.

Complex numbers provide a very useful way in representing oscillations. The amplitude and phase of the oscillation can be combined into a single complex number which we shall refer to as the complex amplitude

$$\tilde{A} = Ae^{i\phi}. \quad (1.6)$$

Note that we have introduced the symbol \sim (tilde) to denote complex numbers. The property that

$$e^{i\phi} = \cos \phi + i \sin \phi \quad (1.7)$$

allows us to represent any oscillating quantity $x(t) = A \cos(\omega_0 t + \phi)$ as the real part of the complex number $\tilde{x}(t) = \tilde{A}e^{i\omega_0 t}$,

$$\tilde{x}(t) = Ae^{i(\omega_0 t + \phi)} = A[\cos(\omega_0 t + \phi) + i \sin(\omega_0 t + \phi)]. \quad (1.8)$$

We calculate the velocity v in the complex representation $\tilde{v} = \dot{\tilde{x}}$, which gives us

$$\tilde{v}(t) = i\omega_0 \tilde{x} = -\omega_0 A[\sin(\omega_0 t + \phi) - i \cos(\omega_0 t + \phi)]. \quad (1.9)$$

Taking only the real part we calculate the particle's velocity

$$v(t) = -\omega_0 A \sin(\omega_0 t + \phi). \quad (1.10)$$

The complex representation is a very powerful tool which, as we shall see later, allows us to deal with oscillating quantities in a very elegant fashion.

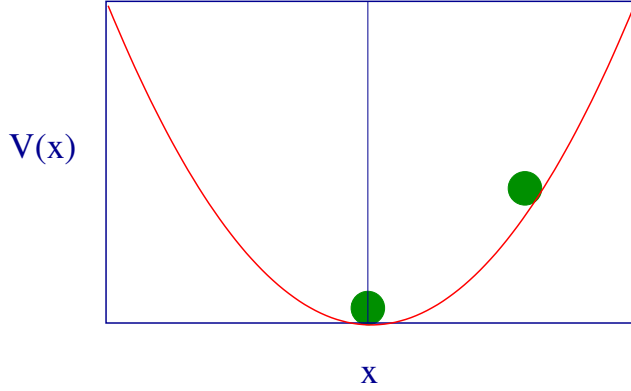


Figure 1.4:

Problem 3: A SHO has position x_0 and velocity v_0 at the initial time $t = 0$. Calculate the complex amplitude \tilde{A} in terms of the initial conditions and use this to determine the particle's position $x(t)$ at a later time t .

Solution The initial conditions tell us that $\text{Re}(\tilde{A}) = x_0$ and $\text{Re}(i\omega_0\tilde{A}) = v_0$. Hence $\tilde{A} = x_0 - iv_0/\omega_0$ which implies that $x(t) = x_0 \cos(\omega_0 t) + (v_0/\omega_0) \sin(\omega_0 t)$.

1.3 Energy.

In a spring-mass system the particle has a potential energy $V(x) = kx^2/2$ as shown in Figure 1.4. This energy is stored in the spring when it is either compressed or stretched. The potential energy of the system

$$U = \frac{1}{2}kA^2 \cos^2(\omega_0 t + \phi) = \frac{1}{4}m\omega_0^2 A^2 \{1 + \cos[2(\omega_0 t + \phi)]\} \quad (1.11)$$

oscillates with angular frequency $2\omega_0$ as the spring is alternately compressed and stretched. The kinetic energy $mv^2/2$

$$T = \frac{1}{2}m\omega_0^2 A^2 \sin^2(\omega_0 t + \phi) = \frac{1}{4}m\omega_0^2 A^2 \{1 - \cos[2(\omega_0 t + \phi)]\} \quad (1.12)$$

shows similar oscillations which are exactly π out of phase.

In a spring-mass system the total energy oscillates between the potential energy of the spring (U) and the kinetic energy of the mass (T). The total energy $E = T + U$ has a value $E = m\omega_0^2 A^2/2$ which remains constant.

The average value of an oscillating quantity is often of interest. We denote the time average of any quantity $Q(t)$ using $\langle Q \rangle$ which is defined as

$$\langle Q \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} Q(t) dt. \quad (1.13)$$

The basic idea here is to average over a time interval T which is significantly larger than the oscillation time period.

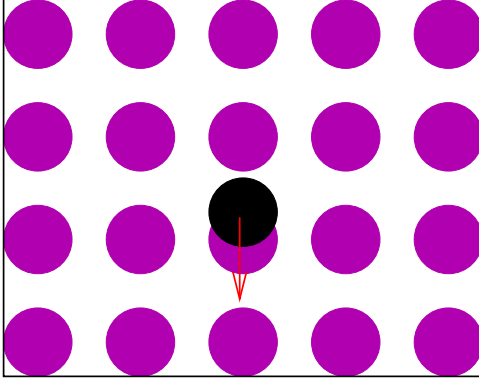


Figure 1.5:

It is very useful to remember that $\langle \cos(\omega_0 t + \phi) \rangle = 0$. This can be easily verified by noting that the values $\sin(\omega_0 t + \phi)$ are bound between -1 and $+1$. We use this to calculate the average kinetic and potential energies both of which have the same values

$$\langle U \rangle = \langle T \rangle = \frac{1}{4} m \omega_0^2 A^2. \quad (1.14)$$

The average kinetic and potential energies, and the total energy are all very conveniently expressed in the complex representation as

$$E/2 = \langle U \rangle = \langle T \rangle = \frac{1}{4} m \tilde{v} \tilde{v}^* = \frac{1}{4} k \tilde{x} \tilde{x}^* \quad (1.15)$$

where $*$ denotes the conjugate of a complex number.

Problem 3: The mean displacement of a SHO $\langle x \rangle$ is zero. The root mean square (rms.) displacement $\sqrt{\langle x^2 \rangle}$ is useful in quantifying the amplitude of oscillation. Verify that the rms. displacement is $\sqrt{\tilde{x} \tilde{x}^* / 2}$.

Solution: $\sqrt{\langle x^2(t) \rangle} = \sqrt{A^2 \langle \cos^2(\omega_0 t + \phi) \rangle} = \sqrt{A^2 / 2} = \sqrt{\tilde{A} e^{i\omega t} \tilde{A}^* e^{-i\omega t} / 2} = \sqrt{\tilde{x} \tilde{x}^* / 2}$

1.4 Why study the SHO?

What happens to a system when it is disturbed from stable equilibrium? This question that arises in a large variety of situations. For example, the atoms in many solids (eg. NaCl, diamond and steel) are arranged in a periodic crystal as shown in Figure 1.5. The periodic crystal is known to be an equilibrium configuration of the atoms. The atoms are continuously disturbed from their equilibrium positions (shown in Figure 1.5) as a consequence of random thermal motions and external forces which may happen to act on the solid. The study of oscillations in the atoms disturbed from their equilibrium position is very interesting. In fact the oscillations of the different atoms are coupled, and

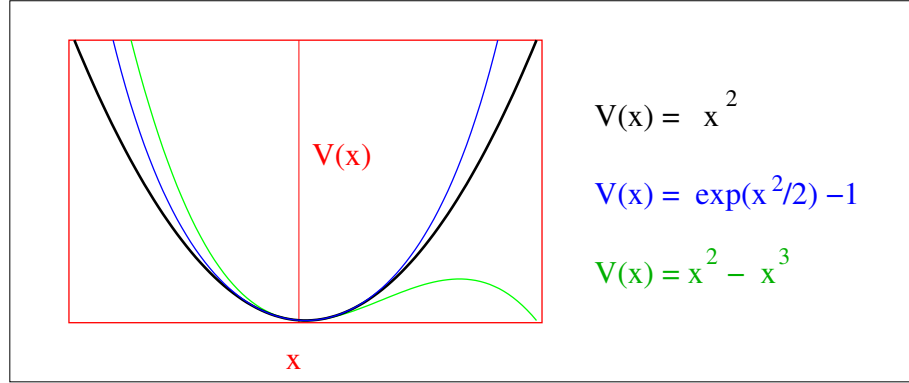


Figure 1.6:

this gives rise to collective vibrations of the whole crystal which can explain properties like the specific heat capacity of the solid. We shall come back to this later, right now the crucial point is that each atom behaves like a SHO if we assume that all the other atoms remain fixed. This is generic to all systems which are slightly disturbed from stable equilibrium.

We now show that any potential $V(x)$ is well represented by a SHO potential in the neighbourhood of points of stable equilibrium. The origin of x is chosen so that the point of stable equilibrium is located at $x = 0$. For small values of x it is possible to approximate the function $V(x)$ using a Taylor series

$$V(x) \approx V(x)_{x=0} + \left(\frac{dV(x)}{dx} \right)_{x=0} x + \frac{1}{2} \left(\frac{d^2V(x)}{dx^2} \right)_{x=0} x^2 + \dots \quad (1.16)$$

where the higher powers of x are assumed to be negligibly small. We know that at the points of stable equilibrium the force vanishes *ie.* $F = -dV(x)/dx = 0$ and $V(x)$ has a minima

$$k = \left(\frac{d^2V(x)}{dx^2} \right)_{x=0} > 0. \quad (1.17)$$

This tells us that the potential is approximately

$$V(x) \approx V(x)_{x=0} + \frac{1}{2} k x^2 \quad (1.18)$$

which is a SHO potential. Figure 1.6 shows two different potentials which are well approximated by the same SHO potential in the neighbourhood of the point of stable equilibrium. The oscillation frequency is exactly the same for particles slightly disturbed from equilibrium in these three different potentials.

The study of SHO is important because it occurs in a large variety of situations where the system is slightly disturbed from equilibrium. We discuss a few simple situations.

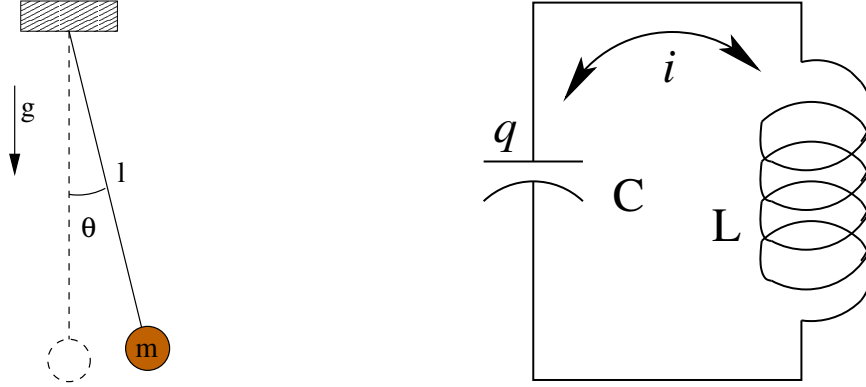


Figure 1.7: (a) and (b)

Simple pendulum

The simple pendulum shown in Figure 1.7(a) is possibly familiar to all of us. A mass m is suspended by a rigid rod of length l , the rod is assumed to be massless. The gravitational potential energy of the mass is

$$V(\theta) = mgl[1 - \cos \theta]. \quad (1.19)$$

For small θ we may approximate $\cos \theta \approx 1 - \theta^2/2$ whereby the potential is

$$V(\theta) = \frac{1}{2}mgl\theta^2 \quad (1.20)$$

which is the SHO potential. Here $dV(\theta)/d\theta$ gives the torque not the force. The pendulum's equation of motion is

$$I\ddot{\theta} = -mgl\theta \quad (1.21)$$

where $I = ml^2$ is the moment of inertia. This can be written as

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \quad (1.22)$$

which allows us to determine the angular frequency

$$\omega_0 = \sqrt{\frac{g}{l}} \quad (1.23)$$

LC Oscillator

The LC circuit shown in Figure 1.7(b) is an example of an electrical circuit which is a SHO. It is governed by the equation

$$L\dot{I} + \frac{Q}{C} = 0 \quad (1.24)$$

where L refers to the inductance, C capacitance, I current and Q charge. This can be written as

$$\ddot{Q} + \frac{1}{LC}Q = 0 \quad (1.25)$$

which allows us to identify

$$\omega_0 = \sqrt{\frac{1}{LC}} \quad (1.26)$$

as the angular frequency.

Problems

1. An empty tin can floating vertically in water is disturbed so that it executes vertical oscillations. The can weighs 100 gm, and its height and base diameter are 20 and 10 cm respectively. [a.] Determine the period of the oscillations. [b.] How much water need one pour into the can to make the time period 1s?
2. A SHO with $\omega_0 = 2 \text{ s}^{-1}$ has initial displacement and velocity 0.1 m and 2.0 ms^{-1} respectively. [a.] At what distance from the equilibrium position does it come to rest? [b.] What are the rms. displacement and rms. velocity? What is the displacement at $t = \pi/4 \text{ s}$?
3. A SHO with $\omega_0 = 3 \text{ s}^{-1}$ has initial displacement and velocity 0.2 m and 2 ms^{-1} respectively. [a.] Expressing this as $\tilde{x}(t) = \tilde{A}e^{i\omega_0 t}$, determine $\tilde{A} = a + ib$ from the initial conditions. [b.] Using $\tilde{A} = Ae^{i\phi}$, what are the amplitude A and phase ϕ for this oscillator? [c.] What are the initial position and velocity if the phase is increased by $\pi/3$?
4. A particle of mass $m = 0.3 \text{ kg}$ in the potential $V(x) = 2e^{x^2/L^2} \text{ J}$ ($L = 0.1 \text{ m}$) is found to behave like a SHO for small displacements from equilibrium. Determine the period of this SHO.
5. Calculate the time average $\langle x^4 \rangle$ for the SHO $x = A \cos \omega t$.

Chapter 2

The Damped Oscillator.

Damping usually comes into play whenever we consider motion. We study the effect of damping on the spring-mass system. The damping force is assumed to be proportional to the velocity, acting to oppose the motion. The total force acting on the mass is

$$F = -kx - c\dot{x} \quad (2.1)$$

where in addition to the restoring force $-kx$ due to the spring we also have the damping force $-c\dot{x}$. The equation of motion for the damped spring mass system is

$$m\ddot{x} = -kx - c\dot{x}. \quad (2.2)$$

Recasting this in terms of more convenient coefficients, we have

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \quad (2.3)$$

This is a second order homogeneous equation with constant coefficients. Both ω_0 and β have dimensions $(\text{time})^{-1}$. Here $1/\omega_0$ is the time-scale of the oscillations that would occur if there was no damping, and $1/\beta$ is the time-scale required for damping to bring any motion to rest. It is clear that the nature of the motion depends on which time-scale $1/\omega_0$ or $1/\beta$ is larger.

We proceed to solve equation (2.4) by taking a trial solution

$$x(t) = Ae^{\alpha t}. \quad (2.4)$$

Putting the trial solution into equation (2.4) gives us the quadratic equation

$$\alpha^2 + 2\beta\alpha + \omega^2 = 0 \quad (2.5)$$

This has two solutions

$$\alpha_1 = -\beta + \sqrt{\beta^2 - \omega^2} \quad (2.6)$$

and

$$\alpha_2 = -\beta - \sqrt{\beta^2 - \omega^2} \quad (2.7)$$

The nature of the solution depends critically on the value of the damping coefficient β , and the behaviour is quite different depending on whether $\beta < \omega_0$, $\beta = \omega_0$ or $\beta > \omega_0$.

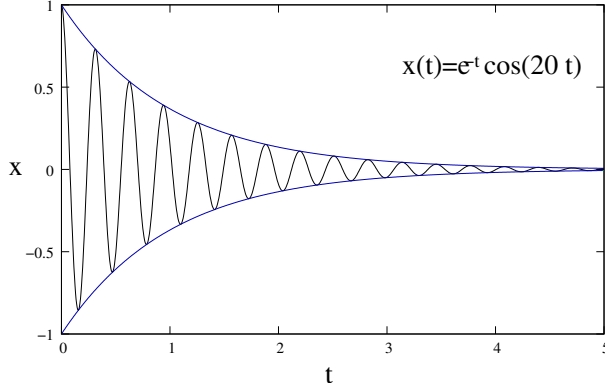


Figure 2.1:

2.1 Underdamped Oscillations

We first consider the situation where $\beta < \omega_0$ which is referred to as underdamped. Defining

$$\omega = \sqrt{\omega_0^2 - \beta^2} \quad (2.8)$$

the two roots which are both complex have values

$$\alpha_1 = -\beta + i\omega \quad \text{and} \quad \alpha_2 = -\beta - i\omega \quad (2.9)$$

The resulting solution is a superposition of the two roots

$$x(t) = e^{-\beta t} [A_1 e^{i\omega t} + A_2 e^{-i\omega t}] \quad (2.10)$$

where A_1 and A_2 are constants which have to be determined from the initial conditions. The term $[A_1 e^{i\omega t} + A_2 e^{-i\omega t}]$ is a superposition of sin and cos which can be written as

$$x(t) = A e^{-\beta t} \cos(\omega t + \phi) \quad (2.11)$$

This can also be expressed in the complex notation as

$$\tilde{x}(t) = \tilde{A} e^{(i\omega - \beta)t} \quad (2.12)$$

where $\tilde{A} = A e^{i\phi}$ is the complex amplitude which has both the amplitude and phase information. Figure 2.1 shows the underdamped motion $x(t) = e^{-t} \cos(20t)$.

In all cases damping reduces the frequency of the oscillations *ie.* $\omega < \omega_0$. The main effect of damping is that it causes the amplitude of the oscillations to decay exponentially with time. It is often useful to quantify the decay in the amplitude during the time period of a single oscillation $T = 2\pi/\omega$. This is quantified by the logarithmic decrement which is defined as

$$\lambda = \ln \left[\frac{x(t)}{x(t+T)} \right] = \frac{2\pi\beta}{\omega} \quad (2.13)$$

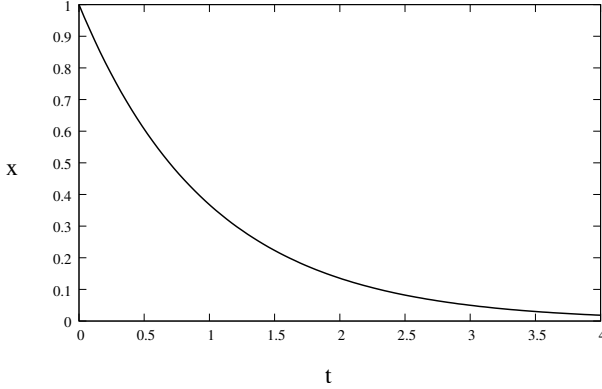


Figure 2.2:

Problem 1.: An under-damped oscillator with $\tilde{x}(t) = \tilde{A}e^{(i\omega-\beta)t}$ has initial displacement and velocity x_0 and v_0 respectively. Calculate \tilde{A} and obtain $x(t)$ in terms of the initial conditions.

Solution: $\tilde{A} = x_0 - i(v_0 + \beta x_0)/\omega$ and $x(t) = e^{-\beta t} [x_0 \cos \omega t + ((v_0 + \beta x_0)/\omega) \sin \omega t]$.

2.2 Over-damped Oscillations.

This refers to the situation where

$$\beta > \omega_0 \quad (2.14)$$

The two roots are

$$\alpha_1 = -\beta + \sqrt{\beta^2 - \omega_0^2} = -\gamma_1 \quad (2.15)$$

and

$$\alpha_2 = -\beta - \sqrt{\beta^2 - \omega_0^2} = -\gamma_2 \quad (2.16)$$

where both $\gamma_1, \gamma_2 > 0$ and $\gamma_2 > \gamma_1$. The two roots give rise to exponentially decaying solutions, one which decays faster than the other

$$x(t) = A_1 e^{-\gamma_1 t} + A_2 e^{-\gamma_2 t}. \quad (2.17)$$

The constants A_1 and A_2 are determined by the initial conditions. For initial position x_0 and velocity v_0 we have

$$x(t) = \frac{v_0 + \gamma_2 x_0}{\gamma_2 - \gamma_1} e^{-\gamma_1 t} - \frac{v_0 + \gamma_1 x_0}{\gamma_2 - \gamma_1} e^{-\gamma_2 t} \quad (2.18)$$

The overdamped oscillator does not oscillate. Figure 2.2 shows a typical situation.

In the situation where $\beta \gg \omega_0$

$$\sqrt{\beta^2 - \omega_0^2} = \beta \sqrt{1 - \frac{\omega_0^2}{\beta^2}} \approx \beta \left[1 - \frac{1}{2} \frac{\omega_0^2}{\beta^2} \right] \quad (2.19)$$

and we have $\gamma_1 = \omega_0^2/2\beta$ and $\gamma_2 = 2\beta$.

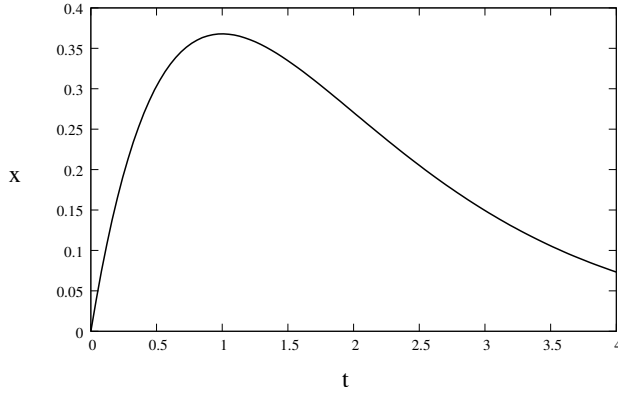


Figure 2.3:

2.3 Critical Damping.

This corresponds to a situation where $\beta = \omega_0$ and the two roots are equal. The governing equation is second order and there still are two independent solutions. The general solution is

$$x(t) = e^{-\beta t}[A_1 + A_2 t] \quad (2.20)$$

The solution

$$x(t) = x_0 e^{-\beta t}[1 + \beta t] \quad (2.21)$$

is for an oscillator starting from rest at x_0 while

$$x(t) = v_0 e^{-\beta t} t \quad (2.22)$$

is for a particle starting from $x = 0$ with speed v_0 . Figure 2.3 shows the latter situation.

2.4 Summary

There are two physical effects at play in a damped oscillator. The first is the damping which tries to bring any motion to a stop. This operates on a time-scale $T_d \approx 1/\beta$. The restoring force exerted by the spring tries to make the system oscillate and this operates on a time-scale $T_0 = 1/\omega_0$. We have overdamped oscillations if the damping operates on a shorter time-scale compared to the oscillations *ie.* $T_d < T_0$ which completely destroys the oscillatory behaviour.

Figure 2.4 shows the behaviour of a damped oscillator under different combinations of damping and restoring force. The plot is for $\omega_0 = 1$, it can be used for any other value of the natural frequency by suitably scaling the values of β . It shows how the decay rate for the two exponentially decaying overdamped solutions varies with β . Note that for one of the modes the decay rate tends to zero as β is increased. This indicates that for very large damping a particle may get stuck at a position away from equilibrium.

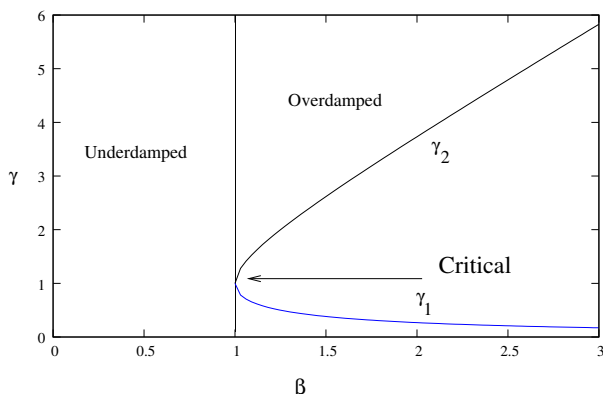


Figure 2.4:

Problems

1. Obtain solution (2.20) for critical damping as a limiting case ($\beta \rightarrow \omega_0$) of overdamped solution (2.18).
2. Find out the conditions for the initial displacement $x(0)$ and the initial velocity $\dot{x}(0)$ at $t = 0$ such that an overdamped oscillator crosses the mean position once in a finite time.
3. An under-damped oscillator has a time period of 2s and the amplitude of oscillation goes down by 10% in one oscillation. [a.] What is the logarithmic decrement λ of the oscillator? [b.] Determine the damping coefficient β . [c.] What would be the time period of this oscillator if there was no damping? [d.] What should be β if the time period is to be increased to 4s? ([a.] 1.05×10^{-1} [b.] $2.7 \times 10^{-2}\text{s}^{-1}$ [c.] 2s [d.] 2.72s^{-1})
4. Two identical under-damped oscillators have damping coefficient and angular frequency β and ω respectively. At $t = 0$ one oscillator is at rest with displacement a_0 while the other has velocity v_0 and is at the equilibrium position. What is the phase difference between these two oscillators. ($\pi/2 - \tan^{-1}(\beta/\omega)$)
5. A door-shutter has a spring which, in the absence of damping, shuts the door in 0.5s. The problem is that the door bangs with a speed 1m/s at the instant that it shuts. A damper with damping coefficient β is introduced to ensure that the door shuts gradually. What are the time required for the door to shut and the velocity of the door at the instant it shuts if $\beta = 0.5\pi$ and $\beta = 0.9\pi$? Note that the spring is unstretched when the door is shut. (0.57s, $4.67 \times 10^{-1}\text{m/s}$; 1.14s, $8.96 \times 10^{-2}\text{m/s}$)
6. An LCR circuit has an inductance $L = 1\text{ mH}$, a capacitance $C = 0.1\text{ }\mu\text{F}$ and resistance $R = 250\Omega$ in series. The capacitor has a voltage 10 V at the instant $t = 0$ when the circuit is completed. What is the voltage across the capacitor after $10\mu\text{s}$ and $20\mu\text{s}$? (7.64 V, 4.84 V)

7. A highly damped oscillator with $\omega_0 = 2 \text{ s}^{-1}$ and $\beta = 10^4 \text{ s}^{-1}$ is given an initial displacement of 2 m and left at rest. What is the oscillator's position at $t = 2 \text{ s}$ and $t = 10^4 \text{ s}$? (2.00 m, $2.70 \times 10^{-1} \text{ m}$)
8. A critically damped oscillator with $\beta = 2 \text{ s}^{-1}$ is initially at $x = 0$ with velocity 6 m s^{-1} . What is the furthest distance the oscillator moves from the origin? (1.10 m)
9. A critically damped oscillator is initially at $x = 0$ with velocity v_0 . What is the ratio of the maximum kinetic energy to the maximum potential energy of this oscillator? (e^2)
10. An overdamped oscillator is initially at $x = x_0$. What initial velocity, v_0 , should be the given to the oscillator that it reaches the mean position ($x=0$) in the minimum possible time.
11. We have shown that the general solution, $x(t)$, with two constants can describe the motion of damped oscillator satisfying given initial conditions. Show that there does not exist any other solution satisfying the same initial conditions.

Chapter 3

Oscillator with external forcing.

In this chapter we consider an oscillator under the influence of an external sinusoidal force $F = \cos(\omega t + \psi)$. Why this particular form of the force? This is because nearly any arbitrary time varying force $F(t)$ can be decomposed into the sum of sinusoidal forces of different frequencies

$$F(t) = \sum_{n=1, \dots}^{\infty} F_n \cos(\omega_n t + \psi_n) \quad (3.1)$$

Here F_n and ψ_n are respectively the amplitude and phase of the different frequency components. Such an expansion is called a Fourier series. The behaviour of the oscillator under the influence of the force $F(t)$ can be determined by separately solving

$$m\ddot{x}_n + kx_n = F_n \cos(\omega_n t + \psi_n) \quad (3.2)$$

for a force with a single frequency and then superposing the solutions

$$x(t) = \sum_n x_n(t). \quad (3.3)$$

We shall henceforth restrict our attention to equation (3.2) which has a sinusoidal force of a single frequency and drop the subscript n from x_n and F_n . It is convenient to switch over to the complex notation

$$\ddot{\tilde{x}} + \omega_0^2 \tilde{x} = \tilde{f} e^{i\omega t} \quad (3.4)$$

where $\tilde{f} = F e^{i\psi} / m$.

3.1 Complementary function and particular integral

The solution is a sum of two parts

$$\tilde{x}(t) = \tilde{A} e^{i\omega_0 t} + \tilde{B} e^{i\omega t}. \quad (3.5)$$

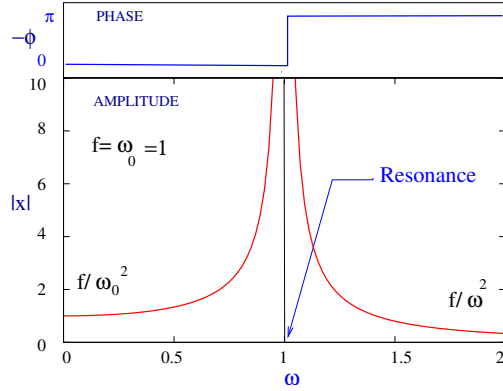


Figure 3.1: Amplitude and phase as a function of forcing frequency

The first term $\tilde{A}e^{i\omega_0 t}$, called the complementary function, is a solution to equation (3.4) without the external force. This oscillates at the natural frequency of the oscillator ω_0 . This part of the solution is exactly the same as when there is no external force. This has been discussed extensively earlier, and we shall ignore this term in the rest of this chapter.

The second term $\tilde{B}e^{i\omega t}$, called the particular integral, is the extra ingredient in the solution due to the external force. This oscillates at the frequency of the external force ω . The amplitude \tilde{B} is determined from equation (3.4) which gives

$$[-\omega^2 + \omega_0^2]\tilde{B} = \tilde{f} \quad (3.6)$$

whereby we have the solution

$$\tilde{x}(t) = \frac{\tilde{f}}{\omega_0^2 - \omega^2} e^{i\omega t}. \quad (3.7)$$

The amplitude and phase of the oscillation both depend on the forcing frequency ω . The amplitude is

$$|\tilde{x}| = \frac{f}{|\omega_0^2 - \omega^2|}. \quad (3.8)$$

and the phase of the oscillations relative to the applied force is $\phi = 0$ for $\omega < \omega_0$ and $\phi = -\pi$ for $\omega > \omega_0$.

Note: One cannot decide here whether the oscillations lag or lead the driving force, i.e. whether $\phi = -\pi$ or $\phi = \pi$ as both of them are consistent with $\omega > \omega_0$ case ($e^{\pm i\pi} = -1$). The zero resistance limit, $\beta \rightarrow 0$, of the damped forced oscillations (which is to be done in the next section) would settle it for $\phi = -\pi$ for $\omega > \omega_0$. So in this case there is an abrupt change of $-\pi$ radians in the phase as the forcing frequency, ω , crosses the natural frequency, ω_0 .

The amplitude and phase are shown in Figure 3.1. The first point to note is that the amplitude increases dramatically as $\omega \rightarrow \omega_0$ and the amplitude blows up at $\omega = \omega_0$. This is the phenomenon of resonance. The response of

the oscillator is maximum when the frequency of the external force matches the natural frequency of the oscillator. In a real situation the amplitude is regulated by the presence of damping which ensures that it does not blow up to infinity at $\omega = \omega_0$.

We next consider the low frequency $\omega \ll \omega_0$ behaviour

$$\tilde{x}(t) = \frac{\tilde{f}}{\omega_0^2} e^{i\omega t} = \frac{F}{k} e^{i(\omega t + \phi)}, \quad (3.9)$$

The oscillations have an amplitude F/k and are in phase with the external force.

This behaviour is easy to understand if we consider $\omega = 0$ which is a constant force. We know that the spring gets extended (or contracted) by an amount $x = F/k$ in the direction of the force. The same behaviour goes through if F varies very slowly with time. The behaviour is solely determined by the spring constant k and this is referred to as the “Stiffness Controlled” regime.

At high frequencies $\omega \gg \omega_0$

$$\tilde{x}(t) = -\frac{\tilde{f}}{\omega^2} e^{i\omega t} = -\frac{F}{m\omega^2} e^{i(\omega t + \phi)}, \quad (3.10)$$

the amplitude is F/m and the oscillations are $-\pi$ out of phase with respect to the force. This is the “Mass Controlled” regime where the spring does not come into the picture at all. It is straight forward to verify that equation (3.10) is a solution to

$$m\ddot{x} = F e^{i(\omega t + \phi)} \quad (3.11)$$

when the spring is removed from the oscillator. Interestingly such a particle moves exactly out of phase relative to the applied force. The particle moves to the left when the force acts to the right and vice versa.

3.2 Effect of damping

Introducing damping, the equation of motion

$$m\ddot{x} + c\dot{x} + kx = F \cos(\omega t + \psi) \quad (3.12)$$

written using the notation introduced earlier is

$$\ddot{\tilde{x}} + 2\beta\dot{\tilde{x}} + \omega_0^2\tilde{x} = \tilde{f}e^{i\omega t}. \quad (3.13)$$

Here again we separately discuss the complementary functions and the particular integral. The complementary functions are the decaying solutions that arise when there is no external force. These are short lived transients which are not of interest when studying the long time behaviour of the oscillations.

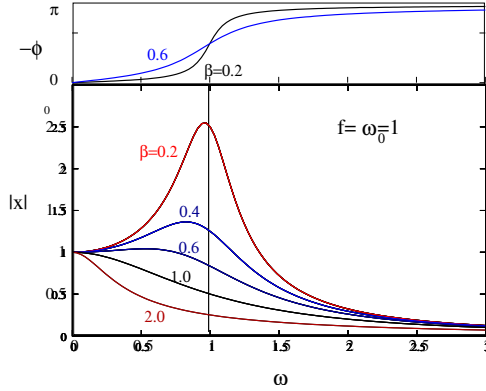


Figure 3.2: Amplitudes and phases for various damping coefficients as a function of driving frequency

These have already been discussed in considerable detail and we do not consider them here. The particular integral is important when studying the long time or steady state response of the oscillator. This solution is

$$\tilde{x}(t) = \frac{\tilde{f}}{(\omega_0^2 - \omega^2) + 2i\beta\omega} e^{i\omega t} \quad (3.14)$$

which may be written as $\tilde{x}(t) = C e^{i\phi} \tilde{f} e^{i\omega t}$ where ϕ is the phase of the oscillation relative to the force \tilde{f} .

This has an amplitude

$$|\tilde{x}| = \frac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \quad (3.15)$$

and the phase ϕ is

$$\phi = \tan^{-1} \left(\frac{-2\beta\omega}{\omega_0^2 - \omega^2} \right) \quad (3.16)$$

Figure 3.2 shows the amplitude and phase as a function of ω for different values of the damping coefficient β . The damping ensures that the amplitude does not blow up at $\omega = \omega_0$ and it is finite for all values of ω . The change in the phase also is more gradual.

The low frequency and high frequency behaviour are exactly the same as the situation without damping. The changes due to damping are mainly in the vicinity of $\omega = \omega_0$. The amplitude is maximum at

$$\omega = \sqrt{\omega_0^2 - 2\beta^2} \quad (3.17)$$

For mild damping ($\beta \ll \omega_0$) this is approximately $\omega = \omega_0$.

We next shift our attention to the energy of the oscillator. The average energy $E(\omega)$ is the quantity of interest. Calculating this as a function of ω we have

$$E(\omega) = \frac{mf^2}{4} \frac{(\omega^2 + \omega_0^2)}{[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2]} \quad (3.18)$$

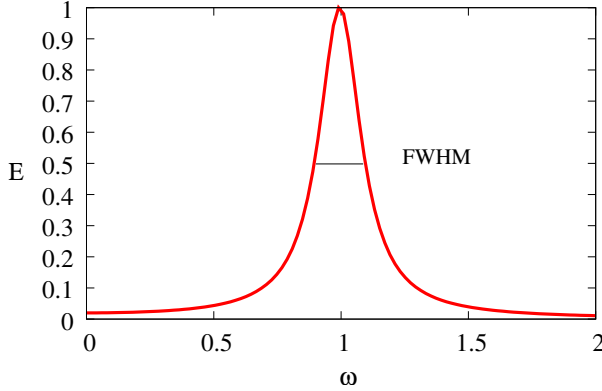


Figure 3.3: Energy resonance

The response to the external force shows a prominent peak or resonance (Figure 3.3) only when $\beta \ll \omega_0$, the mild damping limit. This is of great utility in modelling the phenomena of resonance which occurs in a large variety of situations. In the weak damping limit $E(\omega)$ peaks at $\omega \approx \omega_0$ and falls rapidly away from the peak. As a consequence we can use

$$(\omega_0^2 - \omega^2)^2 = (\omega_0 + \omega)^2(\omega_0 - \omega)^2 \approx 4\omega_0^2(\omega_0 - \omega)^2 \quad (3.19)$$

which gives

$$E(\omega) \approx \frac{k}{8} \frac{f^2}{\omega_0^2[(\omega_0 - \omega)^2 + \beta^2]} \quad (3.20)$$

in the vicinity of the resonance. This has a maxima at $\omega \approx \omega_0$ and the maximum value is

$$E_{max} \approx \frac{k f^2}{8 \omega_0^2 \beta^2}. \quad (3.21)$$

We next estimate the width of the peak or resonance. This is quantified using the FWHM (Full Width at Half Maxima) defined as $\text{FWHM} = 2\Delta\omega$ where $E(\omega_0 + \Delta\omega) = E_{max}/2$ *ie.* half the maximum value. Using equation (3.20) we see that $\Delta\omega = \beta$ and $\text{FWHM} = 2\beta$. as shown in Figure 3.3. The FWHM quantifies the width of the curve and it records the fact that the width increases with the damping coefficient β .

The peak described by equation (3.20) is referred to as a Lorentzian profile. This is seen in a large variety of situations where we have a resonance.

We finally consider the power drawn by the oscillator from the external force. The instantaneous power $P(t) = F(t)\dot{x}(t)$ has a value

$$P(t) = [F \cos(\omega t)][-\dot{x} \mid \omega \sin(\omega t + \phi)]. \quad (3.22)$$

The average power is the quantity of interest, we study this as a function of the frequency. Calculating this we have

$$\langle P \rangle(\omega) = -\frac{1}{2} \omega F \mid \dot{x} \mid \sin \phi. \quad (3.23)$$

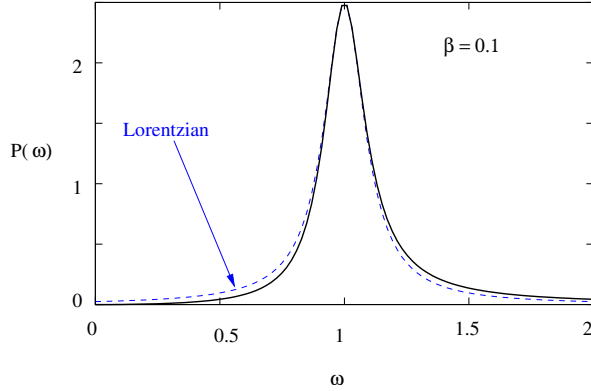


Figure 3.4: Power resonance

Using equation (3.14) we have

$$|\tilde{x}| \sin \phi = \frac{-2\beta\omega}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \left(\frac{F}{m} \right) \quad (3.24)$$

which gives the average power

$$\langle P \rangle(\omega) = \frac{\beta\omega^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \left(\frac{F^2}{m} \right). \quad (3.25)$$

The solid curve in Figure 3.4 shows the average power as a function of ω . Here again, a prominent, sharp peak is seen only if $\beta \ll \omega_0$. In the mild damping limit, in the vicinity of the maxima we have

$$\langle P \rangle(\omega) \approx \frac{\beta}{(\omega_0 - \omega)^2 + \beta^2} \left(\frac{F^2}{4m} \right). \quad (3.26)$$

which again is a Lorentzian profile. For comparison we have also plotted the Lorentzian profile as a dashed curve in Figure 3.4.

Problem 1: Plot the response, $x(t)$, of a forced oscillator with a forcing $3 \cos 2t$ and natural frequency $\omega_0 = 3$ Hz with initial conditions, $x(0) = 3$ and $\dot{x}(0) = 0$, for two different resistances, $\beta = 1$ and $\beta = 0.5$. Plot also for fixed resistance, $\beta = 0.5$ and different forcing amplitudes $f_0 = 1, 3, 5$ and 9 .

Solution 1: The evolution is shown in the Fig. 3.5. Notice that the transients die and the steady state is achieved relatively sooner in the case of larger resistance, $\beta = 1$. Furthermore, the steady state is reached quicker in the case of larger forcing amplitude. See the variation of steady state amplitudes for different parameters.

Problem 2: The galvanometer: A galvanometer is connected with a constant-current source through a switch. At time $t=0$, the switch is closed. After some time the galvanometer deflection reaches its final value θ_{max} . Taking damping torque proportional to the angular velocity draw deflection of the galvanometer from the initial position of rest (i.e. $\theta = 0$, $\dot{\theta} = 0$) to its final

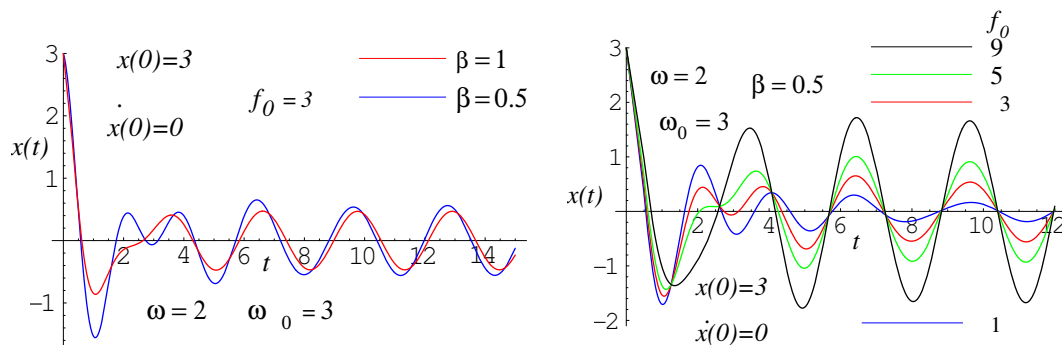


Figure 3.5: Forced oscillations with different resistances and forcing amplitudes

position $\theta = \theta_{max}$, for the underdamped, critically damped and overdamped cases.

Solution 2: We solve the forced oscillator equation with constant forcing (i.e. driving frequency = 0) and given initial conditions and plot the various evolutions. Figure 3.6 shows the galvanometer deflection as a function of time for some arbitrary values of θ_{max} , damping coefficient and natural frequency.

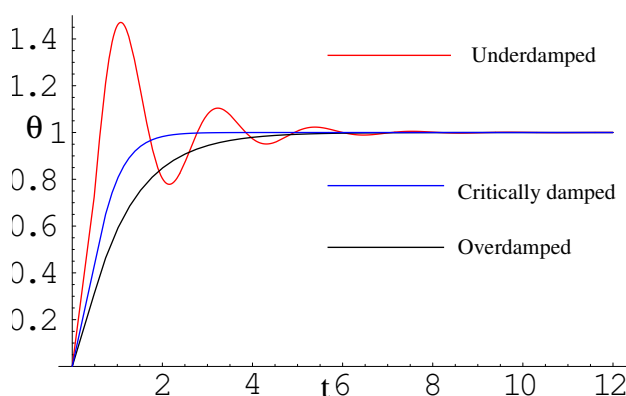


Figure 3.6: Galvanometer deflection

Problems

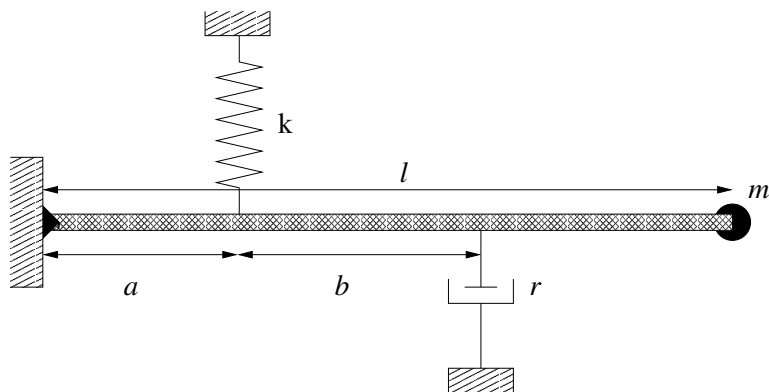
1. An oscillator with $\omega_0 = 2\pi \text{ s}^{-1}$ and negligible damping is driven by an external force $F(t) = a \cos \omega t$. By what percent do the amplitude of oscillation and the energy change if ω is changed from $\pi \text{ s}^{-1}$ to $3\pi/2 \text{ s}^{-1}$? (71.4%, 114%)
2. An oscillator with $\omega_0 = 10^4 \text{ s}^{-1}$ and $\beta = 1 \text{ s}^{-1}$ is driven by an external force $F(t) = a \cos \omega t$. [a.] Determine ω_{max} where the power drawn by the oscillator is maximum? [b.] By what percent does the power fall if ω is changed by $\Delta\omega = 0.5 \text{ s}^{-1}$ from ω_{max} ? [c.] Consider $\beta = 0.1 \text{ s}^{-1}$ instead of $\beta = 1 \text{ s}^{-1}$. (([a.] 10^4 s^{-1} , 33.3%, 96.2%)

3. A mildly damped oscillator driven by an external force is known to have a resonance at an angular frequency somewhere near $\omega = 1\text{MHz}$ with a quality factor of 1100. Further, for the force (in Newtons)

$$F(t) = 10 \cos(\omega t)$$

the amplitude of oscillations is 8.26 mm at $\omega = 1.0\text{ KHz}$ and $1.0\text{ }\mu\text{m}$ at 100 MHz .

- a. What is the spring constant of the oscillator?
 - b. What is the natural frequency ω_0 of the oscillator?
 - c. What is the FWHM?
 - d. What is the phase difference between the force and the oscillations at $\omega = \omega_0 + \text{FWHM}/2$?
4. Show that, $x(t) = \frac{f}{\omega_0^2 - \omega^2}(\cos \omega t - \cos \omega_0 t)$, is a solution of the undamped forced system, $\ddot{x} + \omega_0^2 x = f \cos \omega t$, with initial conditions, $x(0) = \dot{x}(0) = 0$. Show that near resonance, $\omega \rightarrow \omega_0$, $x(t) \approx \frac{f}{2\omega_0} t \sin \omega_0 t$, that is the amplitude of the oscillations grow linearly with time. Plot the solution near resonance. (Hint: Take $\omega = \omega_0 - \Delta\omega$ and expand the solution taking $\Delta\omega \rightarrow 0$.)
5. Find the driving frequencies corresponding to the half-maximum power points and hence find the FWHM for the power curve of Fig. 3.4.
6. Show that the average power loss due to the resistance dissipation is equal to the average input power calculated in the expression (3.25).
7. (a) Evaluate average energies at frequencies, $\omega_{AmRes} = \sqrt{\omega_0^2 - 2\beta^2}$ (at the amplitude resonance) and $\omega_{PoRes} = \omega_0$ (at the power resonance). Show that they are equal and independent of ω_0 .
 (b) Find the value of the forcing frequency, ω_{EnRes} , for which the energy of the oscillator is maximum.
 (c) What is the value of the maximum energy?
 ((a) $mf^2/8\beta^2$,
 (b) $\omega_{EnRes}^2 = 2\omega_0\sqrt{\omega_0^2 - \beta^2} - \omega_0^2$, $\omega_{AmRes} < \omega_{EnRes} < \omega_{PoRes}$,
 (c) $mf^2/16(\omega_0\sqrt{\omega_0^2 - \beta^2} - \omega_0^2 + \beta^2)$.)
8. A massless rigid rod of length l is hinged at one end on the wall. (see figure). A vertical spring of stiffness k is attached at a distance a from the hinge. A damper is fixed at a further distance of b from the spring providing a resistance proportional to the velocity of the attached point of the rod. Now a mass $m(< 0.1ka^2/gl)$ is plugged at the other end of the rod. Write down the condition for critical damping (treat all angular displacements small). If mass is displaced θ_0 from the horizontal, write down the subsequent motion of the mass for the above condition.



9. A critically damped oscillator has mass 1 kg and the spring constant equal to 4 N/m. It is forced with a periodic forcing $F(t) = 2 \cos t \cdot \cos 2t$ N. Write the steady state solution for the oscillator. Find the average power per cycle drawn from the forcing agent.
10. A horizontal spring with a stiffness constant 9 N/m is fixed on one end to a rigid wall. The other end of the spring is attached with a mass of 1 kg resting on a frictionless horizontal table. At $t = 0$, when the spring-mass system is in equilibrium and is perpendicular to the wall, a force $F(t) = 8 \cos 5t$ N starts acting on the mass in a direction perpendicular to the wall. Plot the displacement of the mass from the equilibrium position between $t = 0$ and $t = 2\pi$ neatly.

Chapter 4

Resonance.

4.1 Electrical Circuits

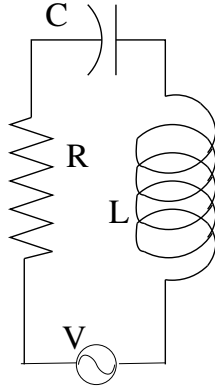


Figure 4.1:

Electrical Circuits are the most common technological application where we see resonances. The LCR circuit shown in Figure 4.1 characterizes the typical situation. The circuit includes a signal generator which produces an AC signal of voltage amplitude V at frequency ω . Applying Kirchoff's Law to this circuit we have,

$$Ve^{i\omega t} + L\frac{dI}{dt} + \frac{Q}{C} + RI = 0, \quad (4.1)$$

which may be written solely in terms of the charge q as

$$L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = Ve^{i\omega t}, \quad (4.2)$$

We see that this is a damped oscillator with an external sinusoidal force. The equation governing this is

$$\ddot{Q} + 2\beta\dot{Q} + \omega_0^2 Q = \tilde{v}e^{i\omega t} \quad (4.3)$$

where $\omega_0^2 = 1/LC$, $\beta = R/2L$ and $v = (V/L)$.

We next consider the power dissipated in this circuit. The resistance is the only circuit element which draws power. We proceed to calculate this by calculating the impedance

$$\tilde{Z}(\omega) = i\omega L - \frac{i}{\omega C} + R \quad (4.4)$$

which varies with frequency. The voltage and current are related as $\tilde{V} = \tilde{I}\tilde{Z}$, which gives the current

$$\tilde{I} = \frac{\tilde{V}}{i(\omega L - 1/\omega C) + R}. \quad (4.5)$$

The average power dissipated may be calculated as $\langle P(\omega) \rangle = R\tilde{I}\tilde{I}^*/2$ which is

$$\langle P(\omega) \rangle = \frac{\omega^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \left(\frac{RV^2}{2L^2} \right). \quad (4.6)$$

Problem For an Electrical Oscillator with $L = 10mH$ and $C = 1\mu F$,

- what is the natural (angular) frequency ω_0 ? ($10KHz$)
- Choose R so that the oscillator is critically damped. (200Ω)
- For $R = 2\Omega$, what is the maximum power that can be drawn from a $10V$ source? ($25W$)
- What is the FWHM of the peak? ($200Hz$)
- At what frequency is half the maximum power drawn? ($10.1KHz$ and $9.9KHz$)
- What is the value of the quality factor Q ? ($Q = \omega_0/2\beta = 50$)
- What is the time period of the oscillator? ($T = 2\pi/\omega$ where $\omega = \omega\sqrt{1 - \beta^2/\omega_0^2} \approx 10KHz$ and $T = 2\pi 10^{-4}sec.$)
- What is the value of the log decrement λ ? ($\tilde{x}(t) = [\tilde{A}e^{-\beta t}]e^{i\omega t}$, $\lambda = \ln(x_n/x_{n+1}) = \beta T = 2\pi 10^{-2}$)

4.2 The Raman Effect

Light of frequency ν is incident on a target. If the emergent light is analysed through a spectrometer it is found that there are components at two new frequencies $\nu - \Delta\nu$ and $\nu + \Delta\nu$ known as the Stokes and anti-Stokes lines respectively. This phenomenon was discovered by Sir. C.V. Raman and it is known as the Raman Effect.

As an example, we consider light of frequency $\nu = 6.0 \times 10^{14}Hz$ incident on benzene which is a liquid. It is found that there are three different pairs of Stokes and anti-Stokes lines in the spectrum. It is possible to associate each of these new pair of lines with different oscillations of the benzene molecule. The vibrations of a complex system like benzene can be decomposed into different normal modes, each of which behaves like a simple harmonic oscillator with its own natural frequency. There is a separate Raman line associated with

each of these different modes. A closer look at these spectral lines shows them to have a finite width, the shape being a Lorentzian corresponding to the resonance of a damped harmonic oscillator. Figure 4.2 shows the Raman line corresponding to the bending mode of benzene.

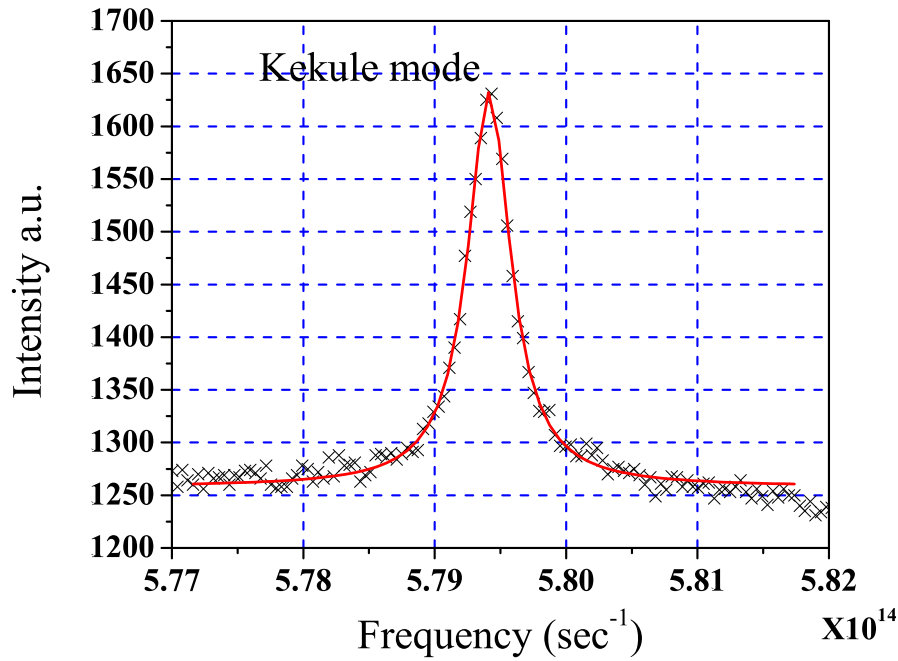


Figure 4.2:

Problem For the Raman line shown in Figure 4.2

- What is the natural frequency and the corresponding ω_0 ?
- What is the FWHM?
- What is the value of the quality factor Q ?

Chapter 5

Coupled Oscillators

Consider two identical simple harmonic oscillators of mass m and spring constant k as shown in Figure 5.1 (a.). The two oscillators are independent with

$$x_0(t) = a_0 \cos(\omega t + \phi_0) \quad (5.1)$$

and

$$x_1(t) = a_1 \cos(\omega t + \phi_1) \quad (5.2)$$

where they both oscillate with the same frequency $\omega = \sqrt{\frac{k}{m}}$. The amplitudes a_0, a_1 and the phases ϕ_0, ϕ_1 of the two oscillators are in no way interdependent. The question which we take up for discussion here is what happens if the two masses are coupled by a third spring as shown in Figure 1 (b.).

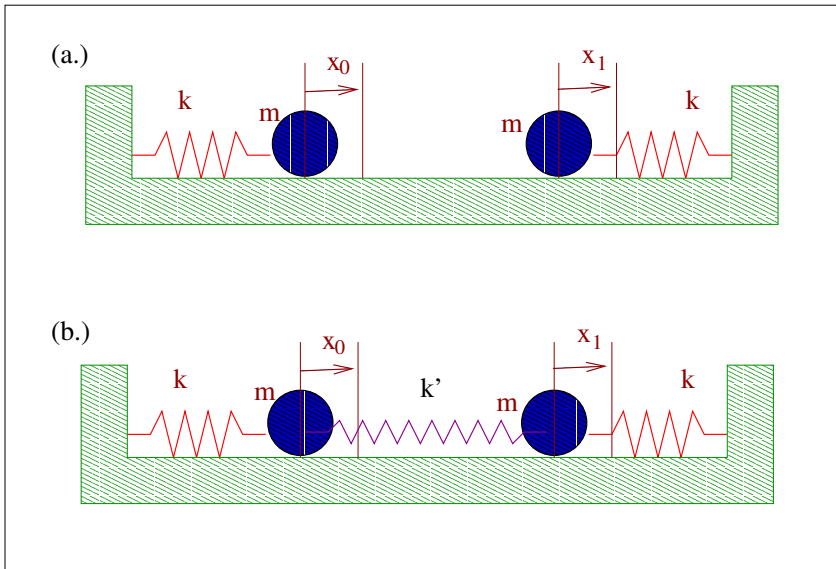


Figure 5.1: This shows two identical spring-mass systems. In (a.) the two oscillators are independent whereas in (b.) they are coupled through an extra spring.

The motion of the two oscillators is now coupled through the third spring of spring constant k' . It is clear that the oscillation of one oscillator affects

the second. The phases and amplitudes of the two oscillators are no longer independent and the frequency of oscillation is also modified. We proceed to calculate these effects below.

The equations governing the coupled oscillators are

$$m \frac{d^2 x_0}{dt^2} = -kx_0 - k'(x_0 - x_1) \quad (5.3)$$

and

$$m \frac{d^2 x_1}{dt^2} = -kx_1 - k'(x_1 - x_0) \quad (5.4)$$

5.1 Normal modes

The technique to solve such *coupled differential equations* is to identify linear combinations of x_0 and x_1 for which the equations become decoupled. In this case it is very easy to identify such variables

$$q_0 = \frac{x_0 + x_1}{2} \text{ and } q_1 = \frac{x_0 - x_1}{2}. \quad (5.5)$$

These are referred to as the *normal modes* (or *eigen modes*) of the system and the equations governing them are

$$m \frac{d^2 q_0}{dt^2} = -kq_0 \quad (5.6)$$

and

$$m \frac{d^2 q_1}{dt^2} = -(k + 2k')q_1. \quad (5.7)$$

The two normal modes execute simple harmonic oscillations with respective angular frequencies

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \text{and} \quad \omega_1 = \sqrt{\frac{k + 2k'}{m}} \quad (5.8)$$

In this case the normal modes lend themselves to a simple physical interpretation where.

The normal mode q_0 represents the center of mass. The center of mass behaves as if it were a particle of mass $2m$ attached to two springs (Figure 5.2) and its oscillation frequency is the same as that of the individual decoupled oscillators $\omega_0 = \sqrt{\frac{2k}{2m}}$.

The normal mode q_1 represents the relative motion of the two masses which leaves the center of mass unchanged. This can be thought of as the motion of two particles of mass m connected to a spring of spring constant $\tilde{k} = (k + 2k')/2$ as shown in Figure 5.3. The oscillation frequency of this normal mode $\omega_1 = \sqrt{\frac{k + 2k'}{m}}$ is always higher than that of the individual uncoupled oscillators (or

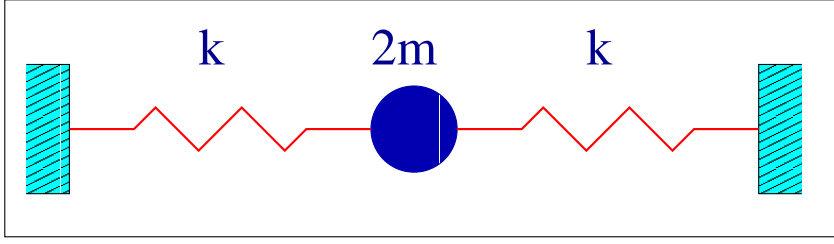


Figure 5.2: This shows the spring mass equivalent of the normal mode q_0 which corresponds to the center of mass.

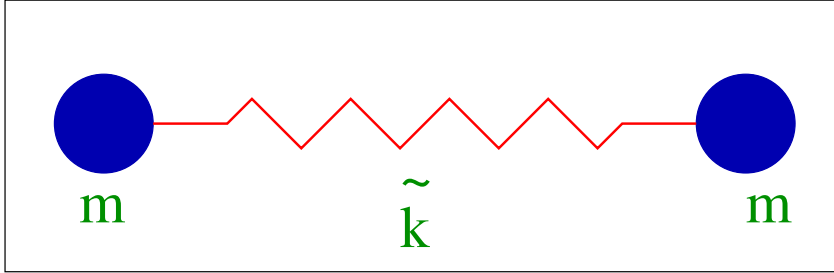


Figure 5.3: This shows the spring mass equivalent of the normal mode q_1 which corresponds to two particles connected through a spring.

the center of mass). The modes q_0 and q_1 are often referred to as the slow mode and the fast mode respectively.

We may interpret q_0 as a mode of oscillation where the two masses oscillate with exactly the same phase, and q_1 as a mode where they have a phase difference of π (Figure 5.4). Recollect that the phases of the two masses are independent when the two masses are not coupled. Introducing a coupling causes the phases to be interdependent.

The normal modes have solutions

$$\tilde{q}_0(t) = \tilde{A}_0 e^{i \omega_0 t} \quad (5.9)$$

$$\tilde{q}_1(t) = \tilde{A}_1 e^{i \omega_1 t} \quad (5.10)$$

where it should be borne in mind that \tilde{A}_0 and \tilde{A}_1 are complex numbers with both amplitude and phase ie. $\tilde{A}_0 = A_0 e^{i\psi_0}$ etc. We then have the solutions

$$\tilde{x}_0(t) = \tilde{A}_0 e^{i \omega_0 t} + \tilde{A}_1 e^{i \omega_1 t} \quad (5.11)$$

$$\tilde{x}_1(t) = \tilde{A}_0 e^{i \omega_0 t} - \tilde{A}_1 e^{i \omega_1 t} \quad (5.12)$$

The complex amplitudes \tilde{A}_1 and \tilde{A}_2 have to be determined from the initial conditions, four initial conditions are required in total.

5.2 Resonance

As an example we consider a situation where the two particles are initially at rest in the equilibrium position. The particle x_0 is given a small displacement

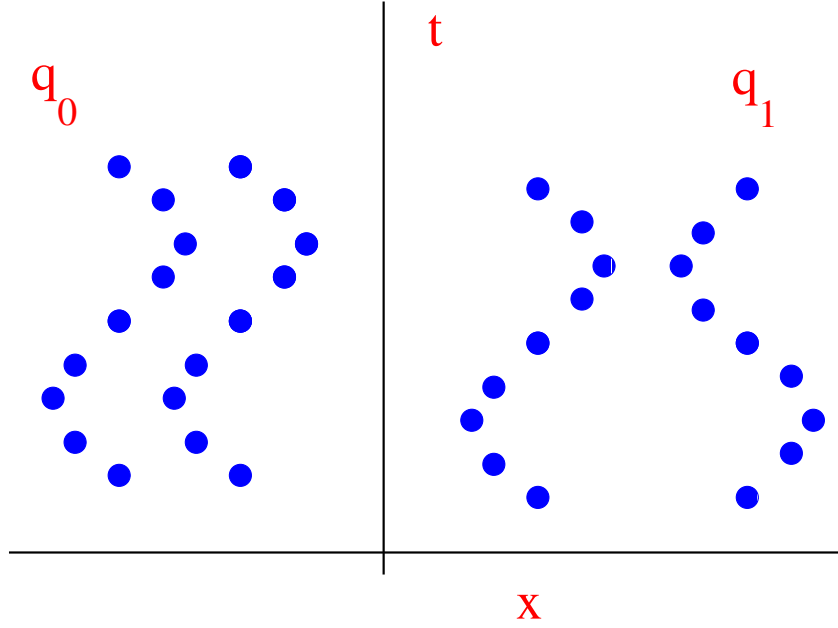


Figure 5.4: This shows the motion corresponding to the two normal modes q_0 and q_1 respectively.

a_0 and then left to oscillate. Using this to determine \tilde{A}_1 and \tilde{A}_2 , we finally have

$$x_0(t) = \frac{a_0}{2} [\cos \omega_0 t + \cos \omega_1 t] \quad (5.13)$$

and

$$x_1(t) = \frac{a_0}{2} [\cos \omega_0 t - \cos \omega_1 t] \quad (5.14)$$

The solution can also be written as

$$x_0(t) = a_0 \cos \left[\left(\frac{\omega_1 - \omega_0}{2} \right) t \right] \cos \left[\left(\frac{\omega_0 + \omega_1}{2} \right) t \right] \quad (5.15)$$

$$x_1(t) = a_0 \sin \left[\left(\frac{\omega_1 - \omega_0}{2} \right) t \right] \sin \left[\left(\frac{\omega_0 + \omega_1}{2} \right) t \right] \quad (5.16)$$

It is interesting to consider $k' \ll k$ where the two oscillators are weakly coupled. In this limit

$$\omega_1 = \sqrt{\frac{k}{m} \left(1 + \frac{2k'}{k} \right)} \approx \omega_0 + \frac{k'}{k} \omega_0 \quad (5.17)$$

and we have solutions

$$x_0(t) = \left[a_0 \cos \left(\frac{k'}{2k} \omega_0 t \right) \right] \cos \omega_0 t \quad (5.18)$$

and

$$x_1(t) = \left[a_0 \sin \left(\frac{k'}{2k} \omega_0 t \right) \right] \sin \omega_0 t. \quad (5.19)$$

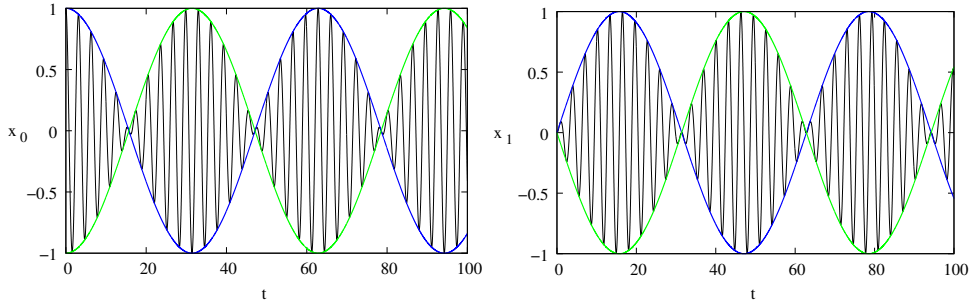


Figure 5.5: This shows the motion of x_0 and x_1 .

The solution is shown in Figure 5.5. We can think of the motion as an oscillation with ω_0 where the amplitude undergoes a slow modulation at angular frequency $\frac{k'}{2k}\omega_0$. The oscillations of the two particles are out of phase and are slowly transferred from the particle which receives the initial displacement to the particle originally at rest, and then back again.

Problems

- For the coupled oscillator shown in Figure 5.1 with $k = 10 \text{ Nm}^{-1}$, $k' = 30 \text{ Nm}^{-1}$ and $m = 1 \text{ kg}$, both particles are initially at rest. The system is set into oscillations by displacing x_0 by 40 cm while $x_1 = 0$.
[a.] What is the angular frequency of the faster normal mode? [b.] Calculate the average kinetic energy of x_1 ? [c.] How does the average kinetic energy of x_1 change if the mass of both the particles is doubled? ([a.] 8.37 s^{-1} [b.] $8.00 \times 10^{-1} \text{ J}$ [c.] No change)
- For a coupled oscillator with $k = 8 \text{ Nm}^{-1}$, $k' = 10 \text{ Nm}^{-1}$ and $m = 2 \text{ kg}$, both particles are initially at rest. The system is set into oscillations by displacing x_0 by 10 cm while $x_1 = 0$.
[a.] What are the angular frequencies of the two normal modes of this system? [b.] With what time period does the instantaneous potential energy of the middle spring oscillate? [c.] What is the average potential energy of the middle spring?
- Consider a coupled oscillator with $k = 9 \text{ Nm}^{-1}$, $k' = 8 \text{ Nm}^{-1}$ and $m = 1 \text{ kg}$. Initially both particles have zero velocity with $x_0 = 10 \text{ cm}$ and $x_1 = 0$. [a.] After how much time does the system return to the initial configuration? [b.] After how much time is the separation between the two masses maximum? [c.] What are the average kinetic and potential energy? ([a.] $2\pi \text{ s}$, [b.] $\pi/5 \text{ s}$ [c.] $14.25 \times 10^{-2} \text{ J}$)
- A coupled oscillator has $k = 9 \text{ Nm}^{-1}$, $k' = 0.1 \text{ Nm}^{-1}$ and $m = 1 \text{ kg}$. Initially both particles have zero velocity with $x_0 = 5 \text{ cm}$ and $x_1 = 0$. After how many oscillations in x_0 does it completely die down? (45)
- Find out the frequencies of the normal modes for the following coupled pendula (see figure 5.6) for small oscillations. Calculate time period for beats.

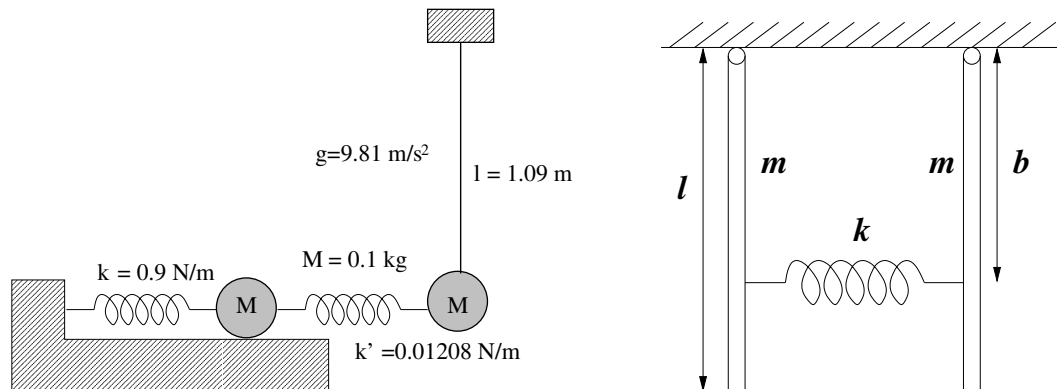


Figure 5.6: Problem 5 and 6

6. A coupled system is in a vertical plane. Each rod is of mass m and length l and can freely oscillate about the point of suspension. The spring is attached at a length b from the points of suspensions (see figure 5.6). Find the frequencies of normal(eigen) modes. Find out the ratios of amplitudes of the two oscillators for exciting the normal(eigen) modes.
7. **Mechanical filter:** *Damped-forced-coupled oscillator*- Suppose one of the masses in the system (say mass 1) is under sinusoidal forcing $F(t) = F_0 \cos \omega t$. Include also resistance in the system such that the damping term is equal to $-2r \times \text{velocity}$. Write down the equations of motion for the above system.

Solution 7:

$$m \frac{d^2 x_0}{dt^2} = -kx_0 - k'(x_0 - x_1) - 2r\dot{x}_0 + F_0 \cos \omega t, \quad (5.20)$$

$$m \frac{d^2 x_1}{dt^2} = -kx_1 - k'(x_1 - x_0) - 2r\dot{x}_1. \quad (5.21)$$

Rearranging the terms we have (with notations of forced oscillations),

$$\ddot{x}_0 + 2\beta\dot{x}_0 + \omega_0^2 x_0 + \frac{k'}{m}(x_0 - x_1) = f_0 \cos \omega t, \quad (5.22)$$

$$\ddot{x}_1 + 2\beta\dot{x}_1 + \omega_0^2 x_1 + \frac{k'}{m}(x_1 - x_0) = 0. \quad (5.23)$$

8. Solve the equations by identifying the normal modes.

Solution 8: Decouple the equations using q_0 and q_1 .

$$\ddot{q}_0 + 2\beta\dot{q}_0 + \omega_0^2 q_0 = \frac{f_0}{2} \cos \omega t, \quad (5.24)$$

$$\ddot{q}_1 + 2\beta\dot{q}_1 + \omega_1^2 q_1 = \frac{f_0}{2} \cos \omega t. \quad (5.25)$$

9. Write down the solutions of q_0 and q_1 as $q_0 = z_0 \cos \omega t$ and $q_1 = z_1 \cos \omega t$ respectively, with $z_0 = |z_0| \exp(i\phi_0)$ and $z_1 = |z_1| \exp(i\phi_1)$. Find $|z_0|$, ϕ_0 , $|z_1|$ and ϕ_1 .
10. Find amplitudes of the original masses, viz x_0 and x_1 .

$$x_0 = q_0 + q_1 = z_0 \cos \omega t + z_1 \cos \omega t = (z_0 + z_1) \cos \omega t \equiv |A_0| \cos(\omega t + \Phi_0)$$

$$x_1 = q_0 - q_1 = z_0 \cos \omega t - z_1 \cos \omega t = (z_0 - z_1) \cos \omega t \equiv |A_1| \cos(\omega t + \Phi_1)$$

Do phasor addition and subtraction to evaluate amplitudes $|A_0|$ and $|A_1|$. Find also the phases Φ_0 and Φ_1 .

11. Using above results show that:

$$\frac{|A_1|^2}{|A_0|^2} = \frac{(\omega_1^2 - \omega_0^2)^2}{(\omega_1^2 + \omega_0^2 - 2\omega^2)^2 + 16\beta^2\omega^2}.$$

12. Plot the above ratio of amplitudes of two coupled oscillators as a function of forcing frequency ω using very small damping (i.e. neglecting the β term). From there observe that the ratio of amplitudes dies as the forcing frequency goes below ω_0 or above ω_1 . So the system works as a band pass filter, i.e. the unforced mass has large amplitude only when the forcing frequency is in between ω_0 and ω_1 . Otherwise it does not respond to forcing.

Chapter 6

Sinusoidal Waves.

We shift our attention to oscillations that propagate in space as time evolves. This is referred to as a wave. The sinusoidal wave

$$a(x, t) = A \cos(\omega t - kx + \psi) \quad (6.1)$$

is the simplest example of a wave, we shall consider other possibilities later in the course. It is often convenient to represent the wave in the complex notation introduced earlier. We have

$$\tilde{a}(x, t) = \tilde{A} e^{i(\omega t - kx)} \quad (6.2)$$

6.1 What is $a(x, t)$?

The wave phenomena is found in many different situations, and $a(x, t)$ represents a different physical quantity in each situation. For example, it is well known that disturbances in air propagate from one point to another as waves and are perceived by us as sound. Any source of sound (eg. a loud speaker) produces compressions and rarefactions in the air, and the patterns of compressions and rarefactions propagate from one point to another. Using $\rho(x, t)$ to denote the air density, we can express this as $\rho(x, t) = \bar{\rho} + \Delta\rho(x, t)$ where $\bar{\rho}$ is the density in the absence of the disturbance and $\Delta\rho(x, t)$ is the change due to the disturbance. We can use equation (6.1) to represent a sinusoidal sound wave if we identify $a(x, t)$ with $\Delta\rho(x, t)$.

The transverse vibrations of a stretched string is another example. In this situation $a(x, t)$ corresponds to $y(x, t)$ which is the displacement of the string shown in Figure 6.1.

6.2 Angular frequency and wave number

The sinusoidal wave in equation (6.2) has a complex amplitude $\tilde{A} = A e^{i\psi}$. Here A , the magnitude of \tilde{A} determines the magnitude of the wave. We refer

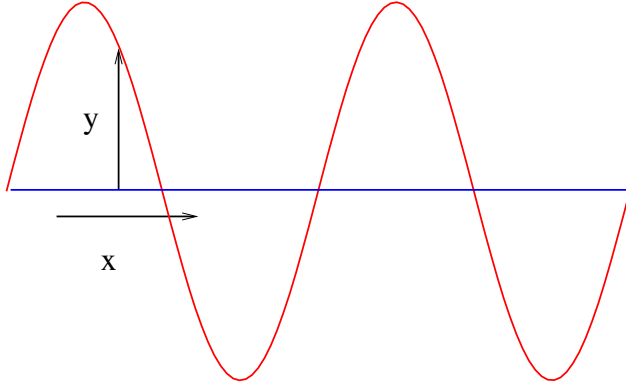


Figure 6.1:

to $\phi(x, t) = \omega t - kx + \psi$ as the phase of the wave, and the wave can be also expressed as

$$\tilde{a}(x, t) = Ae^{i\phi(x, t)} \quad (6.3)$$

If we study the behaviour of the wave at a fixed position x_1 , we have

$$\tilde{a}(t) = [\tilde{A}e^{-ikx_1}]e^{i\omega t} = \tilde{A}'e^{i\omega t}. \quad (6.4)$$

We see that this is the familiar oscillation (SHO) discussed in detail in Chapter 1. The oscillation has amplitude $\tilde{A}' = [\tilde{A}e^{-ikx_1}]$ which includes an extra constant phase factor. The value of $a(t)$ has sinusoidal variations. Starting at $t = 0$, the behaviour repeats after a time period T when $\omega T = 2\pi$. We identify ω as the angular frequency of the wave related to the frequency ν as

$$\omega = \frac{2\pi}{T} = 2\pi\nu. \quad (6.5)$$

We next study the wave as a function of position x at a fixed instant of time t_1 . We have

$$\tilde{a}(x) = \tilde{A}e^{i\omega t_1}e^{-ikx} = \tilde{A}''e^{-ikx} \quad (6.6)$$

where we have absorbed the extra phase $e^{i\omega t_1}$ in the complex amplitude \tilde{A}'' . This tells us that the spatial variation is also sinusoidal as shown in Figure 6.2. The wavelength λ is the distance after which $a(x)$ repeats itself. Starting from $x = 0$, we see that $a(x)$ repeats when $kx = 2\pi$ which tells us that $k\lambda = 2\pi$ or

$$k = \frac{2\pi}{\lambda} \quad (6.7)$$

where we refer to k as the wave number. We note that the wave number and the angular frequency tell us the rate of change of the phase $\phi(x, t)$ with position and time respectively

$$k = -\frac{\partial\phi}{\partial x} \text{ and } \omega = \frac{\partial\phi}{\partial t} \quad (6.8)$$

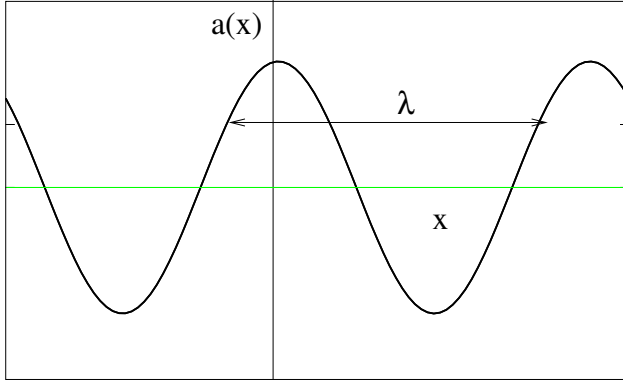


Figure 6.2:

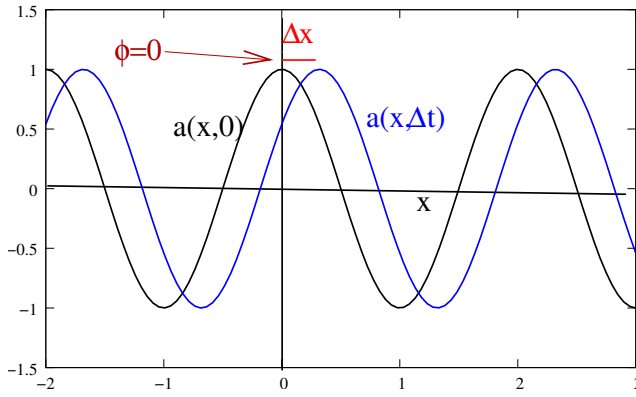


Figure 6.3:

6.3 Phase velocity.

We now consider the evolution of the wave in both position and time together. We consider the wave

$$\tilde{a}(x, t) = Ae^{i(\omega t - kx)} \quad (6.9)$$

which has phase $\phi(x, t) = \omega t - kx$. Let us follow the motion of the position where the phase has value $\phi(x, t) = 0$ as time increases. We see that initially $\phi = 0$ at $x = 0, t = 0$ and after a time Δt this moves to a position

$$\Delta x = \left(\frac{\omega}{k}\right) \Delta t \quad (6.10)$$

shown in Figure 6.3. The point with phase $\phi = 0$ moves at speed

$$v_p = \left(\frac{\omega}{k}\right). \quad (6.11)$$

It is not difficult to convince oneself that this is true for any constant value of the phase, and the whole sinusoidal pattern propagates along the $+x$ direction (Figure 6.4) at the speed v_p which is called the phase velocity of the wave.

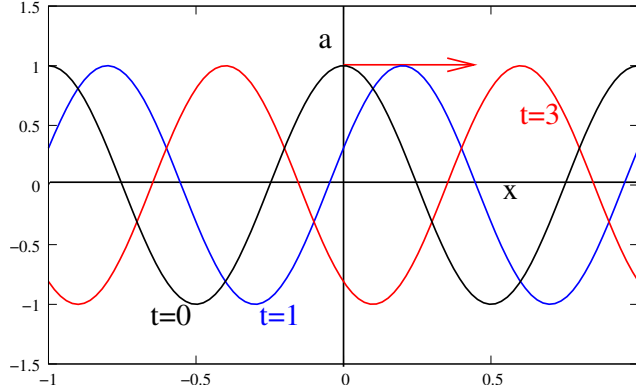


Figure 6.4:

6.4 Waves in three dimensions.

We have till now considered waves which depend on only one position coordinate x and time t . This is quite adequate when considering waves on a string as the position along a string can be described by a single coordinate. It is necessary to bring three spatial coordinates (x, y, z) into the picture when considering a wave propagating in three dimensional space. A sound wave propagating in air is an example.

We use the vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ to denote a point in three dimensional space. The solution which we have been discussing

$$\tilde{a}(\vec{r}, t) = Ae^{i(\omega t - kx)} \quad (6.12)$$

can be interpreted in the context of a three dimensional space. Note that $\tilde{a}(\vec{r}, t)$ varies only along the x direction and not along y and z . Considering the phase $\phi(\vec{r}, t) = \omega t - kx$ we see that at any particular instant of time t , there are surfaces on which the phase is constant. The constant phase surfaces of a wave are called wave fronts. In this case the wave fronts are parallel to the $y - z$ plane as shown in Figure 6.5. The wave fronts move along the $+x$ direction with speed v_p as time evolves. You can check this by following the motion of the $\phi = 0$ surface shown in Figure 6.5.

6.5 Waves in an arbitrary direction.

Let us now discuss how to describe a sinusoidal plane wave in an arbitrary direction denoted by the unit vector \vec{n} . A wave propagating along the \hat{i} direction can be written as

$$\tilde{a}(\vec{r}, t) = \tilde{A}e^{i(\omega t - \vec{k} \cdot \vec{r})} \quad (6.13)$$

where $\vec{k} = k\hat{i}$ is called the wave vector. Note that \vec{k} is different from \hat{k} which is the unit vector along the z direction. It is now obvious that a wave along an arbitrary direction \hat{n} can also be represented by eq. (6.13) if we change the

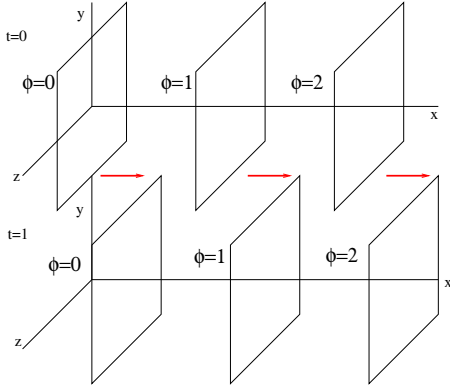


Figure 6.5:

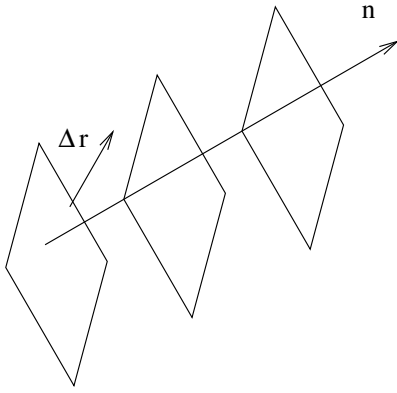


Figure 6.6:

wave vector to $\vec{k} = k\hat{n}$. The wave vector \vec{k} carries information about both the wavelength λ and the direction of propagation \hat{n} .

For such a wave, at a fixed instant of time, the phase $\phi(\vec{r}, t) = \omega t - \vec{k} \cdot \vec{r}$ changes only along \hat{n} . The wave fronts are surfaces perpendicular to \hat{n} as shown in Figure 6.6.

Problem: Show the above fact, that is the surface swapped by a constant phase at a fixed instant is a two dimensional plane and the wave vector \vec{k} is normal to that plane.

The phase difference between two point (shown in Figure 6.6) separated by $\Delta\vec{r}$ is $\Delta\phi = -\vec{k} \cdot \Delta\vec{r}$.

Problems

1. What are the wave number and angular frequency of the wave $a(x, t) = A \cos^2(2x - 3t)$ where x and t are in m and s respectively? (4 m^{-1} , 6 s^{-1})
2. What is the wavelength corresponding to the wave vector $\vec{k} = 3\hat{i} + 4\hat{j} \text{ m}^{-1}$? (0.4 m)

3. A wave with $\omega = 10 \text{ s}^{-1}$ and $\vec{k} = 7\hat{i} + 6\hat{j} - 3\hat{k} \text{ m}^{-1}$ has phase $\phi = \pi/3$ at the point $(0, 0, 0)$ at $t = 0$. [a.] At what time will this value of phase reach the point $(1, 1, 1) \text{ m}$? [b.] What is the phase at the point $(1, 0, 0) \text{ m}$ at $t = 1 \text{ s}$? [c.] What is the phase velocity of the wave? ([a.] 10 s [b.] 24 rad [c.] 1.03 m s^{-1})
4. For a wave with $\vec{k} = (4\hat{i} + 5\hat{j}) \text{ m}^{-1}$ and $\omega = 10^8 \text{ m}^{-1}$, what are the values of the following? [a.] wavelength, [b.] frequency [c.] phase velocity, [d.] phase difference between the two points $(x, y, z) = (3, 4, 7) \text{ m}$ and $(4, 2, 8) \text{ m}$.
5. The phase of a plane wave is the same at the points $(2, 7, 5)$, $(3, 10, 6)$ and $(4, 12, 5)$. and the phase is $\pi/2$ ahead at $(3, 7, 5)$. Determine the wave vector for the wave.[All coordinates are in m.]
6. Two waves of the same frequency have wave vectors $\vec{k}_1 = 3\hat{i} + 4\hat{j} \text{ m}^{-1}$ and $\vec{k}_2 = 4\hat{i} + 3\hat{j} \text{ m}^{-1}$ respectively. The two waves have the same phase at the point $(2, 7, 8) \text{ m}$, what is the phase difference between the waves at the point $(3, 5, 8) \text{ m}$? (3 rad)

Chapter 7

Electromagnetic Waves.

What is light, particle or wave? Much of our daily experience with light, particularly the fact that light rays move in straight lines tells us that we can think of light as a stream of particles. This is further borne out when place an opaque object in the path of the light rays. The shadow, as shown in Figure 7.1 is a projected image of the object, which is what we expect if light were a stream of particles. But a closer look at the edges of the shadow reveals a very fine pattern of dark and bright bands or fringes. Such a pattern can also be seen if we stretch out our hand and look at the sky through a thin gap produced by bringing two of our fingers close. This cannot be explained unless we accept that light is some kind of a wave.

It is now well known that light is an electromagnetic wave. We shall next discuss what we mean by an electromagnetic wave or radiation.

7.1 Electromagnetic Radiation.

What is the electric field produced at a point P by a charge q located at a distance r as shown in Figure 7.2? Anybody with a little knowledge of physics

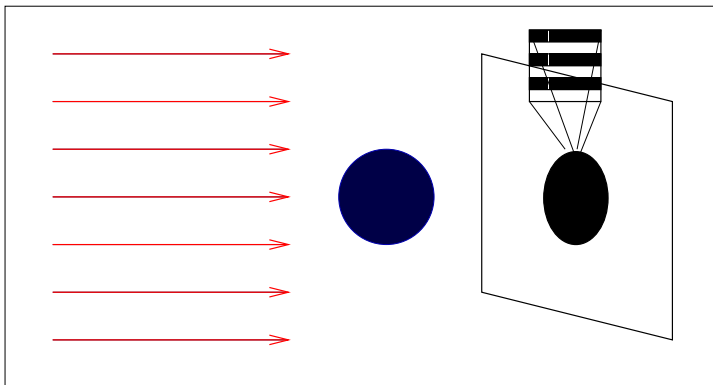


Figure 7.1:

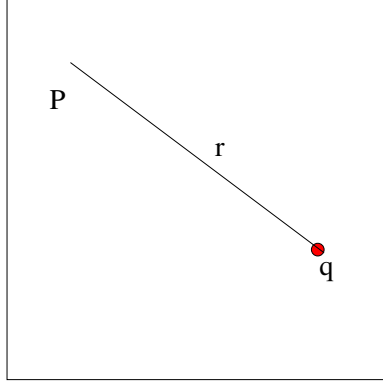


Figure 7.2:

will tell us that this is given by Coulomb's law

$$\vec{E} = \frac{-q}{4\pi\epsilon_0} \frac{\hat{e}_r}{r^2} \quad (7.1)$$

where \hat{e}_r is an unit vector from P to the position of the charge. In what follows we shall follow the notation used in Feynman Lectures (Vol. I, Chapter 28, Electromagnetic radiation).

In the 1880s J.C. Maxwell proposed a modification in the laws of electricity and magnetism which were known at that time. The change proposed by Maxwell unified our ideas of electricity and magnetism and showed both of them to be manifestations of a single underlying quantity. Further it implied that Coulomb's law did not tells us the complete picture. The correct formula for the electric field is

$$\vec{E} = \frac{-q}{4\pi\epsilon_0} \left[\frac{\hat{e}_{r'}}{r'^2} + \frac{r'}{c} \frac{d}{dt} \left(\frac{\hat{e}_{r'}}{r'^2} \right) + \frac{1}{c^2} \frac{d^2}{dt^2} \hat{e}_{r'} \right] \quad (7.2)$$

This formula incorporates several new effects. The first is the fact that no information can propagate instantaneously. This is a drawback of Coulomb's law where the electric field at a distant point P changes the moment the position of the charge is changed. This should actually happen after some time. The new formula incorporates the fact that the influence of the charge propagates at a speed c . The electric field at the time t is determined by the position of the charge at an earlier time. This is referred to as the retarded position of the charge r' , and $\hat{e}_{r'}$ also refers to the retarded position.

The first term in eq. (7.2) is Coulomb's law with the retarded position. In addition there are two new terms which arise due to the modification proposed by Maxwell. These two terms contribute only when the charge moves. The magnetic field produced by the charge is

$$\vec{B} = -\hat{e}_{r'} \times \vec{E}/c \quad (7.3)$$

A close look at eq. (7.2) shows that the contribution from the first two terms falls off as $1/r'^2$ and these two terms are of not of interest at large

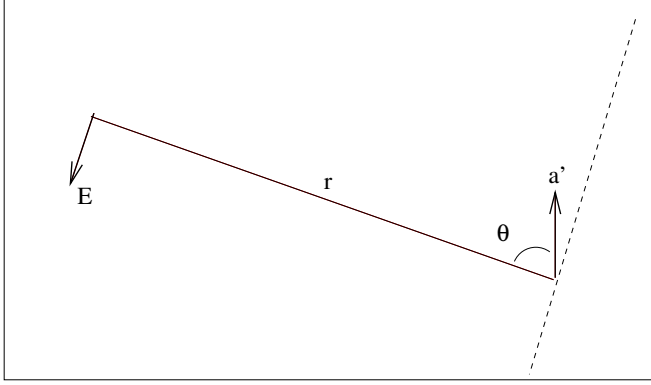


Figure 7.3:

distances from the charge. It is only the third term which has a $1/r$ behaviour that makes a significant contribution at large distances. This term permits the a charged particle to influence another charged particle at a great distance through the $1/r$ electric field. This is referred to as electromagnetic radiation and light is a familiar example of this phenomena. It is obvious from the formula that only accelerating charges produce radiation.

The interpretation of the formula is substantially simplified if we assume that the motion of the charge is relatively slow, and is restricted to a region which is small in comparison to the distance r to the point where we wish to calculate the electric field. We then have

$$\frac{d^2}{dt^2} \hat{e}_{r'} = \frac{d^2}{dt^2} \left(\frac{\vec{r}'}{r'} \right) \approx \frac{\ddot{\vec{r}}_{\perp}}{r} \quad (7.4)$$

where $\ddot{\vec{r}}_{\perp}$ is the acceleration of the charge in the direction perpendicular to $\hat{e}_{r'}$. The parallel component of the acceleration does not effect the unit vector $\hat{e}_{r'}$ and hence it does not make a contribution here. Further, the motion of the charge makes a very small contribution to r' in the denominator, neglecting this we replace r' with the constant distance r .

The electric field at a time t is related to $a(t - r/c)$ which is the retarded acceleration as

$$E(t) = \frac{-q}{4\pi\epsilon_0 c^2 r} a(t - r/c) \sin \theta \quad (7.5)$$

where θ is the angle between the line of sight \hat{e}_r to the charge and the direction of the retarded acceleration vector. The electric field vector is in the direction obtained by projecting the retarded acceleration vector on the plane perpendicular to \hat{e}_r as shown in Figure 7.3.

Problem 1: Show that the second term inside the bracket of eq.(7.2) indeed falls off as $1/r'^2$. Also show that the expression for electric field for an accelerated charge i.e. eq. (7.5) follows from it.

Solution 1: See fig. 7.4 (r and θ can be treated as constants with respect to time).

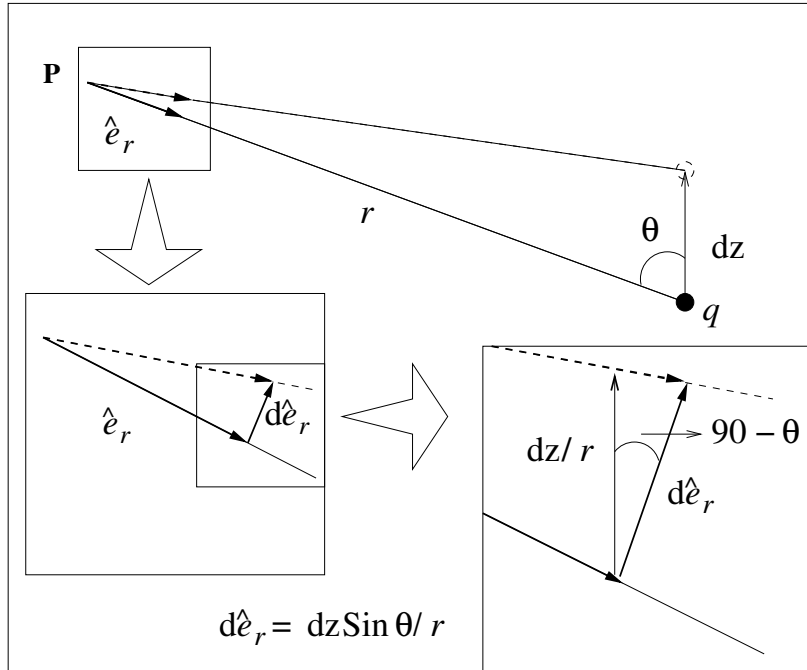


Figure 7.4:

7.2 Electric dipole radiation.

We next consider a situation where a charge accelerates up and down along a straight line. The analysis of this situation using eq. (7.5) has wide applications including many in technology. We consider the device shown in Figure 7.5 which has two wires A and B connected to an oscillating voltage generator. Consider the situation when the terminal of the voltage generator connected to A is positive and the one connected to B is negative. There will be an accumulation of positive charge at the tip of the wire A and negative charge at the tip of B respectively. The electrons rush from B to A when the voltage is reversed. The oscillating voltage causes charge to oscillate up and down the



Figure 7.5:

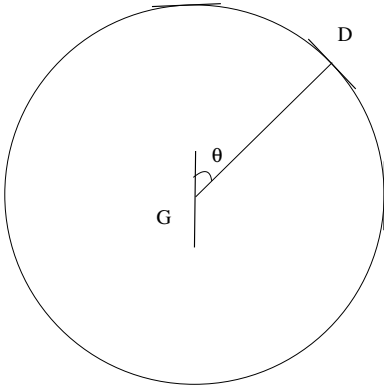


Figure 7.6:

two wire A and B as if they were a single wire. In the situation where the time taken by the electrons to move up and down the wires is much larger than the time taken for light signal to cross the wire, this can be thought of as an oscillating electric dipole. Note that here we have many electrons oscillating up and down the wire. Since all the electrons have the same acceleration, the electric fields that they produce adds up.

The electric field produced is inversely proportional to the distance r from the oscillator. At any time t , the electric field is proportional to the acceleration of the charges at a time $t - r/c$ in the past.

It is possible to measure the radiation using a another electric dipole oscillator where the voltage generator is replaced by a detector, say an oscilloscope. An applied oscillating electric field will give rise to an oscillating current in the wires which can be converted to a voltage and measured. A dipole can measure oscillating electric fields only if the field is parallel to the dipole and not if they are perpendicular. A dipole is quite commonly used as an antenna to receive radio waves which is a form of electromagnetic radiation.

Figure 7.6 shows an experiment where we use a dipole with a detector (D) to measure the electromagnetic radiation produced by another oscillating electric dipole (G). The detected voltage is maximum at $\theta = 90^\circ$ and falls as $\sin \theta$ in other directions. At any point on the circle, the direction of the electric field vector of the emitted radiation is along the tangent. Often we find that placing the antenna of a transistor radio in a particular orientation improves the reception. This is roughly aligning the antenna with the incoming radiation which was transmitted by a transmitter.

7.3 Sinusoidal Oscillations.

We next consider a situation where a sinusoidal voltage is applied to the dipole oscillator. The dipole is aligned with the y axis (Figure 7.7). The voltage

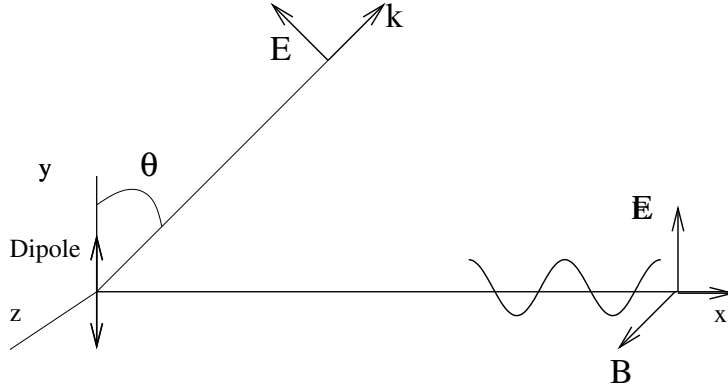


Figure 7.7:

causes the charges to move up and down as

$$y(t) = y_0 \cos(\omega t) \quad (7.6)$$

with acceleration

$$a(t) = -\omega^2 y_0 \cos(\omega t). \quad (7.7)$$

producing an electric field

$$E(t) = \frac{q y_0 \omega^2}{4\pi\epsilon_0 c^2 r} \cos[\omega(t - r/c)] \sin \theta. \quad (7.8)$$

It is often useful and interesting to represent the oscillating charge in terms of other equivalent quantities namely the dipole moment and the current in the circuit. Let us replace the charge q which moves up and down as $y(t)$ by two charges, one charge $q/2$ which moves as $y(t)$ and another charge $-q/2$ which moves in exactly the opposite direction as $-y(t)$. The electric field produced by the new configuration is exactly the same as that produced by the single charge considered earlier. This allows us to interpret eq. (7.8) in terms of an oscillating dipole

$$d_y(t) = q y(t) = d_0 \cos(\omega t) \quad (7.9)$$

which allows us to write eq. (7.8) as

$$E(t) = \frac{-1}{4\pi\epsilon_0 c^2 r} \ddot{d}_y(t - r/c) \sin \theta. \quad (7.10)$$

Returning once more to the dipole oscillator shown in Figure 7.5, we note that the excess electrons which rush from A to B when B has a positive voltage reside at the tip of the wire B. Further, there is an equal excess of positive charge in A which resides at the tip of A. The fact that excess charge resides at the tips of the wire is property of charges on conductors which should be familiar from the study of electrostatics. Now the dipole moment is $d_y(t) = l q(t)$ where l is the length of the dipole oscillator and $q(t)$ is the excess charge

accumulated at one of the tips. This allows us to write $\ddot{d}_y(t)$ in terms of the current in the wires $I(t) = \dot{q}(t)$ as

$$\ddot{d}_y(t) = l\dot{I}(t). \quad (7.11)$$

We can then express the electric field produced by the dipole oscillator in terms of the current. This is particularly useful when considering technological applications of the electric dipole oscillator. For a current

$$I(t) = -I \sin(\omega t) \quad (7.12)$$

the electric field is given by

$$E(t) = \frac{\omega l I}{4\pi\epsilon_0 c^2 r} \cos[\omega(t - r/c)] \sin \theta. \quad (7.13)$$

where I refers to the peak current in the wire.

We now end the small detour where we discussed how the electric field is related to the dipole moment and the current, and return to our discussion of the electric field predicted by eq. (7.8). We shall restrict our attention to points along the x axis. The electric field of the radiation is in the y direction and has a value

$$E_y(x, t) = \frac{qy_0\omega^2}{4\pi\epsilon_0 c^2 x} \cos\left[\omega t - \left(\frac{\omega}{c}\right)x\right] \quad (7.14)$$

Let us consider a situation where we are interested in the x dependence of the electric field at a great distance from the emitter. Say we are 1 km away from the oscillator and we would like to know how the electric field varies at two points which are 1 m apart. This situation is shown schematically in Figure 7.7. The point to note is that a small variation in x will make a very small difference to the $1/x$ dependence of the electric field which we can neglect, but the change in the cos term cannot be neglected. This is because x is multiplied by a factor ω/c which could be large and a change in $\omega x/c$ would mean a different phase of the oscillation. Thus at large distances the electric field of the radiation can be well described by

$$E_y(x, t) = E \cos[\omega t - kx] \quad (7.15)$$

where the wave number is $k = \omega/c$. This is the familiar sinusoidal plane wave which we have studied in the previous chapter and which can be represented in the complex notation as

$$\tilde{E}_y(x, t) = \tilde{E} e^{i[\omega t - kx]} \quad (7.16)$$

We next calculate the magnetic field \vec{B} . Referring to Figure 7.7 we see that we have $\hat{e}_r = -\hat{i}$. Using this in eq. (7.3) with $\vec{E} = E_y(x, t)\hat{j}$ we have

$$\vec{B}(x, t) = \hat{i} \times \vec{E}_y(x, t)/c = \frac{E}{c} \cos(\omega t - kx) \hat{k} \quad (7.17)$$

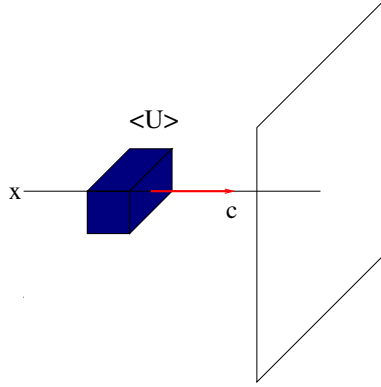


Figure 7.8:

The magnetic field is perpendicular to $\vec{E}(x, t)$ and its amplitude is a factor $1/c$ smaller than the electric field. The magnetic field oscillates with the same frequency and phase as the electric field.

Although our previous discussion was restricted to points along the x axis, the facts which we have learnt about the electric and magnetic fields hold at any position (Figure 7.7). At any point the direction of the electromagnetic wave is radially outwards with wave vector $\hat{k} = k\hat{r}$. The electric and magnetic fields are mutually perpendicular, they are also perpendicular to the wave vector \hat{k} .

7.4 Energy density, flux and power.

We now turn our attention to the energy carried by the electromagnetic wave. For simplicity we shall initially restrict ourselves to points located along the x axis for the situation shown in Figure 7.7.

The energy density in the electric and magnetic fields is given by

$$U = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \quad (7.18)$$

For an electromagnetic wave the electric and magnetic fields are related. The energy density can then be written in terms of only the electric field as

$$U = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2c^2\mu_0} E^2 \quad (7.19)$$

The speed of light c is related to ϵ_0 and μ_0 as $c^2 = 1/\epsilon_0\mu_0$. Using this we find that the energy density has the form

$$U = \epsilon_0 E^2 \quad (7.20)$$

The instantaneous energy density of the electromagnetic wave oscillates with time. The time average of the energy density is often a more useful quantity.

We have already discussed how to calculate the time average of an oscillating quantity. This is particularly simple in the complex notation where the electric field

$$\tilde{E}_y(x, t) = \tilde{E}e^{i(\omega t - kx)} \quad (7.21)$$

has a mean squared value $\langle E^2 \rangle = \tilde{E}\tilde{E}^*/2$. Using this we find that the average energy density is

$$\langle U \rangle = \frac{1}{2}\epsilon_0\tilde{E}\tilde{E}^* = \frac{1}{2}\epsilon_0 E^2 \quad (7.22)$$

where E is the amplitude of the electric field.

We next consider the energy flux of the electromagnetic radiation. The radiation propagates along the x axis at the point where we want to calculate the flux. Consider a surface which is perpendicular to direction in which the wave is propagating as shown in Figure 7.8. The energy flux S refers to the energy which crosses an unit area of this surface in unit time. It has units Watt m^{-2} . The flux S is the power that would be received by collecting the radiation in an area 1 m^2 placed perpendicular to the direction in which the wave is propagating as shown in Figure 7.8.

The average flux can be calculated by noting that the wave propagates along the x axis with speed c . The average energy $\langle U \rangle$ contained in an unit volume would take a time $1/c$ to cross the surface. The flux is the energy which would cross in one second which is

$$\langle S \rangle = \langle U \rangle c = \frac{1}{2}c\epsilon_0 E^2 \quad (7.23)$$

The energy flux is actually a vector quantity representing both the direction and the rate at which the wave carries energy. Referring back to equation (7.8) and (7.15) we see that at any point the average flux $\langle \vec{S} \rangle$ is pointed radially outwards and has a value

$$\langle \vec{S} \rangle = \left(\frac{q^2 y_0^2 \omega^4}{32\pi^2 c^3 \epsilon_0} \right) \frac{\sin^2 \theta}{r^2} \hat{r}. \quad (7.24)$$

Note that the flux falls as $1/r^2$ as we move away from the source. This is a property which may already be familiar to some of us from considerations of the conservation of energy. Note that the total energy crossing a surface enclosing the source will be constant irrespective of the shape and size of the surface.

Let us now shift our point of reference to the location of the dipole and ask how much power is radiated in any given direction. This is quantified using the power emitted per solid angle. Consider a solid angle $d\Omega$ along a direction \hat{r} at an angle θ to the dipole as shown in Figure 7.9. The power dP radiated into this solid angle can be calculated by multiplying the flux with the area corresponding to this solid angle

$$dP = \langle \vec{S} \rangle \cdot \hat{r} r^2 d\Omega \quad (7.25)$$

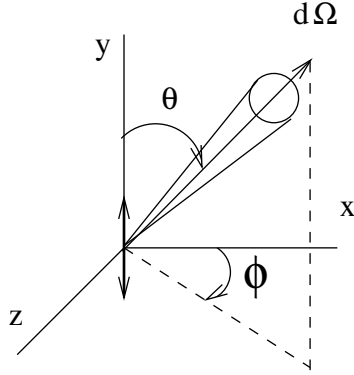


Figure 7.9:

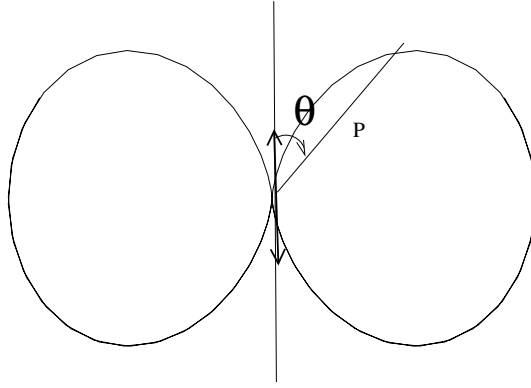


Figure 7.10:

which gives us the power radiated per unit solid angle to be

$$\frac{d}{d\Omega}\langle\vec{P}\rangle(\theta) = \left(\frac{q^2 y_0^2 \omega^4}{32\pi^2 c^3 \epsilon_0}\right) \sin^2 \theta. \quad (7.26)$$

This tells us the radiation pattern of the dipole radiation, *ie.* the directional dependence of the radiation is proportional to $\sin^2 \theta$. The radiation is maximum in the direction perpendicular to the dipole while there is no radiation emitted along the direction of the dipole. The radiation pattern is shown in Figure 7.10. Another important point to note is that the radiation depends on ω^4 which tells us that the same dipole will radiate significantly more power if it is made to oscillate at a higher frequency, doubling the frequency will increase the power sixteen times.

The total power radiated can be calculated by integrating over all solid angles. Using $d\Omega = \sin\theta d\theta d\phi$ and

$$\int_0^{2\pi} d\phi \int_0^\pi \sin^3 \theta d\theta = \frac{8\pi}{3} \quad (7.27)$$

gives the total power P to be

$$\langle\vec{P}\rangle = \frac{q^2 y_0^2 \omega^4}{12\pi c^3 \epsilon_0}. \quad (7.28)$$

It is often convenient to express this in terms of the amplitude of the current in the wires of the oscillator as

$$\langle \vec{P} \rangle = \frac{I^2 l^2 \omega^2}{12\pi c^3 \epsilon_0}. \quad (7.29)$$

The power radiated by the electric dipole is proportional to the square of the current. This behaviour is exactly the same as that of a resistance except that the oscillator emits the power as radiation while the resistance converts it to heat. We can express the radiated power in terms of an equivalent resistance with

$$\langle \vec{P} \rangle = \frac{1}{2} R I^2 \quad (7.30)$$

where

$$R = \left(\frac{l}{\lambda} \right)^2 790 \, \Omega \quad (7.31)$$

l being the length of the dipole (Figure 7.5) and λ the wavelength of the radiation.

Problems

1. An oscillating current of amplitude 2 Amps and $\omega = 3 \text{ GHz}$ is fed into a dipole antenna of length 1 m oriented along the y axis and located at $(0, 0, 0)$. All coordinates are in km and $\epsilon_0 = 8.85 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2}$.
 - a. What is the amplitude of the electric field at the point $(1, 2, 0)$? ($1.2 \times 10^{-1} \text{ Vm}^{-1}$)
 - b. In which direction does the electric oscillate at the point $(1, 2, 0)$? [Give the angles with respect to the x, y and z axis.] ($26.6^\circ, 63.4^\circ, 0^\circ$)
 - c. What is the total power radiated? ($4.00 \times 10^3 \text{ W}$)
 - d. How does the total power change if the frequency is doubled? (4 times)
2. A charge q moves in a circular orbit with period T in the $y - z$ plane with center at the origin $(0, 0, 0)$. The charge is at $(0, 0, a)$ at $t = 0$, and the motion is counter-clockwise as seen from the point P $(r, 0, 0)$ where $r \gg a$. At a later time t
 - a. What is the acceleration of the charge?
 - b. For the point P, what is the retarded acceleration of the charge?
 - c. What is the electric field at P?
 - d. What is the time averaged power radiated by the charge?
3. In which direction does an oscillating electric dipole radiate maximum power? What is the FWHM of the radiation pattern?

4. For the same peak current and frequency, how does the total power change if the length of the dipole is halved.
5. Consider a conducting wire of length 1m and 1 mm diameter. At what frequency does the radiative resistance become comparable to the usual Ohmic resistance. Use $\sigma = 10^{-7}(\Omega m)^{-1}$ which is the typical value of conductivity for conducting metals.
6. An electric dipole oscillator radiates 1KW power. What is the flux 1km away in the direction perpendicular to the dipole and at 30° to the dipole.
7. Particles of charge q and mass m with kinetic energy E_0 are injected perpendicular to a magnetic field B . The charges experiences an acceleration as they go around in circles in the magnetic field. Calculate the rate at which the energy is radiated. Show that the energy of the particles falls as $E(t) = E_0 e^{-t/\tau}$, where τ the decay time is related to q, B and m .

Cyclotrons typically have magnetic fields of 1Tesla or higher. Using this value, calculate the frequency at which the radiation will be emitted for electrons $(q, m) = (1.6^{-19}C, 9.1^{-31} kg)$ and protons $(q, m) = (1.6^{-19}C, 1.67^{-27} kg)$. Calculate τ for protons and electrons, and use this to determine how long it will take for them to radiate away half the initial kinetic energy.

Chapter 8

The vector nature of electromagnetic radiation.

Consider a situation where the same electrical signal is fed to two mutually perpendicular dipoles, one along the y axis and another along the z axis as shown in Figure 19.1.



Figure 8.1: The resultant electric field at x

We are interested in the electric field at a distant point along the x axis. The electric field is a superposition of two components

$$\vec{E}(x, t) = E_y(x, t)\hat{j} + E_z(x, t)\hat{k} \quad (8.1)$$

one along the y axis produced by the dipole which is aligned along the y axis, and another along the z axis produced by the dipole oriented along the z axis.

8.1 Linear polarization

In this situation where both dipoles receive the same signal, the two components are equal $E_y = E_z$ and

$$\vec{E}(x, t) = E(\hat{j} + \hat{k}) \cos(\omega t - kx) \quad (8.2)$$

If we plot the time evolution of the electric field at a fixed position (Figure 19.2) we see that it oscillates up and down along a direction which is at 45° to the y and z axis.

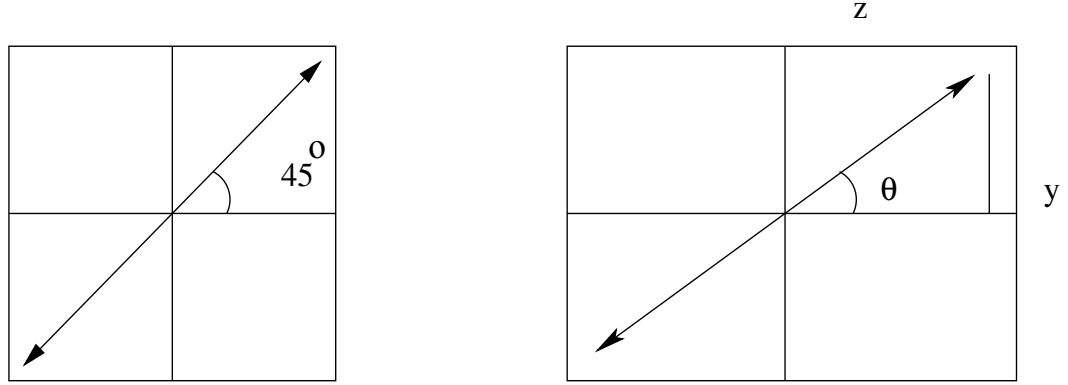


Figure 8.2: Linear Polarization

The point to note is that it is possible to change the relative amplitudes of E_y and E_z by changing the currents in the oscillators. The resultant electric field is

$$\vec{E}(x, t) = (E_y \hat{k} + E_z \hat{j}) \cos(\omega t + kx) \quad (8.3)$$

The resultant electric field vector has magnitude $E = \sqrt{E_y^2 + E_z^2}$ and it oscillates along a direction at an angle $\theta = \tan^{-1} \left(\frac{E_z}{E_y} \right)$ with respect to the y axis (Figure 19.2).

Under no circumstance does the electric field have a component along the direction of the wave i.e along the x axis. The electric field can be oriented along any direction in the $y - z$ plane. In the cases which we have considered until now, the electric field oscillates up and down a fixed direction in the $y - z$ plane. Such an electromagnetic wave is said to be linearly polarized.

8.2 Circular polarization

The polarization of the wave refers to the time evolution of the electric field vector. An interesting situation occurs if the same signal is fed to the two dipoles, but the signal to the z axis is given an extra $\pi/2$ phase. The electric field now is

$$\vec{E}(x, t) = E \left[\cos(\omega t - kx) \hat{j} + \cos(\omega t - kx + \pi/2) \hat{k} \right] \quad (8.4)$$

$$= E \left[\cos(\omega t - kx) \hat{j} - \sin(\omega t - kx) \hat{k} \right] \quad (8.5)$$

If we now follow the evolution of $\vec{E}(t)$ at a fixed point, we see that the tip of the vector $\vec{E}(t)$ moves on a circle of radius E clockwise (when the observer looks towards the source) as shown in Figure 19.3

We call such a wave right circularly polarized. The electric field would have rotated in the opposite direction had we applied a phase lag of $\pi/2$. We would then have obtained a left circularly polarized wave.

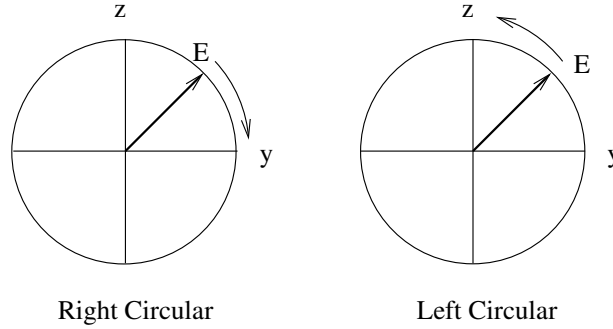


Figure 8.3: Circular polarization

8.3 Elliptical polarization

Oscillations of different amplitude combined with a phase difference of $\pi/2$ produces elliptically polarized wave where the ellipse is aligned with the $y - z$ axis as shown in Figure 19.4. The ellipse is not aligned with the $y - z$ axis for an arbitrary phase difference between the y and z components of the electric field. This is the most general state of polarization shown in the last diagram of the Figure 19.4. Linear and circularly polarized waves are specific cases of elliptically polarized waves.

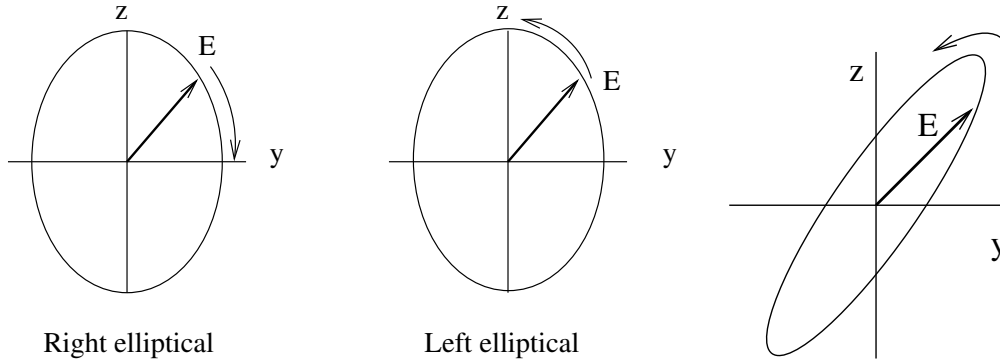


Figure 8.4: Elliptical polarization

Problems

1. Find the plane of polarization of a light which is moving in the positive x direction and having amplitudes of electric field in the y and z directions, 3 and $\sqrt{3}$ respectively in same units. The oscillating components of the electric field along y and z have the same frequency and wavelength and the z component is leading with a phase π .
2. Find the state of polarization of a light which is moving in the positive x direction with electric field amplitudes same along the y and z directions.

The oscillating components of the electric field along the y and z have the same frequency and wavelength and the z component is lagging with a phase $\pi/3$.

(Ans: Left elliptically polarized and the major axis is making an angle $\pi/4$ with the y axis.)

3. Find the state of polarization of a light which is moving in the positive x direction and having amplitudes of electric field in the y and z directions, 1 and $3^{1/4}$ respectively in same units. The oscillating components of the electric field along y and z have the same frequency and wavelength and the y component is leading with a phase $\pi/4$.

(Ans: Left elliptically polarized and the major axis is making an angle $\tan^{-1} \sqrt{1 + (2/\sqrt{3})}$ with the y axis.)

4. Find out the maximum and minimum values of electric field at point x for the previous problem. (Ans: $E_{max}^2 = (3 + \sqrt{3})/2$ and $E_{min}^2 = (\sqrt{3} - 1)/2$.)

Chapter 9

The Spectrum of Electromagnetic Radiation.

Electromagnetic waves come in a wide range of frequency or equivalently wavelength. We refer to different bands of the frequency spectrum using different names. These bands are often overlapping. The nomenclature of the bands and the frequency range are typically based on the properties of the waves and the techniques to generate or detect them.

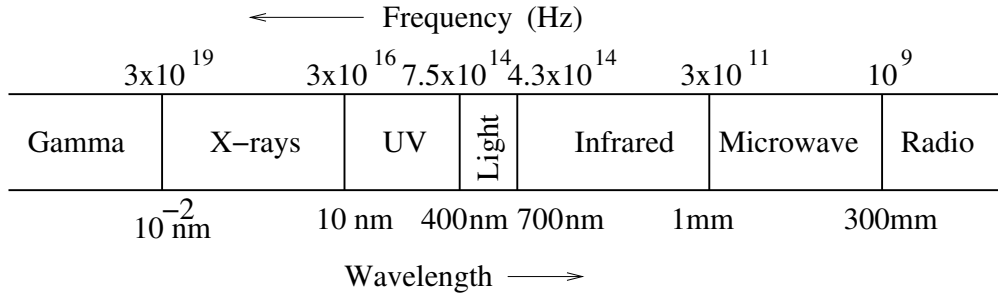


Figure 9.1: Electromagnetic spectrum

9.1 Radiowave and Microwave

Radiowaves span frequencies less than 1GHz . The higher frequency part of this band is used for radio and television transmission. There is no theoretical lower limit, and the familiar 50Hz power supply lines emits radio waves at this frequency which corresponds to a wavelength.

$$\lambda = \frac{c}{\nu} = \frac{3 \times 10^8 \text{ m s}^{-1}}{50 \text{ s}^{-1}} = 6 \times 10^6 \text{ m} \quad (9.1)$$

The upper frequency part of this band, VHF(30 MHz–300 MHz) and UHF(300 MHz–3 GHz), is used for FM and television transmission whereas

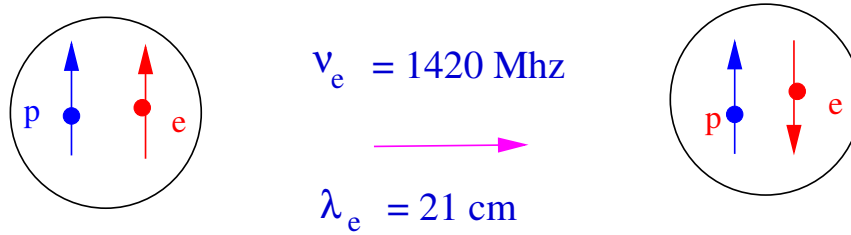


Figure 9.2: Hyperfine splitting in neutral hydrogen

the medium waves (0.5 MHz – 1.5 MHz) and short waves (3 MHz–30 MHz) are used for radio transmission.

Microwave refers to electromagnetic waves in the frequency range 1GHz to 300GHz or wavelengths 30cm to 1mm. The communication band extends into the microwave.

The communication band extends into the microwave. Global system for mobile (GSM) operates in 900/1800/1900/MHz bands.

The earth's atmosphere is largely transparent to electromagnetic waves from 1cm to 30cm. Consequently, radiowaves and microwaves are both very useful for space communication and astronomy. This branch of astronomy is called radio astronomy.

In these two bands there are a large variety of electrical circuits and antennas that are used to produce waves for communication. We briefly discuss below a few of the astronomical sources of radio and microwave radiation.

9.1.1 21cm radiation.

Hydrogen is the most abundant element in the universe. Atomic neutral hydrogen, has two states, one where the proton and electron spins are aligned and another where they are opposite as shown in Figure 9.2. The separation between these two energy states is known as hyperfine splitting in hydrogen. A transition between these two states causes radiation at 1.42GHz or 21cm to be emitted. This is a very important source of information about our Galaxy and external galaxies, neutral hydrogen being found in many galaxies including our own. The Giant Meterwave Radio Telescope (GMRT, Figure 9.3) located in Narayangaon near Pune is currently the world's largest low frequency radio telescope operating in several frequency bands from 1.42 GHz to 50 MHz. Figure 9.3 also shows the image of a dwarf galaxy DDO210 made with the GMRT using the 21cm radiation from neutral hydrogen. The contours show how the neutral hydrogen is distributed while the distribution of stars is shown in black. It is clear that the hydrogen gas which is referred to as the interstellar medium is spread out over a much larger region compared to the stars.

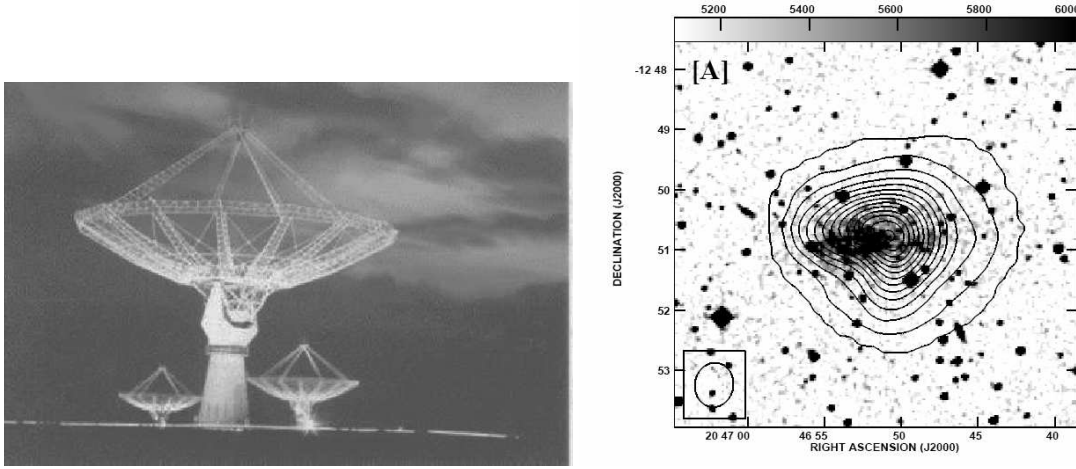


Figure 9.3: GMRT and radio image of a dwarf galaxy

9.1.2 Cosmic Microwave Background Radiation.

An object which is equally efficient in absorbing and emitting radiation of all frequencies is referred to as a black body. Consider a cavity enclosed inside a black body at a temperature T . The electromagnetic waves inside this cavity will be repeatedly absorbed and re-emitted by the walls of this cavity until the radiation is in thermal equilibrium with the black body. It is found that the radiation spectrum inside this black body cavity is completely specified by T the temperature of the black body. This radiation is referred to as **black body radiation**. Writing the energy density du_ν of the black body radiation in a frequency interval $d\nu$ as

$$du_\nu = u_\nu d\nu$$

the spectral energy density u_ν is found to be given by

$$u_\nu = \frac{8\pi h\nu^3}{c^3} \frac{1}{\left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]} \quad (9.2)$$

where $h = 6.63 \times 10^{-34}$ Joule-sec is the Planck constant and $k = R/N = 8.314/(6.022 \times 10^{23}) = 1.38 \times 10^{-23}$ Joules/Kelvin is the Boltzmann Constant. The spectral energy density can equivalently be defined in terms of the wavelength interval $d\lambda$ as $du_\lambda = u_\lambda d\lambda$. Figure 9.4 (a) shows the energy density of black body radiation for different values of the temperature T . The curves for different temperatures are unique and the curves corresponding to different values of T do not intersect. The wavelength λ_m at which the energy density peaks decreases with T , and the relation is given by the Wien's displacement law

$$\lambda_m T = 2.898 \times 10^{-3} \text{ m K} \quad (9.3)$$

Radio observations carried out by pointing a radio receiver in different directions on the sky show that there is a radiation with a black body spectrum

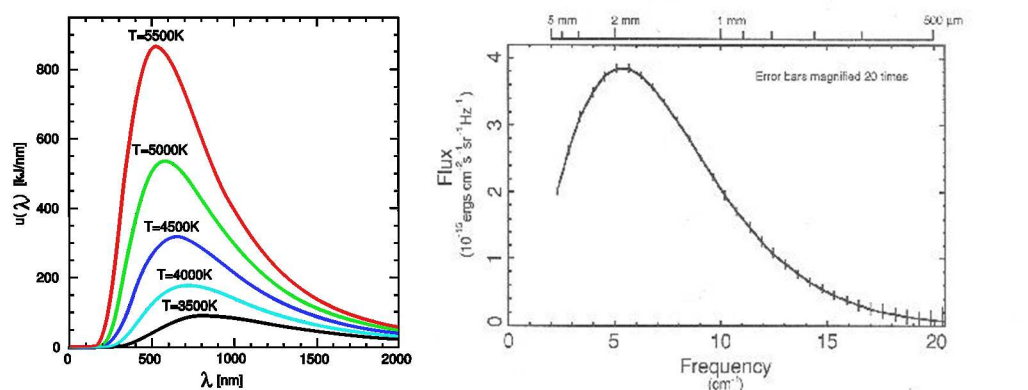
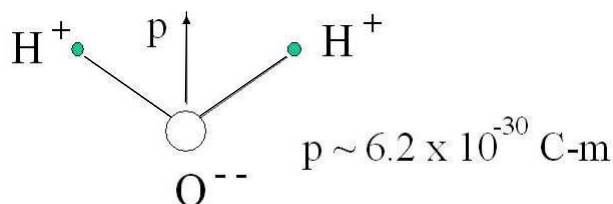


Figure 9.4: (a) Black body spectrum and (b) CMBR spectrum

(Figure 9.4 (b)) at $T = 2.735 \pm 0.06\text{K}$ arriving from all directions in the sky. This radiation is not terrestrial in origin. It is believed that we are actually seeing a radiation which pervades the whole universe and is a relic of a hot past referred to as the hot Big Bang. This black body radiation is referred to as the Cosmic Microwave Background Radiation (CMBR) which peaks in the microwave region of the Spectrum.

9.1.3 Molecular lines.



Polar molecules like water are influenced by any incident electromagnetic radiation, and such molecules can be set into vibrations or rotation by the external, oscillating electric field. The water molecules try to align their dipole movement with the external electric field which is oscillating. This sets the molecule into rotation. The rotational energy levels are quantized, and the molecules have distinct frequencies at which there are resonances where the molecule emits or absorbs maximum energy. The Microwave ovens utilize a 2.45GHz rotational transition of water. The water molecules absorb the incident electromagnetic radiation and start rotating. The rotational energy of the molecules is converted to random motions or heat.

Most of the rotational and vibrational transitions of molecules lie in the microwave and Infrared (IR) bands. The frequency range 50 GHz to 10 THz is often called Tera hertz radiation or T-rays. The water vapour in the atmosphere is opaque at much of these frequencies. Dry substances like paper, plastic which do not have water molecules are transparent to T-rays, whereas

it is absorbed by substances with water and is reflected by metals which makes it suitable for various imaging applications.

9.2 Infrared

The frequency range from 3×10^{11} Hz to $\sim 4 \times 10^{14}$ Hz is referred to as the infrared band (IR). The frequency is just below that of red light. The IR band is again subdivided as follows

near IR	780 - 3000nm
intermediate IR	3000 - 6000 nm
far IR IR	6000 - 15000 nm
extreme IR	15,000 nm - 1mm

The division are quite loose and it varies.

There are many molecular vibrational and rotational transitions which produce radiation in the IR. Vibrational transitions of CO_2 and H_2O fall in the energy 0.2–0.8 eV range. The energy of a photon is calculated by multiplying its frequency with Planck constant h and converting the Joules in electron-volts(eV)($1\text{eV} = 1.6 \times 10^{-19}$ Joules). Usually small wavelength (or the large frequency) bands of the electromagnetic radiation are distinguished by their energies. Further, the black body radiation from hot objects emit copious amounts of IR. as the spectrum typically peaks in this band .

For example the black body radiation from human beings peaks at around 10,000nm making the IR suitable for “night vision”. Some snakes sense their preys using infrared vision in night. Stars like the sun and incandescent lamps emit copiously in the IR. In fact an incandescent lamp radiates away more than 50% of its energy in the IR.

IR spy satellites are used to monitor events of sudden heat generation indicating a rocket launch or possibly nuclear explosion.

Much of fiber optical communication also works in the IR.

9.3 Visible light

This corresponds to electromagnetic radiation in the wavelength range 3.84×10^{14} Hz to 7.69×10^{14} Hz The atmosphere is transparent in this band, which is possibly why we have developed vision in this frequency range.

Visible radiation is produced when electrons in the outer shell of atoms jump from a higher energy level to a lower energy level. These are used in sources of visible radiation as well as detectors. Black body radiation from objects at a temperature of a few thousand Kelvin is another source of visible light.

Colour is our perception of the relative contribution from different frequencies. Different combinations of the frequencies can produce the same colour.

9.4 Ultraviolet (UV)

These are electromagnetic waves with frequency just higher than blue light, and this band spans from 8×10^{14} Hz to 3.4×10^{16} Hz. Typical energy range corresponds to 3–100 eV.

One of the main sources of UV radiation is Sun. UV radiation has sufficient energy to ionize the atoms in the upper atmosphere, producing the ionosphere. Exposure to UV radiation is harmful to living organisms, and it is used to kill bacteria in many water purification systems. Wavelengths less than 300 nm are well suited for this killing. Fortunately Ozone (O_3) in the atmosphere absorbs UV radiation protecting life on the earth's surface.

The human eyes cannot see in UV as it is absorbed by the lens. People whose lens have been removed due to cataract can see in UV to some extent. Many insects can see in part of the UV band.

UV is produced through more energetic electronic transitions in atoms. For example the transition of the hydrogen atom from the first excited state to the ground state (Lyman- α) produces UV.

UV is used in lithography for making ICS, erasing EPROMS, etc.

9.5 X-rays

Electromagnetic waves with frequency in the range 2.4×10^{16} Hz to 5×10^{19} Hz is referred to as X-ray. The energy range of the X-rays are between 0.1 and 200 KeV. These are produced by very fast moving electrons when they encounter positively charged nuclei of atoms and are accelerated as a consequence. This occurs when electrons are bombarded on a copper plate which is typically how X-rays are produced. Figure 9.5 shows a X-ray tube where electrons are accelerated by a voltage in the range 30 to 150 KeV and then bombarded onto a copper plate.

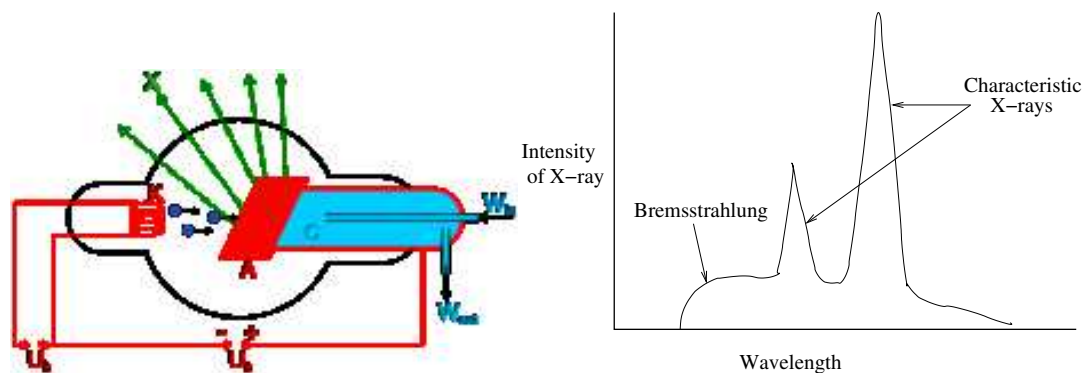


Figure 9.5: (a) An X-ray tube (b) Characteristic X-rays and Bremsstrahlung

Characteristic X-rays are produced through inner shell electron transitions in atoms. Characteristic X-rays are discrete in wavelengths. Different elements

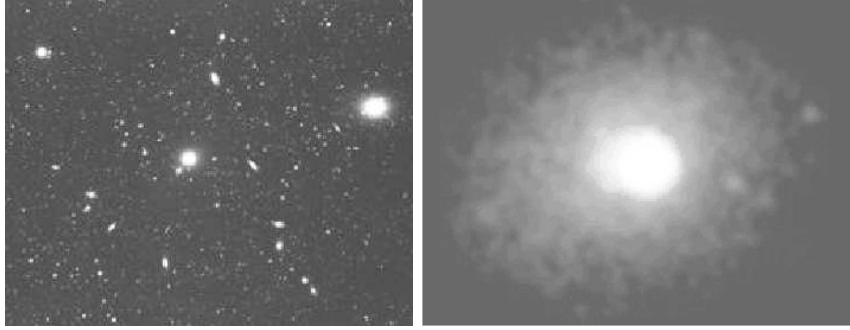


Figure 9.6: Centaurus cluster (a) Optical image (b) X-ray image

present in a substance can be identified using X-ray diffraction which will show the X-ray peaks at particular wavelengths. Accelerated or decelerated electrons produce continuous X-rays (Bremsstrahlung). When a metal target is fired with electrons both Characteristic X-rays of the metal are produced with continuous Bremsstrahlung background.

Since the wavelengths of the X-rays are of atomic dimensions or less they are good probes for finding the structures of substances. Human body is transparent for X-rays but the bones are not. 10-100 KeV X-rays are used for diagnostic purpose to locate fractures etc.

Hot ionized gas (plasma) found in many astrophysical situations like around our sun or around black holes also produces X-ray. Figure 9.6 shows the Centaurus cluster, a cluster of galaxies more than hundred in number as seen in the optical image on the left. The X-ray image of the same cluster shown on the right reveals that in addition to the galaxies there is a hot ionized gas at a temperature of few tens of million Kelvin which emits copious amounts of X-ray.

Inner shell electronic transitions in atoms also produce X-ray. X-ray is also produced in particle accelerators like the synchrotron.

9.6 Gamma Rays

The highest frequency ($> 5^{19}$ Hz) electromagnetic radiation is referred to as Gamma Rays. These are produced in nuclear transitions when the nucleus goes from an excited state to a lower energy state. These are produced in copious amounts in nuclear reactors and in nuclear explosions. Electron positron annihilation also produces Gamma rays. Typical energy range for Gamma radiation is in MeVs (million electron-volts). They penetrate through almost any material. One needs thick lead walls to stop Gamma rays. Gamma rays will ionize most gases, and is not very difficult to detect this radiation through this. Exposure to gamma rays can mutate and even kill living cells, and is very harmful for humans. Controlled exposure is used to kill cancerous cells

and this is used for cancer treatment.

There are mysterious astrophysical sources which emit a very intense burst of Gamma rays. The exact nature of these sources is still a subject of intense study, and one of the theories is that these are hypernovae in distant galaxies.

Problems

1. Show that in the limit $h\nu/(kT) \ll 1$ the spectral energy density u_ν of blackbody radiation is proportional to T *ie.* $U_\nu \propto T$. What is the constant of proportionality?
2. Use the Wien's displacement law to estimate the wavelength and corresponding frequency at which the radiation from the earth peaks. Assume it to be a blackbody at $T = 300$ K. Repeat the same for the sun assuming $T = 6,000$ K. For what value of T does the peak shift to UV?
3. What is the wavelength range corresponding to X-ray?
4. Evaluate approximately the energy of a Sodium yellow photon ($\lambda=589.3$ nm).

Chapter 10

Interference.

Consider a situation where we superpose two waves. Naively, we would expect the intensity (energy density or flux) of the resultant to be the sum of the individual intensities. For example, a room becomes twice as bright if we switch on two lamps instead of one. This actually does not always hold. A wave, unlike the intensity, can have a negative value. If we add two waves whose values have opposite signs at the same point, the total intensity is less than the intensities of the individual waves. This is an example of a phenomena referred to as interference.

10.1 Young's Double Slit Experiment.

We begin our discussion of interference with a situation shown in Figure 10.1. Light from a distant point source is incident on a screen with two thin slits. The separation between the two slits is d . We are interested in the image of the two slits on a screen which is at a large distance from the slits. Note that the point source is aligned with the center of the slits as shown in Figure 10.2. Let us calculate the intensity at a point P located at an angle θ on the screen.

The radiation from the point source is well described by a plane wave by the time the radiation reaches the slits. The two slits lie on the same wavefront of this plane wave, thus the electric field oscillates with the same phase at both

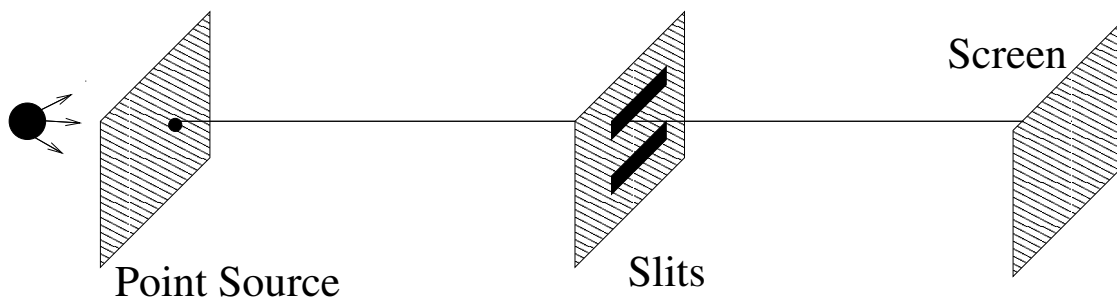


Figure 10.1: Young's double slit experiment I

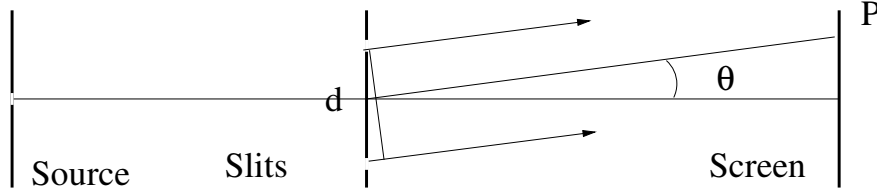


Figure 10.2: Young's double slit experiment II

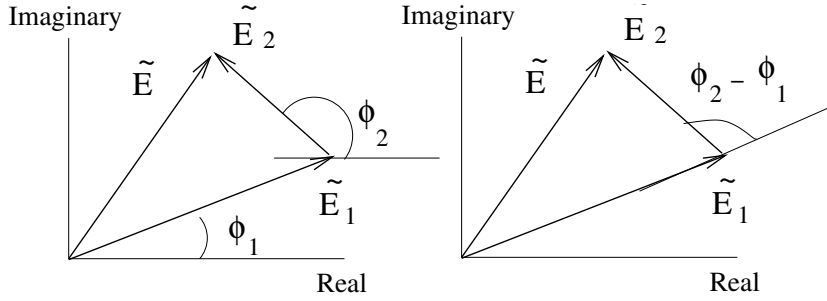


Figure 10.3: Summation of two phasors

the slits. If $\tilde{E}_1(t)$ and $\tilde{E}_2(t)$ be the contributions from slits 1 and 2 to the radiation at the point P on the screen, the total electric field will be

$$\tilde{E}(t) = \tilde{E}_1(t) + \tilde{E}_2(t) \quad (10.1)$$

Both waves originate from the same source and they have the same frequency. We can thus express the electric fields as $\tilde{E}_1(t) = \tilde{E}_1 e^{i\omega t}$, $\tilde{E}_2(t) = \tilde{E}_2 e^{i\omega t}$ and $\tilde{E}(t) = \tilde{E} e^{i\omega t}$. We then have a relations between the amplitudes

$$\tilde{E} = \tilde{E}_1 + \tilde{E}_2$$

. It is often convenient to represent this addition of complex amplitudes graphically as shown in Figure 10.3. Each complex amplitude can be represented by a vector in the complex plane, such a vector is called a phasor. The sum is now a vector sum of the phasors.

The intensity of the wave is

$$I = \langle E(t)E(t) \rangle = \frac{1}{2} \tilde{E} \tilde{E}^* \quad (10.2)$$

where we have dropped the constant of proportionality in this relation. It is clear that the square of the length of the resultant phasor gives the intensity. Geometrically, the resultant intensity I is the square of the vector sum of two vectors of length $\sqrt{I_1}$ and $\sqrt{I_2}$ with angle $\phi_2 - \phi_1$ between them as shown in Figure 10.3. Consequently, the resulting intensity is

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos(\phi_2 - \phi_1). \quad (10.3)$$

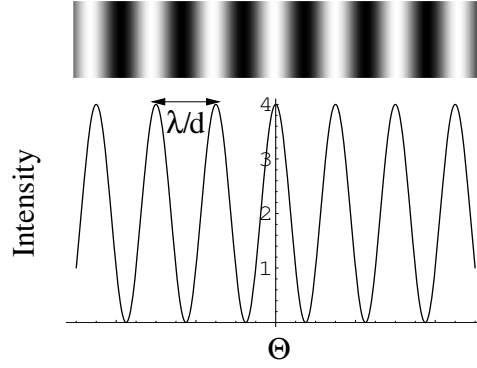


Figure 10.4: Young double slit interference fringes with intensity profile

Calculating the intensity algebraically, we see that it is

$$I = \frac{1}{2}[\tilde{E}_1 \tilde{E}_1^* + \tilde{E}_2 \tilde{E}_2^* + \tilde{E}_1 \tilde{E}_2^* + \tilde{E}_1^* \tilde{E}_2] \quad (10.4)$$

$$= I_1 + I_2 + \frac{1}{2}E_1 E_2 [e^{i(\phi_1 - \phi_2)} + e^{i(\phi_2 - \phi_1)}] \quad (10.5)$$

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos(\phi_2 - \phi_1) \quad (10.6)$$

The intensity is maximum when the two waves have the same phase

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \quad (10.7)$$

and it is minimum when $\phi_2 - \phi_1 = \pi$ i.e the two waves are exactly out of phase

$$I = I_1 + I_2 - 2\sqrt{I_1 I_2}. \quad (10.8)$$

The intensity is the sum of the two intensities when the two waves are $\pi/2$ out of phase.

In the Young's double slit experiment the waves from the two slits arrive at P with a time delay because the two waves have to traverse different paths. The resulting phase difference is

$$\phi_1 - \phi_2 = 2\pi \frac{d \sin \theta}{\lambda}. \quad (10.9)$$

If the two slits are of the same size and are equidistant from the the original source, then $I_1 = I_2$ and the resultant intensity,

$$I(\theta) = 2I_1 \left[1 + \cos \left(\frac{2\pi d \sin \theta}{\lambda} \right) \right] \quad (10.10)$$

$$= 4I_1 \cos^2 \left(\frac{\pi d \sin \theta}{\lambda} \right) \quad (10.11)$$

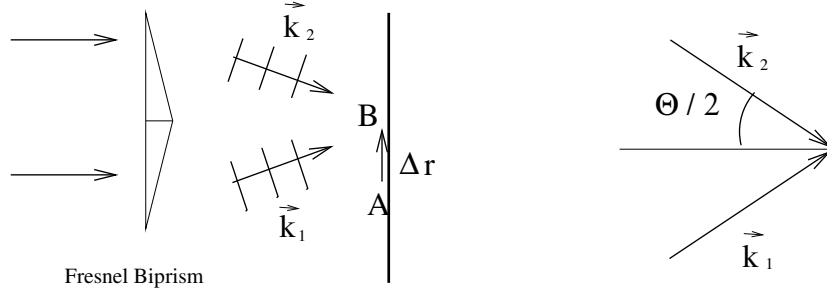


Figure 10.5: Fresnel biprism to realise the Young's double slit

For small θ we have

$$I(\theta) = 2I_1 \left[1 + \cos \left(\frac{2\pi d\theta}{\lambda} \right) \right] \quad (10.12)$$

There will be a pattern of bright and dark lines, referred to as fringes, that will be seen on the screen as in Figure 10.4. The fringes are straight lines parallel to the slits, and the spacing between two successive bright fringes is λ/d radians.

10.1.1 A different method of analysis.

A Fresnel biprism is constructed by joining two identical thin prisms as shown in Figure 10.5. Consider a plane wave from a distant point source incident on the Fresnel biprism. The part of the wave that passes through the upper half of the biprism propagates in a slightly different direction from the part that passes through the lower half of the biprism. The light emanating from the biprism is equivalent to that from two exactly identical sources, the sources being located far away and there being a small separation between the sources. The Fresnel biprism provides a method for implementing the Young's double slit experiment.

The two waves emanating from the biprism will be coplanar and in different directions with wave vectors \vec{k}_1 and \vec{k}_2 as shown in Figure 10.5. We are interested in the intensity distribution on the screen shown in the figure. Let A be a point where both waves arrive at the same phase, i.e. $\phi(A)$ i.e. $\tilde{E}_1 = \tilde{E}_2 = Ee^{i\phi(A)}$. The intensity at this point will be a maximum. Next consider a point B at a displacement $\Delta\vec{r}$ from the point A. The phase of the two waves are different at this point. The phase of the first wave at the point B is given by

$$\phi_1(B) = \phi(A) - \vec{k}_1 \cdot \Delta\vec{r} \quad (10.13)$$

and for the second wave

$$\phi_2(B) = \phi(A) - \vec{k}_2 \cdot \Delta\vec{r} \quad (10.14)$$

The phase difference is

$$\phi_2 - \phi_1 = (\vec{k}_1 - \vec{k}_2) \cdot \Delta\vec{r} \quad (10.15)$$

Using eq. (10.6), the intensity pattern on the screen is given by

$$I(\Delta\vec{r}) = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos[(\vec{k}_1 - \vec{k}_2) \cdot \Delta\vec{r}] \quad (10.16)$$

where I_1 and I_2 are the intensities of the waves from the upper and lower half of the biprism respectively. Assuming that the wave vectors make a small angle $\theta/2 \ll 1$ to the horizontal we have

$$\vec{k}_1 = k[\hat{i} + \frac{\theta}{2}\hat{j}] \quad \text{and} \quad \vec{k}_2 = k[\hat{i} - \frac{\theta}{2}\hat{j}] \quad (10.17)$$

where θ is the angle between the two waves. Using this and assuming that $I_1 = I_2$ we have

$$I(\Delta\vec{r}) = 2I_1 \left[1 + \cos\left(\frac{2\pi\theta \Delta y}{\lambda}\right) \right]. \quad (10.18)$$

There will be straight line fringes on the screen, these fringes are perpendicular to the y axis and have a fringe spacing $\Delta y = \lambda/\theta$.

The analysis presented here is another way of analysing the Young's double slit experiment. It is left to the reader to verify that eq. (10.12) and eq. (10.18) are equivalent.

Like Fresnel biprism one can also realise double slit experiment with 'Fresnel mirrors'. Here one uses two plane mirrors and one of the mirrors is tilted slightly ($\theta < 1^\circ$) and glued with the other as shown in Figure 10.6.

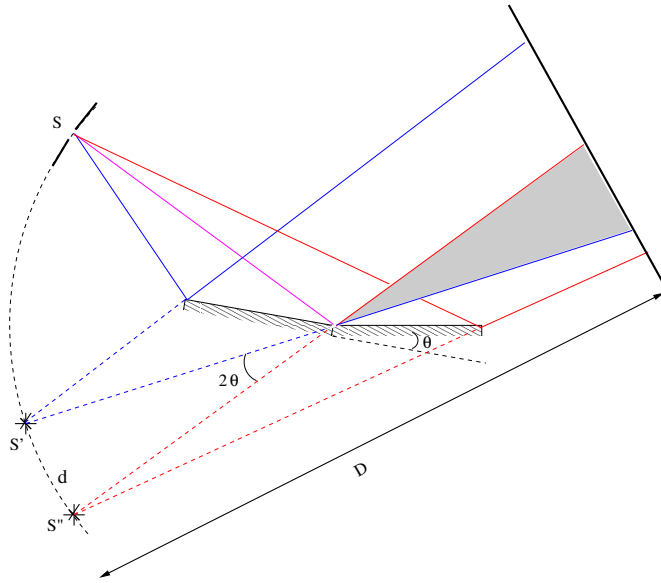


Figure 10.6: Fresnel mirrors

10.2 Michelson Interferometer

Figure 10.7 shows a typical Michelson interferometer setup. A ground glass plate G is illuminated by a light source S . The ground glass plate has the property that it scatters the incident light into all directions. Each point on the ground glass plate acts like a source that emits light in all directions.

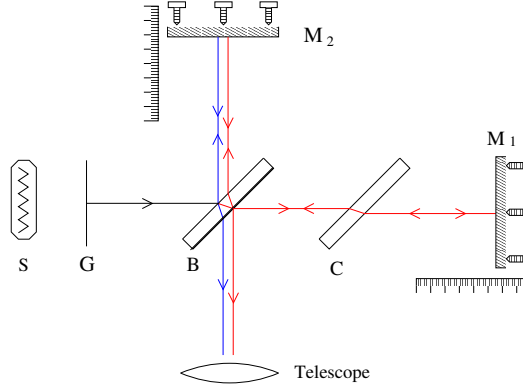


Figure 10.7: Michelson Interferometer

The light scattered forward by G is incident on a beam splitter B which is at 45° . The beam splitter is essentially a glass slab with the lower surface semi-silvered to increase its reflectivity. It splits the incident wave into two parts \tilde{E}_1 and \tilde{E}_2 , one which is transmitted (\tilde{E}_1) and another (\tilde{E}_2) which is reflected. The two beams have nearly the same intensity. The transmitted wave \tilde{E}_1 is reflected back to B by a mirror M_1 , and a part of it is reflected into the telescope T . The reflected wave \tilde{E}_2 travels in a perpendicular direction. The mirror M_2 reflects this back to B where a part of it is transmitted into T . An observer at T would see two images of G , namely G_1 and G_2 (shown in Figure 10.8) produced by the two mirrors M_1 and M_2 respectively. The two images are at a separation $2d$ where d is the difference in the optical paths from B to G_1 and from B to G_2 . Note that \tilde{E}_2 traverses the thickness of the beam splitter thrice whereas \tilde{E}_1 traverses the beam splitter only once. This introduces an extra optical path for \tilde{E}_2 even when M_1 and M_2 are at the same

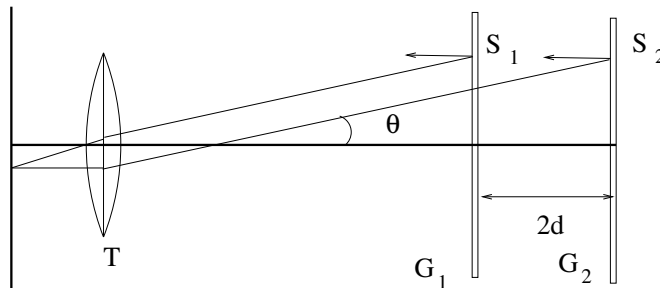


Figure 10.8: Effective set-up for Michelson Interferometer

radiation distance from B. It is possible to compensate for this by introducing an extra displacement in M_1 , but this would not serve to compensate for the extra path over a range of frequencies as the refractive index of the glass in B is frequency dependent. A compensator C, which is a glass block identical to B (without the silver coating), is introduced along the path to M_1 to compensate for this.

S_1 and S_2 are the two images of the same point S on the ground glass plate. Each point on the ground glass plate acts as a source emitting radiation in all directions. Thus S_1 and S_2 are coherent sources which emit radiation in all direction. Consider the wave emitted at an angle θ as shown in Figure 10.8. The telescope focuses both waves to the same point. The resultant electric field is

$$\tilde{E} = \tilde{E}_1 + \tilde{E}_2 \quad (10.19)$$

and the intensity is

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos(\phi_2 - \phi_1) \quad (10.20)$$

The phase difference arises because of the path difference in the two arms of the interferometer. Further, there is an additional phase difference of π because \tilde{E}_2 undergoes internal reflection at B whereas \tilde{E}_1 undergoes external reflection. We then have

$$\phi_2 - \phi_1 = \pi + 2d \cos \theta \frac{2\pi}{\lambda} \quad (10.21)$$

So we have the condition

$$2d \cos \theta_m = m\lambda \quad (m = 0, 1, 2, \dots) \quad (10.22)$$

for a minima or a dark fringe. Here m is the order of the fringe, and θ_m is the angle of the m^{th} order fringe. Similarly, we have

$$2d \cos \theta_m = \left(m + \frac{1}{2}\right) \lambda \quad (m = 0, 1, 2, \dots) \quad (10.23)$$

as the condition for a bright fringe. The fringes will be circular as shown in Figure 10.9. When the central fringe is dark, the order of the fringe is

$$m = \frac{2d}{\lambda}. \quad (10.24)$$

Let us follow a fringe of a fixed order, say m , as we increase d the difference in the length of the two arms. The value of $\cos \theta_m$ has to decrease which implies that θ_m increases. As d is increased, new fringes appear at the center, and the existing fringes move outwards and finally move out of the field of view. For any value of d , the central fringe has the largest value of m , and the value of m decreases outwards from the center.

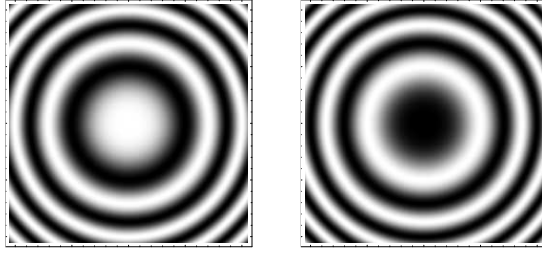


Figure 10.9: Michelson fringes

Considering the situation where there is a central dark fringe as shown in the left of Figure 10.9, let us estimate θ the radius of the first dark fringe. The central dark fringe satisfies the condition

$$2d = m\lambda \quad (10.25)$$

and the first dark fringe satisfies

$$2d \cos \theta = (m - 1)\lambda \quad (10.26)$$

For small θ *ie.* $\theta \ll 1$ we can write eq. (10.26) as

$$2d \left(1 - \frac{\theta^2}{2}\right) = (m - 1)\lambda \quad (10.27)$$

which with eq. (10.25) gives

$$\theta = \sqrt{\frac{\lambda}{d}} \quad (10.28)$$

Compare this with the Young's double slit where the fringe separation is λ/d .

The Michelson interferometer can be used to determine the wavelength of light. Consider a situation where we initially have a dark fringe at the center. This satisfies the condition given by eq. 10.25 where λ , d and m are all unknown. One of the mirrors is next moved so as to increase d the difference in the lengths of the two arms of the interferometer. As the mirror is moved, the central dark fringe expands and moves out while a bright fringe appears at the center. A dark fringe reappears at the center if the mirror is moved further. The mirror is moved a distance Δd so that N new dark fringes appear at the center. Although initially d and m were unknown for the central dark fringe, it is known that finally the difference in lengths is $d + \Delta d$ and the central dark fringe is of order $N + m$ and hence it satisfies

$$2(d + \Delta d) = (m + N)\lambda \quad (10.29)$$

Subtracting eq. 10.25 from this gives the wavelength of light to be

$$\lambda = \frac{2 \Delta d}{N} \quad (10.30)$$



Figure 10.10: Laser Interferometer Gravitational-Wave Observatory

We next consider a situation where there are two very close spectral lines λ_1 and $\lambda_1 + \Delta\lambda$. Each wavelength will produce its own fringe pattern. Concordance refers to the situation where the two fringe patterns coincide at the center

$$2d = m_1\lambda_1 = m_2(\lambda_1 + \Delta\lambda) \quad (10.31)$$

and the fringe pattern is very bright. As d is increased, m_1 and m_2 increase by different amounts with $\Delta m_2 < \Delta m_1$. When $m_2 = m_1 - 1/2$, the bright fringes of λ_1 coincide with the dark fringes of $\lambda_1 + \Delta\lambda$ and vice-versa, and consequently the fringe pattern is washed away. The two set of fringes are now said to be discordant.

It is possible to measure $\Delta\lambda$ by increasing d to $d + \Delta d$ so that the two sets of fringes that are initially concordant become discordant and are finally concordant again. It is clear that if m_1 changes to $m_1 + \Delta m$, m_2 changes to $m_2 + \Delta m - 1$ when the fringes are concordant again. We then have

$$2(d + \Delta d) = (m_1 + \Delta m)\lambda_1 = (m_2 + \Delta m - 1)(\lambda_1 + \Delta\lambda) \quad (10.32)$$

which gives

$$\lambda_1 = \left(\frac{2\Delta d}{\lambda_1} - 1 \right) \Delta\lambda \quad (10.33)$$

where on assuming that $2\Delta d/\lambda_1 = m_1 \gg 1$ we have

$$\delta\lambda = \frac{\lambda_1^2}{\Delta d}. \quad (10.34)$$

The Michelson interferometer finds a variety of other application. It was used by Michelson and Morley in 1887 to show that the speed of light is the same in all directions. The armlength of their interferometer was 11 m. Since the Earth is moving, we would expect the speed of light to be different along the direction of the Earth's motion. Michelson and Morley established that the speed of light does not depend on the motion of the observer, providing a direct experimental basis for Einstein's Special Theory of Relativity.

The fringe patter in the Michelson interferometer is very sensitive to changes in the mirror positions, and it can be used to measure very small displacements of the mirrors. A Michelson interferometer whose arms are 4 km long

(Figure 10.10) is being used in an experiment called Laser Interferometer Gravitational-Wave Observatory (LIGO¹) which is an ongoing effort to detect Gravitational Waves, one of the predictions of Einstein's General Theory of Relativity. Gravitational waves are disturbances in space-time that propagate at the speed of light. A gravitational wave that passes through the Michelson interferometer will produce displacements in the mirrors and these will cause changes in the fringe pattern. These displacements are predicted to be extremely small. LIGO is sensitive enough to detect displacements of the order of 10^{-16} cm in the mirror positions.

Problems

1. An electromagnetic plane wave with $\lambda = 1$ mm is normally incident on a screen with two slits with spacing $d = 3$ mm.
 - a. How many maxima will be seen, at what angles to the normal?
 - b. Consider the situation where the wave is incident at 30^{circ} to the normal.
2. Two radio antennas separated by a distance $d = 10$ m emit the same signal at frequency ν with phase difference ϕ . Determine the values of ν and ϕ so that the radiation intensity is maximum in one direction along the line joining the two antennas while it is minimum along exactly the opposite direction. How do the maxima and minima shift if ϕ is reduced to half the earlier value?
3. A lens of diameter 5.0 cm and focal length 20 cm is cut into two identical halves. A layer 1 mm in thickness is cut from each half and the two lenses joined again. The lens is illuminated by a point source located at the focus and a fringe pattern is observed on a screen 50 cm away. What is the fringe spacing and the maximum number of fringes that will be observed?
4. Two coherent monochromatic point sources are separated by a small distance, find the shape of the fringes observed on the screen when, a) the screen is at one side of the sources and normal to the screen is along the line joining the two sources and b) when the normal to the screen is perpendicular to the line joining the sources.
5. The radiation from two very distant sources A and B shown in the Figure 10.11 is measured by the two antennas 1 and 2 also shown in the figure. The antennas operate at a wavelength λ . The antennas produce voltage outputs \tilde{V}_1 and \tilde{V}_2 which have the same phase and amplitude as the

¹<http://www.ligo.caltech.edu/>

7. Calculate the separation between the secondary sources if the primary source is placed at a distance r from the mirror-joint and the tilt angle is θ .
8. Two coherent plane waves with wave vectors $\vec{k}_1 = k[\cos 30^\circ \hat{i} + \sin 30^\circ \hat{j}]$ and $\vec{k}_2 = k[\sin 30^\circ \hat{i} + \cos 30^\circ \hat{j}]$ with $k = 1.2 \times 10^{-6} \text{m}^{-1}$ are incident on a screen which is perpendicular to the x axis to produce straight line fringes. Determine the spacing between two successive dark lines in the fringe pattern.
9. Starting from a central dark fringe, eighth successive bright and dark fringes are observed at the center when one of the mirrors of a Michelson interferometer is moved $2.2 \mu\text{m}$. Determine the wavelength of the light which is being used. (5.5 Å)
10. A Sodium lamp emits light at two neighbouring wavelengths 5890 Å and 5896 Å. A Michelson interferometer is adjusted so that the fringes are in concordance. One of the mirrors is moved a distance Δd so that the fringes become discordant and concordant again. For what displacement Δd are the fringes most discordant *ie.* the fringe pattern becomes the faintest, and for what Δd does it become concordant again?
11. A Michelson interferometer illuminated by sodium light is adjusted so that the fringes are concordant with a central dark fringe. What is the angular radius of the first dark fringe if the order of the central fringe is $m = 100$ and $m = 1000$?
12. What happens if a Michelson interferometer is illuminated by white light? Also consider the situation where $d = 0$ *ie.* the two arms have the same length.

Chapter 11

Coherence

We shall separately discuss “spatial coherence” and “temporal coherence”.

11.1 Spatial Coherence

The Young’s double slit experiment (Figure 11.1) essentially measures the spatial coherence. The wave $\tilde{E}(t)$ at the point P on the screen is the superposition of $\tilde{E}_1(t)$ and $\tilde{E}_2(t)$ the contributions from slits 1 and 2 respectively. Let us now shift our attention to the values of the electric field $\tilde{E}_1(t)$ and $\tilde{E}_2(t)$ at the positions of the two slits. We define the spatial coherence of the electric field at the two slit positions as

$$C_{12}(d) = \frac{\frac{1}{2} \langle \tilde{E}_1(t) \tilde{E}_2^*(t) + \tilde{E}_1^*(t) \tilde{E}_2(t) \rangle}{2\sqrt{I_1 I_2}} \quad (11.1)$$

The waves from the two slits pick up different phases along the path from the slits to the screen. The resulting intensity pattern on the screen can be written as

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} C_{12}(d) \cos(\phi_2 - \phi_1) \quad (11.2)$$

where $\phi_2 - \phi_1$ is the phase difference in the path from the two slits to the screen. The term $\cos(\phi_2 - \phi_1)$ gives rise to a fringe pattern.

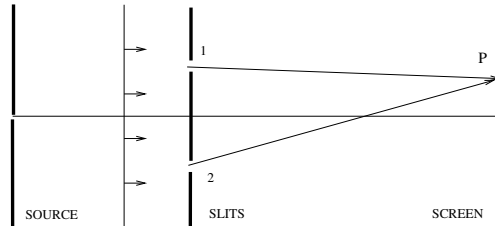


Figure 11.1: Young’s double slit with a point source

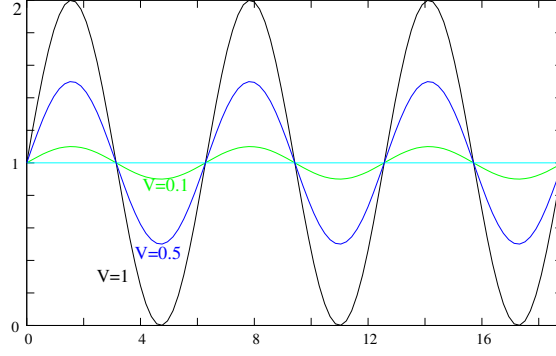


Figure 11.2: Fringe intensity for different visibilities

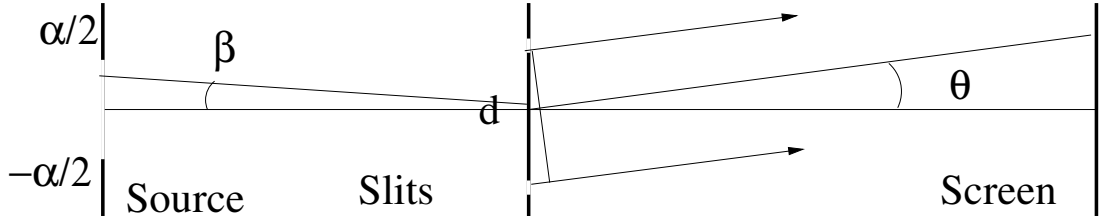


Figure 11.3: Double slit with a wide source

The fringe visibility defined as

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} \quad (11.3)$$

quantifies the contrast of the fringes produced on the screen. It has values in the range $1 \geq V \geq 0$. A value $V = 1$ implies very high contrast fringes, the fringes are washed away when $V = 0$. Figure 11.2 shows the fringe pattern for different values of V . It can be easily checked that the visibility is related to the spatial coherence as

$$V = \frac{2\sqrt{I_1 I_2} |C_{12}(d)|}{I_1 + I_2} \quad (11.4)$$

and the visibility directly gives the spatial coherence $V = |C_{12}(d)|$ when $I_1 = I_2$.

Let us first consider the situation when the two slits are illuminated by a distant point source as shown in Figure 11.1. Here the two slits lie on the same wavefront, and $\tilde{E}_1(t) = \tilde{E}_2(t)$. We then have

$$\frac{1}{2} \langle \tilde{E}_1(t) \tilde{E}_2^*(t) \rangle = \frac{1}{2} \langle \tilde{E}_1^*(t) \tilde{E}_2(t) \rangle = I_1 = I_2. \quad (11.5)$$

whereby $C_{12}(d) = 1$ and the fringes have a visibility $V = 1$.

We next consider the effect of a finite source size. It is assumed that the source subtends an angle α as shown in Figure 11.3. This situation can be

analyzed by first considering a source at an angle β as shown in the figure. This produces an intensity

$$I(\theta, \beta) = 2I_1 \left[1 + \cos \left(\frac{2\pi d}{\lambda} (\theta + \beta) \right) \right] \quad (11.6)$$

at a point at an angle θ on the screen where it is assumed that $\theta, \beta \ll 1$. Integrating β over the angular extent of the source

$$\begin{aligned} I(\theta) &= \frac{1}{\alpha} \int_{-\alpha/2}^{\alpha/2} I(\theta, \beta) d\beta \\ &= 2I_1 \left[1 + \frac{\lambda}{\alpha 2\pi d} \left\{ \sin \left[\frac{2\pi d}{\lambda} \left(\theta + \frac{\alpha}{2} \right) \right] - \sin \left[\frac{2\pi d}{\lambda} \left(\theta - \frac{\alpha}{2} \right) \right] \right\} \right] \\ &= 2I_1 \left[1 + \frac{\lambda}{\pi d \alpha} \cos \left(\frac{2\pi d \theta}{\lambda} \right) \sin \left(\frac{\pi d \alpha}{\lambda} \right) \right] \end{aligned} \quad (11.7)$$

It is straightforward to calculate the spatial coherence by comparing eq. (11.7) with eq. (11.2). This has a value

$$C_{12}(d) = \sin \left(\frac{\pi d \alpha}{\lambda} \right) / \left(\frac{\pi d \alpha}{\lambda} \right) \quad (11.8)$$

and the visibility is $V = |C_{12}(d)|$. Thus we see that the visibility which quantifies the fringe contrast in the Young's double slit experiment gives a direct estimate of the spatial coherence. The visibility, or equivalently the spatial coherence goes down if the angular extent of the source is increased. It is interesting to note that the visibility becomes exactly zero when the argument of the Sine term the expression (11.8) becomes integral multiple of π . So when the width of the source is equal to $m\lambda/d$, $m = 1, 2, \dots$ the visibility is zero.

Why does the fringe contrast go down if the angular extent of the source is increased? This occurs because the two slits are no longer illuminated by a single wavefront, There now are many different wavefronts incident on the slits, one from each point on the source. As a consequence the electric fields at the two slits are no longer perfectly coherent $|C_{12}(d)| < 1$ and the fringe contrast is reduced.

Expression (11.8) shows how the Young's double slit experiment can be used to determine the angular extent of sources. For example consider a situation where the experiment is done with starlight. The variation of the visibility V or equivalently the spatial coherence $C_{12}(d)$ with varying slit separation d is governed by eq. (11.8). Measurements of the visibility as a function of d can be used to determine α the angular extent of the star.

11.2 Temporal Coherence

The Michelson interferometer measures the temporal coherence of the wave. Here a single wave front $\tilde{E}(t)$ is split into two $\tilde{E}_1(t)$ and $\tilde{E}_2(t)$ at the beam splitter. This is referred to as division of amplitude. The two waves are then superposed, one of the waves being given an extra time delay τ through the difference in the arm lengths. The intensity of the fringes is

$$\begin{aligned} I &= \frac{1}{2} \langle [\tilde{E}_1(t) + \tilde{E}_2(t + \tau)] [\tilde{E}_1(t) + \tilde{E}_2(t + \tau)]^* \rangle \\ &= I_1 + I_2 + \frac{1}{2} \langle \tilde{E}_1(t) \tilde{E}_2^*(t + \tau) + \tilde{E}_1^*(t) \tilde{E}_2(t + \tau) \rangle \end{aligned} \quad (11.9)$$

where it is last term involving $\tilde{E}_1(t) \tilde{E}_2^*(t + \tau) \dots$ which is responsible for interference. In our analysis of the Michelson interferometer in the previous chapter we had assumed that the incident wave is purely monochromatic *ie.* $\tilde{E}(t) = \tilde{E}e^{i\omega t}$ whereby

$$\frac{1}{2} \langle \tilde{E}_1(t) \tilde{E}_2^*(t + \tau) + \tilde{E}_1^*(t) \tilde{E}_2(t + \tau) \rangle = 2\sqrt{I_1 I_2} \cos(\omega\tau) \quad (11.10)$$

The above assumption is an idealization that we adopt because it simplifies the analysis. In reality we do not have waves of a single frequency, there is always a finite spread in frequencies. How does this affect eq. 11.10?

As an example let us consider two frequencies $\omega_1 = \omega - \Delta\omega/2$ and $\omega_2 = \omega + \Delta\omega/2$ with $\Delta\omega \ll \omega$

$$\tilde{E}(t) = \tilde{a} [e^{i\omega_1 t} + e^{i\omega_2 t}] . \quad (11.11)$$

This can also be written as

$$\tilde{E}(t) = \tilde{A}(t)e^{i\omega t} \quad (11.12)$$

which is a wave of angular frequency ω whose amplitude $\tilde{A}(t) = 2\tilde{a} \cos(\Delta\omega t/2)$ varies slowly with time. We now consider a more realistic situation where we have many frequencies in the range $\omega - \Delta\omega/2$ to $\omega + \Delta\omega/2$. The resultant will again be of the same form as eq. (11.12) where there is a wave with angular frequency ω whose amplitude $\tilde{A}(t)$ varies slowly on the timescale

$$T \sim \frac{2\pi}{\Delta\omega} .$$

Note that the amplitude $A(t)$ and phase $\phi(t)$ of the complex amplitude $\tilde{A}(t)$ both vary slowly with timescale T . Figure 11.4 shows a situation where $\Delta\omega/\omega = 0.2$, a pure sinusoidal wave of the same frequency is shown for comparison. What happens to eq. (11.10) in the presence of a finite spread in frequencies? It now gets modified to

$$\frac{1}{2} \langle \tilde{E}_1(t) \tilde{E}_2^*(t + \tau) + \tilde{E}_1^*(t) \tilde{E}_2(t + \tau) \rangle = 2\sqrt{I_1 I_2} C_{12}(\tau) \cos(\omega\tau) \quad (11.13)$$

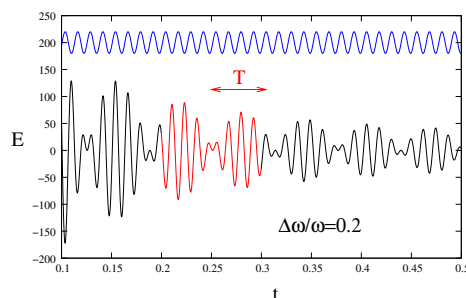


Figure 11.4: Variation of E with time for monochromatic and polychromatic light

where $C_{12}(\tau) \leq 1$. Here $C_{12}(\tau)$ is the temporal coherence of the two waves $\tilde{E}_1(t)$ and $\tilde{E}_2(t)$ for a time delay τ . Two waves are perfectly coherent if $C_{12}(\tau) = 1$, partially coherent if $0 < C_{12}(\tau) < 1$ and incoherent if $C_{12}(\tau) = 0$. Typically the coherence time τ_c of a wave is decided by the spread in frequencies

$$\tau_c = \frac{2\pi}{\Delta\omega}. \quad (11.14)$$

The waves are coherent for time delays τ less than τ_c *ie.* $C_{12}(\tau) \sim 1$ for $\tau < \tau_c$, and the waves are incoherent for larger time delays *ie.* $C_{12}(\tau) \sim 0$ for $\tau > \tau_c$. Interference will be observed only if $\tau < \tau_c$. The coherence time τ_c can be converted to a length-scale $l_c = c\tau_c$ called the coherence length.

An estimate of the frequency spread $\Delta\nu = \Delta\omega/2\pi$ can be made by studying the intensity distribution of a source with respect to frequency. Full width at half maximum (FWHM) of the intensity profile gives a good estimate of the frequency spread.

The Michelson interferometer can be used to measure the temporal coherence $C_{12}(\tau)$. Assuming that $I_1 = I_2$, we have $V = C_{12}(\tau)$. Measuring the visibility of the fringes varying d the difference in the arm lengths of a Michelson interferometer gives an estimate of the temporal coherence for $\tau = d/c$. The fringes will have a good contrast $V \sim 1$ only for $d < l_c$. The fringes will be washed away for d values larger than l_c .

Problems

1. Consider a situation where Young's double slit experiment is performed using light of wavelength 550 nm and $d = 1$ m. Calculate the visibility assuming a source of angular width $1'$ and 1° . Plot $I(\theta)$ for both these cases.
2. A small aperture of diameter 0.1 mm at a distance of 1m is used to illuminate two slits with light of wavelength $\lambda = 550$ nm. The slit separation is $d = 1$ mm. What is the fringe spacing and the expected visibility of the fringe pattern? (5.5×10^{-4} rad, $V = 0.95$)

3. A source of unknown angular extent α emitting light at $\lambda = 550 \text{ nm}$ is used in a Young's double slit experiment where the slit spacing d can be varied. The visibility is measured for different values of d . It is found that the fringes vanish ($V = 0$) for $d = 10 \text{ cm}$. [a.] What is the angular extent of the source? (5.5×10^{-6})
4. Estimate the coherence time τ_c and coherence length l_c for the following sources

Source	$\lambda \text{ nm}$	$\Delta\lambda \text{ nm}$
White light	550	300
Mercury arc	546.1	1.0
Argon ion gas laser	488	0.06
Red Cadmium	643.847	0.0007
Solid state laser	785	10^{-5}
He-Ne laser	632.8	10^{-6}

5. Assume that Kr^{86} discharge lamp has roughly the following intensity distribution at various wavelengths, λ (in nm),

$$I(\lambda) = \frac{36I_0}{36 + (\lambda - 605.616)^2 \times 10^8} \quad .$$

Estimate the coherence length of Kr^{86} source. (Ans. 0.3m)

6. An ideal Young's double slit (i.e. identical slits with negligible slit width) is illuminated with a source having two wavelengths, $\lambda_1 = 418.6 \text{ nm}$ and $\lambda_2 = 421.4 \text{ nm}$. The intensity at λ_1 is double of that at λ_2 .
 - a) Compare the visibility of fringes near order $m = 0$ and near order $m = 50$ on the screen [visibility = $(I_{\max} - I_{\min}) / (I_{\max} + I_{\min})$]. (Ans. 1:0.5)
 - b) At what order(s) on the screen visibility of the fringes is poorest and what is this minimum value of the visibility. (Ans. 75, 225 etc. and $1/3$)
7. An ideal Young's double slit (separation d between the slits) is illuminated with two identical strong monochromatic point sources of wavelength λ . The sources are placed symmetrically and far away from the double slit. The angular separation of the sources from the mid point of the double slit is θ_s . Estimate θ_s so that the visibility of the fringes on the screen is zero. Can one have visibility almost 1 for a non zero θ_s .

Hint: See the following figure 11.5,

(Further reading: Michelson's stellar interferometer for estimating angular separation of double stars and diameters of distant stars)

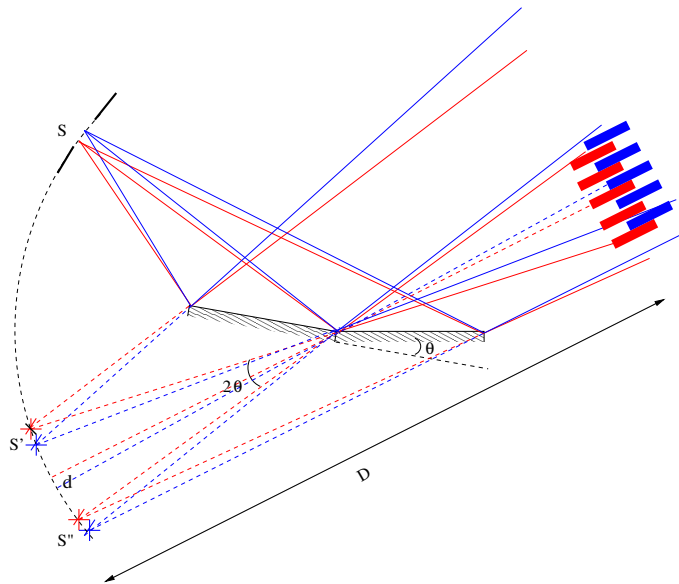


Figure 11.5: Two source vanishing visibility condition

Chapter 12

Diffraction

Our discussion of interference in the previous chapter considered the superposition of two waves. The discussion can be generalized to a situation where there are three or even more waves. Diffraction is essentially the same as interference, except that we have a superposition of a very large number of waves. In some situations the number is infinite. An obstruction placed along the path of a wave (Figure 12), is an example of a situation where diffraction occurs. In simple terms the deviation of light rays from straight path due to some obstacle is known as diffraction. In the case of diffraction the longer wavelengths deviate more compared to the shorter wavelengths, whereas in refraction the shorter wavelengths deviate more compare to the longer ones. In the case of diffraction it is necessary to solve the equation governing the wave in the complicated geometry produced by the obstruction. This is usually very complicated, and beyond analytic treatment in most situations. The Huygens-Fresnel principle is a heuristic approach which allows such problems to be handled with relative ease.

Huygens' principle, proposed around 1680, states that every point on a wavefront acts as the source for secondary spherical wavelets such that the wavefront at a later time is the envelope of these wavelets. Further, the frequency and propagation speed of the wavelets is the same as that of the wave. Applying Huygens' principle to the propagation of a plane wave, we see that a

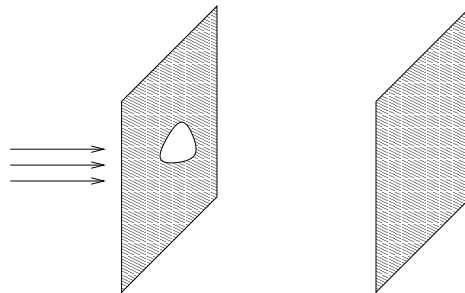


Figure 12.1: Diffraction due to an obstruction



Figure 12.2: Wavefronts and Huygens' secondary wavelets

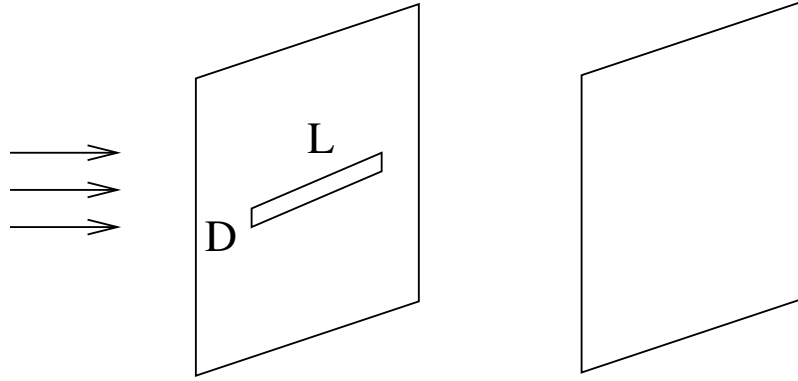


Figure 12.3: Single slit diffraction

plane wave front evolves into a shifted plane wave at a later time (Figure 12.2). We also show how it can be applied to the propagation of a spherical wavefront. It is also useful for studying the propagation of waves through a refracting medium where the light speed changes with position and is also different in different directions.

Huygens' principle, in its original form cannot be used to explain interference or diffraction. It was modified in the early 19th century by Fresnel to explain diffraction. The modified version is referred to as the Huygens-Fresnel principle. Kirchoff later showed that the Huygens-Fresnel principle is actually consistent with the wave equation that governs the propagation of light.

The Huygens-Fresnel principle states that every unobstructed point on the wavefront acts like a source for a secondary wavelet. The contribution from all these wavelets are to be superposed to find the resultant at any point.

12.1 Single slit Diffraction Pattern.

Consider a situation where light from a distant source falls on a rectangular slit of width D and length L as shown in Figure 12.3. We shall assume that L is much larger than D . We are interested in the image on a screen which is at a great distance from the slit.

As the source is very far away, we can treat the incident light as plane

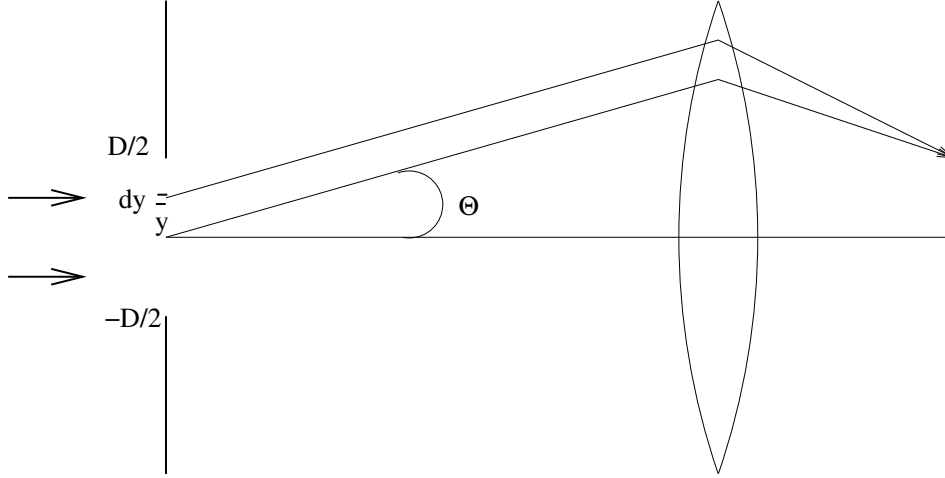


Figure 12.4: Single slit effective one dimensional arrangement

wave. Further the source is aligned so that the incident wavefronts are parallel to the plane of the slit. Every point in the slit emits a spherical secondary wavelet. These secondary wavelets are well described by plane waves by the time they reach the distant screen. This situation where the incident wave and the emergent secondary waves can all be treated as plane waves is referred to as Fraunhofer diffraction. In this case both, the source as well as the screen are effectively at infinity from the obstacle.

We assume L to be very large so that the problem can be treated as one dimensional as shown in Figure 12.4. Instead of placing the screen far away, it is equivalent to introduce a lens and place the screen at the focal plane. Each point on the slit acts like a secondary source. These secondary sources are all in phase and they all emit secondary wavelets with the same phase. Let us calculate the total radiation at a point at an angle θ . If

$$d\tilde{E} = \tilde{A} dy \quad (12.1)$$

be the contribution from a small element dy at the center of the slit, the contribution from an element a distance y away will be at a different phase

$$d\tilde{E} = \tilde{A} e^{i\delta} dy \quad (12.2)$$

where

$$\delta = \frac{2\pi}{\lambda} y \sin \theta = ky \quad (12.3)$$

The total electric field can be calculated by adding up the contribution from all points on the slit. This is an integral

$$\tilde{E} = \int_{-\frac{D}{2}}^{\frac{D}{2}} d\tilde{E} = \tilde{A} \int_{-\frac{D}{2}}^{\frac{D}{2}} e^{iky} dy$$

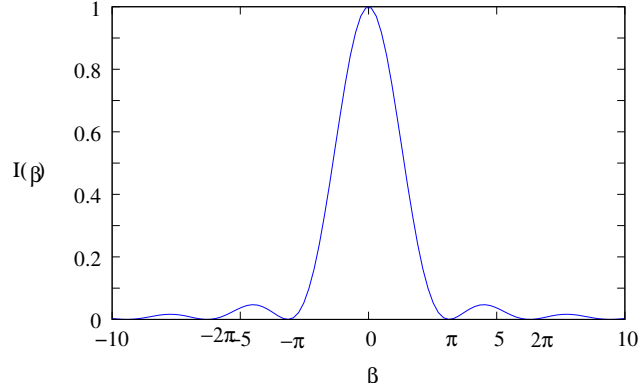


Figure 12.5: Single slit intensity pattern

$$= \tilde{A}D \sin\left(\frac{kD}{2}\right) \left(\frac{kD}{2}\right)^{-1} = \tilde{A}D \operatorname{sinc}\left(\frac{kD}{2}\right) \quad (12.4)$$

where $\operatorname{sinc}(x) = \sin(x)/x$. We use eq. (12.4) to calculate the intensity of light at an angle θ . Defining

$$\beta = \frac{\pi D \sin \theta}{\lambda}, \quad (12.5)$$

the intensity is given by

$$I(\beta) = \frac{1}{2}EE^* = I_0 \operatorname{sinc}^2 \beta. \quad (12.6)$$

Figure 12.5 shows the intensity as a function of β . The intensity is maximum at the center where $\beta = 0$. In addition to the oscillations, the intensity falls off proportional to β^2 away from the center. The analysis of the intensity pattern is further simplified in the situation where $\theta \ll 1$ as

$$\beta = \frac{\pi D \theta}{\lambda}. \quad (12.7)$$

The zeros of the intensity pattern occur at $\beta = \pm m\pi$ ($m, = 1, 2, 3, \dots$), or in terms of the angle θ , the zeros are at

$$\theta = m \frac{\lambda}{D}. \quad (12.8)$$

There are intensity maxima located between the zeros. The central maximum at $\theta = 0$ is the brightest, and its angular separation from the nearest zero is λ/D . This gives an estimate of the angular width of the central maximum. The intensities of the other maxima fall away from the center.

Let us compare the intensity pattern $I(\beta)$ shown in Figure 12.5 with the predictions of geometrical optics. Figure 12.4 shows a beam of parallel rays incident on a slit of dimension D . In geometrical optics the only effect of the

slit is to cut off some of the rays in the incident beam and reduce the transverse extent of the beam. We expect a beam of parallel rays with transverse dimension D to emerge from the slit. This beam is now incident on a lens which will focus all the rays to a single point on the screen. Thus in geometrical optics the image is a single bright point on the screen. In reality the wave nature of light manifests itself through the phenomena of diffraction, and we see a pattern of bright spots as shown in Figure 12.5. The central spot is the brightest and it has an angular extent $\pm\lambda/D$. The other spots located above and below the central spot are fainter. Taking into account both dimensions of the slit we have

$$I(\beta_x, \beta_y) = I_0 \text{sinc}^2(\beta_x) \text{sinc}^2(\beta_y) \quad (12.9)$$

where

$$\beta_x = \frac{\pi L \theta_x}{\lambda} \quad \text{and} \quad \beta_y = \frac{\pi D \theta_y}{\lambda}. \quad (12.10)$$

and θ_x and θ_y are the angles along the x and y axis respectively. The diffraction effects are important on angular scales $\theta_x \sim \lambda/L$ and $\theta_y \sim \lambda/D$. In the situation where $L \gg D$ the diffraction effects along θ_x will not be discernable, and we can treat it as a one dimensional slit of dimension D .

12.1.1 Angular resolution

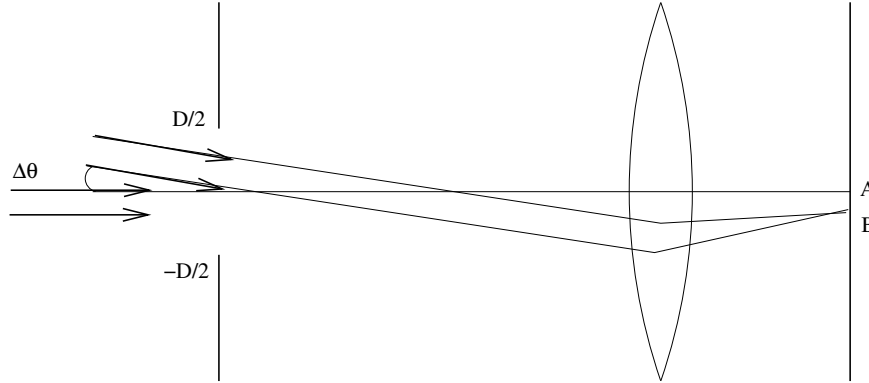


Figure 12.6: Light from two distant sources incident on single slit and their images

We consider a situation where the light from two distant sources is incident on a slit of dimension D . The sources are at an angular separation $\Delta\theta$ as shown in Figure 12.6. The light from the two sources is focused onto a screen. In the absence of diffraction the image of each source would be distinct point on the screen. The two images are shown as A and B respectively in Figure 12.6.

In reality we shall get the superposition of the diffraction patterns produced by the two sources as shown in Figure 12.7. The two diffraction patterns are

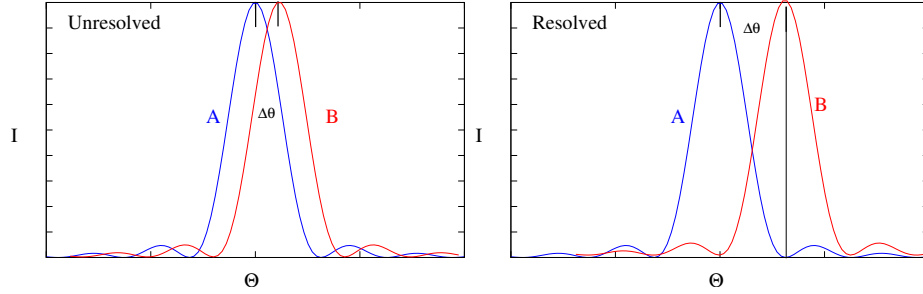


Figure 12.7: The Rayleigh criterion for resolution

centered on the positions A and B respectively where we expect the geometrical image. In case the angular separation $\Delta\theta$ is very small the two diffraction patterns will have a significant overlap. In such a situation it will not be possible to make out that there are two sources as it will appear that there is a single source. Two sources at such small angular separations are said to be unresolved. The two sources are said to be resolved if their diffraction patterns do not have a significant overlap and it is possible to make out that there are two sources and not one. *What is the smallest angle $\Delta\theta$ for which it is possible to make out that there are two sources and not one?*

Lord Rayleigh had proposed a criterion that the smallest separation at which it is possible to distinguish two diffraction patterns is when the maximum of one coincides with the minimum of the other (Figure 12.7). It follows that two sources are resolved if their angular separation satisfies

$$\Delta\theta \geq \frac{\lambda}{D} \quad (12.11)$$

The smallest angular separation $\Delta\theta$ at which two sources are resolved is referred to as the “angular resolution” of the aperture. A slit of dimension D has an angular resolution of λ/D .

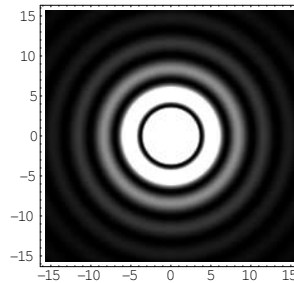


Figure 12.8: Circular aperture diffraction pattern

A circular aperture produces a circular diffraction pattern as shown in the Figure 12.8. The mathematical form is a little more complicated than the

sinc function which appears when we have a rectangular aperture, but it is qualitatively similar. The first minima is at an angle $\theta = 1.22 \lambda/D$ where D is the diameter of the aperture. It then follows that the “angular resolution” of a circular aperture is $1.22\lambda/D$. When a telescope of diameter D is used to observe a star, the image of the star is basically the diffraction pattern corresponding to the circular aperture of the telescope. Suppose there are two stars very close in the sky, what is the minimum angular separation at which it will be possible to distinguish the two stars? It is clear from our earlier discussion that the two stars should be at least $1.22\lambda/D$ apart in angle, other they will not be resolved. The Figure 12.9 shows not resolved, barely resolved and well resolved cases for a circular aperture.

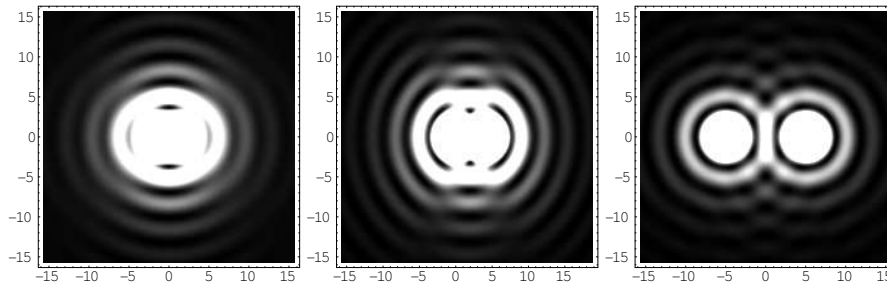


Figure 12.9: Not resolved, barely resolved and well resolved cases

Diffraction determines the angular resolution of any imaging instrument. This is typically of the order of λ/D where D is the size of the instrument’s aperture.

12.2 Chain of sources

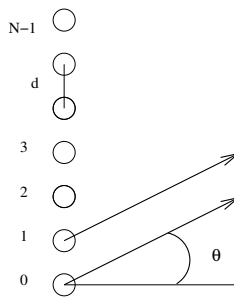


Figure 12.10: Chain of coherent dipoles

Consider N dipole oscillators arranged along a linear chain as shown in Figure 12.10, all emitting radiation with identical amplitude and phase. How much radiation will a distant observer at an angle θ receive? If $\tilde{E}_0, \tilde{E}_1, \tilde{E}_2, \dots$, and \tilde{E}_{N-1}

are the radiations from the 0th, 1st, 2nd, ..., and the $N - 1$ th oscillator respectively, \tilde{E}_1 is identical to \tilde{E}_0 except for a phase difference as it travels a shorter path. We have

$$\tilde{E}_1 = \tilde{E}_0 e^{i2\alpha} \quad (12.12)$$

where $2\alpha = \frac{2\pi}{\lambda} d \sin \theta$ is the phase difference that arises due to the path difference. Similarly $\tilde{E}_2 = [e^{i2\alpha}]^2 \tilde{E}_0$. The total radiation is obtained by summing the contributions from all the sources and we have

$$\tilde{E} = \sum_{n=0}^{N-1} \tilde{E}_n = \sum_{n=0}^{N-1} [e^{i2\alpha}]^n \tilde{E}_0 \quad (12.13)$$

This is a geometric progression, on summing this we have

$$\tilde{E} = \tilde{E}_0 \frac{e^{i2N\alpha} - 1}{e^{i2\alpha} - 1}. \quad (12.14)$$

This can be simplified further

$$\tilde{E} = \tilde{E}_0 \frac{e^{iN\alpha}}{e^{i\alpha}} \frac{e^{iN\alpha} - e^{-iN\alpha}}{e^{i\alpha} - e^{-i\alpha}} = \tilde{E}_0 e^{i(N-1)\alpha} \frac{\sin(N\alpha)}{\sin(\alpha)} \quad (12.15)$$

which gives the intensity to be

$$I = 0.5 \tilde{E} \tilde{E}^* = I_0 \frac{\sin^2(N\alpha)}{\sin^2(\alpha)} \quad (12.16)$$

where

$$\alpha = \frac{\pi d \sin \theta}{\lambda} \quad (12.17)$$

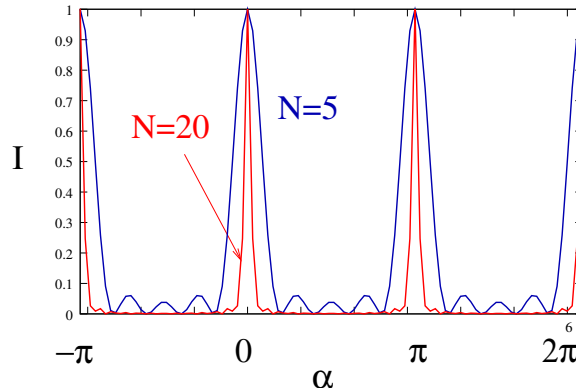


Figure 12.11: Intensity pattern for chain of dipoles

Plotting the intensity as a function of α (Figure 12.11) we see that it has a value $I = N^2 I_0$ at $\alpha = 0$. Further, it has the same value $I = I_0 N^2$ at

all other α values where both the numerator and denominator are zero i.e $\alpha = m\pi (m = 0, \pm 1, \pm 2, \dots)$ or

$$d \sin \theta = m\lambda \quad (m = 0, \pm 1, \pm 2, \pm 3 \dots) \quad (12.18)$$

The intensity is maximum whenever this condition is satisfied. These are referred to as the primary maxima of the diffraction pattern and m gives the order of the maximum.

The intensity drops away from the primary maxima. The intensity becomes zero $N - 1$ times between any two successive primary maxima and there are $N - 2$ secondary maxima in between. The number of secondary maxima increases and the primary maxima becomes increasingly sharper (Figure 12.11) if the number of sources N is increased. Let us estimate the width of the m th order principal maximum. The m th order principal maximum occurs at an angle θ_m which satisfies,

$$d \sin \theta_m = m\lambda. \quad (12.19)$$

If $\Delta\theta_m$ is the width of the maximum, the intensity should be zero at $\theta_m + \Delta\theta_m$ i.e.

$$\frac{\pi d \sin(\theta_m + \Delta\theta_m)}{\lambda} = m\pi + \frac{\pi}{N} \quad (12.20)$$

which implies that

$$\sin(\theta_m + \Delta\theta_m) = \frac{\lambda}{d} \left(m + \frac{1}{N} \right) \quad (12.21)$$

Expanding

$$\sin(\theta_m + \Delta\theta_m) = \sin \theta_m \cos \Delta\theta_m + \cos \theta_m \sin \Delta\theta_m$$

and assuming that $\Delta\theta_m \ll 1$ we have

$$\Delta\theta_m \cos \theta_m = \frac{\lambda}{dN} \quad (12.22)$$

which gives the width to be

$$\Delta\theta_m = \frac{\lambda}{dN \cos \theta_m}. \quad (12.23)$$

Thus we see that the principal maxima get sharper as the number of sources increases. Further, the 0th order maximum is the sharpest, and the width of the maximum increases with increasing order m .

The chain of radiation sources serves as an useful model for many applications.

12.2.1 Phased array

Consider first any one of the dipole radiators shown in Figure 12.10. For a dipole oriented perpendicular to the plane of the page, the radiation is uniform

in all directions on the plane of the page. Suppose we want to construct something like a radar that sends out radiation in only a specific direction, and not in all directions. It is possible to do so using a chain of N dipoles with spacing $d < \lambda$. The maxima of the radiation pattern occur at

$$\sin \theta = m \frac{\lambda}{d} \quad (12.24)$$

and as $\lambda/d > 1$ the only solutions are at $\theta = 0$ and $\theta = \pi$ (i.e. for $m = 0$). Thus the radiation is sent out only in the forward and backward directions, the radiation from the different dipoles cancel out in all other directions. The width of this maxima is

$$\Delta\theta = \frac{\lambda}{Nd} \quad (12.25)$$

which is determined by the separation between the two extreme dipoles in the chain.

The direction at which the maximum radiation goes out can be changed by introducing a constant phase difference 2ϕ between every pair of adjacent oscillators. The phase difference in the radiation received from any two adjacent dipoles is now given by

$$2\alpha = \frac{2\pi}{\lambda} d \sin \theta + 2\phi \quad (12.26)$$

and the condition at which the maxima occurs is now given by

$$\sin \theta = \left(\frac{\lambda}{d} \right) \left[m + \frac{\phi}{\pi} \right] \quad (12.27)$$

The device discussed here is called a “phased array”, and it can be used to send out or receive radiation from only a specific region of the sky. This has several applications in communications, radars and radio-astronomy.

12.2.2 Diffraction grating

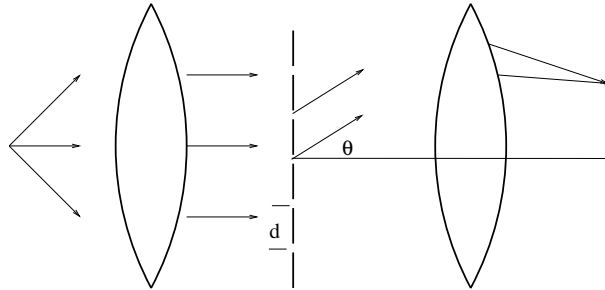


Figure 12.12: Diffraction grating

We consider the transmission diffraction grating shown in Figure 12.12. The transmission grating is essentially a periodic arrangement of N slits, each

slit of width D and slit spacing d . The spacing between successive slits d is referred to as the “grating element” or as the “period of the grating”. Each slit acts like a source, and the diffraction grating is equivalent to the chain of sources shown in Figure 12.10.

The intensity pattern of a diffraction grating is the product of the intensity pattern of a single slit and the intensity pattern of a periodic arrangement of emitters

$$I = I_0 \frac{\sin^2(N\alpha)}{\sin^2(\alpha)} \text{sinc}^2(\beta) \quad (12.28)$$

where

$$\alpha = \frac{\pi d \sin \theta}{\lambda} \quad \text{and} \quad \beta = \frac{\pi D \sin \theta}{\lambda}$$

Typically the slit spacing d is larger than the slit width *ie.* $d > D$. Figure 12.13 shows the intensity pattern for a diffraction grating. The finite slit width causes the higher order primary maximas to be considerably fainter than the low order ones.

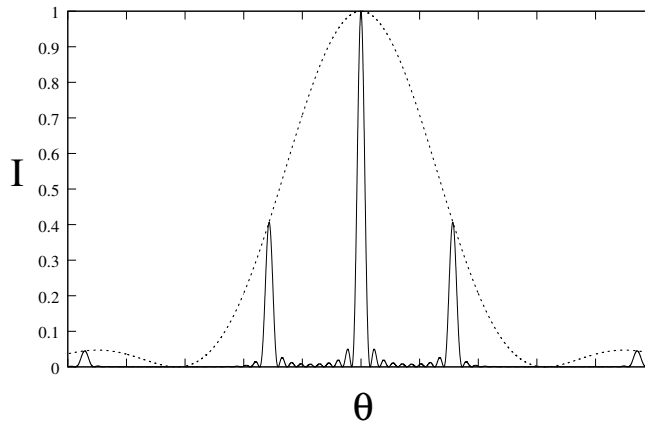


Figure 12.13: Intensity pattern of a diffraction grating

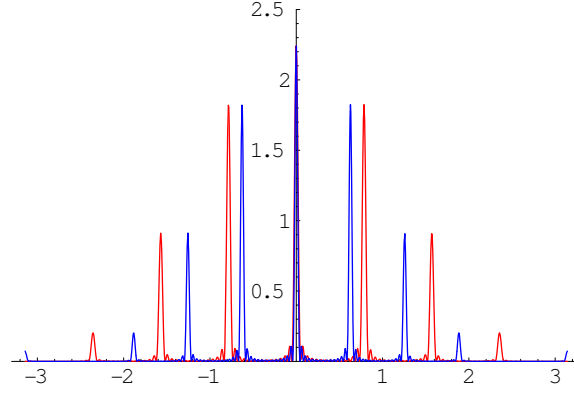
The transmission grating is a very useful device in spectroscopy. The grating is very effective in dispersing the light into different wavelength components. For each wavelength the m th order primary maximum occurs at a different angle determined by

$$\sin \theta_m = m \frac{\lambda}{d} \quad (12.29)$$

The diffraction pattern, when two different wavelengths are incident on a grating, is shown in the Figure 12.14.

The dispersive power of a grating is defined as

$$\mathcal{D} = \left(\frac{d\theta_m}{d\lambda} \right) = \frac{m}{d \cos \theta_m} \quad (12.30)$$

Figure 12.14: Diffraction pattern for two wavelengths (Intensity vs β)

We see that it increases with the order m and is inversely proportional to d . The finer the grating (small d) the more its dispersive power. Also, the higher orders have a greater dispersive power, but the intensity of these maxima is also fainter.

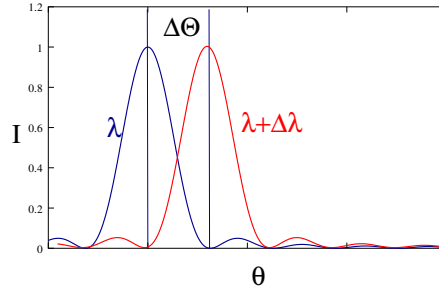


Figure 12.15: Chromatic resolution

The Chromatic Resolving Power(CRP) quantifies the ability of a grating to resolve two spectral lines of wavelengths λ and $\lambda + \Delta\lambda$. Applying Rayleigh's criterion, it will be possible to resolve the lines if the maximum of one coincides with the minimum of the other.

The minimum corresponding to λ (Figure 12.15) is at

$$\Delta\theta = \frac{\lambda}{Nd \cos \theta_m}, \quad (12.31)$$

from the maximum and the maximum corresponding to $\lambda + \Delta\lambda$ is at

$$\Delta\theta = \frac{m\Delta\lambda}{d \cos \theta_m}. \quad (12.32)$$

Equating these gives the CRP to be

$$\mathcal{R} \equiv \frac{\lambda}{\Delta\lambda} = Nm \quad (12.33)$$

The chromatic resolving power increases with the number of surfs or rulings in the grating. This makes the grating a very powerful dispersive element in spectrometers.

Problems

1. For a slit of dimensions $1\text{ mm} \times 1\text{ cm}$, what are the positions of the first three minima's on either side of the central maxima? Use $\lambda = 550\text{ nm}$ and 0.1 mm .
2. For a rectangular slit whose smaller dimension is D , what are the positions of the maxima for light of wavelength λ ? (Ans. $\beta \sim \pm 1.43\pi$, $\pm 2.46\pi$, $\pm 3.47\pi$, etc.)
3. Calculate the ratio of intensities of the first intensity maximum and central maximum for the previous problem. (Ans. ~ 21)
4. Compare the angular resolutions of two circular apertures a. $D = 1\text{ mm}$ and $\lambda = 550\text{ nm}$ and b. $D = 45\text{ m}$ and $\lambda = 1\text{ m}$.
5. A plane wave of light with wavelength $\lambda = 0.5\text{ }\mu\text{m}$ falls on a slit of width $= 10\text{ }\mu\text{m}$ at an angle 30° to the normal. Find the angular position, with respect to the normal, of the first minima on both sides of the central maxima.
6. The collimator of a spectrometer has a diameter of 2 cm . What would be the largest grating element for a grating, which would just resolve the Sodium doublet at the second order, using this spectrometer. (Sodium doublet: $D_1 = 589.0\text{ nm}$ and $D_2 = 589.6\text{ nm}$, Ans. $d \sim 0.04\text{ mm}$)
7. Obtain the expression of intensity for a double slit with separation d between the slits and individual slit width D , as a special case of $N=2$.
8. Plot intensity profile as a function of θ for a double slit with $d = 0.1\text{ mm}$ and $D = 0.025\text{ mm}$. Assume wavelength of the incident monochromatic light to be equal to 500 nm . Keep $|\theta| < 2.5^\circ$. Notice that there are no 4th order and the 8th order intensity maxima in the above. These are called the missing orders in the pattern.
9. Missing orders: Find the ratio of d and D for the following double slit diffraction, Figure 12.16.
10. Find an expression for the intensity of a double slit diffraction when one of the slits is having a width C and the other is having a width D and the separation between them is d .

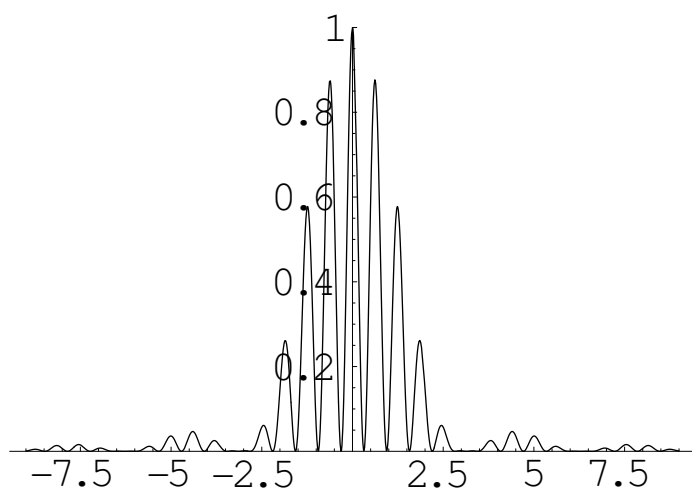


Figure 12.16: A double slit diffraction pattern

Chapter 13

X-ray Diffraction

We have seen that a one-dimensional periodic arrangement of coherent radiation sources (chain of sources) produces a diffraction pattern. The diffraction grating is an example. Atoms and molecules have a three dimensional periodic arrangement inside crystalline solids. A diffraction pattern is produced if the atoms or molecules act like a three dimensional grating. Inside crystalline solids the inter-atomic spacing is of the order of 1 Å. Crystals can produce a diffraction pattern with X-ray whose wavelength is comparable to inter atomic spacings. The wavelength of visible light is a few thousand times larger and this does not serve the purpose.

X-ray is incident on a crystal as shown in Figure 13.1. The oscillating electric field of this electromagnetic wave induces a oscillating dipole moment in every atom or molecule inside the crystal. These dipoles oscillate at the same frequency as the incident X-ray. The oscillating dipoles emit radiation in all directions at the same frequency as the incident radiation, this is known as Thomson scattering. Every atom scatters the incident X-ray in all directions. The radiation scattered from different atoms is coherent. The total radiation scattered in any particular direction is calculated by superposing the contribution from each atom.

For a crystal where the atoms have a periodic arrangement, it is convenient to think of the three-dimensional grating as a set of planes arranged in a one-dimensional grating as shown in Figure 13.2.

Consider X-ray incident at a grazing angle of θ as shown in Figure 13.2. The intensity of the reflected X-ray will be maximum when the phase difference

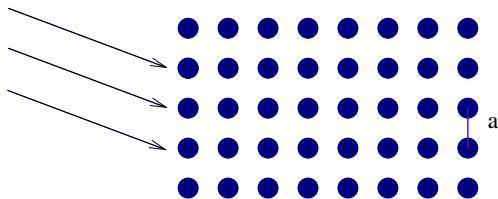


Figure 13.1:

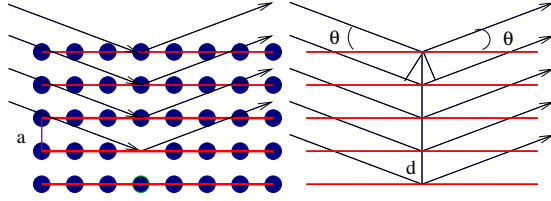


Figure 13.2:

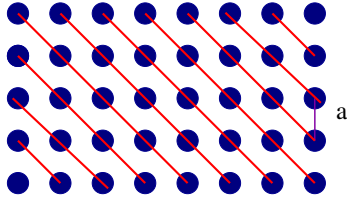


Figure 13.3:

between the waves reflected from two successive planes is λ or its integer multiple. This occurs when

$$2d \sin \theta = m\lambda. \quad (13.1)$$

This formula is referred to as Bragg's Law. The diffraction can also occur from other planes as shown in Figure 13.3.

The spacing between the planes d is different in the two cases and the maxima will occur at a different angle. The first set of planes are denoted by the indices $(1, 0, 0)$ and the second set by $(1, 1, 0)$. It is, in principle, possible to have a large number of such planes denoted by the indices (h, k, l) referred to as the Miller indices. The distance between the planes is

$$d(h, k, l) = \frac{a}{\sqrt{h^2 + k^2 + l^2}} \quad (13.2)$$

where a is the lattice constant or lattice spacing. Note that the crystal has been assumed to be cubic.

Figure 13.4 shows a schematic diagram of an X-ray diffractometer. This essentially allows us to measure the diffracted X-ray intensity as a function of 2θ as shown in Figure 13.4.

Figure 13.5 shows the unit cell of $La_{0.66}Sr_{0.33}MnO_3$. X-ray of wavelength $\lambda = 1.542\text{\AA}$ is used in an X-ray diffractometer, the resulting diffraction pattern with intensity as a function of 2θ is shown in Figure 13.6.

The 2θ values of the first three peaks have been tabulated below. The question is how to interpret the different peaks. All the peaks shown correspond to $m = 1$ *ie.* first order diffraction maximas, the higher orders $m = 2, 3, \dots$ are much fainter. The different peaks correspond to different Miller indices which give different values of d (eq. 13.2). The maxima at the smallest θ arises from the largest d value which corresponds to the indices $(h, k, l) = (1, 0, 0)$. The

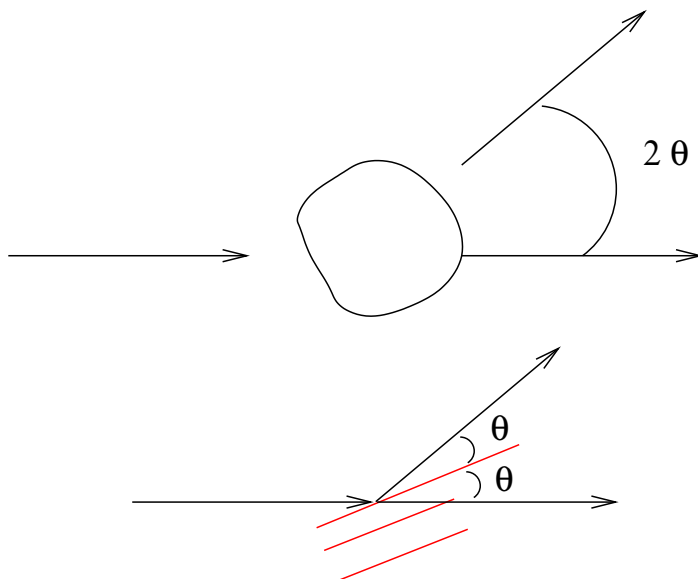


Figure 13.4:

other maxima may be interpreted using the fact that θ and d are inversely related.

h, k, l	2θ
1,0,0	23.10°
1,1,0	32.72°
1,1,1	40.33°

Problems

1. Determine the lattice spacing for $La_{0.66}Sr_{0.33}MnO_3$ using the data given above. Check that all the peaks give the same lattice spacing.
2. For a particular crystal which has a cubic lattice the smallest angle θ at which the X-ray diffraction pattern has a maxima is $\theta = 15^\circ$. At what angle is the next maxima expected? [Ans: 21.5°]
3. For a cubic crystal with lattice spacing $a = 2\text{\AA}$ and X-ray with $\lambda = 1.5\text{\AA}$, what are the two smallest angles where a diffraction maxima will be observed?

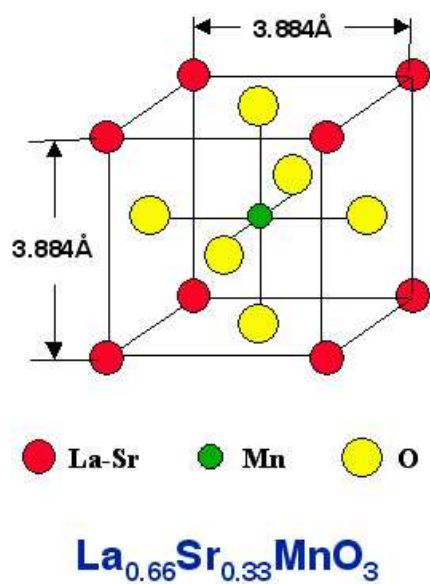


Figure 13.5:

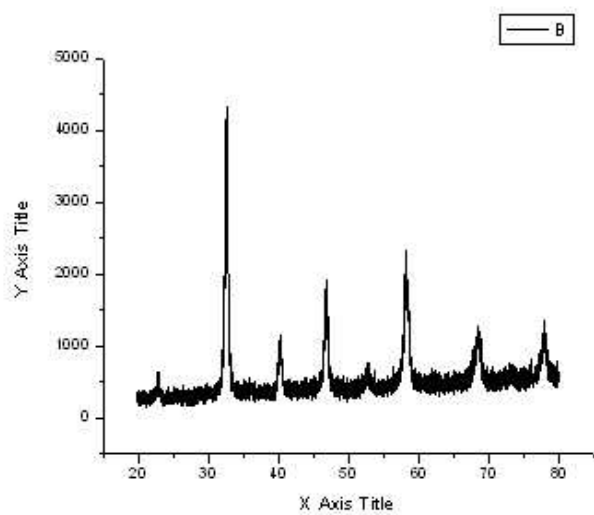


Figure 13.6:

Chapter 14

Beats

In this chapter we consider the superposition of two waves of different frequencies. At a fixed point along the propagation direction of the waves, the time evolution of the two wave are,

$$\tilde{A}_1(t) = A_0 e^{i\omega_1 t}, \quad (14.1)$$

and

$$\tilde{A}_2(t) = A_0 e^{i\omega_2 t}. \quad (14.2)$$

The superposition of these two waves of equal amplitude is

$$\tilde{A}(t) = \tilde{A}_1(t) + \tilde{A}_2(t), \quad (14.3)$$

which can be written as,

$$\begin{aligned} \tilde{A}(t) &= 2A_0 [e^{-i(\omega_2 - \omega_1)t/2} e^{i(\omega_1 + \omega_2)t/2} + e^{i(\omega_2 - \omega_1)t/2} e^{i(\omega_1 + \omega_2)t/2}], \\ &= 2A_0 \cos(\Delta\omega t/2) e^{i\bar{\omega}t}, \end{aligned} \quad (14.4)$$

where $\Delta\omega = \omega_2 - \omega_1$ and $\bar{\omega} = (\omega_1 + \omega_2)/2$ are respectively the difference and the average of the two frequencies. If the two frequencies ω_1 and ω_2 are very close, and the difference in frequencies $\Delta\omega$ is much smaller than $\bar{\omega}$, we can think of the resultant as a fast varying wave with frequency $\bar{\omega}$ whose amplitude varies slowly at a frequency $\Delta\omega/2$. The intensity of the resulting wave is modulated at a frequency $\Delta\omega$. This slow modulation of the intensity is referred to as beats. This modulation is heard when two strings of a musical instrument are nearly tuned and this is useful in tuning musical instruments.

In the situation where the two amplitudes are different we have,

$$\tilde{A}(t) = a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t} = [a_1 e^{-i\Delta\omega t/2} + a_2 e^{i\Delta\omega t/2}] e^{i\bar{\omega}t}. \quad (14.5)$$

Again we see that we have a fast varying component whose amplitude is modulated slowly. The intensity of the resultant wave is,

$$I = AA^* = a_1^2 + a_2^2 + 2a_1 a_2 \cos(\Delta\omega t). \quad (14.6)$$

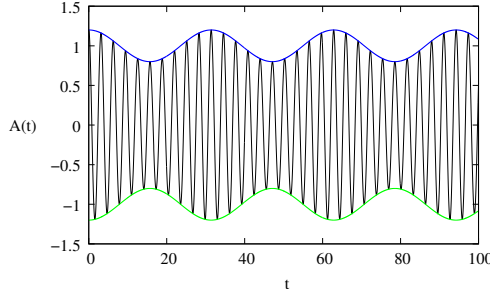


Figure 14.1: A modulated wave

We see that the intensity oscillates at the frequency difference $\Delta\omega$, and it never goes to zero if the two amplitudes are different.

Radio transmission based on “amplitude modulation” is the opposite of this. The transmitter has a generator which produces a sinusoidal electrical wave at a high frequency, say 800 kHz which is the transmission frequency. This is called the carrier wave. The signal which is to be transmitted, say sound, is a relatively slowly varying signal in the frequency range 20 Hz to 20 kHz. The sound is converted to an electrical signal and the amplitude of the carrier wave is modulated by the slowly varying signal. Mathematically,

$$\tilde{A}(t) = [1 + f(t)] e^{i\omega_c t}, \quad (14.7)$$

where ω_c is the angular frequency of the carrier wave, and $f(t)$ is the slowly varying signal. As an example we consider a situation where the signal has a single frequency component,

$$f(t) = a_m \cos(\omega_m t), \quad (14.8)$$

where a_m is the amplitude and ω_m the frequency of the modulating signal. The transmitted signal $A(t)$ is shown in Figure 14.1. The envelope contains the signal. This can be recovered at the receiver by discarding the carrier and retaining only the envelope. The transmitted signal can be expressed as,

$$A(t) = e^{i\omega_c t} + \frac{a_m}{2} e^{i(\omega_c + \omega_m)t} + \frac{a_m}{2} e^{i(\omega_c - \omega_m)t}. \quad (14.9)$$

We see that though the transmitter originally produces output only at a single frequency ω_c when there is no modulation, it starts transmitting two other frequencies $\omega_c + \omega_m$ and $\omega_c - \omega_m$ once the signal is modulated. These new frequencies are referred to as sidebands. If we plot the spectrum of the radiation from the transmitter, we see that the energy is distributed in three frequencies as shown in Figure 14.2.

For a more complicated sound signal, the sidebands will be spread over a range of frequencies instead of a few discrete frequencies. The audible frequency range extends upto 20 KHz but the transmitters and receivers usually do not work beyond 10 KHz. So a radio station transmitting at 800 KHz will

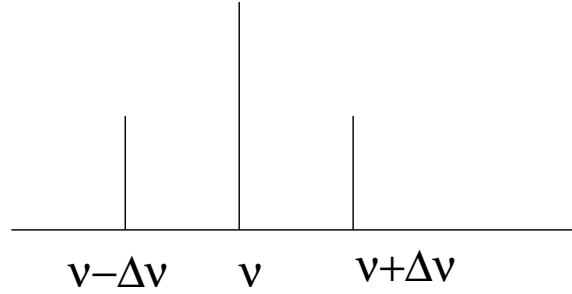


Figure 14.2: Sidebands

actually be transmitting modulated signal in the frequency range 790 KHz to 810 KHz.

If our radio receiver were so sensitive that it picks up only a very small range of frequency around 800 KHz, we would not be able to hear the sound that is being transmitted as the higher frequency components would be missing.

Further, if there were two stations one at 800 KHz and another at 805 KHz, the transmissions from the two stations would overlap and we would get a garbage sound from our receivers. The stations should transmit at frequencies that are sufficiently apart so that they do not overlap. Typically the frequency range 500 KHz to 1500 KHz is available for AM transmission and it is possible to accommodate a large number of stations.

We next consider the full position and time dependence of the superposition of two waves of different frequencies. Assuming equal amplitudes for the two waves we have,

$$A(t) = A[e^{i(\omega_1 t - k_1 x)} + e^{i(\omega_2 t - k_2 x)}]. \quad (14.10)$$

Proceeding in exactly the same way as when we considered only the time dependence, we now have,

$$A(t) = 2A \cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right) e^{i(\bar{\omega}t - \bar{k}x)}, \quad (14.11)$$

where $\bar{\omega} = (\omega_2 + \omega_1)/2$ and $\bar{k} = (k_2 + k_1)/2$ are the mean angular frequency and wave number respectively, and $\Delta\omega = \omega_2 - \omega_1$ and $\Delta k = k_2 - k_1$ are the difference in the angular frequency and wave number respectively.

Let us consider a situation where the two frequencies are very close such that $\Delta\omega \ll \bar{\omega}$ and $\Delta k \ll \bar{k}$ the resultant (equation (14.11)) can then be interpreted as a travelling wave with angular frequency and wave number $\bar{\omega}$ and \bar{k} respectively. This wave has a phase velocity,

$$v_p = \bar{\omega}/\bar{k}. \quad (14.12)$$

The amplitude of this wave undergoes a slow modulation. The modulation itself is a travelling wave that propagates at a speed $\frac{\Delta\omega}{\Delta k}$. The speed at which

the modulation propagates is called the group velocity, and we have

$$v_g = \frac{d\omega}{dk}. \quad (14.13)$$

As discussed earlier, it is possible to transmit signals using waves by modulating the amplitude. Usually (but not always) signals propagate at the group velocity.

There are situations where the phase velocity is greater than the speed of light in vacuum, but the group velocity usually comes out to be smaller. In all cases it is found that no signal propagates at a speed faster than the speed of light in vacuum. This is one of the fundamental assumptions in Einstein's Theory of Relativity.

Problems

1. Consider the superposition of two waves with different angular frequencies $\nu_1 = 200$ Hz and $\nu_2 = 202$ Hz. The two waves are in phase at time $t = 0$. [a.] After how much time are the two waves exactly out of phase and when are they exactly in phase again? [b.] What happens to the intensity of the superposed wave?
2. Consider the superposition of two fast oscillating signals

$$A(t) = 2 \cos(\omega_1 t) + 3 \sin(\omega_2 t)$$

with $\omega_1 = 1.0 \times 10^3 \text{s}^{-1}$ and $\omega_2 = 1.01 \times 10^3 \text{s}^{-1}$. The intensity of the resulting signal $A(t)$ is found to have beats where the intensity oscillates slowly.

- (i.) What is the time period of the beats?
 - (ii.) What is the ratio of the minimum intensity to the maximum intensity?
3. The amplitude of a carrier wave of frequency $\nu = 1$ MHz is modulated with the signal $f(t) = \cos^3(\omega t)$ where $\omega = 2\pi \times 20 \text{s}^{-1}$. What are the frequencies of the different side bands?
 4. The two strings of a guitar which is being tuned are found to produce beats of time period 1/10 s. Also, the minimum intensity is 20% of the maximum intensity. What is the frequency difference between the two strings? What is the ratio of the vibration amplitudes in the two strings?
 5. The refractive index of x-rays inside materials, is

$$n = 1 - \frac{a}{\omega^2}, \quad (14.14)$$

where a is a constant whose value depends on the properties of the material. Calculate the phase velocity and the group velocity.

Solution

$$v_p = \frac{c}{n} = \frac{c}{1 - \frac{a}{\omega^2}} > c, \quad (14.15)$$

$$k = \frac{\omega}{c} - \frac{a}{c\omega}, \quad (14.16)$$

$$v_g = \frac{c}{1 + \frac{a}{\omega^2}} < c. \quad (14.17)$$

6. Consider the superposition of two waves,

$$\tilde{A}(x, t) = e^{i(\omega_1 t - k_1 x)} + e^{i(\omega_2 t - k_2 x)},$$

with wavelengths $\lambda_1 = 1$ m and $\lambda_2 = 1.2$ m. The wave has a dispersion relation,

$$\omega = c\sqrt{k^2 + 0.1k^4}.$$

Treating the wave as a slow modulation on a faster carrier wave,

- a. What are the angular frequency, wave number and phase velocity of the carrier wave?
 - b. What are the angular frequency and wave number of the modulation?
 - c. At what speed does the modulation propagate?
7. The dispersion relation for free relativistic electron waves is $\omega = (c^2 k^2 + \frac{m_e^2 c^4}{\hbar^2})^{1/2}$. Show that the product of phase and group velocity of the wave is a constant.
8. A wave packet in a certain medium is represented by the following

$$\tilde{A}(x, t) = 4 \cos(0.1x - 0.2t) \cos(x - 10t) \cos(0.05x - 0.1t).$$

Find group velocity and phase velocity for the packet. Plot the phase velocity in the medium as a function of wave number k , near $k=1$.

Chapter 15

The wave equation.

15.1 Longitudinal elastic waves

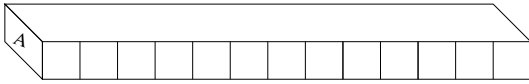


Figure 15.1: An elastic beam

Consider a beam made of elastic material with cross sectional area A as shown in Figure 15.1. Lines have been shown at a uniform spacing along the length of the beam .

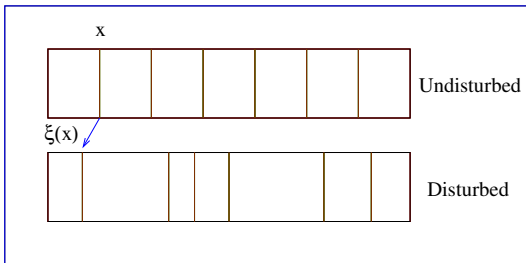


Figure 15.2: Compression and rarefaction in a disturbed beam

A disturbance is introduced in this beam as shown in Figure 15.2. This also shows the undisturbed beam. The disturbance causes the rod to be compressed at some places (where the lines have come closer) and to get rarefied at some other places (where the lines have moved apart). The beam is made of elastic material which tries to oppose the deformation i.e. the compressed region tries to expand again to its original size and same with the rarefied region. We would like to study the behaviour of these disturbances in an elastic beam.

Let us consider the material originally at the point x of the undisturbed beam (Figure15.2). This material is displaced to $x + \xi(x)$ where $\xi(x)$ is the horizontal displacement of the point x on the rod. What happens to an elastic solid when it is compressed or extended?

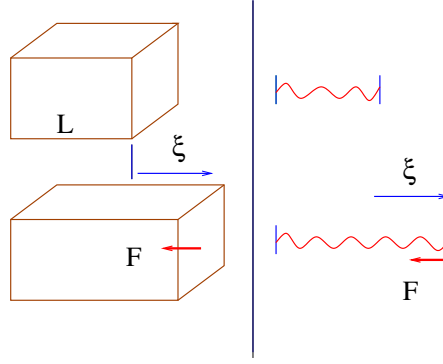


Figure 15.3: A rarefied section of the beam

$$\text{Stress} = F/A$$

$$\text{Strain} = \frac{\xi}{L}$$

$$Y = \frac{\text{Stress}}{\text{Strain}} \quad (\text{Young's modulus}) \quad (15.1)$$

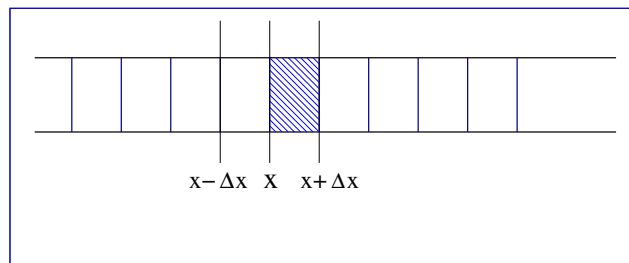
$$F = \left(\frac{YA}{L} \right) \xi \quad (15.2)$$

$$F = k\xi \rightarrow \text{Spring} \quad (15.3)$$

Coming back to our disturbed beam, let us divide it into small slabs of length Δx each. Each slab acts like a spring with spring constant

$$k = \frac{YA}{\Delta x}.$$

We focus our attention to one particular slab (shaded below).



Writing the equation of motion for this slab we have,

$$\Delta x \rho A \frac{\partial^2 \xi(x, t)}{\partial t^2} = F \quad (15.4)$$

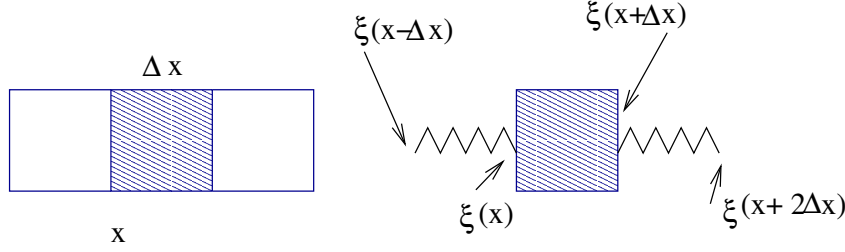


Figure 15.4: One particular slab in the beam

where ρ is the density of the rod, $\rho A \Delta x$ the mass of the slab, $\partial^2 \xi(x, t) / \partial t^2$ its acceleration. F denotes the total external forces acting on this slab.

The external forces arise from the adjacent slabs which are like springs.

This force from the spring on the left is

$$F_L = -k [\xi(x, t) - \xi(x - \Delta x, t)] \quad (15.5)$$

$$\approx -YA \frac{\partial \xi}{\partial x}(x, t). \quad (15.6)$$

The force from the spring on the right is

$$F_R = -k [\xi(x + \Delta x, t) - \xi(x + 2\Delta x, t)] \quad (15.7)$$

$$\approx YA \frac{\partial \xi}{\partial x}(x + \Delta x, t). \quad (15.8)$$

The total force acting on the shaded slab $F = F_L + F_R$ is

$$F = YA \frac{\partial}{\partial x} [\xi(x + \Delta x, t) - \xi(x, t)] \quad (15.9)$$

$$\approx YA \Delta x \frac{\partial^2}{\partial x^2} \xi(x, t) \quad (15.10)$$

Using this in the equation of motion of the slab (eq. 15.4) we have

$$\rho A \Delta x \frac{\partial^2 \xi}{\partial t^2} = YA \Delta x \frac{\partial^2 \xi}{\partial x^2} \quad (15.11)$$

which gives us

$$\frac{\partial^2 \xi}{\partial x^2} - \left(\frac{\rho}{Y} \right) \frac{\partial^2 \xi}{\partial t^2} = 0 \quad (15.12)$$

This is a wave equation. Typically the wave equation is written as

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{1}{c_s^2} \frac{\partial^2 \xi}{\partial t^2} = 0 \quad (15.13)$$

where c_s is the phase velocity of the wave. In this case

$$c_s = \sqrt{\frac{Y}{\rho}}$$

In three dimensions the wave equation is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \xi - \frac{1}{c_s^2} \frac{\partial^2 \xi}{\partial t^2} = 0. \quad (15.14)$$

This is expressed in a compact notation as

$$\nabla^2 \xi - \frac{1}{c_s^2} \frac{\partial^2 \xi}{\partial t^2} = 0 \quad (15.15)$$

where ∇^2 denotes the Laplacian operator defined as

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

We next check that the familiar sinusoidal plane wave discussed earlier

$$\xi(x, t) = \tilde{a} e^{i(\omega t - kx)} \quad (15.16)$$

is a solution of the wave equation. Substituting this in the wave equation (15.15) gives us

$$k^2 = \frac{\omega^2}{c_s^2}. \quad (15.17)$$

Such a relation between the wave vector \vec{k} and the angular frequency ω is called a dispersion relation. We have

$$\omega = \pm c_s k \quad (15.18)$$

which tells us that the constant c_s which appears in the wave equation is the phase velocity of the wave.

15.2 Transverse waves in stretched strings

In this section we discuss the transverse vibrations in stretched strings. We consider a uniform stretched string having a tension T . We take a particular section of this string which is disturbed from its mean position as shown in the figure 15.5 below. The displacement of the point x on the string at time t is denoted with $\xi(x, t)$. Further we assume that the disturbances are small and strictly orthogonal to the undisturbed string.

The horizontal component of the force is,

$$F_x = T_2 \cos \theta_2 - T_1 \cos \theta_1 = 0, \quad (15.19)$$

where T_1 and T_2 are the new tensions at points x and $x + \Delta x$ respectively. Hence,

$$T_2 \cos \theta_2 = T_1 \cos \theta_1 = T. \quad (15.20)$$

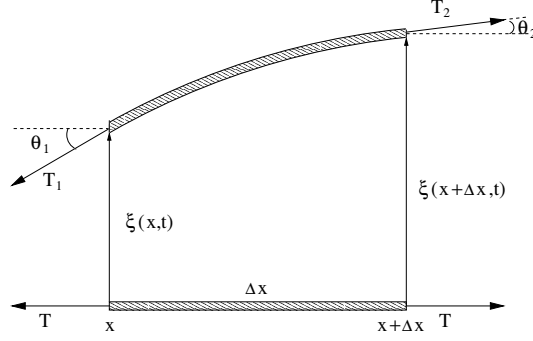


Figure 15.5: A particular section of the string

Now coming to the vertical component of the force, we have,

$$F_y = T_2 \sin \theta_2 - T_1 \sin \theta_1, \quad (15.21)$$

$$F_y = T_2 \cos \theta_2 \tan \theta_2 - T_1 \cos \theta_1 \tan \theta_1. \quad (15.22)$$

Using the equation (15.20), we obtain,

$$F_y = T \tan \theta_2 - T \tan \theta_1. \quad (15.23)$$

Now $\tan \theta$ at a particular point is nothing but the slope at that point of the disturbed string, so we can write,

$$F_y = T \left(\frac{\partial \xi(x + \Delta x, t)}{\partial x} \right) - T \left(\frac{\partial \xi(x, t)}{\partial x} \right), \quad (15.24)$$

$$F_y = T \frac{\partial}{\partial x} (\xi(x + \Delta x, t) - \xi(x, t)), \quad (15.25)$$

$$F_y = T \left(\frac{\partial^2 \xi}{\partial x^2} \right) \Delta x. \quad (15.26)$$

The above force would produce the vertical acceleration in that particular section of the string. Hence,

$$F_y = \mu \Delta x \left(\frac{\partial^2 \xi}{\partial t^2} \right), \quad (15.27)$$

where μ is the mass per unit length of the string. Now from equations (15.26) and (15.27) we have,

$$\left(\frac{\partial^2 \xi}{\partial x^2} \right) = \frac{\mu}{T} \left(\frac{\partial^2 \xi}{\partial t^2} \right), \quad (15.28)$$

which can be again written as equation (15.13), that is,

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{1}{c_s^2} \frac{\partial^2 \xi}{\partial t^2} = 0, \quad (15.29)$$

where now the phase velocity, c_s , of the wave is equal to $\sqrt{T/\mu}$.

15.3 Solving the wave equation

15.3.1 Plane waves

We consider a disturbance which depends on only one position variable x . For example x could be along the length of the beam. We have the wave equation

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \xi}{\partial t^2} = 0. \quad (15.30)$$

In solving the wave equation it is convenient to introduce two new variables

$$\begin{aligned} w_1 &= x + ct \text{ and } w_2 = x - ct \\ x &= \frac{w_1 + w_2}{2} \text{ and } t = \frac{w_1 - w_2}{2c} \end{aligned}$$

We can represent $\xi(x, t)$ as a function of w_1 and w_2 i.e. $\xi(w_1, w_2)$
Also

$$\begin{aligned} \frac{\partial \xi}{\partial x} &= \frac{\partial w_1}{\partial x} \frac{\partial \xi}{\partial w_1} + \frac{\partial w_2}{\partial x} \frac{\partial \xi}{\partial w_2} \\ &= \frac{\partial \xi}{\partial w_1} + \frac{\partial \xi}{\partial w_2} \end{aligned}$$

Similarly

$$\frac{1}{c} \frac{\partial \xi}{\partial t} = \frac{\partial \xi}{\partial w_1} - \frac{\partial \xi}{\partial w_2} \quad (15.31)$$

Using these the wave equation becomes

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \xi}{\partial t^2} = \left(\frac{\partial}{\partial w_1} + \frac{\partial}{\partial w_2} \right)^2 \xi - \left(\frac{\partial}{\partial w_1} - \frac{\partial}{\partial w_2} \right)^2 \xi = 0 \quad (15.32)$$

which gives us the condition

$$\frac{\partial^2}{\partial w_1 \partial w_2} \xi(w_1, w_2) = 0 \quad (15.33)$$

Possible solutions:-

1. $\xi(w_1, w_2) = \text{Constant}$ (not of interest)
2. $\xi(w_1, w_2) = f_1(w_1)$ (function of w_1 alone.)
3. $\xi(w_1, w_2) = f_2(w_2)$ (function of w_2 alone.)

Any linear superposition of 2 and 3 above is also a solution.

$$\xi(w_1, w_2) = c_1 f_1(w_1) + c_2 f_2(w_2) \quad (15.34)$$

To physically interpret these solutions we revert back to (x, t) .

Let us first consider solution 2 i.e. any arbitrary function of $w_1 = x + ct$.

$$\xi(x, t) = f_1(x + ct) \quad (15.35)$$

At $t = 0$ we have

$$\xi(x, 0) = f_1(x) \quad (15.36)$$

at $t = 1$ we have $\xi(x, 1) = f_1(x + c)$ i.e the origin $x = 0$ has now shifted to $x = -c$. and at $t = \tau$, the origin shift to $x = -\tau/c$. The form of the disturbance remains unchanged and the disturbance propagates to the left i.e. along the $-x$ direction with speed c as shown in the left panel of Figure 15.6.

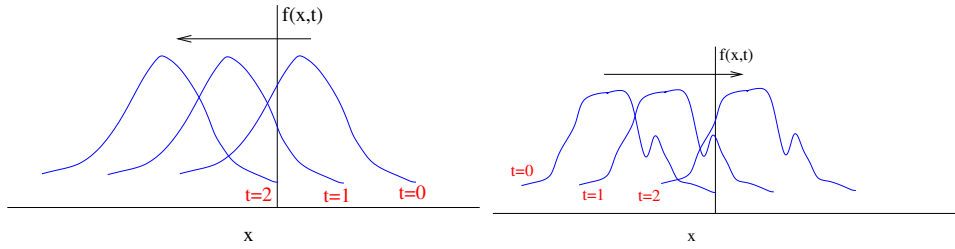


Figure 15.6: A left moving wave and a right moving wave

Similarly, the solution 3 corresponds to a right travelling wave

$$\xi(x, t) = f_2(x - ct) \quad (15.37)$$

which propagates along $+x$ with speed c as shown in the right panel of Figure 15.6

$$\xi(x, t) = a_1 f_1(x + ct) + a_2 f_2(x - ct) \quad (15.38)$$

Any arbitrary combination of a left travelling solution and a right travelling solution is also a solution to the wave equation

The value of ξ is constant on planes perpendicular to the x axis and these solutions are plane wave solutions. The sinusoidal plane wave

$$\xi(x, t) = a \cos[k(x - ct)] \quad (15.39)$$

that we have studied earlier is a special case of the more general plane wave solution.

15.3.2 Spherical waves

We now consider spherically symmetric solutions to the wave equation

$$\nabla^2 \xi - \frac{1}{c_s^2} \frac{\partial^2 \xi}{\partial t^2} = 0. \quad (15.40)$$

where $\xi(\vec{r}, t) = \xi(r, t)$ depends only on the distance r from the origin. It is now convenient to use the spherical polar coordinates (r, θ, ϕ) instead of (x, y, z) , and the wave equation becomes

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \xi}{\partial r} \right) - \frac{1}{c_s^2} \frac{\partial^2 \xi}{\partial t^2} = 0. \quad (15.41)$$

Substituting $\xi(r, t) = u(r, t)/r$ we have

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} - u \right) - \frac{1}{rc_s^2} \frac{\partial^2 u}{\partial t^2} = 0. \quad (15.42)$$

which gives us the wave equation in a single variable r ,

$$\frac{\partial^2 u}{\partial r^2} - \frac{1}{c_s^2} \frac{\partial^2 u}{\partial t^2} = 0. \quad (15.43)$$

whose solutions are already known. Using these we have

$$\xi(r, t) = \frac{f_1(r + ct)}{r} + \frac{f_1(r - ct)}{r} \quad (15.44)$$

which is the most general spherical wave solution. The first part of the solution $f_1(r + ct)$ represents a spherical wave travelling towards the origin and the second part $f_1(r - ct)$ represents a wave travelling out from the origin as shown in the left and right panels of Figure 15.7 respectively. In both cases the amplitude varies as $1/r$ and the solution is singular at $r = 0$.

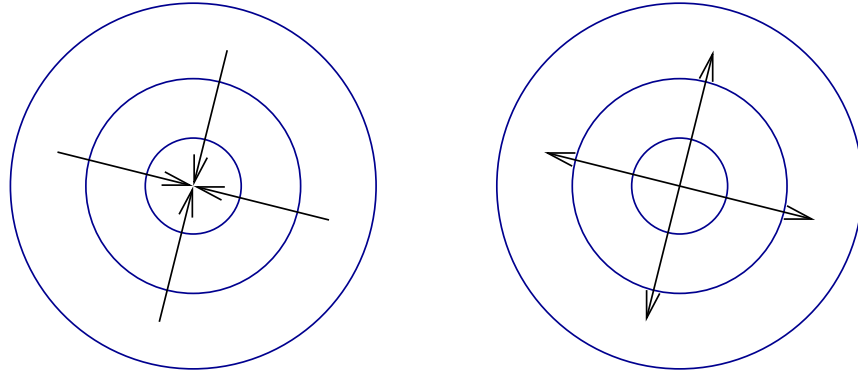


Figure 15.7: Spherical waves

15.3.3 Standing Waves

We consider a stretched string of length L as shown in Figure 15.8. The string is plucked and left to vibrate. In this case we have a transverse wave where $\xi(x, t)$ the displacement of the string is perpendicular to the direction of the string which is along the x axis.

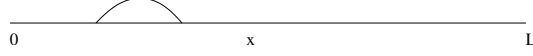


Figure 15.8: A string fixed at the ends

As we have seen in the earlier section that the evolution of $\xi(x, t)$ is governed by the wave equation,

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{1}{c_s^2} \frac{\partial^2 \xi}{\partial t^2} = 0, \quad (15.45)$$

where $c_s = \sqrt{T/\mu}$. Here T is the tension in the string and μ is the mass per unit length of the string. The two ends of the string are fixed. This imposes the boundary conditions $\xi(0, t) = 0$ and $\xi(L, t) = 0$. We could proceed by taking the general form of the solution and imposing the boundary conditions. Instead of doing this we proceed to introduce a different method of solving the wave equation. We take a trial solution of the form,

$$\xi(x, t) = X(x)T(t), \quad (15.46)$$

ie. $\xi(x, t)$ is the product of two functions, the function $X(x)$ depends only on x and the function $T(t)$ depends only on t . This is referred to as the separation of variables. The wave equation now reads,

$$T \frac{d^2 X}{dx^2} - \frac{X}{c_s^2} \frac{d^2 T}{dt^2} = 0. \quad (15.47)$$

We divide this equation by XT , which gives,

$$\frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{T c_s^2} \frac{d^2 T}{dt^2}. \quad (15.48)$$

The left hand side of this equation is a function of x alone whereas the right hand side is a function of t alone. This implies that each of these two should be separately equal to a constant *ie.*

$$\frac{1}{X} \frac{d^2 X}{dx^2} = \alpha, \quad (15.49)$$

and

$$\frac{1}{T c_s^2} \frac{d^2 T}{dt^2} = \alpha. \quad (15.50)$$

Let us first consider the solution to $X(x)$. These are of the form,

$$X(x) = B_1 e^{\sqrt{\alpha}x} + B_2 e^{-\sqrt{\alpha}x}. \quad (15.51)$$

In the situation where $\alpha > 0$, it is not possible to simultaneously satisfy the two boundary condition that $\xi(0, t) = 0$ and $\xi(L, t) = 0$. We therefore consider $\alpha < 0$ and write it as,

$$\alpha = -k^2, \quad (15.52)$$

and the equation governing $X(x)$ is

$$\frac{d^2 X}{dx^2} = -k^2 X. \quad (15.53)$$

This is the familiar differential equation of a Simple Harmonic Oscillator whose solution is,

$$X(x) = A \cos(kx + \psi). \quad (15.54)$$

The boundary condition $X(0) = 0$ implies that $\psi = \pm\pi/2$ whereby

$$X(x) = A \sin(kx). \quad (15.55)$$

The boundary condition $X(L) = 0$ is satisfied only if,

$$k = \frac{N\pi}{L}, \quad (N = 1, 2, 3, \dots). \quad (15.56)$$

We see that there are a large number of possible solutions, one corresponding to each value of the integer $N = 1, 2, 3, \dots$. Let us next consider the time dependence which is governed by,

$$\frac{d^2 T}{dt^2} = -c_s^2 k^2 T, \quad (15.57)$$

which has a solution,

$$T(t) = B \cos(\omega t + \phi), \quad (15.58)$$

where $\omega = c_s k$. Combining $X(x)$ and $T(t)$ we obtain the solution,

$$\xi(x, t) = A_N \sin(k_N x) \cos(\omega_N t + \phi_N), \quad (15.59)$$

corresponding to each possible value of the integer N . These are standing waves and each value of N defines a different mode of the standing wave. The solution with $N = 1$ is called the fundamental mode or first harmonic. We have,

$$\xi(x, t) = A_1 \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{c_s \pi t}{L} + \phi_1\right), \quad (15.60)$$

which is shown in the left panel of Figure 15.9. The fundamental mode has wavelength $\lambda_1 = 2L$ and frequency $\nu_1 = c_s/2L$.

The second harmonic,

$$\xi(x, t) = A_2 \sin\left(\frac{2\pi x}{L}\right) \cos\left(\frac{c_s 2\pi t}{L} + \phi_2\right), \quad (15.61)$$

which is shown in the right panel of Figure 15.9 has wavelength $\lambda_2 = L$ and frequency $\nu_2 = c_s/L$. The higher harmonics have wavelengths $\lambda_3 = \lambda_1/3$, $\lambda_4 = \lambda_1/4$, $\lambda_5 = \lambda_1/5$ and frequencies $\nu_3 = 3\nu_1$, $\nu_4 = 4\nu_1$, $\nu_5 = 5\nu_1$ respectively.

Each standing wave is a superposition of a left travelling and a right travelling wave. For example,

$$\sin(\omega_1 t + k_1 x) - \sin(\omega_1 t - k_1 x) = 2 \sin(k_1 x) \cos(\omega_1 t), \quad (15.62)$$

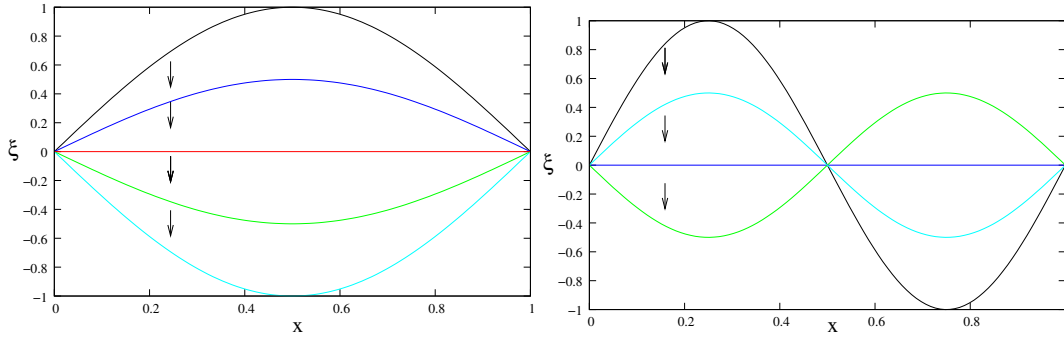
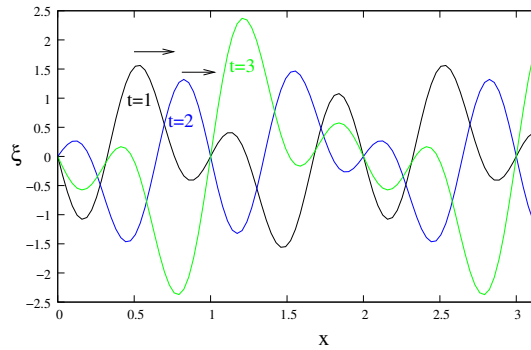
Figure 15.9: Standing wave modes: 1st and 2st harmonics

Figure 15.10: Arbitrary travelling wave in a string

gives the fundamental mode. At all times the left travelling and right travelling waves exactly cancel at $x = 0$ and $x = L$.

Any arbitrary disturbance of the string can be expressed as a sum of standing waves

$$\xi(x, t) = \sum_{N=1}^{\infty} A_N \sin(k_N x) \cos(\omega_N t + \phi_N) \quad (15.63)$$

The resultant disturbance will in general not be a standing wave but will travel along the string as shown in Figure 15.10.

Question : If a string which is fixed at both the ends is plucked at an arbitrary point then which of the modes will not be excited?

Answer : Read about Young-Helmholtz law.

Problems

1. Consider the wave equations

$$\left[\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \psi(\vec{r}, t) = \frac{1}{L^2} \psi(\vec{r}, t)$$

- What is the dispersion relation for this wave equation?
 - Calculate the phase velocity and the group velocity.
 - Analyze the behaviour when $k \ll 1/L$ and $k \gg 1/L$
2. For the wave equation given below (where c_s is a constant)

$$\left[4 \frac{\partial^2}{\partial x^2} + 9 \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c_s^2} \frac{\partial^2}{\partial t^2} \right] \xi(\vec{r}, t) = 0$$

- What is the speed of a travelling wave solution propagating along the x axis?
- What is the speed of a travelling wave solution propagating along the y axis?
- For what value of b is the travelling wave given below a solution of the wave equation given above?

$$\xi(\vec{r}, t) = e^{-[bx+y-5c_s t]^2}$$

- What is the speed of the travelling wave given above ?
3. Consider the longitudinal disturbance

$$\xi(x) = \frac{4}{4 + x^2}$$

inside an elastic rod where $c_s = 2\text{m/s}$.

- Plot the disturbance as a function of x .
 - What is $\xi(x, t)$ if the disturbance is a travelling wave moving along $+x$?
 - Plot the disturbance as a function of x at $t = 3$ s.
4. Which of the following are travelling waves? If yes, what is the speed?
- $\xi(x, t) = \sin^2[\pi(ax + bt)]$
 - $\xi(x, t) = \sin[\pi(ax + bt)^2]$
 - $\xi(x, t) = \sin^2[\pi(ax^2 + bt)]$
 - $\xi(x, t) = e^{-[ax^2 + bt^2 + 2\sqrt{ab}xt]}$
 - $\xi(\vec{r}, t) = e^{-[ax+by-t]^2/L^2}$
5. Consider a spherical wave $\xi(r, t) = a \sin[k(r - c_s t)]/r$ with $k = 3, \text{ m}^{-1}$ and $c_s = 330 \text{ ms}^{-1}$.

- a. What is the frequency of this wave ?
- b. How much does the amplitude of this wave change over $\Delta r = 2\pi/k$?
- c. In which direction does this wave propagate?

6. A longitudinal travelling wave

$$\xi(x, t) = Ae^{-[x - c_s t]^2/L^2} \quad [A = 10^{-4}\text{m}, L = 1\text{m}]$$

passes through a long steel rod for which $\rho = 8000\text{ kg/m}^3$ and $Y = 200 \times 10^9\text{ N/m}^2$.

- a. At which point is the displacement maximum at $t = 10^{-3}\text{s}$?
- b. Plot the displacement at $x = 5\text{ m}$ as a function of time.
- c. Plot the velocity of the steel at $x = 5\text{ m}$ as a function of time.
- d. When is the velocity of the steel zero at $x = 5\text{ m}$?

7. Consider a longitudinal wave

$$\xi(x, t) = A[\cos(\omega_1 t) \sin\left(\frac{\pi x}{L}\right) + \cos(\omega_2 t) \sin\left(\frac{2\pi x}{L}\right)]$$

in a steel rod of length $L = 10\text{ cm}$.

- a. What are the values of the angular frequencies ω_1 and ω_2 of the fundamental mode and the second harmonic respectively ?
- b. After what time period does the whole displacement profile repeat?

8. Consider a longitudinal standing wave

$$\xi(x, t) = A \cos(\omega_1 t) \sin\left(\frac{\pi x}{L}\right) \quad [A = 10^{-4}\text{m}]$$

in a steel rod of length $L = 5\text{ cm}$.

- a. What is the instantaneous kinetic energy per unit volume at $x = 2.5\text{cm}$?
 - b. What is the instantaneous potential energy per unit volume at $x = 2.5\text{cm}$?
 - c. What is the time averaged kinetic energy per unit volume at $x = 2.5\text{cm}$?
 - d. What is the time averaged potential energy per unit volume at $x = 2.5\text{cm}$?
9. Two steel rods, one 1m and another 1 cm longer are both vibrating in the fundamental mode of longitudinal standing waves. What is the time period of the beats that will be produced?
10. Write the three dimensional Laplacian operator of equation (15.40) in spherical polar co-ordinates, and hence obtain the equation (15.41).

Chapter 16

Polarization

We have learnt that light is an electromagnetic wave that can have different states of polarization. In this chapter we shall discuss how to produce polarized light and manipulate the polarization state.

16.1 Natural Radiation

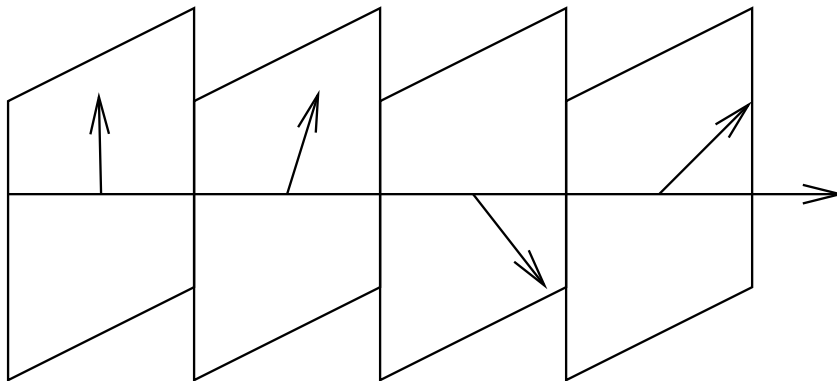
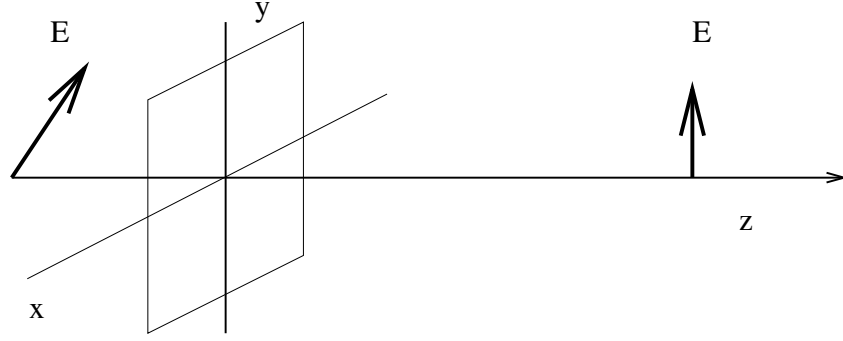


Figure 16.1: The electric field in an ordinary light

What is the polarization state of the light from a natural source like the sun or from an incandescent lamp? Such sources may be thought of as a collection of a large number of incoherent sources. The radiation is the sum of the contributions from all of these sources. The electric field \vec{E} of the resulting light wave varies randomly (Figure 16.1) in the plane perpendicular to the direction in which the wave propagates. This randomly polarized light is referred to as unpolarized light.

Figure 16.2: A linear polarizer producing a polarized light along the y direction

16.2 Producing polarized light

There are devices called polarizers which produce linearly polarized light. Figure 16.2 shows a polarizer placed in the path of light travelling along the z axis. The electric field \vec{E} of the incident light can oscillate in any direction in the $x - y$ plane. The polarizer permits only the \vec{E} component parallel to the transmission axis of the polarizer to pass through. Here we consider a polarizer whose transmission axis is along the y direction. So the light that emerges from the polarizer is linearly polarized along the y axis.

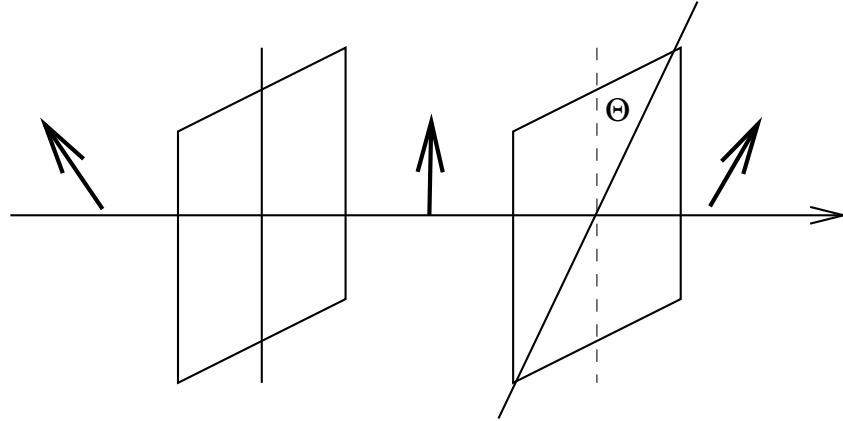


Figure 16.3: Consecutive polarizers

We next consider two polarizers whose transmission axes have an angle θ between them as shown in Figure 16.3. The light is linearly polarized after the first polarizer with

$$\vec{E}(z, t) = E_0 \cos(\omega t - kz) \hat{j} \quad (16.1)$$

and intensity $I_0 = E_0^2/2$. The second polarizer allows only the \vec{E} component along its transmission axis to pass through. The light that emerges is

linearly polarized along \hat{e} the transmission axis of the second polarizer. The transmitted wave is

$$\vec{E}(z, t) = E_0 \cos \theta \cos(\omega t - kz) \hat{e} \quad (16.2)$$

The amplitude of the wave goes down by $\cos \theta$ and the intensity is $I = E_0^2 \cos^2 \theta / 2 = \cos^2 \theta I_0$. We see that linearly polarized light of intensity I_0 has intensity $I_0 \cos^2 \theta$ after it passes through a polarizer. Here θ is the angle between the transmission axis of the polarizer and the polarization direction of the incident light. This is known as Malus's Law.

There are essentially four methods to produce polarized light from unpolarized light: [a.] Dichroism [b] Scattering [c.] Reflection [d.] Birefringence.

16.2.1 Dichroism or selective absorption

A dichroic polarizer absorbs the \vec{E} component perpendicular to the transmission axis. Unpolarized light gets converted to polarized light. A wire mesh polarizer is an example. Consider unpolarized light incident on a wire mesh as shown in Figure 16.4.

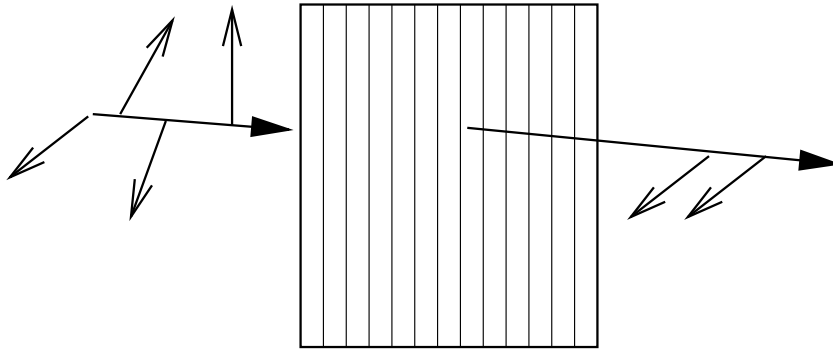


Figure 16.4: A wire mesh polarizer

The \vec{E} component parallel to the wires in the mesh sets up currents in the wires and this component is not allowed to go through the mesh. The perpendicular component is allowed to go through unaffected. The light that emerges is linearly polarized perpendicular to the wires in the mesh. It is relatively easy to construct a wire mesh with a few mm spacing, and this is a very effective polarizer for radiowaves and microwaves. Microscopic wire meshes produced by depositing gold or aluminium act like a polarizer for infrared waves.

There are dichroic crystals like tourmaline. These crystals have a preferred direction called the optic axis. For incident light the \vec{E} component perpendicular to the optic axis is strongly absorbed, the component parallel to the optic axis is allowed to pass through as shown in the Fig. 16.5. The light that emerges is linearly polarized parallel to the optic axis. The absorption

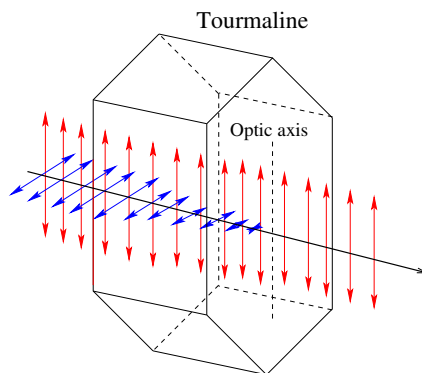


Figure 16.5: A dichroic crystal

properties of such crystals have a strong wavelength dependence whereby the crystal appears coloured.

In 1938 E.H. Land invented the polaroid H-sheet, possibly the most commonly used linear polarizer. A sheet of polyvinyl alcohol is heated and stretched in a particular direction causing the long-chained hydrocarbons to get aligned. The sheet is then dipped into ink that is rich in iodine. The sheet absorbs iodine which forms chains along the polymer chains. These iodine chains act like conducting wires and the whole sheet acts like a wire mesh. The \vec{E} component parallel to the chains is absorbed, and the light that passes through is linearly polarized in the direction perpendicular to that in which the sheet was stretched.

16.2.2 Scattering

Unpolarized light is incident on a molecule. The \vec{E} field of the incident light induces a dipole moment in the molecule. This oscillating dipole sends out radiation in different directions. This is the process of scattering.

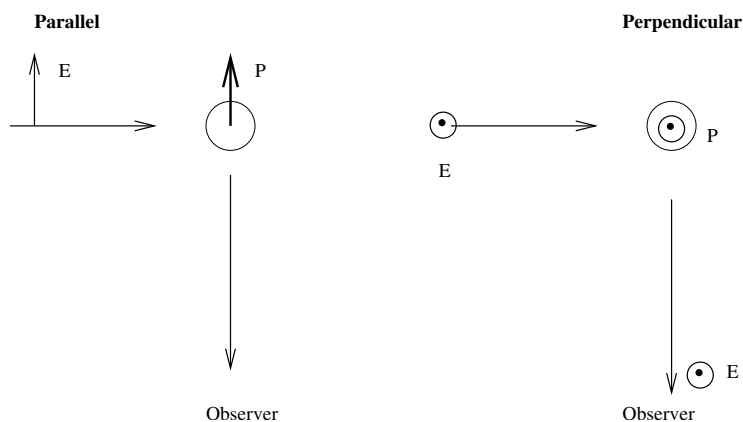


Figure 16.6: Scattering of light by a molecule

The incident light can be decomposed into two orthogonal linear polarizations respectively parallel and perpendicular to the plane of the page as shown in Figure 16.6. An observer O located at 90° to the incident direction observes the scattered light. The dipole produced by the parallel component of the incident \vec{E} does not produce any \vec{E} at the observer O. The observer O receives radiation only from the dipole oscillations perpendicular to the plane of the paper. The scattered radiation received at O is linearly polarized perpendicular to the plane of the paper.

During daytime the sky appears illuminated because of the sunlight scattered by the atmosphere (Figure 16.7). The scattered light is polarized by

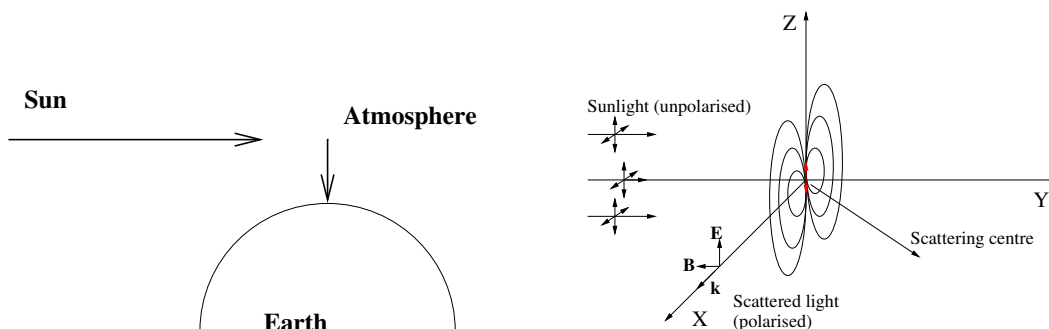


Figure 16.7: Polarization by scattering

the mechanism discussed here. The polarization is maximum (although not completely plane polarized) when the light is scattered at 90° to the incident direction. It is less partially polarized at other scattering angles.

16.2.3 Reflection

Light reflected from the surface of dielectric materials like glass or water is partially linearly polarized. We decompose the incident light into \vec{E} components parallel and perpendicular to the plane of the paper as shown in Figure 16.8.

To model the reflection at the surface we assume that the dielectric is a collection of dipoles which are set into oscillation by the electric field of the radiation inside the dielectric. The reflected wave is produced by the combined radiation of these oscillating dipoles. In the situation where

$$\theta_r + \theta_t = 90^\circ \quad (16.3)$$

the dipole produced by the component of \vec{E} parallel to the plane of the paper is aligned with the direction of the reflected wave. As a consequence the intensity of the reflected wave is zero for this component of linear polarization. The reflected wave is linearly polarized perpendicular to the plane of the paper. The angle of incidence at which this occurs is called the Brewster's angle θ_B (Figure 16.9) (also known as *polarization angle*). This can be calculated using

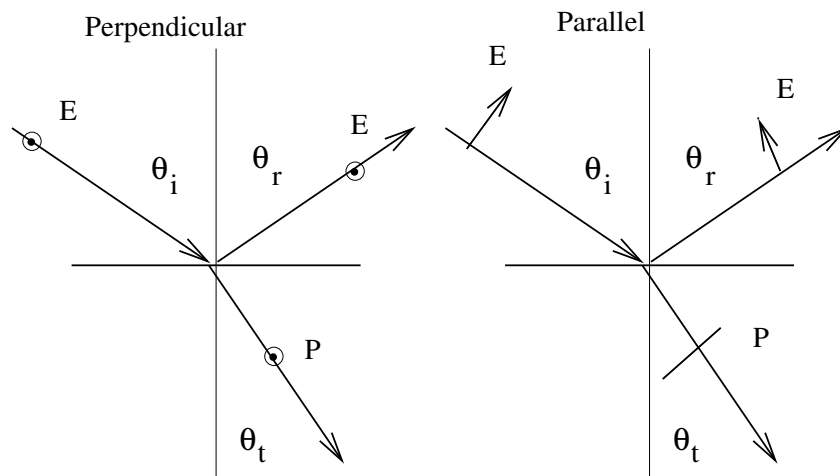


Figure 16.8: Polarization by reflection

$$n_i \sin \theta_B = n_r \sin \theta_t \quad (16.4)$$

and $\theta_t = 90^\circ - \theta_B$ whereby

$$n_i \sin \theta_B = n_r \cos \theta_B \quad (16.5)$$

and

$$\tan \theta_B = \frac{n_r}{n_i} \quad (16.6)$$

This is known as Brewster's Law.

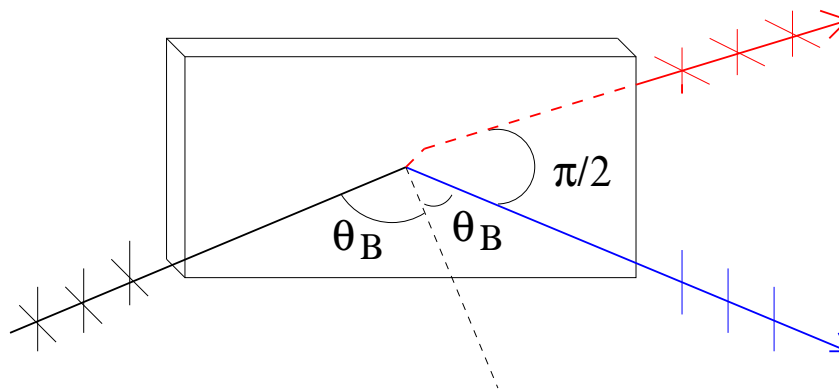


Figure 16.9: Brewster's angle

16.2.4 Birefringence or double refraction

Birefringence usually occurs in anisotropic crystals. In these crystals the arrangement of atoms is different in different directions. As a consequence the

crystal properties are direction dependent. As an example consider a crystal where the refractive index is n_e when the \vec{E} vector is along the x axis and it is n_o when \vec{E} is perpendicular to the x axis. Here the x axis is referred to as the optic axis. The optical properties are the same in all directions perpendicular to the optic axis and it is different if \vec{E} is along the optic axis. Crystals may have more than one optic axis. Here we only consider a situation where there is a single optic axis. Such birefringent crystals are referred to as uniaxial crystals. Through birefringent crystals one sees two images of the object unless the direction of viewing is particularly well chosen (in which case there is only one image) as shown in the Figure 16.10. So there are two refracted rays for each incident beam and hence the name. Optic axis of the crystal is in the vertical direction in the Figure. We find that the second image is slightly displaced from the original one. The image which is not displaced (refractive index n_o) is having a polarization perpendicular to the optic axis whereas the displaced image (with refractive index n_e) has a polarization along the optic axis of the crystal.

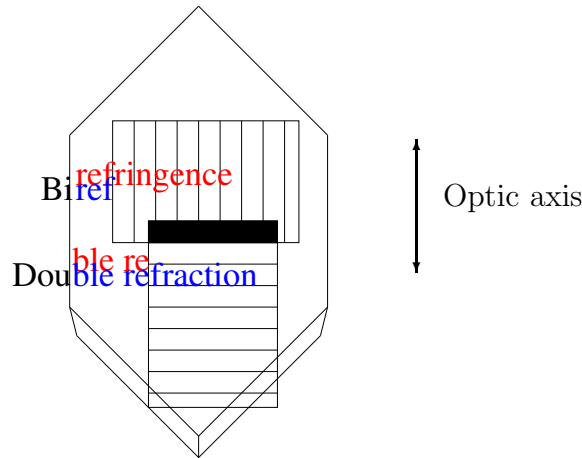


Figure 16.10: Birefringence in Calcite

The difference $\Delta n = n_e - n_o$ is a measure of the birefringence and is often called the birefringence. A material is referred to as a positive material if $\Delta n > 0$ and a negative material if $\Delta n < 0$. We tabulate below the refractive indices for some birefringent materials.

Crystal $\lambda_0 = 589.3 \text{ nm}$	n_o	n_e
Tourmaline	1.669	1.638
Calcite	1.6584	1.4864
Quartz	1.5443	1.5534
Sodium Nitrate	1.5854	1.3369

Nicol prism is a smart device often used in the laboratories to produce linearly polarized beam. Here two calcite pieces cut in a special way (SQP

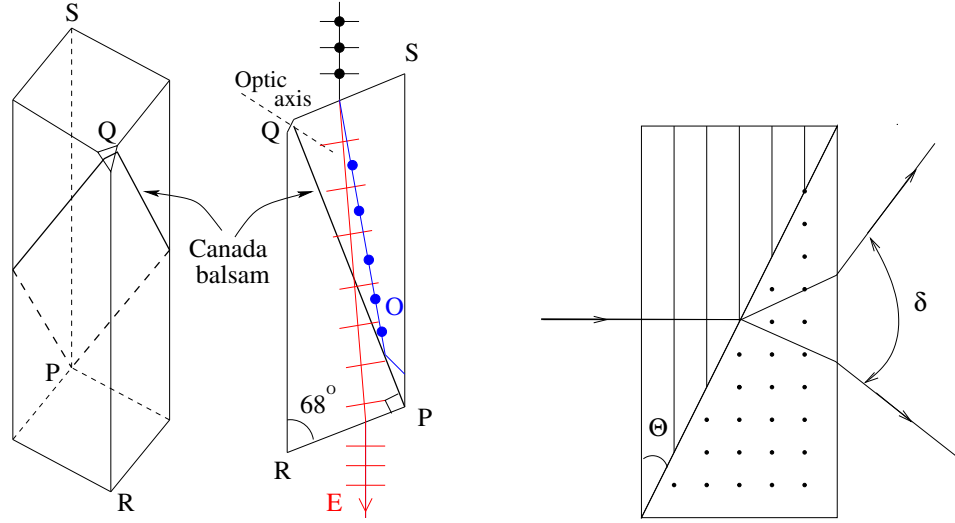


Figure 16.11: a) Nicol and 16.11: b) Wollaston prisms

and PQR) are cemented in a manner shown in the left of Figure 16.11. The two prisms are glued with a material called Canada balsam, a transparent material having a refractive index, $n = 1.55$, which is between the $n_o = 1.6584$ and $n_e = 1.4864$ of calcite. Once the unpolarized light is incident on the surface QS of the prism as shown in the middle of the Figure 16.11, it is divided into two rays due to birefringence. One of the rays is totally internally reflected by the layer of Canada balsam, QP, and is absorbed by a black paint on the wall, SP, of the crystal. The other ray transmits through the Canada balsam to produce a plane polarized light.

The Wollaston prism is a device that uses birefringence to separate the unpolarized incident light into two linearly polarized components. Two triangular prisms are glued together as shown in Figure 16.11. Both prisms are made of the same birefringent material. The optic axis of the two prisms are mutually perpendicular as shown in the Figure. We decompose the unpolarized incident light into \vec{E}_{\parallel} and \vec{E}_{\perp} respectively parallel and perpendicular to the plane of the paper. In the first prism \vec{E}_{\parallel} has refractive index n_e and \vec{E}_{\perp} has n_o . The situation reverses when the light enters the second prism where \vec{E}_{\parallel} has refractive index n_o and \vec{E}_{\perp} has n_e . Figure 16.11 shows the paths of the two polarizations through the Wollaston prism. The two polarizations part ways at the interface of the two prisms and they emerge in different directions as shown in Figure 16.11.

16.2.5 Quarter wave plate

Consider a birefringent crystal of thickness h with its optic axis along the y direction as shown in Figure 16.12. Consider light that passes through the crystal as shown in the figure. If we decompose the incident light into two

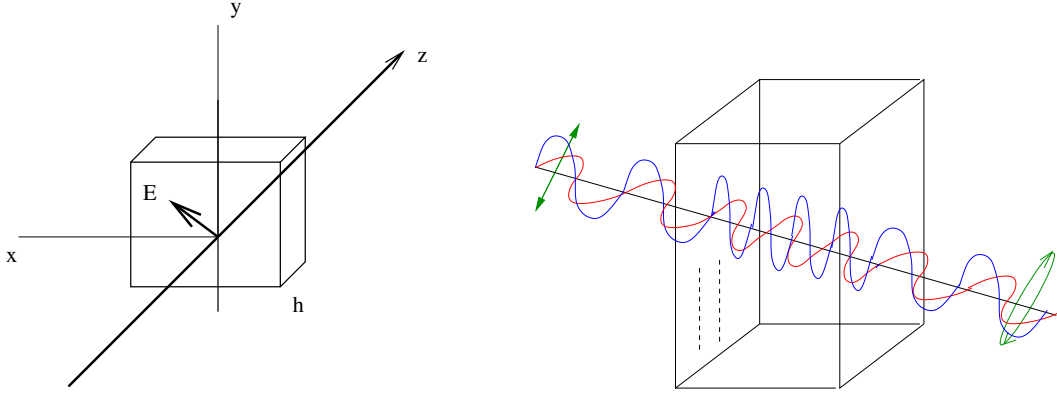


Figure 16.12: A quarter wave plate

perpendicular polarizations along the x and y axes respectively, these two polarizations will traverse different optical path lengths through the crystal. The optical path length is $n_o h$ for the x component and $n_e h$ for the y component respectively. The crystal is called a quarter wave plate if this difference in the optical path lengths is $\lambda/4$, *ie.*

$$|n_e - n_o| h = \frac{\lambda}{4} \quad (16.7)$$

and it is called a half wave plate if,

$$|n_e - n_o| h = \frac{\lambda}{2}. \quad (16.8)$$

A quarter wave plate can be used to convert linearly polarized light to circularly polarized light as shown in the right Figure 16.12. The incident light should be at 45° to the optic axis. The incident wave can be expressed as

$$\vec{E}(z, t) = E_0(\hat{i} + \hat{j}) \cos(\omega t - kz) \quad (16.9)$$

The quarter wave plate introduces a phase difference between the x and y polarizations. An optical path difference of $\lambda/4$ corresponds to a phase difference of 90° . The wave that comes out of the quarter wave plate can be expressed as

$$\vec{E}(z, t) = E_0[\hat{i} \cos(\omega t - kz) + \hat{j} \cos(\omega t - kz + 90^\circ)] \quad (16.10)$$

which is circularly polarized light.

The output is elliptically polarized if the angle between the incident linear polarization and the optic axis is different from 45° . The quarter wave plate can also be used to convert circularly polarized or elliptically polarized light to linearly polarized light. In contrast a half wave plate rotates the plane of polarization as shown in the left of the Figure 16.13.

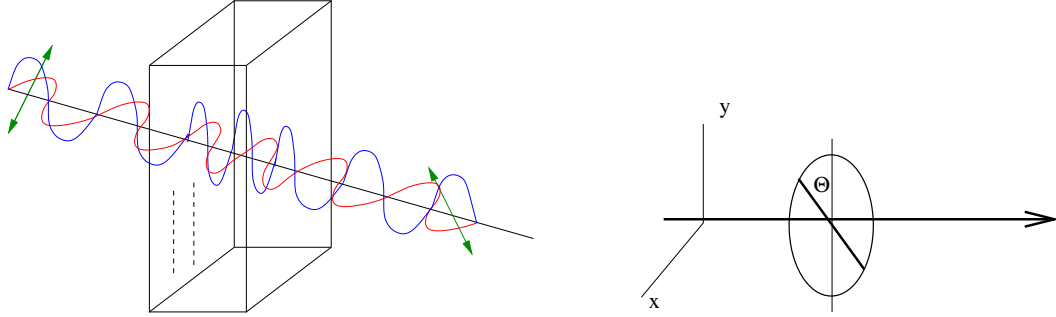


Figure 16.13: a) Half wave plate and 16.13: b) Analysis of partially polarized light

16.3 Partially polarized light

Many of the polarizers discussed earlier produce partially polarized light. This is a mixture of polarized and unpolarized light. Consider a mixture with unpolarized light of intensity I_U and light that is linearly polarized along the y axis of intensity I_P . A polaroid sheet, referred to as an analyzer, is used to analyze this light (right in the Figure 16.13). The intensity of the unpolarized light becomes $I_U/2$ after passing through the analyzer whereas the intensity of the polarized light is $I_P \cos^2 \theta$ where θ is the angle between the transmission axis of the analyzer and the y axis. The resulting intensity is

$$I = \frac{I_U}{2} + I_P \cos^2 \theta. \quad (16.11)$$

The intensity of the transmitted light changes as the analyzer is rotated and the maximum and minimum intensity respectively are

$$I_{max} = \frac{I_U}{2} + I_P \quad I_{min} = \frac{I_U}{2}. \quad (16.12)$$

The degree of polarization is defined as

$$P = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \frac{I_P}{I_U + I_P} \quad (16.13)$$

This quantifies the fraction of the light intensity in the linearly polarized component.

Problems

1. Consider polarized light whose \vec{E} is given to be

$$\vec{E}(z, t) = E_0 [A \hat{i} \cos(\omega t - kz) + B \hat{j} \sin(\omega t - kz)].$$

Calculate the degree of polarization P for [a.] $A = 1, B = 0$ [b.] $A = 1, B = 1$ [c.] $A = 1, B = 2$. What are the corresponding polarization states. What happens if these waves are sent through a quarter wave plate that adds an extra 90° delay to the y component?

2. Incident light is scattered by an electron. What is the degree of polarization for scattering angles [a.] 90° [b.] 45° ?
3. A birefringent crystal of thickness d has its optic axis parallel to the surface of the crystal. What should be the value of d (in μm) if the crystal is to be used as a quarter wave plate for light of wavelength $\lambda = 589.3\text{ nm}$? ($n_e = 1.5334$, $n_o = 1.5443$).
4. Linearly polarized light with intensity I is normally incident on a polarizer. The plane of polarization of the incident light is at 30° to the transmission axis of the polarizer. What is the intensity of the transmitted light?
5. For the Wollaston prism in Figure 16.11 with $\theta = 30^\circ$, $\lambda = 589\text{ nm}$, $n_e = 1.486$ and $n_o = 1.658$, calculate the angle δ between the two rays that come out.
6. Calculate the Brewster's angle θ_B for glass $n = 1.5$.

Chapter 17

Wave-particle duality

The phenomena of interference and diffraction lead us to believe that light is a wave. Further we have learnt that it is a transverse electromagnetic wave. But there are several experiments like the photo-electric effect and the Compton effect which cannot be explained by the wave theory of light. Here we briefly discuss the Compton effect.

17.1 The Compton effect

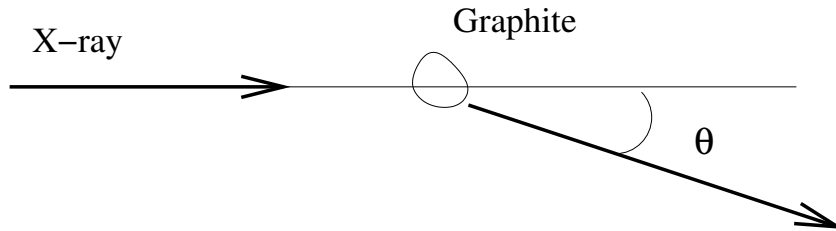


Figure 17.1: X-ray scattering

A nearly monochromatic X-ray beam of frequency ν is incident on a graphite sample as shown in Figure 17.1. The oscillating electric field of the incident electromagnetic wave causes the electrons in the graphite to oscillate at the same frequency ν . These oscillating electrons will emit radiation in all directions, the frequency of this radiation is expected to also be ν , same as the incident frequency. This process where the incident X-ray is scattered in different directions, its frequency being unchanged is called Thomson scattering. In addition to this, it is found that there is a component of scattered X-ray which has a smaller frequency ν' or larger wavelength λ' . The situation where there is a change in the frequency of the incoming light is referred to as the Compton effect. It is not possible to explain the Compton effect if we think of the incident X-ray as a wave.

To explain the Compton effect it is necessary to associate a particle called a photon (γ) with the incident electromagnetic wave,

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\omega t - \vec{k} \cdot \vec{r})}. \quad (17.1)$$

The momentum \vec{p} and energy E of the photon are related to the wave number and angular frequency respectively of the wave as

$$\vec{p} = \hbar \vec{k} \quad E = \hbar \omega. \quad (17.2)$$

where $\hbar = h/2\pi$ and $h = 6.63 \times 10^{-34}$ J s is the Planck's constant.

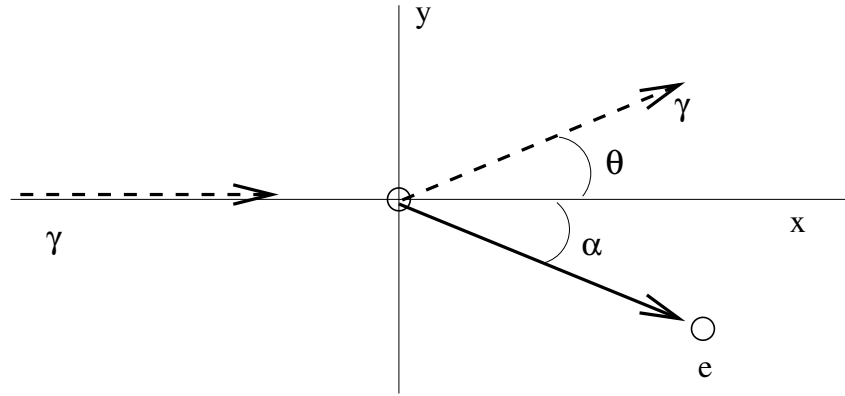


Figure 17.2: The Compton scattering

It is possible to explain the Compton effect if we think of it as the elastic scattering of a photon (γ) and an electron (e) as shown in Figure 17.2. The electron's energy is related to its momentum, \vec{p}_e , as

$$E^2 = p_e^2 c^2 + m_e^2 c^4. \quad (17.3)$$

This relativistic formula taken into account the rest mass energy $m_e c^2$ of the electron and is valid even if the electron moves at a high speed approaching the speed of light. Applying the conservation of energy to the γ, e collision we have,

$$\hbar \omega + m_e c^2 = \hbar \omega' + \sqrt{p_e^2 c^2 + m_e^2 c^4}. \quad (17.4)$$

The x and y components of the conservation of momentum are respectively,

$$\hbar k - \hbar k' \cos \theta = p_e \cos \alpha \quad (17.5)$$

and

$$\hbar k' \sin \theta = p_e \sin \alpha. \quad (17.6)$$

Squaring and adding equations (17.5) and (17.6) we have

$$p_e^2 = \hbar^2 (k^2 + k'^2 - 2kk' \cos \theta). \quad (17.7)$$

Multiplying this by c^2 and writing it in terms of wavelengths we have,

$$c^2 p_e^2 = c^2 h^2 \left(\frac{1}{\lambda} - \frac{1}{\lambda'} \right)^2 + \frac{2c^2 h^2}{\lambda \lambda'} (1 - \cos \theta). \quad (17.8)$$

Squaring the conservation of energy (eq. 17.5) and writing it in terms of wavelength gives us

$$c^2 p_e^2 = \left(\frac{ch}{\lambda} - \frac{ch}{\lambda'} + m_e c^2 \right)^2 - m_e^2 c^4. \quad (17.9)$$

Subtracting equations (17.8) and (17.9) and rearranging the terms we have the difference in wavelength,

$$\lambda' - \lambda = \lambda_c (1 - \cos \theta), \quad (17.10)$$

where $\lambda_c = h/m_e c = 2.4 \times 10^{-12}$ m is the Compton wavelength. The difference, $\Delta\lambda = \lambda' - \lambda$, is known as the Compton shift. It is interesting to note that the Compton shift is independent of initial wavelength of the light and depends only on the scattering angle θ of the light. The maximum difference in the wavelength $\Delta\lambda = \lambda' - \lambda$ is in the backwards direction $\theta = 180^\circ$ where $\Delta\lambda = 2\lambda_c$.

In this picture we think of the incident X-ray as particles called photons which lose energy when they collide with the electrons. This results in the increase in wavelength observed in the Compton effect. The change in wavelength is very small, of the order of λ_c . This change will be significant only when the incident wavelength λ is comparable to λ_c , which is the case in X-rays where $\lambda \sim 10^{-10}$ m.

The photoelectric effect and the Compton effect require us to think of electromagnetic radiation in terms of a particle called the photon. This does not mean that we can abandon the wave theory. We cannot explain interference or diffraction without this. This basically tells us that light has a dual nature. It is sometimes necessary to think of it as a wave and sometimes as a particle, depending on the phenomenon that we are trying to explain. This dual wave-particle behaviour is not restricted to light alone, and it actually extends to the whole of nature.

17.2 The wave nature of particles

In 1924 de Broglie first hypothesized that associated with every particle there is a wave. In particular, we can associate a wave

$$\psi(\vec{r}, t) = \tilde{A} e^{-i(\omega t - \vec{k} \cdot \vec{r})} \quad (17.11)$$

With a particle which has energy $E = \hbar\omega$ and momentum $\vec{p} = \hbar\vec{k}$. The corresponding wavelength $\lambda = h/p$ is referred to as the de Broglie wavelength.

It should be noted that at any instant of time t a particle has an unique well defined position \vec{r} and momentum \vec{p} . Unlike the particle, at any time t the wave $\psi(\vec{r}, t)$ is defined all over space. This is the crucial difference between a particle and a wave. While the wave incorporates the particle's momentum, it does not contain any information about the particle's position. This is an issue which we shall return to when we discuss how to interpret the wave associated with a particle.

What is the dispersion relation of the de Broglie wave? A particle's energy and momentum are related as

$$E = \frac{p^2}{2m} \quad (17.12)$$

which gives the dispersion relation

$$\omega = \frac{\hbar k^2}{2m}. \quad (17.13)$$

Note that the relativistic relation $E = \sqrt{p^2 c^2 + m^2 c^4}$ should be used at velocities comparable to c .

Problem: An electron is accelerated by a voltage $V = 100$ V inside an electron gun. [a.] What is the de Broglie wavelength of the electron when it emerges from the gun? [b.] When do the relativistic effects become important? (Ans: a. 1.25 \AA , [b] $\sim 10^4$ V)

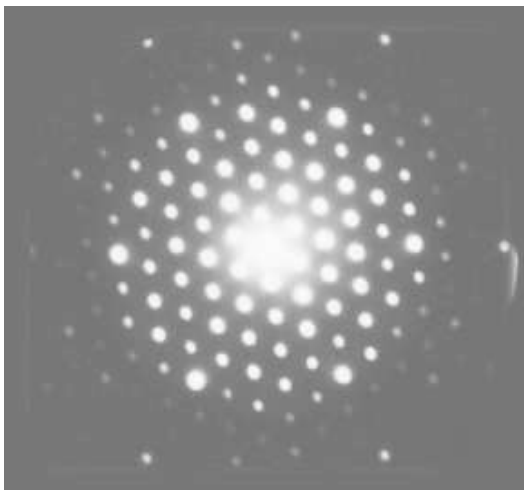


Figure 17.3:

The wave nature of particles was verified by Davison and Germer in 1927 who demonstrated electron diffraction from a large metal crystal. A pattern of maxima (Figure 17.3) is observed when a beam of electrons is scattered from a crystal. This is very similar to the diffraction pattern observed when X-ray are scattered from a crystal. This clearly demonstrates that particles like electrons also exhibit wave properties in some circumstances.

Problems

1. Through how much voltage difference should an electron be accelerated so that it has a wavelength of 0.5 \AA ?
2. In a Compton effect experiment a photon which is scattered at 180° to the incident direction has half the energy of the incident photon.
 - a. What is the wavelength of the incident photon?
 - b. What is the energy of the scattered photon?
 - c. Determine the total relativistic energy of the scattered electron.
 - d. What is the momentum of the scattered electron?
3. For what momentum is the de Broglie wavelength of an electron equal to its Compton wavelength.

Chapter 18

Interpreting the electron wave

18.1 An experiment with bullets

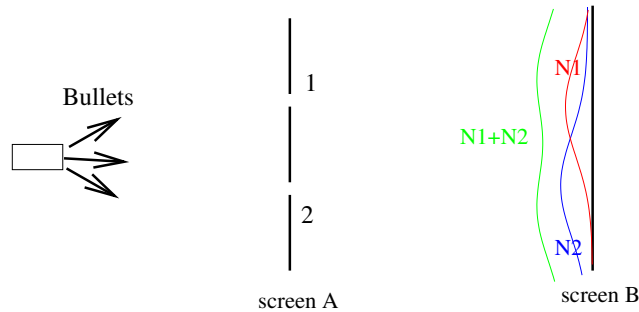


Figure 18.1: Double slit experiment with bullets

We consider an experiment where bullets are sprayed randomly on screen A which has two slits 1 and 2 as shown in Figure 18.1. The bullets that hit the screen are stopped, the bullets that pass through the slits reach screen B. The number of bullets arriving at different points on screen B is recorded.

The experiment is first performed with slit 2 closed. The bullets can now arrive at screen B only through slit 1. We use $N1$ to denote the number of bullets arriving at different points on screen B through slit 1. As shown in Figure 18.1, $N1$ peaks at the point just behind slit 1, and falls off as we move away from this point. The experiment is next repeated with slit 2 open and slit 1 closed. The bullets can now reach screen B only through slit 2, and $N2$ is very similar to $N1$ except that the peak is shifted.

Finally the experiment is performed with both slits open. Bullets can now reach screen B through slits 1 or slit 2, and the number of bullets at any point on screen B is $N1 + N2$.

Salient points that should be noted are

1. Any point on screen B is hit by either one, or two or a larger integer number of bullets. Half bullets or other fractions are never detected.

2. If the rate of shooting is increased, the frequency of arrival increases.
3. The counts can be converted to probability of a bullet arriving at a point x on the screen.
4. A bullet arrives at a point on screen B either through slit 1 or slit 2. Denoting the associated probabilities as P_1 and P_2 respectively, the probability when both slits are open is $P_{12} = P_1 + P_2$.

18.2 An experiment with waves.

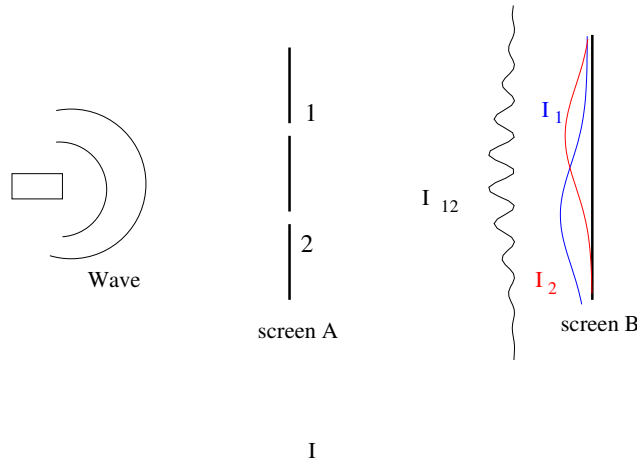


Figure 18.2: Double slit experiment with waves

The experiment that was being done using bullets is now repeated replacing the device that sprays bullets with a source that produces waves. The wave amplitude A can be increased continuously and the intensity $I \propto A^2$ also increases continuously. This is to be contrasted with the bullet count which can increase only by an integer number.

Further, using I_1 to denote the intensity of the waves at screen B when only slit 1 is open (Figure 18.2) and I_2 when only slit 2 is open, as shown in the figure the intensity I_{12} when both slits are open is not the sum of I_1 and I_2 *ie.* $I_{12} \neq I_1 + I_2$. At some points the intensity I_{12} is more than $I_1 + I_2$ whereas it is less than this at some other points on screen B. The intensity at any point x on screen B is given by

$$I_{12} = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos [\delta(x)] . \quad (18.1)$$

The value of the wave at x is calculated by superposing \tilde{A}_1 and \tilde{A}_2 from slits 1 and 2 respectively, and we use this to calculate the intensity

$$I_{12} = \left| \tilde{A}_1 + \tilde{A}_2 \right|^2 . \quad (18.2)$$

These two contributions have a phase difference which varies with x

$$\delta(x) = \phi_2(x) - \phi_1(x). \quad (18.3)$$

The two waves add constructively when \tilde{A}_1 and \tilde{A}_2 are in phase $\delta(x) = 0, 2\pi, \dots$ and I_{12} is more than $I_1 + I_2$, they add destructively when they are out of phase $\delta(x) = \pi, 3\pi, \dots$ and I_{12} is less than $I_1 + I_2$. This is the familiar phenomenon of interference discussed in detail earlier.

18.3 An experiment with electrons

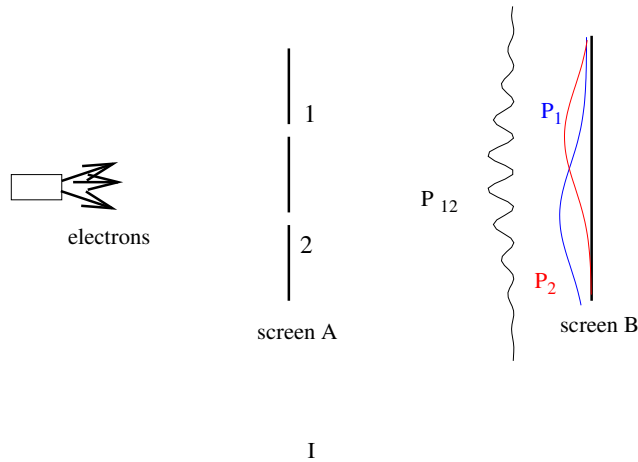


Figure 18.3: Double slit experiment with electrons

We consider an experiment where an electron gun fires electrons at screen A with two slits as shown in Figure 18.3. Like bullets, electrons are discrete objects and a detector on screen B either detects the arrival of one electron or does not detect an electron. A fraction of an electron is never detected. If the firing is decreased or increased, the electron count rate at different points on screen B also decreases or increases. This count rate can be converted to a probability. If we assume that an electron arrives at a point x on screen B either through slit 1 or slit 2 we then expect that

$$P_{12} = P_1 + P_2. \quad (18.4)$$

As shown in the figure this does not actually hold and at some points $P_{12} < P_1 + P_2$, just like an interference pattern. The probability is a positive quantity and it is not possible to add two probabilities and have the result go down.

18.4 Probability amplitude

We associate a wave ψ with an electron such that the $|\psi(x, t)|^2$ gives that probability of finding the electron at x . The probability is mathematically

equivalent to the intensity of the wave.

At any point on the screen

$$\psi = \psi_1 + \psi_2, \quad (18.5)$$

where ψ_1 is the contribution from slit 1 and ψ_2 from slit 2. ψ_1 and ψ_2 represents the two alternatives by which the electron can reach the point x on screen B. In quantum mechanics the two alternatives interfere.

The probability of finding the electron on any point on the screen is

$$P_{12} = |\psi_1 + \psi_2|^2 = P_1 + P_2 + 2\sqrt{P_1 P_2} \cos \delta. \quad (18.6)$$

Although the wave is defined everywhere, we always detect the electron at only one point and in whole.

The wave function $\psi(x, t)$ is the probability amplitude which is necessarily complex.

What happens if we try to determine through which slit the electron reaches screen B.

This can be done by placing a light at each slit so as to illuminate it. The electrons will scatter the light as it passes through the slit. So if we see a flash of light from slit 1, we will know that the electron has passed through it, similarly if the electron passes through slit 2 we will get a flash from that direction.

If we know through which slit the electron reaches screen B then

$$P_{12} = P_1 + P_2, \quad (18.7)$$

ie it came through either slit 1 or slit 2.

It is found that once we determine through which slit the electron passes, the two possibilities no longer interfere. The probability is then given by equation (18.7) instead of (18.6).

The act of measurement disturbs the electron. The momentum is changed in the scattering, and we no longer have any information where it goes and hits the screen.

There is a fundamental restriction on the accuracy to which we can simultaneously determine a particle's position and momentum. The product of the uncertainties in the position and the momentum satisfies the relation,

$$\Delta x \Delta p \geq \hbar/2, \quad (18.8)$$

known as **Heisenberg's uncertainty principle**.

Chapter 19

Probability

Consider a process whose outcome is uncertain. For example, we throw a dice. The value the dice returns can have 6 values,

$$x_1 = 1 \quad x_2 = 2 \quad x_3 = 3 \quad x_4 = 4 \quad x_5 = 5 \quad x_6 = 6 \quad (19.1)$$

We can ask the question “What is the probability of getting an outcome x_i when you throw the dice?” This probability can be calculated using,

$$P(x_i) = p_i = \frac{N_i \rightarrow \text{Number of time } x_i \text{ occurs.}}{N \rightarrow \text{Total number of events.}}. \quad (19.2)$$

For a fair dice $P(x_i) = \frac{1}{6}$ for all the six possible outcomes as shown in Figure 19.1.

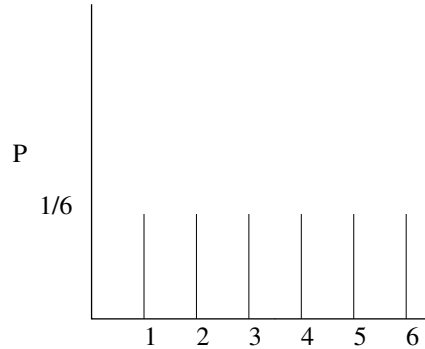


Figure 19.1: Probabilities of different outcomes for a fair dice

What is the expected outcome when we throw the dice? We next calculate the expectation value which again is an integral,

$$\begin{aligned} \langle x \rangle &= \sum_i \frac{x_i N_i}{N} = \sum_{i=1}^6 x_i p_i = (1 + 2 + 3 + 4 + 5 + 6) \frac{1}{6}, \\ &= \frac{21}{6} = \frac{7}{2} = 3.5. \end{aligned} \quad (19.3)$$

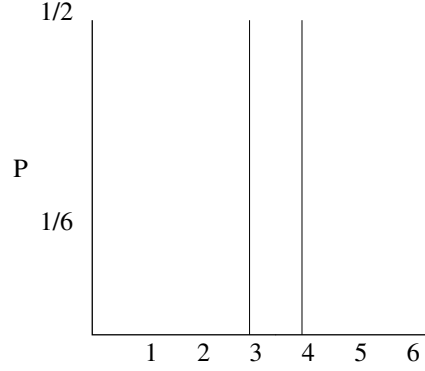


Figure 19.2: Probabilities of different outcomes for a biased dice

Suppose we have a biased dice which produces only 3 and 4. We have $P(x_3) = \frac{1}{2}$ $p(x_4) = \frac{1}{2}$, the rest are all zero as shown in Figure 19.2.

Calculating the expectation value for the biased dice we have

$$\langle x \rangle = 3.5. \quad (19.4)$$

The expectation value is unchanged even though the probability distributions are different. Whether we throw the unbiased dice or the biased dice, the expected value is the same. Each time we throw the dice, we will get a different value. In both cases the values will be spread around 3.5. The values will have a larger spread for the fair dice as compared to the biased one. How to characterize this? The root mean square or standard deviation σ tells this. The variance σ^2 is calculated as,

$$\sigma^2 = \langle (x - \langle x \rangle)^2 \rangle = \sum_i (x_i - \langle x \rangle)^2 p_i. \quad (19.5)$$

Larger the value of σ the more is the spread. Let us calculate the variance for the fair dice,

$$\sigma^2 = \frac{1}{6} \left[(1 - 3.5)^2 + (2 - 3.5)^2 + \dots + (6 - 3.5)^2 \right], \quad (19.6)$$

which gives the standard deviation $\sigma = 1.7$. For the biased dice we have

$$\sigma^2 = \frac{1}{2} \left[(3 - 3.5)^2 + (4 - 3.5)^2 \right], \quad (19.7)$$

which gives $\sigma = 0.5$. We see that the fair dice has a larger variance and standard deviation than the unbiased one. The uncertainty in the outcome is larger for the fair dice and is smaller for the biased one.

We now shift over attention to continuous variables. For example, x is a random variable which can have any value between 0 and 10 (Figure 19.3). What is the probability that x has a value 2.36?

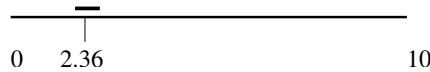


Figure 19.3:

This probability is zero. We see this as follows. The probability of any value between 0 and 10 is the same. There are infinite points between 0 and 10 and the probability of getting exactly 2.36 is,

$$P = \frac{1}{\text{Total number of points between 0 and 10}}. \quad (19.8)$$

The denominator is infinitely large and the probability is thus zero. A correct question would be, what is the probability that x has a value between $2.36 - 0.001$ and $2.36 + 0.001$. We can calculate this as,

$$\frac{2.361 - 2.359}{10. - 0.} = 2 \times 10^{-4}. \quad (19.9)$$

For a continuous variable it does not make sense to ask for the probability of its having a particular value x . A meaningful question is, what is the probability that it has a value in the interval Δx around the value x .

Making Δx an infinitesimal we have,

$$dP(x) = \rho(x) dx, \quad (19.10)$$

where dP is probability of getting a value in the interval $x - \frac{dx}{2}$ to $x + \frac{dx}{2}$. If all values in the range 0 to 10 are equally probable then,

$$\rho(x) = \frac{1}{10}. \quad (19.11)$$

The function $\rho(x)$ is called the probability density. It has the properties:

1. It is necessarily positive $\rho(x) \geq 0$
2. $P(a \leq x \leq b) = \int_a^b \rho(x) dx$ gives the probability that x has a value in the range a to b .
3. $\int_{-\infty}^{\infty} \rho(x) dx = 1$ Total probability is 1

Let us consider an example (Figure 19.4) where

$$\begin{aligned} 0 \leq x \leq \pi \quad \rho(x) &= A \sin^2 x, \\ \text{outside} &= 0. \end{aligned} \quad (19.12)$$

We first normalize the probability density. This means to ensure that $\int_{-\infty}^{\infty} \rho(x) dx =$

1. Applying this condition we have,

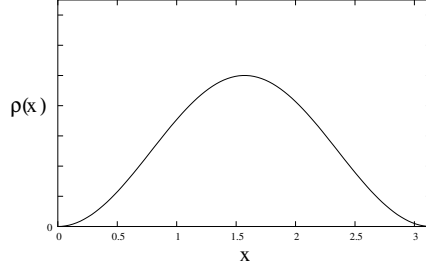


Figure 19.4:

$$\begin{aligned}
 A \int_0^{\pi} \sin^2 x \, dx &= \frac{A}{2} \int_0^{\pi} [1 - \cos 2x] \, dx, \\
 &= \frac{A}{2} \left[\pi - \frac{1}{2} \sin(2x) \right]_0^{\pi} = \frac{\pi A}{2} = 1,
 \end{aligned} \tag{19.13}$$

which gives us,

$$A = \frac{2}{\pi}. \tag{19.14}$$

We next calculate the expectation value $\langle x \rangle$. The sum in equation (19.3) is now replaced by an integral and,

$$\langle x \rangle = \int x \rho(x) dx. \tag{19.15}$$

Evaluating this we have,

$$\langle x \rangle = \frac{2}{\pi} \int_0^{\pi} x \sin^2(x) dx = \frac{\pi}{2}. \tag{19.16}$$

We next calculate the variance,

$$\sigma^2 = \langle (x - \langle x \rangle)^2 \rangle = \int_{-\infty}^{\infty} (x - \langle x \rangle)^2 \rho(x) dx. \tag{19.17}$$

This can be simplified as,

$$\begin{aligned}
 \sigma^2 &= \int_{-\infty}^{\infty} (x^2 - x\langle x \rangle + \langle x \rangle^2) \rho(x) \, dx, \\
 &= \int_{-\infty}^{\infty} x^2 \rho(x) \, dx - \langle x \rangle^2.
 \end{aligned} \tag{19.18}$$

Problem Calculate the variance σ^2 for the probability distribution in equation (19.12).

We now return to the wave $\psi(x, t)$ associated with every particle. The laws governing this wave are referred to as Quantum Mechanics. We have already learnt that this wave is to be interpreted as the probability amplitude. The probability amplitude can be used to calculate the probability density $\rho(x, t)$ using,

$$\rho(x, t) = \psi(x, t)\psi^*(x, t), \quad (19.19)$$

and

$$dP = \rho(x, t) dx,$$

gives the probability of finding the particle in an interval dx around the point x . The expectation value of the particle's position can be calculated using,

$$\langle x \rangle = \int x \rho(x) dx.$$

This is the expected value if we measure the particle's position. Typically, a measurement will not yield this value. If the position of many identical particles all of which have the “wave function” $\psi(x, t)$ are measured these values will be centered around $\langle x \rangle$. The spread in the measured values of x is quantified through the variance,

$$\sigma^2 = \langle (\Delta x)^2 \rangle = \int_{-\infty}^{\infty} x^2 \rho(x, t) dx - \langle x \rangle^2 = \langle x^2 \rangle - \langle x \rangle^2. \quad (19.20)$$

The standard deviation which is also denoted as $\Delta x \equiv \sqrt{\langle (\Delta x)^2 \rangle} = \sigma$ gives an estimate of the uncertainty in the particle's position.

Problems

1. The measurement of a particle's position x in seven identical replicas of the same system we get values 1.3, 2.1, 7.8, 2.56, 6.12, 3.12 and 9.1. What are the expectation value and uncertainty in x ?
2. In an experiment the value of x is found to be different each time the experiment is performed. The values of x are found to be always positive, and the distribution is found to be well described by a probability density

$$\rho(x) = \frac{1}{L} \exp[-x/L] \quad L = 0.2 \text{ m}$$

- a. What is the expected value of x if the experiment is performed once?
- b. What is the uncertainty Δx ?
- c. What is the probability that x is less than 0.2 m?
- d. The experiment is performed ten times. What is the probability that all the values of x are less than 0.2 m?

3. For a Gaussian probability density distribution

$$\rho(x) = A \exp \left[-x^2 / (2L^2) \right] \quad -\infty \leq x \leq \infty$$

- a. Calculate the normalization coefficient A .
- b. What is the expectation value $\langle x \rangle$?
- c. What is the uncertainty $\Delta x = \sqrt{\langle (x - \langle x \rangle)^2 \rangle}$?
- d. What is the probability that x has a positive value?

Chapter 20

Quantum Mechanics

We have learnt that there is a wave associated with every particle. This is not evident for macroscopic particles like a bullet or an elephant because the wavelength $\lambda = h/p$ is extremely small. The wave nature becomes important when dealing with microscopic particles like an electron for which the wavelength can be of the order of 1 \AA . We have also learnt that this wave $\psi(x, t)$ may be interpreted as the probability amplitude and this wave is often referred to as the wavefunction. The laws governing the evolution of the wavefunction and its interpretation are referred to as Quantum Mechanics.

20.1 The Laws of Quantum Mechanics

1. In Newtonian Mechanics the motion of a particle under the influence of a potential $V(\vec{r}, y)$ is described by the particle's trajectory $\vec{r}(t)$. There are different trajectories corresponding to different possible states of the particle.

In **Quantum Mechanics** there is a different wavefunction for every state of a particle. For example consider three different states of a particle referred to as states 1, 2 and 3 with wave functions $\psi_1(x, t)$, $\psi_2(x, t)$ and $\psi_3(x, t)$ respectively.

2. In Newtonian Mechanics the trajectory is determined by solving Newton's equation of motion

$$m \frac{d^2 \vec{r}}{dt^2} = -\vec{\nabla} V(\vec{r}, t) \quad (20.1)$$

In **Quantum Mechanics** the wavefunction is governed by Schrodinger's equation

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{r}, t) = \frac{-\hbar^2}{2m} \nabla^2 \psi(\vec{r}, t) + V(\vec{r}, t) \psi(\vec{r}, t) \quad (20.2)$$

For a particle free to move only in one dimension along the x axis we have

$$i\hbar \frac{\partial}{\partial t} \psi(x, t) = \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x, t) + V(x, t) \psi(x, t). \quad (20.3)$$

Here we shall only consider time independent potentials $V(x)$. Applying the method of separation of variables we take a trial solution

$$\psi(x, t) = X(x)T(t) \quad (20.4)$$

whereby the Schrodinger's equation is

$$Xi\hbar \frac{dT}{dt} = \frac{-\hbar^2}{2m} T \frac{d^2 X}{dx^2} + V(x)XT \quad (20.5)$$

which on dividing by XT gives

$$i\hbar \frac{1}{T} \frac{dT}{dt} = \frac{-\hbar^2}{2m} \frac{1}{X} \frac{d^2 X}{dx^2} + V(x) = E \quad (20.6)$$

The first term is a function of t alone whereas the second term is a function of x alone. It is clear that both terms must have a constant value if they are to be equal for all values of x and t . Denoting this constant as E we can write the solution for the time dependent part as

$$T(t) = Ae^{-iEt/\hbar}. \quad (20.7)$$

The x dependence has to be determined by solving

$$\frac{\hbar^2}{2m} \frac{d^2 X}{dx^2} = -[E - V(x)]X. \quad (20.8)$$

The general solution can be written as

$$\psi(x, t) = Ae^{-iEt/\hbar} X(x) \quad (20.9)$$

where the t dependence is known and $X(x)$ has to be determined from equation (20.8).

Free particle The potential $V(x) = 0$ for a free particle. It is straightforward to verify that $X(x) = e^{ipx/\hbar}$ with $p = \pm\sqrt{2mE}$ satisfies equation (20.8). This gives the solution

$$\psi(x, t) = Ae^{-i(Et - px)/\hbar}. \quad (20.10)$$

This is a plane wave with angular frequency $\omega = E/\hbar$ and wave number $k = p/\hbar$ where E and p are as yet arbitrary constant related as $E = p^2/2m$. This gives the wave's dispersion relation $\omega = \hbar k^2/2m$.

Here different value of E will give different wavefunctions. For example p_1 and p_2 are different constants with $E_1 = p_1^2/2m$ and $E_2 = p_2^2/2m$, then

$$\psi_1(x, t) = A_1 e^{-i(E_1 t - p_1 x)/\hbar} \quad (20.11)$$

and

$$\psi_2(x, t) = A_2 e^{-i(E_2 t - p_2 x)/\hbar} \quad (20.12)$$

are two different wavefunctions corresponding to two different states of the particle.

3. In **Quantum Mechanics** the superposition of two different solutions $\psi_1(x, t)$ and $\psi_2(x, t)$ corresponding to two different states of the particle is also a solution of the Schödinger's equation, an example being

$$\psi(x, t) = A_1 e^{-i(E_1 t - p_1 x)/\hbar} + A_2 e^{-i(E_2 t - p_2 x)/\hbar}. \quad (20.13)$$

This can be generalized to a superposition of three, four and more states. In the continuum we have

$$\psi(x, t) = \int_{-\infty}^{\infty} A(p) e^{-i(E(p)t - px)/\hbar} dp \quad (20.14)$$

4. What happens when we make a measurement? We have already discussed what happens when we measure a particle's position. In **Quantum Mechanics** it is not possible to predict the particle's position. We can only predict probabilities for finding the particle at different positions. The probability density $\rho(x, t) = \psi(x, t)\psi^*(x, t)$ gives the probability $dP(x, t)$ of finding the particle in the interval dx around the point x to be $dP(x, t) = \rho(x, t) dx$. But there are other quantities like momentum which we could also measure. What happens when we measure the momentum?

In **Quantum Mechanics** there is a Hermitian Operator corresponding to every observable dynamical quantity like the momentum, angular momentum, energy, etc.

What is an operator? An operator \hat{O} acts on a function $\psi(x)$ to give another function $\phi(x)$.

$$\hat{O}\psi(x) = \phi(x) \quad (20.15)$$

We consider an example where $\hat{O} = \frac{d}{dx}$

$$\hat{O} \sin x = \frac{d}{dx} \sin x = \cos x. \quad (20.16)$$

Consider another example where $\hat{O} = 2$ so that

$$\hat{O}\psi(x) = 2\psi(x) = \phi(x) \quad (20.17)$$

What is a Hermitian operator? We do not go into the definition here, instead we only state a relevant, important property of a Hermitian operators. Given an operator \hat{O} , a function $\psi(x)$ is an eigenfunction of \hat{O} with eigenvalue λ if

$$\hat{O}\psi(x) = \lambda\psi(x) \quad (20.18)$$

As an example consider

$$\hat{O} = -i\hbar \frac{d}{dx} \text{ and } \psi(x) = e^{ikx} \quad (20.19)$$

we see that

$$\hat{O}\psi(x) = \hbar k \psi(x) \quad (20.20)$$

The function e^{ikx} is an eigenfunction of the operator $-i\hbar \frac{d}{dx}$ with eigenvalue $\hbar k$. As another example for the same operator we consider the function

$$\psi(x) = \cos(kx) \quad (20.21)$$

we see that

$$\hat{O}\psi = -i\hbar \frac{d}{dx} \cos(kx) = i\hbar k \sin(kx). \quad (20.22)$$

This is not an eigenfunction of the operator.

Hermitian operators are a special kind of operators all of whose eigenvalues are real.

We present the Hermitian operators corresponding to a few observable quantities.

Position - $x \rightarrow$ Operator \hat{x}

$$\hat{x}\psi(x, t) = x\psi(x, t) \quad (20.23)$$

Momentum - $p \rightarrow$ Operator \hat{p}

$$\hat{p}\psi(x, t) = -i\hbar \frac{\partial \psi}{\partial x} \quad (20.24)$$

Hamiltonian $H(p, x, t) \rightarrow$ Operator \hat{H}

$$\hat{H}\psi(x, t) = i\hbar \frac{\partial}{\partial t} \psi(x, t) \quad (20.25)$$

The eigenvalues of the Hamiltonian \hat{H} correspond to the Energy.

Let us return to what happens when we make a measurement. Consider a particle in a state with wavefunction $\psi(x, t)$. We measure its momentum p . There are two possibilities

A. If $\psi(x, t)$ is an eigenfunction of \hat{p} . For example

$$\psi(x, t) = A_1 e^{-i(E_1 t - p_1 x)/\hbar} \quad (20.26)$$

$$\hat{p}\psi = p_1 \psi \quad (20.27)$$

We will get the value p_1 for the momentum The measurement does not disturb the state and after the measurement the particle continues to be in the same state with wavefunction $\psi(x, t)$.

B. If $\psi(x, t)$ is not an eigenfunction of \hat{p} . For example

$$\psi(x, t) = \left(\frac{3}{5}\right) e^{-i(E_1 t - p_1 x)/\hbar} + \left(\frac{4}{5}\right) e^{-i(E_2 t - p_2 x)/\hbar} \quad (20.28)$$

On measuring the momentum we will get either p_1 or p_2 . The probability of getting p_1 is $\left(\frac{3}{5}\right)^2$ and probability of getting p_2 is $\left(\frac{4}{5}\right)^2$. The measurement changes the wavefunction.

In case we get p_1 the wavefunction will be changed to

$$\psi(x, t) = A_1 e^{-i(E_1 t - p_1 x)} \quad (20.29)$$

and in case we get p_2 - the wavefunction will be changed to

$$\psi(x, t) = A_2 e^{-i(E_2 t - p_2 x)}. \quad (20.30)$$

If we measure p again we shall continue to get the same value in every successive measurement.

In general, if

$$\psi(x, t) = c_1 \psi_1(x, t) + c_2 \psi_2(x, t) \quad (20.31)$$

such that

$$\hat{O}\psi_1 = O_1\psi_1 \quad \hat{O}\psi_2 = O_2\psi_2. \quad (20.32)$$

On measuring O

$$P_1 = \frac{|c_1|^2}{|c_1|^2 + |c_2|^2} \text{ and } P_2 = \frac{|c_2|^2}{|c_1|^2 + |c_2|^2} \quad (20.33)$$

are probabilities of getting the values O_1 and O_2 respectively. We can now interpret the solution

$$\psi(x, t) = A_1 e^{-i(E_1 t - p_1 x)}. \quad (20.34)$$

This corresponds to a particle with momentum p_1 and energy E_1 . We will get these value however many times we repeat the measurement .

What happens if we measure the position of a particle whose wavefunction $\psi(x, t)$ is given by equation (20.34)? Calculating the probability density

$$\rho(x, t) = |A_1|^2 \quad (20.35)$$

we see that this has no x dependence. The probability of finding the particle is equal at all points. This wave function has no position information.

5. Consider many identical replicas of a particle in a state $\psi(x, t)$ as shown below. We measure the particle's momentum p . This is done independently on all the replicas and the result recorded. The outcome cannot

be exactly predicted unless $\psi(x, t)$ is an eigenfunction of \hat{p} . Let us consider a general situation where this is not the case. As shown below, a different value of momentum will then be measured for each of the particle.

$$\boxed{\psi} \rightarrow p_1 \quad \boxed{\psi} \rightarrow p_2 \quad \boxed{\psi} \rightarrow p_2 \quad (20.36)$$

$$\boxed{\psi} \rightarrow p_3 \quad \boxed{\psi} \rightarrow p_1 \quad \boxed{\psi} \rightarrow p_2$$

What is the expectation value of the momentum? This can be determined after the experiment is performed as

$$\langle p \rangle = \frac{\sum_{i=1}^n p_i N_i}{N} \quad (20.37)$$

where N_i is the number of times the momentum p_i occurs and $N = \sum_{i=1}^n N_i$. In **Quantum Mechanics** it is possible to predict this using the particle's wavefunction as

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^*(x, t) \hat{p} \psi(x, t) dx. \quad (20.38)$$

Similarly, it is possible to predict the variance in the momentum values

$$\begin{aligned} \langle (\Delta p)^2 \rangle &= \int \psi^*(x, t) (\hat{p} - \langle p \rangle)^2 \psi(x, t) dx \\ &= \int \psi^* \hat{p}^2 \psi dx - 2\langle p \rangle \int \psi^* \hat{p} \psi dx + \langle p \rangle^2 \int \psi^* \psi dx \\ &= \langle p^2 \rangle - 2\langle p \rangle^2 + \langle p \rangle^2 = \langle p \rangle^2 - \langle p \rangle^2 \end{aligned} \quad (20.39)$$

The uncertainty in the momentum can be calculated as $\Delta p = \sqrt{\langle (\Delta p)^2 \rangle}$

20.2 Particle in a potential.

Let us consider a general situation where there is a particle in a potential $V(x)$. We have seen that there exist wavefunctions of the form

$$\psi = e^{-iEt/\hbar} X(x) \quad (20.40)$$

where $X(x)$ is a solution to the equation

$$\frac{\hbar^2}{2m} \frac{d^2 X}{dx^2} = -[E - V(x)]X. \quad (20.41)$$

What happens if we measure the energy of a particle with such a wavefunction? This is an eigenfunction of the Hamiltonian operator with eigenvalue E_1 . This

means that this state has a well defined energy E . It is called an energy eigenstate. The probability density of such a state is time independent.

$$\rho(x) = \psi^*(x, t)\psi(x, t) = X^*(x)X(x) \quad (20.42)$$

For this reason a state with a wave function of this form is also called a stationary state.

20.2.1 In Classical Mechanics

In figure 20.1 we show a general potential $V(x)$. A particle of energy E moving

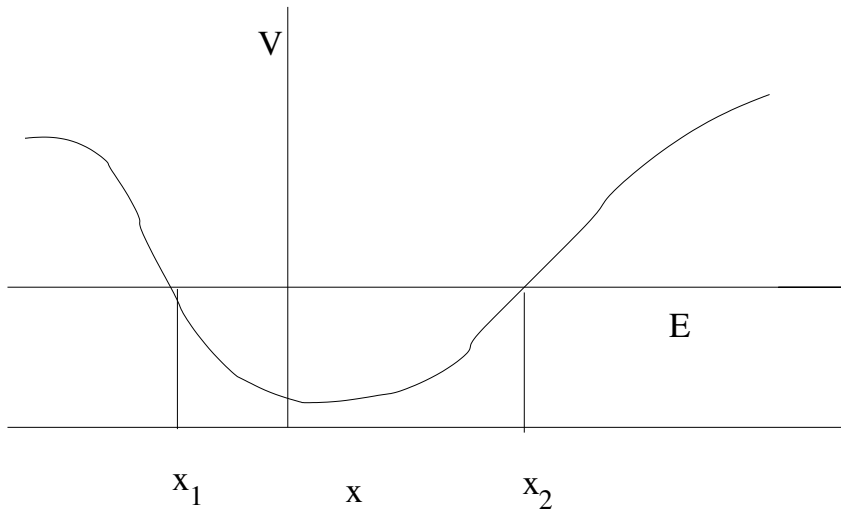


Figure 20.1:

in this potential has momentum p which is given by

$$\frac{p^2}{2m} + V(x) = E \quad (20.43)$$

or

$$p = \pm \sqrt{2m(E - V(x))} \quad (20.44)$$

Since p is real we can see that the particle's motion is restricted to the region where $E \geq V(x)$, or the region $x_a \leq x \leq x_b$ shown in the the figure. In the same situation, what happens in Quantum Mechanics?

20.2.2 Step potentials

Here we shall make a simplification and assume that V is constant over a range of x and it varies in steps as shown in Figure 20.2 instead of varying smoothly as shown in Figure 20.1

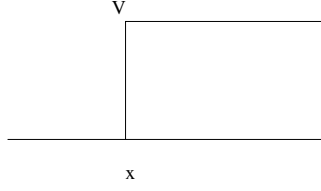


Figure 20.2:

We calculate the wavefunction of a particle of energy E inside a region of constant potential V . We have

$$\frac{d^2 X}{dx^2} = -\frac{2m}{\hbar^2}(E - V)X \quad (20.45)$$

There are two possibilities. The first possibility, shown in Figure 20.3, is where $E > V$. The particle's momentum inside the potential is

$$p' = \sqrt{2m(E - V)}. \quad (20.46)$$

Writing equation (20.45) in terms of this we have

$$\frac{d^2 X}{dx^2} = \frac{-p'^2}{\hbar^2} X \quad (20.47)$$

which has a solution

$$X(x) = A_1 e^{ip'x/\hbar} + A_2 e^{-ip'x/\hbar} \quad (20.48)$$

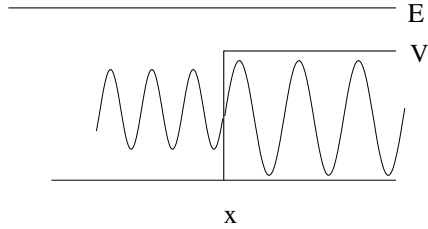


Figure 20.3:

The wave function inside the potential is

$$\psi(x, t) = e^{-iEt/\hbar} [A_1 e^{ip'x/\hbar} + A_2 e^{-ip'x/\hbar}] \quad (20.49)$$

and for the same particle outside where $V = 0$ we have

$$\psi(x, t) = e^{-iEt/\hbar} [A_1 e^{ipx/\hbar} + A_2 e^{-ipx/\hbar}] \quad (20.50)$$

where $p = \sqrt{2mE}$. We see that the wavefunctions frequency is the same both inside and outside the potential, whereas the wave number is different inside

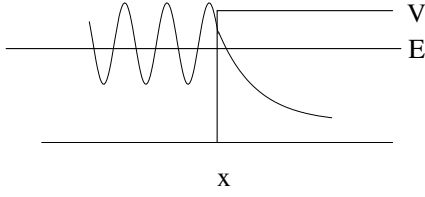


Figure 20.4:

the potential (Figure 20.3) and outside. The potential is like a change in the refractive index, the wavelength changes. We shall discuss matching of boundary conditions later.

The second possibility, shown in Figure 20.4 is where $E < V$. In classical mechanics the particle is never found in this region as the momentum is imaginary which is meaningless. Defining

$$\sqrt{2m(E - V)} = \sqrt{-1}\sqrt{2m(V - E)} = iq \quad (20.51)$$

we write equation (20.45) as

$$\frac{d^2 X}{dx^2} = \frac{q^2}{\hbar^2} X \quad (20.52)$$

which has solutions

$$X(x) = A_1 e^{-qx/\hbar} + A_2 e^{qx/\hbar} \quad (20.53)$$

The second solution blows up as $x \rightarrow \infty$ and if the region to the right extends to infinity then $A_2 = 0$ and we have the solution

$$X(x) = A_1 e^{-qx/\hbar} \quad (20.54)$$

in the region inside the potential. The corresponding wavefunction is

$$\psi(x, t) = A_1 e^{-iEt/\hbar} e^{-qx/\hbar} \quad (20.55)$$

The wave function decays exponentially inside the potential. There is finite probability of finding the particle in a region $E < V$. The probability decays exponentially inside the region where $E < V$. The decay rate increases with V .

20.2.3 Particle in a box

A particle of energy E is confined to $0 \leq x \leq L$ by a very high (infinite) potential as shown in Figure 20.5. This is often called an infinite potential well or a particle in a box. The potential is zero inside the well and it is infinite outside.

We have seen that the wave function decays exponentially inside a region where $V > E$ and the decay occurs more rapidly for higher V . In the limit where $V \rightarrow \infty$ the wavefunction vanishes at the boundary. We then have

$$\frac{d^2 x}{dx^2} = -\frac{2mE}{\hbar^2} X \quad 0 \leq x \leq L \quad (20.56)$$

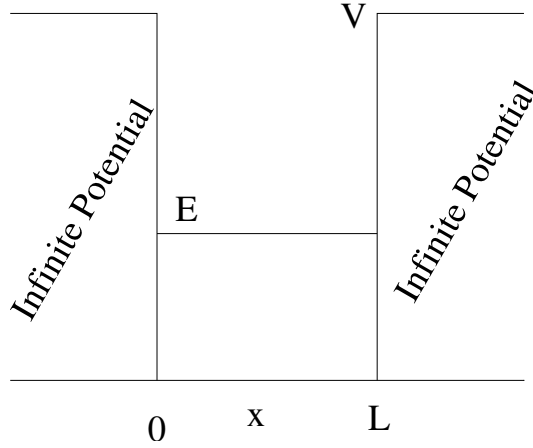


Figure 20.5:

with the boundary condition that $X(x)$ vanishes at $x = 0$ and $x = L$. We have encountered exactly the same situation when studying standing waves and the solution is

$$X(x) = A_n \sin\left(\frac{n\pi x}{L}\right) \quad n = \pm 1, \pm 2, \dots \quad (20.57)$$

where there is a different solution corresponding to each integer $n = 1, 2, 3, \dots$ etc. Substituting this solution in equation (20.56) we have

$$\left(\frac{n\pi}{L}\right)^2 = \frac{2mE_n}{\hbar^2} \quad (20.58)$$

which gives the energy to be

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}. \quad (20.59)$$

We find that there is a discrete set of allowed energies E_1, E_2, E_3, \dots corresponding to different integers 1, 2, 3, Inside the potential well there do not exist states with other values of energies.

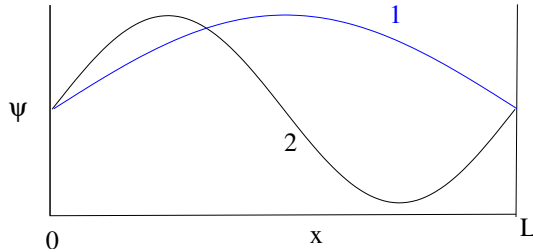


Figure 20.6:

The state with $n = 1$ has the lowest energy

$$E_1 = \frac{\pi^2 \hbar^2}{2mL^2} \quad (20.60)$$

and this is called the ground state. The wavefunction for a particle in this state is

$$\psi_1(x, t) = A_1 e^{-iE_1 t/\hbar} \sin\left(\frac{\pi x}{L}\right), \quad (20.61)$$

This is shown in Figure 20.6. Normalizing this wavefunction $\int_{-\infty}^{\infty} \psi_1^* \psi_1 dx = 1$ determines $A_1 = \sqrt{\frac{2}{L}}$.

The $n = 2$ state is the first excited state. It has energy $E_2 = 4 E_1$ and its wavefunction (shown in Figure 20.6) is

$$\psi_2(x) = A_2 e^{-iE_2 t/\hbar} \sin\left(\frac{2\pi x}{L}\right). \quad (20.62)$$

The wavefunction

$$\psi(x, t) = c_1 \psi_1(x, t) + c_2 \psi_2(x, t) \quad (20.63)$$

is also an allowed state of a particle in the potential well. This is not an energy eigenstate. We will get either E_1 or E_2 if the particle's energy is measured.

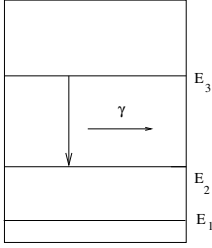


Figure 20.7:

The energy of the higher excited states increases as $E_n = n^2 E_1$. Consider a particle that undergoes a transition from the n state to the $n - 1$ state as shown in Figure 20.7. The particle loses energy in such a process. Such a transition may be accompanied by the emission of a photon (γ) of frequency $\nu = (E_n - E_{n-1})/h$ which carries away the energy lost by the particle.

It is now possible to fabricate microscopic potential wells using modern semiconductor technology. This can be achieved by doping a very small regions of a semiconductor so that an electron inside the doped region has a lower potential than the rest of the semiconductor. An electron trapped inside this potential well will have discrete energy levels E_1, E_2 , etc. like the ones calculated here. Such a device is called a quantum well and photon's are emitted when electron's jump from a higher to a lower energy level inside the quantum well.

20.2.4 Tunnelling

A particle of energy E is incident on a step potential of height $V > E$ as shown in Figure 20.8. The step potential extends from $x = 0$ to $x = a$, the potential

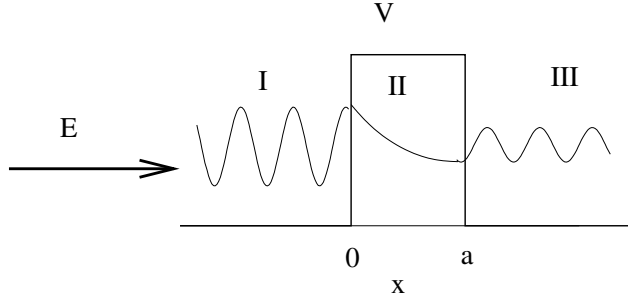


Figure 20.8:

is zero on either side of the step. The particle has wavefunction

$$\psi_I(x, t) = e^{-iEt/\hbar} [A_I e^{ipx/\hbar} + B_I e^{-ipx/\hbar}] \quad (20.64)$$

in region *I* to the left of the step. The first part $A_I e^{ipx/\hbar}$ represents the incident particle i.e. travelling along $+x$ direction and the second part $B_I e^{-ipx/\hbar}$ the reflected particle travelling along the $-x$ axis.

In classical mechanics there is no way that the particle can cross a barrier of height $V > E$. In quantum mechanics the particle's wave function penetrates inside the step and in region *II* we have

$$\psi_{II}(x, t) = e^{-iEt/\hbar} [A_{II} e^{-qx/\hbar} + B_{II} e^{qx/\hbar}] \quad (20.65)$$

In region *III* the wave function is

$$\psi_{III}(x, t) = e^{-iEt/\hbar} [A_{III} e^{ipx/\hbar} + B_{III} e^{-ipx/\hbar}] \quad (20.66)$$

where the term $A_{III} e^{ipx/\hbar}$ represents a particle travelling to the right and $B_{III} e^{-ipx/\hbar}$ represents a particle incident from the right. In the situation that we are analyzing there are no particles incident from the right and hence $B_{III} = 0$.

In quantum mechanics the wave function does not vanish in region *II*. As shown in Figure 20.8 the incident wave function decays exponentially in this region, and there is a non-zero value at the other boundary of the barrier. As a consequence there is a non-zero wavefunction in region *III* implying that there is a non-zero probability that the particle penetrates the potential barrier and gets through to the other side even though its energy is lower than the height of the barrier. This is known as quantum tunnelling. It is as if the particle makes a tunnel through the potential barrier and reaches the other side. The probability that the incident particle tunnels through to the other side depends on the relative amplitude of the incident wave in region *I* and the wave in region *III*. The relation between these amplitudes can be worked out by matching the boundary conditions at the boundaries of the potential barrier.

The wave function and its x derivative should both be continuous at all the boundaries. This is to ensure that the Schrodinger's equation is satisfied at all points including the boundaries.

Matching boundary conditions at $x = 0$ we have

$$\psi_I(0, t) = \psi_{II}(0, t) \quad (20.67)$$

and

$$\left(\frac{\partial \psi_I}{\partial x} \right)_{x=0} = \left(\frac{\partial \psi_{II}}{\partial x} \right)_{x=0} \quad (20.68)$$

We also assume that the step is very high $V \gg E$ so that

$$q = \sqrt{2m(V - E)} \approx \sqrt{2mV} \quad (20.69)$$

and we also know that

$$p = \sqrt{2mE}, \quad \text{so} \quad \frac{p}{q} = \sqrt{\frac{E}{V}} \ll 1 \quad (20.70)$$

Applying the boundary conditions at $x = 0$ we have

$$A_I + B_I = A_{II} + B_{II} \quad (20.71)$$

and

$$ip(A_I - B_I) = -q(A_{II} - B_{II}). \quad (20.72)$$

The latter condition can be simplified to

$$A_I - B_I = \frac{iq}{p}(A_{II} - B_{II}) \quad (20.73)$$

Applying the boundary conditions at $x = a$ we have

$$A_{II}e^{-qa/\hbar} + B_{II}e^{qa/\hbar} = A_{III}e^{ipa/\hbar} \quad (20.74)$$

and

$$-q[A_{II}e^{-qa/\hbar} - B_{II}e^{qa/\hbar}] = ipA_{III}e^{ipa/\hbar}. \quad (20.75)$$

The latter condition can be simplified to

$$A_{II}e^{-qa/\hbar} - B_{II}e^{qa/\hbar} = \frac{-ip}{q}A_{III}e^{ipa/\hbar} \quad (20.76)$$

Considering the $x = a$ boundary first and using the fact that $p/q \ll 1$ we have

$$A_{II}e^{-qa/\hbar} - B_{II}e^{qa/\hbar} = 0 \quad (20.77)$$

which implies that

$$B_{II} = e^{-2qa/\hbar}A_{II} \ll A_{II} \quad (20.78)$$

Using this in equation (20.74) we have

$$A_{III} = 2e^{-ipa/\hbar}e^{-qa/\hbar}A_{II} \quad (20.79)$$

Considering the boundary at $x = 0$ next, we can drop B_{II} as it is much smaller than the other terms. Adding equations (20.71) and (20.73) we have

$$\left(1 + \frac{iq}{p}\right) A_{II} = 2A_I \quad (20.80)$$

and as $q/p \gg 1$, this gives us

$$A_{II} = -\frac{ip}{q} 2A_I. \quad (20.81)$$

Using this in equation (20.79) we have

$$A_{III} = -4i\frac{p}{q} e^{-ipa/\hbar} e^{-qa/\hbar} A_I. \quad (20.82)$$

The transmission coefficient

$$T = \frac{|A_{III}|^2}{|A_I|^2} = 16\frac{p^2}{q^2} e^{-2qa/\hbar} \quad (20.83)$$

gives the probability that an incident particle is transmitted through the potential barrier. This can also be expressed in terms of E and V as

$$T = 16\frac{E}{V} e^{-2a\sqrt{2mV}/\hbar} \quad (20.84)$$

The transmission coefficient drops if either a or V is increased. The reflection coefficient $R = 1 - T$ gives the probability that an incident particle is reflected.

20.2.5 Scanning Tunnelling Microscope

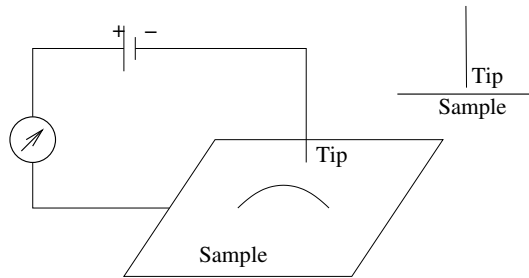


Figure 20.9:

The scanning tunnelling microscope (STM) for which a schematic diagram is shown in Figure 20.9 uses quantum tunnelling for its functioning. A very narrow tip usually made of tungsten or gold and of the size of the order of 1 \AA or less is given a negative bias voltage. The tip scans the surface of the sample which is given a positive bias. The tip is maintained at a small distance from the surface as shown in the figure. Figure 20.10 shows the

potential experienced by an electron respectively in the sample, tip and the vacuum in the gap between the sample and the tip. As the tip has a negative bias, and electron in the tip is at a higher potential than in the sample. As a consequence the electrons will flow from the tip to the sample setting up a current in the circuit. This is provided the electrons can tunnel through the potential barrier separating the tip and the sample. The current in the circuit is proportional to the tunnelling transmission coefficient T calculated earlier. This is extremely sensitive to the size of the gap a .

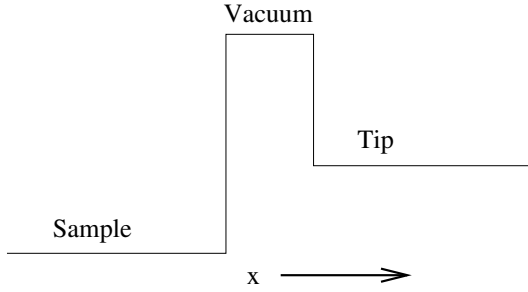


Figure 20.10:

In the STM the tip is moved across the surface of the sample. The current in the circuit differs when the tip is placed over different points on the sample. The tip is moved vertically so that the current remains constant as it scans across the sample. This vertical displacement recorded at different points on the sample gives an image of the surface at the atomic level. Figure 20.11 shows an STM image of a graphite sample.

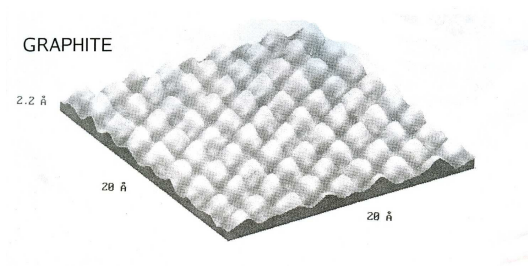


Figure 20.11:

Problems

1. Consider a particle at time $t = 0$ with the wave function

$$\psi(x, t = 0) = A \exp[-(x - a)^2 / (4L^2)]$$

a and L are constants of dimension length. Given $\int_{-\infty}^{\infty} \exp[-x^2/2] dx = \sqrt{2\pi}$.

- a. Determine the normalization constant A .

- b. What is the expectation value $\langle x \rangle$?
 - c. How much is Δx , the uncertainty in x ?
 - d. What is the momentum expectation value $\langle p \rangle$?
 - e. How much is Δp , the uncertainty in p ?
 - f. What happens to the uncertainty in x and the uncertainty p if L is increased?
 - d. How does the product $\Delta x \Delta p$ change if L is varied?
2. A free particle of mass m and energy E is incident on a step potential barrier V . What is the wave function $\psi(x, t)$ of the particle inside the step potential if (a.) $V < E$ (b.) $V > E$.
3. A particle of mass m is confined to $0 \leq x \leq L$ by two very high step potentials (particle in a box).
 - a. What is the wave function $\psi_1(x, t)$ of the particle in the lowest energy state?
 - b. What is the wave function $\psi_2(x, t)$ of the particle in the first excited energy state?
 - c. For $\psi_1(x, t)$, what are the expectation values $\langle x \rangle$ and $\langle p \rangle$?
 - d. For $\psi_1(x, t)$, what are the uncertainties Δx and Δp .
4. An electron trapped in a region of length $L = 10 \text{ } \vec{r}A$. (a.) What are the energies of the ground state and the first two excited states? (b.) An electron in the first excited state emits radiation and de-excites to the ground state. What is the wavelength of the emitted radiation?
5. The wave function of a particle confined in a region of length L is given to be

$$\psi(x, t) = \frac{3}{5}\psi_1(x, t) + \frac{4}{5}\psi_2(x, t)$$

where $\psi_1(x, t)$ and $\psi_2(x, t)$ are the wave functions introduced in Problem

6. What are the possible outcomes and their probabilities if the energy of the particle is measured? What is the expectation value of the energy? What is the uncertainty in the energy?
6. A particle of mass m and energy E is incident from zero potential to a step potential V , where $V = 2E$ as shown in Figure 20.12.

The incident, reflected and transmitted wave functions ψ_I , ψ_R and ψ_T respectively are

$$\psi_I(x, t) = Ae^{-i(Et - px)/\hbar}, \quad \psi_R(x, t) = Be^{-i(Et + px)/\hbar}$$

$$\text{and } \psi_T(x, t) = Ce^{-i(Et + ax)/\hbar}.$$

where p and a are constants.

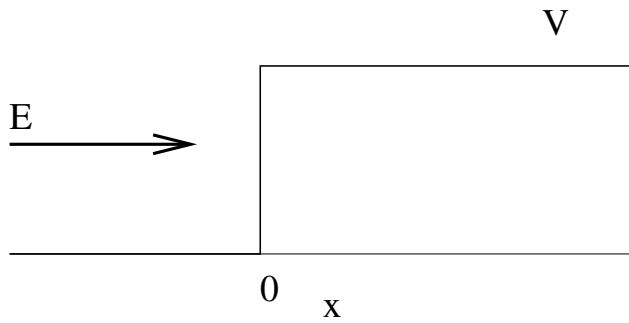


Figure 20.12:

- What is the ratio a/p ?
- Match the boundary conditions at $x = 0$ to determine the ratio $(A + B)/C$.
- Match the boundary conditions at $x = 0$ to determine the ratio $(A - B)/C$.