Physics GRE Comprehensive Notes

These set of notes were written while studying to take the physics GRE. They are based largely on older exams. They summarize most of the necessary topics to succeed in the PGRE. The notes include tips and point out common questions.

Written by: JEFF ASAF DROR

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Preface

These are notes that I wrote up when studying for the physics GREs. The notes are extensive and were meant to include every possible question on the exam. While they are not fully inclusive they come pretty close and were a very big help for me on the GREs. They are largely based on previous GRE exams that ETS distributes. Since ETS constantly repeats questions, the notes are a good study material for anyone taking the exam. When writing these notes I did take a few images from online sources without putting references. This is because I did not initially intend on distributing these notes. If I took your image and did not reference it please let me know and I'd be more then happy to credit you in the bibliography.

One thing you're bound to notice while taking the GRE practice exams (if you have already) is that they love to put questions that you need simple tricks to solve them. In order to avoid getting fooled by these problems I've sprinkled asides which I denote as "**STOP! Common Gre Problem**". I've attempted to describe the common tricks and the solution in these subsections.

I hope those notes will be as useful for you as they were for me. Good luck!

Chapter 2

Classical Mechanics

2.1 Newton's Laws

Def 1. Newton's First Law: An object at rest stays at rest unless acted on by an outside force.

Def 2. Newton's Second Law: $\mathbf{F} = m\mathbf{a}$

Def 3. Newton's Third Law: Every action has an equal and opposite reaction

2.2 Forces

The maximum friction acting on an object is given by

$$\mathbf{F_f} \leq \mu \mathbf{F_N} \tag{2.1}$$

The coefficient of static friction is always equal or greater to kinetic friction. i.e.

$$\mu_{static} \geq \mu_{kinetic} \tag{2.2}$$

STOP! Common GRE Problem 1. *Consider the situation of figure 2.1. In this case the force of friction on the top block is not in general given by the product of the normal force and the coefficient of static friction since the force that acting on the top block that friction opposes is just $m_B a$.*

The work done by a force $\mathbf{F}$ is given by

$$W = \int \mathbf{F} \cdot d\mathbf{r} \tag{2.3}$$

The power is equal to the time rate of change of the work begin done:

$$P = \frac{dW}{dt} = \int \mathbf{F} \cdot d\mathbf{v} \tag{2.4}$$

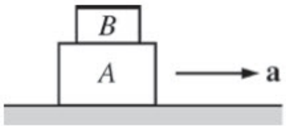


Figure 2.1: Situation shown in definition 1

If a force is conservative it implies that the work done by that force is independent of the path taken. A force is conservative if the curl of the force is equal to zero. i.e.

$$\nabla \times \mathbf{F} = 0 \tag{2.5}$$

A force, $\mathbf{F}$ can be extracted from a potential energy by

$$\mathbf{F} = -\nabla U \tag{2.6}$$

Conversely a potential energy can be extracted from a force using

$$U = -\int \mathbf{F} \cdot d\mathbf{r} \tag{2.7}$$

The potential energy on Earth due to gravity is (where y is the distance from the Earth's surface)

$$U(y) = -\int mg\hat{y} \cdot \hat{y}dy \tag{2.8}$$

$$= \int mgdy \tag{2.9}$$

$$= mgy \tag{2.10}$$

Where the integration path above was chosen to be directly away from the Earth (since the force is conservative the integral is independent of path)

STOP! Common GRE Problem 2. *The force on two block system with one block in the air. The second block is held up by the force of friction. The trick here is to remember what to put as the force of friction. The idea is that the force of friction is equal to the product of the acceleration and mass (this is the magnitude of the normal force in this case)*

STOP! Common GRE Problem 3. *Given that there is an east-west wind. How long does it take an air pilot to travel due North? The pilot must fly at some angle toward the incoming wind direction such that the component of the speed of the pilot in the east-west direction is equal to the speed of the wind. It is then straight forward to calculate the time it take the pilot to travel due North (using the y component of the speed).*

2.3 Projectiles

The height (y)of a particle launched at speed v_0, an angle θ_0 from the horizontal neglecting air resistance is given by (note this is very simple to derive from Newton's second law.

$$y = -\frac{gx^2}{2(v_0 \cos(\theta_0))^2} + x \tan\theta \tag{2.11}$$

The range of the projectile is (the distance the projectile travels until it get back to its initial height) (this follows by finding the points which y is equal to zero.

$$R = \frac{v_0^2}{g} \sin(2\theta_0) \tag{2.12}$$

If we consider air resistance acting on the projectile with drag being proportional to the velocity of the projectile Newton's second law take the form

$$m\mathbf{a} = -mg\hat{y} + kv_y\hat{y} \tag{2.13}$$

At equilibrium (terminal velocity) the projectile ceases to accelerate. In such a case the terminal velocity is (rearranging the equation above with a = 0)

$$v_{term} = \frac{mg}{k} \tag{2.14}$$

In this case of air drag being proportional the square of the speed following the same procedure gives

$$v_{term} = \sqrt{\frac{mg}{k}} \tag{2.15}$$

2.4 Springs

The elastic force is given by Hooke's law which says that the force is proportional to the distance away from the equilibrium point:

$$F = -kx \tag{2.16}$$

k is called the spring constant and represents the stiffness of the spring. If you have two springs connected in parallel the spring constant adds:

$$k_{tot} = k_1 + k_2 \qquad \text{(parallel)} \tag{2.17}$$

If you have two spring connected in series the reciprocals of the constants add:

$$\frac{1}{k_{tot}} = \frac{1}{k_1} + \frac{1}{k_2} \qquad \text{(series)} \tag{2.18}$$

If damping force is applied to the spring it is typically taken to be proportional to either the velocity or velocity squared. The new motion can be underdamped, critically damped, or overdamped. If the motion is underdamped then the particle continues to oscillate with a reduced frequency of

$$\omega_d = \omega_o \sqrt{1 - \frac{b}{4mk}} \tag{2.19}$$

Where b is the damping coefficient, m the mass, and k is the spring constant. Under critical damping and overdamping the particle motion is exponential (while damping is faster with critical damping). The different types of damping are illustrated in figure 2.2

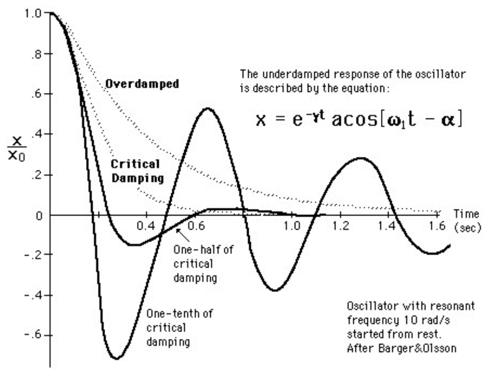


Figure 2.2: Different types of damping

2.5 Systems of Particles and Rigid Bodies

The center of mass of a collection of n particles is given by

$$\mathbf{r}_{COM} = \frac{1}{M} \sum_{i=1}^{n} m_i \mathbf{r_i} \tag{2.20}$$

While the center of mass of a solid body is given by

$$\mathbf{r}_{COM} = \frac{\int \mathbf{r}_i dm}{\int dm} \tag{2.21}$$

In this special case of constant density ρ:

$$\mathbf{r}_{COM} = \frac{\rho \int \mathbf{r} dV}{\rho \int dV} \tag{2.22}$$

$$= \frac{\int \mathbf{r} dV}{V} \tag{2.23}$$

STOP! Common GRE Problem 4. *What is the center of mass doing as a function of time for a system of 3 springs connected in series? Since there are no external forces on the system the center of mass is constant!*

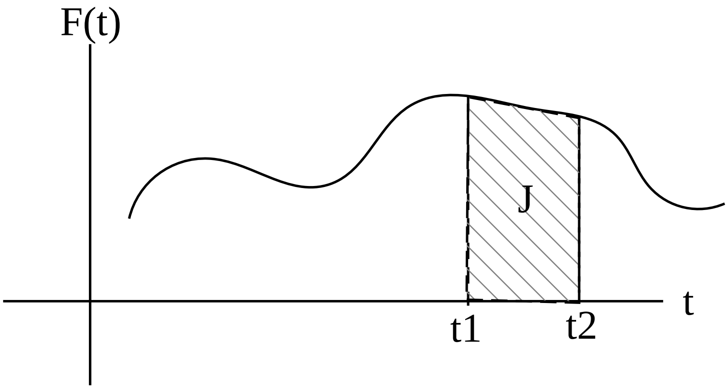


Figure 2.3: Graphical representation of impulse in terms of force.

2.6 Collisions

The impulse ($\mathbf{J}$) is defined as

$$\mathbf{J} = \int_{t_1}^{t_2} \mathbf{F}(t)dt \tag{2.24}$$

This is shown graphically in figure 2.3 In order words impulse is equal to the change in momentum:

$$\Delta \mathbf{p} = \mathbf{J} \tag{2.25}$$

There are three types of collisions: For the following consider a collision between two objects with the ball 2 initially at rest and an incoming ball traveling at $\mathbf{v}_1$

Def 4. Elastic Collision: Collision in which the kinetic energy of the system is constant

If you choose your reference frame such that the initial velocity of the second ball is zero then

$$\mathbf{v_1'} = \frac{m_1 - m_2}{m_1 + m_2}\mathbf{v_1} \qquad \mathbf{v_2'} = \frac{2m_1}{m_1 + m_2}\mathbf{v_1} \tag{2.26}$$

Note that if $m_1 = m_2$ then $\mathbf{v_1'} = 0$ and hence the incoming ball ends up staying still.

STOP! Common GRE Problem 5. *The height of a pendulum after it underwent an elastic collision. You can use equation 2.26 as well as gravitational potential to quickly find the final height (as opposed to working through the lengthy conservation of energy and momentum equations.*

Def 5. Inelastic Collision:Collision in which the kinetic energy of the system is not conserved

Def 6. Completely Inelastic Collision: Collision in which both the particles stick together after the collision.

In such a collision the final velocity is very simple to calculate and is given as (momentum conservation):

$$\mathbf{v}' = \frac{m_1\mathbf{v_1} + m_2\mathbf{v_2}}{m_1 + m_2} \tag{2.27}$$

2.7 Rotational Motion

Rotational motion analogue extends to Newton's second law:

$$\boldsymbol{\tau}_{net} = I\ddot{\theta} \tag{2.28}$$

The centripetal acceleration experienced by an object moving in circular motion is

$$a = \frac{v^2}{r} \tag{2.29}$$

The centripetal force is just the mass times the acceleration:

$$F = \frac{mv^2}{r} \tag{2.30}$$

STOP! Common GRE Problem 6. *A car on the road undergoing circular motion. The way a car works is the tires push the road backwards and the road pushes the tires forwards. Hence the force of the road on the tires provides the centripetal force. Additionally the car may have other forces such as air resistance. The direction of the net force can be inferred by thinking about what direction the tires are angled at.*

The period is given by

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = \frac{2\pi r}{v} \tag{2.31}$$

The angle of a rotating object(θ) through arc length s and radius r is given by

$$\theta = \frac{s}{r} \tag{2.32}$$

The angular velocity is

$$\omega = \frac{d\theta}{dt} \tag{2.33}$$

The angular acceleration is

$$\alpha = \frac{d\omega}{dt} \tag{2.34}$$

The equations of linear and rotational motion are analogous to one another and are shown in table 2.1 The moment of inertia I is given by

$$I = \begin{cases} \sum_i m_i r_i^2 & \text{For system of particles} \\ \int r^2 dm & \text{For a rigid body} \end{cases} \tag{2.35}$$

For the case of a uniform solid the moment of inertia simplifies to

$$I = \frac{M}{V}\int r^2 dV \tag{2.36}$$

The moment of inertia of some common shapes are shown in table 2.2

Torque (τ) is given by

$$\tau = \mathbf{r_{cm}} \times \mathbf{F} \tag{2.37}$$

Linear EOM	Angular EOM
$x - x_0 = v_0 t + \frac{1}{2}at^2$	$\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2$
$x - x_0 = \frac{v^2 - v_0^2}{2a()}$	$\theta - \theta_0 = \frac{\omega^2 - \omega_0^2}{2\alpha}$
$x - x_0 = \frac{1}{2}(v_0 + v)t$	$\theta - \theta_0 = \frac{1}{2}(\omega_0 + \omega)t$
$x - x_0 = vt - \frac{1}{2}at^2$	$\theta - \theta_0 = \omega t - \frac{1}{2}\alpha t^2$

Table 2.1: Equations of Rotational and Linear Motion Undergoing Constant Acceleration

Object	Moment of Inertia About Axis of Symmetry
Point Mass	0
Hoop	mr^2
Sheet (as shown in figure 3.1)	$\frac{1}{3}md^2$

Table 2.2: Moment of inertia of different systems

STOP! Common GRE Problem 7. *Where is the fulcrum on a balance hold two weights? This can be found in two ways the direct way is to equate torque about he pivot point. The alternative way which is far simpler but not as general is to find the center of mass of the system.*

For simple rotations:

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = I\boldsymbol{\alpha} \tag{2.38}$$

Angular momentum is given by

$$\begin{aligned} \mathbf{L} &= \mathbf{r} \times \mathbf{p} && (2.39)\\ &= \mathbf{r} \times \mathbf{v}m && (2.40)\\ &= mr^2\boldsymbol{\omega} \qquad \text{If } \mathbf{r} \perp \mathbf{S} && (2.41)\\ &= I\boldsymbol{\omega} && (2.42) \end{aligned}$$

The kinetic rotational energy (K_{rot}) is

$$K_{rot} = \frac{1}{2}I\omega^2 = \frac{L^2}{2I} \tag{2.43}$$

STOP! Common GRE Problem 8. *Rotation of a single objects in two different orientations. This problem is solved using conservation of angular momentum in the two different situations.*

STOP! Common GRE Problem 9. *A rigid body rolling down a hill. This is solving by equating the sum of rotational kinetic energy, translational kinetic energy, and gravitational potential energy at the top and bottom of the hill. This is shown in equation 2.44.*

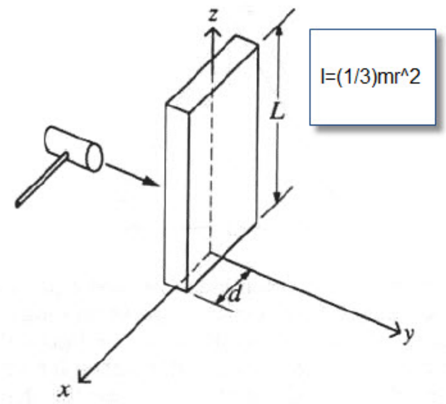


Figure 2.4: The Moment of inertia through a sheet

In that case there are three energies to consider:

$$E_{tot} = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 + mgh \tag{2.44}$$

Def 7. Parallel Axis Theorem: If the moment of inertia about the center of mass of an object is given by I_{com} then the moment of inertia along some parallel axis a distance h away from the center of mass is

$$I = I_{com} + Mh^2 \tag{2.45}$$

Def 8. Principle Axis: The axis in which the object can rotate with constant speed without the need for any torque. It is the axes in which the moment of inertia tensor is diagonalized

Consider a arbitrarily shaped pendulum. The angular frequency of the pendulum is given by

$$\omega = \sqrt{\frac{mgr}{I}} \tag{2.46}$$

Where r is the distance from the center of mass to the pivot point. Note this can be derived directly using the rotational analogue for Newton's second law.
Now consider the special case of a massless rod and a ball with mass m at the end. In this case $I = m\ell^2$, $r = \ell$ and the angular frequency is given by

$$\omega = \sqrt{\frac{g}{\ell}} \tag{2.47}$$

and the period is

$$T = 2\pi\sqrt{\frac{\ell}{g}} \tag{2.48}$$

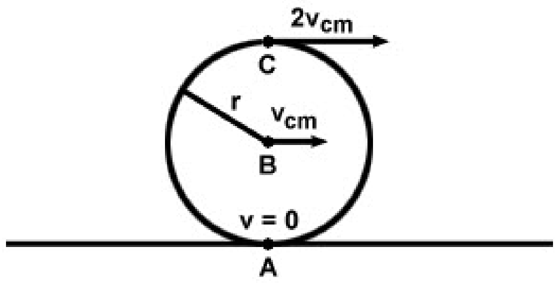


Figure 2.5: Combining Linear and Rotational Motion to Find Odd Axes of Rotation

Consider an object rolling without slipping. Then the motion can be considered a linear combination of rotational and linear motion at every point. Suppose an object is rolling as shown in figure 2.5. If the object is has linear velocity v_{com} and angular velocity ω it is straightforward to show that the speed at the bottom of the object is zero (the translational velocity perfectly cancels the rotational velocity). As opposed to the expected result of the axes of rotation being at the center of the wheel, the part of the wheel that doesn't move and hence the axis of rotation is actually the bottom of the wheel.

STOP! Common GRE Problem 10. *Suppose a ball hits a long stick at the end in an elastic collision. How can you go ahead and find the resultant velocity of the long stick? Even though this collision is elastic you must use conservation of momentum not energy unless you want to take into account rotational kinetic energy.*

The angular momentum of an object is dependent on the point that the angular momentum is about. If an object is rotating and translating then the angular momentum is a sum of both these angular momenta.

$$\mathbf{L} = \mathbf{L}_T + \mathbf{L}_{Rot} \tag{2.49}$$

Where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ and $\mathbf{L}_{Rot} = I\omega$.

2.8 Non-Inertial Rotational Forces

The centrifugal force is a fictitious force felt by objects in a rotating reference frame. The force is radially outward from the axis of rotation. It is given by

$$\mathbf{F} = m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}) \tag{2.50}$$

For objects in rotational motion the centripetal force needs to equal this centrifugal force to keep the object in rotation.

The Coriolis effect is the a non-inertial force that exists in rotating reference frames when a body is moving. Its magnitude and direction are given by

$$\mathbf{F_C} = -2m\boldsymbol{\omega} \times \mathbf{v} \tag{2.51}$$

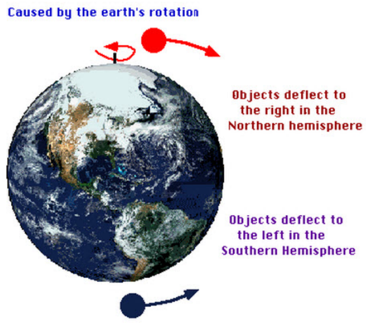


Figure 2.6: The Coriolis effect due to Earth's rotation

Coriolis effect causes interesting effects on Earth. It prohibits wind currents from moving in the expected direction namely from places of high pressure to low pressure. This is shown in figure 2.6

2.9 Orbits

The force due to gravity is

$$\mathbf{F} = \frac{Gm_1m_2}{r^2}\hat{r} \tag{2.52}$$

Where $\mathbf{r}$ is the vector connecting the two masses. The escape speed of an object is easy to derive since it is the speed required such that you will get infinitely far from your planet with no speed. Thus the final energy is zero. The initial energy is just the sum of your kinetic and potential energy. This gives

$$v = \sqrt{\frac{2GM}{R}} \tag{2.53}$$

The minimum speed required in order to orbit a planet is given by setting the centripetal force equal to the force due to gravity

$$\frac{mv^2}{r} = mg \tag{2.54}$$

$$v = \sqrt{gr} \tag{2.55}$$

The velocity of an orbiting planet is found by setting the centripetal force equal to the gravitational force:

$$\frac{mv^2}{r} = \frac{GMm}{r^2} \tag{2.56}$$

$$v = \sqrt{\frac{GM}{r}} \tag{2.57}$$

Type of Orbit	Condition	Bound/Unbound	Required Velocity
Spiral Orbit	$E < V_{min}$	Bound	$v < \sqrt{\frac{GM}{R}}$
Circular Orbit	$E = V_{min}$	Bound	$v = \sqrt{\frac{GM}{R}}$
Ellipse	$V_{min} < E < 0$	Bound	$\sqrt{\frac{GM}{m}} < v < \sqrt{\frac{2GM}{R}}$
Parabolic	$E = 0$	Unbound	$v = \sqrt{\frac{2GM}{R}}$
Hyperbolic	$E > 0$	Unbound	$v > \sqrt{\frac{2GM}{R}}$

Table 2.3: The Different Possible Orbits

The effective potential energy of an orbiting body is

$$V_{eff}(r) = V(r) + \frac{L^2}{2mr^2} \tag{2.58}$$

For a gravitational potential:

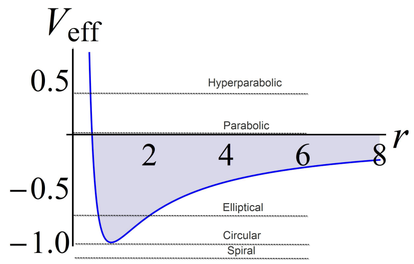


Figure 2.7: The Effective Gravitational Potential With the Shaded Region Representing Energy Values For an Elliptical Orbit

$$V(r) \propto \frac{1}{r} \tag{2.59}$$

The energy of an object in an effective potential $V_{eff}(r)$

$$E = \frac{1}{2}mv^2 + V_{eff}(r) \tag{2.60}$$

The different types of orbits are summarized in table 2.3 A useful theorem for deterining possible orbits is Bertrand's Theorem. The theorem states that only the inverse square law and the radial harmonic oscillator can produce stable closed non-circular orbits. However circular orbits can be produced by any central force.

STOP! Common GRE Problem 11. *Given a velocity which orbit is a planet in? By comparing the velocity the of the planet with the values of* $\sqrt{\frac{GM}{r}}$ *and* $\sqrt{\frac{2GM}{R}}$ *one can determine what type of orbit the planet is in.*

Kepler's laws are as follows.

1. The orbit of every planet is an ellipse with the Sun at one of the two foci
2. A line joining a planet and the Sun sweeps out equal areas during equal intervals of time (i.e. $dA/dt = \frac{L}{2m}$).
3. The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis (half the major axis, the maximum diameter of the ellipse) of the orbit. The proportionality constant is equal for any planet around the sun (only dependent on the mass of the sun). Note that this law is very simple to derive for circular orbits.

Note that in the case of circular orbit Kepler's first and second laws reduce to: the Sun is at the center of a circular orbit and the orbital period squared of the planets is proportional to the radius of the orbit cubed.

2.10 Fluids

Pressure is given by

$$P = \frac{dF}{dA} \tag{2.61}$$

Given a fluid at rest the pressure as a function of height is given by

$$p - p_0 = \rho g h \tag{2.62}$$

Def 9. Pascal's Principle: A change in pressure applied to an enclosed incompressible fluid is transmitted undiminished to every portion of the fluid and to the walls of the fluid

The Bernoulli's equation for fluid flow says that (assume incompressible, nonviscous, laminar flow) is

$$\overbrace{P}^{\text{Pressure}} + \underbrace{\rho g y}_{\text{Gravitational Term}} + \overbrace{\frac{1}{2}\rho v^2}^{\text{Kinetic Term}} = \text{constant} \tag{2.63}$$

Using Bernoulli's equation it is straight forward to derive the speed of water emitted from a small hole at the bottom of a barrel:

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2 \tag{2.64}$$

Assume $v_1 = 0, P_1 = P_2, h_2 = 0$

$$v_2 = \sqrt{2gh_1} \tag{2.65}$$

Given a fluid going through cross sectional areas A_i with velocities v_i, the quantity $v_i A_i$ is always conserved. i.e.

$$v_i A_i = constant \tag{2.66}$$

This effect is shown in figure 2.8

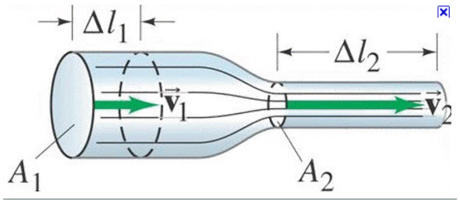


Figure 2.8: Speed of fluid passing through a tube of variable length

Def 10. Archimedes Principle: The buoyant force on an object is equal to the weight of the water displaced by the object.

Thus the force is given by

$$F = mg = \rho_{fluid} V g \tag{2.67}$$

Stoke's law gives the force on a ball passing through a fluid. His law says that:

$$F_{drag} = -6\pi\mu R v_s \tag{2.68}$$

where μ is the viscosity, R is the radius of the ball, and v_s is terminal velocity of the ball was falling in the fluid due to gravity.

2.11 Waves

Any function with the form

$$\psi(x,t) = f\,(kx \pm \omega t) \tag{2.69}$$

is considered a wave in the x direction. The speed (phase velocity) of the wave is

$$v = \frac{\omega}{k} \tag{2.70}$$

The direction of the wave is in the positive x direction if kx ωt carry opposite signs and in the negative direction if kx and ωt carry the same sign.

The speed of a wave moving down a string is given by

$$v = \sqrt{\frac{T}{\mu}} \tag{2.71}$$

Where T is the tension in the rope an μ is the linear mass density of the string. The intensity of a spherical wave falls off with $1/r^2$:

$$I = \frac{P_s}{4\pi r^2} \tag{2.72}$$

This relation is required for energy conservation. Note that the total power emitted by a source is constant by the intensity is not. The sound level in decibels for and intensity I is given by

$$\beta = 10 \text{ dB} \log \frac{I}{I_0} \tag{2.73}$$

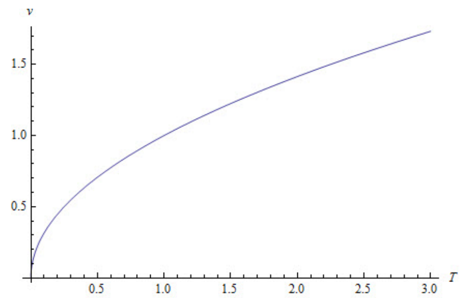


Figure 2.9: Speed of sound in air as a function of temperature

Where $I_0 = 10^{-12}$. Thus if something is amplified by 30 dB it is 1000 (10^3) times more intense.
The speed of sound in an ideal gas is similar to equation 2.71

$$v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{\gamma RT}{M}} \tag{2.74}$$

Where R is the gas constant, M is the molecular weight of the gas, γ is the adiabatic constant of the gas, and T is the temperature. Its important to note that v depends on the square root of the temperature. This is shown in figure 2.9

2.12 Eigenmodes

A system with coupled oscillators can oscillate at different frequencies (eigenfrequencies, normal modes, harmonics, and resonant frequencies are all synonyms). The number of possible frequencies that the system can oscillate in is equal to the number of degrees of freedom of the system. There is always (usually?) a normal mode for the system which corresponds to having a single degree of freedom. For example for two boxes connected by two springs there must exist the normal mode:

$$\omega_1 = \sqrt{\frac{k}{m}} \tag{2.75}$$

There also exists a second normal mode

$$\omega_2 = \sqrt{\frac{3k}{m}} \tag{2.76}$$

Limits can often be used to find the eigenfrequencies. Usually in a certain limit the eigenmodes are degenerate (not true for the case above).

2.13 Sound in a Pipe

Consider a pipe with sound resonating through the pipe as shown in figure 2.10. The

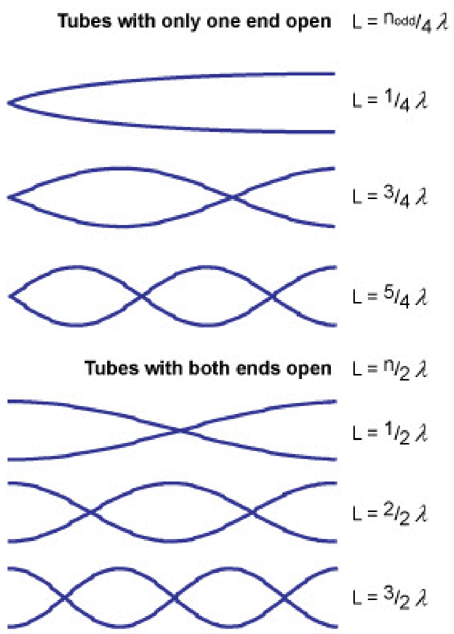


Figure 2.10: Sound waves in a pipe

sound waves have to form a discrete number of wavelengths in the pipe. For a pipe open at both ends:

$$L = \frac{n\lambda}{2}; \qquad n \in \mathbb{Z} \tag{2.77}$$

For a pipe closed at one end there must be a node at the closed end. This forces the constraint:

$$L = \frac{\lambda}{4}(2n+1) \tag{2.78}$$

The number of harmonics (n) separating two sounds of frequencies f_2 and f_1 is

$$n = Round\left(\frac{f_2}{f_1}\right) \tag{2.79}$$

The number of beats separating the sounds is

$$beats = n * f_1 - f_2 \tag{2.80}$$

2.14 Rocket Motion

Motion of a rocket is due to the rocket exhausting fuel backwards and use Newton's third law to accelerate it forwards. The equation of motion is a function of the rockets mass:

$$\overbrace{m\frac{dv}{dt}}^{ma} + \overbrace{u\frac{dm}{dt}}^{\text{Rocket exhaust relative to rocket}} = 0 \tag{2.81}$$

2.15 Lagrangian Mechanics

The Lagrangian is defined as the potential energy (U) subtracted from the kinetic energy (T)

$$\mathcal{L}(q, \dot{q}, t) = T - U \tag{2.82}$$

The action S is defined as:

$$\mathcal{S} = \int \mathcal{L} dt \tag{2.83}$$

Def 11. Hamilton's Principle: Out of all the possible paths taken by a system of particles, the actual path is the one at which the action is an extremum.

The Lagrange-Euler equations representing the motion of a particle are:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \tag{2.84}$$

The conjugate momentum p is defined as:

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}} \tag{2.85}$$

The Hamiltonian is defined as:

$$H(q, p, t) = \sum_i p_i \dot{q}_i - \mathcal{L} \tag{2.86}$$

The Hamiltonian equations of motion are:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{2.87}$$

In order to find conserved quantities from the Hamiltonian we can find (by equations of motion)

$$\dot{p}_i = \frac{\partial H}{\partial q_i} \tag{2.88}$$

If this quantity is zero then the momentum in this coordinate is constant in time. A consequence of this is if the Hamiltonian does not contain explicit dependence on a coordinate q_j then p_j is automatically conserved.

Chapter 3

Electromagnetism

3.1 General Knowledge

The Biot-Savart law is

$$\mathbf{B}(\mathbf{r}) \int \mathbf{dB}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \hat{r}}{r^2} dl' \tag{3.1}$$

The potential difference between two points a and b is given by

$$V(\mathbf{b}) - V(\mathbf{a}) = -\int_a^b \mathbf{E} \cdot d\mathbf{l} \tag{3.2}$$

The electric and magnetic field are related to the vector and scalar potentials by

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \tag{3.3}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{3.4}$$

The Poynting vector is given by:

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) \tag{3.5}$$

Often times the time average Poynting vector is more useful:

$$\langle \mathbf{S} \rangle = \Re(\mathbf{E} \times \mathbf{H}^*) \tag{3.6}$$

The intensity of light (I) is given by the time average of the Poynting vector. Consider light in a vacuum with electric field E:

$$S = \frac{1}{c\mu_0} E^2 \qquad I = \frac{1}{c\mu_0} E_{rms}^2 \tag{3.7}$$

Electromagnetic waves have linear momentum (E/c). Thus pressure can be exerted on an object with light. The expression for this radiation pressure is given by

$$p_r = \begin{cases} \frac{I}{c} \text{ For total absorption} \\ \frac{2I}{c} \text{ For total reflection} \end{cases} \tag{3.8}$$

For electromagnetic waves the B field is related to the E field by:

$$\mathbf{B} = \frac{1}{c}\hat{\mathbf{k}} \times \mathbf{E} \tag{3.9}$$

Furthermore, the magnitude of the B field is related to the magnitude of the E field by

$$B = \frac{E}{c} \tag{3.10}$$

The Lorentz force law is given by

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \tag{3.11}$$

The force on a current wire is given by

$$\mathbf{F} = L\mathbf{I} \times \mathbf{B} \tag{3.12}$$

In a cyclotron a particle goes in a circular path. The centripetal force is the magnetic force. Hence

$$qvB = \frac{mv^2}{r} \tag{3.13}$$

$$r = \frac{mv}{qB} \tag{3.14}$$

The angular frequency of the cyclotron is given by (easy to derive from above)

$$\omega = \frac{qB}{m} \tag{3.15}$$

The magnetic force between two current wires is given by (where L is the length of the wires)

$$F = iLB\sin(90^o) \tag{3.16}$$

$$= i_1 L\left(\frac{\mu_0 i_2}{2\pi d}\right) \tag{3.17}$$

If the current in the wires is in the same direction then there is an attractive force between the wires (and the opposite if the current is going in opposite directions)

The potential inside an N sided polygon without any charges inside is just the average of the potential on each of the sides. In order words:

$$V = \frac{1}{N}\sum_i V_i \tag{3.18}$$

Note that the electric field may indeed be zero but that just means the potential is positive. Alternatively one may be able to impose boundary conditions ($E^{\perp}$ is continuous) to obtain the potential.

The exponent of Coloumb's inverse square law is most accurately tested by finding the electric field inside a charged conducting shell.

3.2 Magnetization and Polarization

The displacement **D** and the auxiliary magnetic field **H** are given by (for linear materials)

$$\mathbf{D} = \epsilon\mathbf{E} \qquad \mathbf{H} = \frac{\mathbf{B}}{\mu} \tag{3.19}$$

The polarization (P) and magnetization (M) are given by:

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E} \qquad \mathbf{M} = \frac{\chi_m\mathbf{B}}{\mu} \tag{3.20}$$

ϵ and μ are given by

$$\epsilon = \epsilon_0(1+\chi_e) \qquad \mu = \mu_0(1+\chi_m) \tag{3.21}$$

The displacement vector in a uniform dielectric material is

$$\mathbf{D} = \epsilon_0\mathbf{E_{vac}} \tag{3.22}$$

Thus the electric field in a uniform dielectric material is

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon} = \frac{\epsilon_0}{\epsilon}\mathbf{E}_{vac} \tag{3.23}$$

Note that these equations do not hold true when inserted a dielectric in between parallel plates connected to a battery since in that case the charges on the plates increases as the dielectric is added. In this case the electric field between the plates is constant as the dielectric is added since the potential on the plates must be constant (determined by the battery). On the other hand the charge on the plates is increasing and the energy stored by the plates is also increasing. These ideas are summarized in table 3.1

Parameter	Initial Condition	Final Condition	Effect
Electric field (E)	$\frac{V}{d}$	$\frac{V}{d}$	$E' = E$
Surface charge density (σ)	$\epsilon_o E = \frac{\epsilon_o V}{d}$	$\frac{\epsilon V}{d}$	$\sigma' > \sigma$
Capacitance per unit area ($\frac{C}{A}$)	$\frac{\sigma}{V} = \frac{\epsilon_o}{d}$	$\frac{\epsilon}{d}$	$C' > C$
Charge (Q)	$\sigma A = \epsilon_o EA$	ϵEA	$Q' > Q$
Energy (U)	$\frac{1}{2}CV^2 = \frac{1}{2}\frac{\epsilon_o V^2}{d}$	$\frac{1}{2}\frac{\epsilon V^2}{d}$	

Table 3.1: Effect of Adding a Dielectric to a Capacitor

The most common magnetic states and their corresponding values of magnetic susceptibility and permeability are shown in table 3.2

Table 3.2: Conditions and Descriptions For Common Magnetic States

State	Values of μ and χ_m	Descriptions
Diamagnetic	$\mu < \mu_0$, $\chi_m < 0$	No unpaired electrons, Field reduced by Lenz's law acting on electron orbits
Paramagnetic	$\mu > \mu_0$, $\chi_m > 0$	Unpaired electrons have spin which align and produce magnetic fields
Ferromagnetic	$\mu >> \mu_0$, $\chi_m >> 0$	Domains are produced

3.3 Maxwell's Equation

The differential form of Maxwell's equations is

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \rightleftharpoons \int \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{enc}}{\epsilon_0} \tag{3.24}$$

$$\nabla \cdot \mathbf{B} = 0 \rightleftharpoons \int \mathbf{B} \cdot d\mathbf{a} = 0 \tag{3.25}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \rightleftharpoons \int \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_B}{dt} \tag{3.26}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{B}}{\partial t} \rightleftharpoons \int \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc} + \overbrace{\frac{1}{c^2}\frac{\partial}{\partial t}\int \mathbf{E} \cdot d\mathbf{a}}^{\text{Typically taken to be zero}} \tag{3.27}$$

Note that if magnetic monopoles existed then there would be magnetic current and magnetic charges. This would require a formulation of $\nabla \cdot \mathbf{B}$ and $\nabla \times \mathbf{E}$ and Maxwell's equations would be perfectly symmetric between B and E.

STOP! Common GRE Problem 12. *What is the electric field a distance R from the center of a sphere with radial charge density that is a function of r? In this case to find the charge enclosed you must integrate infinitely small shells and then Gauss's takes you the rest of the way.*

3.4 E Field Due to Different Charge Configurations

For a charge ring:

$$\mathbf{E} = \frac{qz}{4\pi\epsilon_0 (z^2 + R^2)^{3/2}} \tag{3.28}$$

Note in the large z limit we have the expected $\frac{1}{z^2}$ limit due to Coloumb's law. For an electric dipole:

$$\mathbf{E} = \frac{1}{2\pi\epsilon_0}\frac{p}{z^3} \tag{3.29}$$

Where p is the dipole moment. For a pair of charges with opposite charges the dipole moment is given by

$$\mathbf{p} = q\mathbf{d} \tag{3.30}$$

Where **d** is the vector that connects the two charges. The potential energy of an electric dipole in an electric field is given by

$$U = -\mathbf{p} \cdot \mathbf{E} \tag{3.31}$$

While the potential energy of a magnetic dipole in a magnetic field is given by

$$U = -\mu \cdot \mathbf{B} \tag{3.32}$$

The electric field due to a line of charge is

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{r} \tag{3.33}$$

The electric field due a single infinite sheet of charge is

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0}\hat{z} \tag{3.34}$$

Where z is the direction away from the sheet. The electric field for a spherical shell is zero inside the shell (since the charge enclosed is zero) and the same as if the charge was focused at the center for outside the shell

3.5 Magnetic Field For Different Current Configurations

The direction for the magnetic field in the following cases are given by the right hand rule. For a long straight wire:

$$B = \frac{\mu_0 I}{2\pi r} \tag{3.35}$$

For the center of a circular arc:

$$B = \frac{\mu_0 I \phi}{4\pi R} \tag{3.36}$$

The magnetic field of a solenoid is

$$B = \begin{cases} \mu_0 I n & \text{Inside Solenoid} \\ 0 & \text{Outside Solenoid} \end{cases} \tag{3.37}$$

3.6 Method of Images

Consider a grounded conducting plane next a set of charges. The electric field due to charge induced on the plane is the same as if a charge of opposite sign is behind the plane the same distance the original charge is from the plane. This idea is shown in figure 3.1.

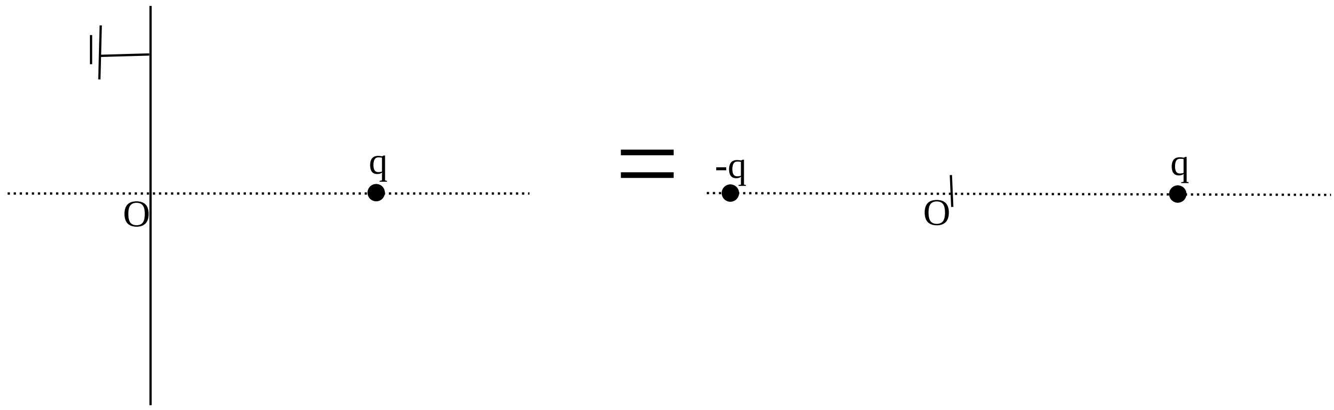


Figure 3.1: The Method of images for a single point charge

3.7 The Hall Effect

If a current is sent through a slab of material and a magnetic field applied perpendicular to the slab then the charge carriers experience a magnetic force pushing them to one side of the slab (the sign of the charge carriers tells you which side). After a little while charge will begin to build on one side of the slab. This will create a potential difference in the slab and hence a force pushing the charge carriers along the slab. When the electric and magnetic force are equal the charge carriers are in equilibrium. At this point:

$$qE = qvB \tag{3.38}$$

$$\frac{V}{d} = vB \tag{3.39}$$

$$\frac{V}{d} = \frac{IB}{dtnq} \tag{3.40}$$

$$n = \frac{IB}{qtV} \tag{3.41}$$

Where V is the potential difference (easy to measure), q is the charge of the charge carriers, d is the width of the slab, t is the thickness of the slab, B is the electric field, n is the number density of the charge carriers, and I is the current.

3.8 Characterizing Material Using Their Conductivity

One way to characterize materials is by how well they conduct electricity. The typical model used for conductivity in solids assumes that the nuclei are immobilized in a lattice. The electrons then form two bands (two “collections” of electrons), the valence and conduction bands. The valence band electrons stay with the nuclei while the conduction

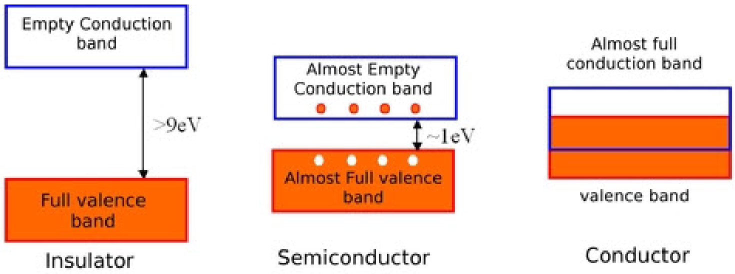


Figure 3.2: Conductors vs. semiconductors vs. insulators

bands are free to move. Since conductivity requires movement of electrons the conduction bands alone contribute to the conductivity. The more electrons in the conduction band, the greater the conductivity of a material. See section 10.3 for more details on the "free electron model". There are three categories for characterizing materials. They are distinguished in figure 3.2

3.8.1 Conductors

Conductor conductivity is approximately $10^6 - 10^7$, though this can vary by an order of magnitude. Conductors work by each atom having a tightly bound set of an electrons and one or two electrons that are very loosely held. These loosely held electrons contribute to the conduction band. The basic properties of a conductor are

1. $E = 0$ inside a conductor
2. $\rho = 0$ insider a conductor (follows from $E = 0$ by Gauss's law)
3. A conductor is an equipotential (By the definition of a potential difference and $E = 0$)
4. Any net charge resides on the surface of the conductor
5. E is perpendicular to the surface just outside the conductor

STOP! Common GRE Problem 13. *A grounded conductor and nearby charges. The total induced charge on this conductor must be equal to negative the nearby charge.*

3.8.2 Semiconductors

The conductivity of semiconductors is approximately $10^{-3} - 10^3$, though can very by a few orders of magnitude. Semiconductors are composed of atoms with typical thermal valence electron energies that are just low enough that they cannot become free electrons (i.e. cannot join the conduction band). However due to the Boltzmann distribution of energies some electrons do have enough energy to go into the free electron band. Thus semiconductors can conduct electricity but nowhere near as well as conductors. However

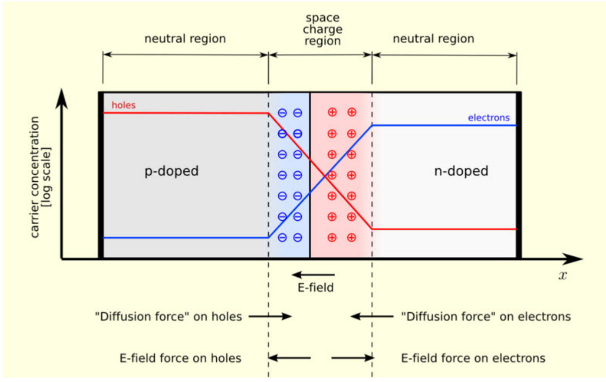


Figure 3.3: A pn junction

semi conductors have purpose for creating other devices such as a p-n junction by doping the semiconductor.

The p-n Junction A p-n junction uses two types of materials:

Def 12. n-type Semiconductors: Semiconductors that are doped with "donor atoms" (atoms which are free to give up electrons). This produces more electrons in the conductor band (making it more negative, hence the n-type)

Def 13. p-type Semiconductors: Semiconductors that are doped with "acceptor atoms" (atoms which are easily accept electrons). This produces more holes in the valence band (making it more positive, hence p-type).

A p-n junction occurs when an n-type and p-type semiconductor are put together side by side. When the two types of semiconductors are put side by side there are more electrons in the p side and more holes in the n side. Thus by diffusion a diffusion current is created, I_{diff}. However recall the p side has excess acceptor atoms. When an electron from the n side finds its way to the p side it combines with the acceptor atoms forming a negatively charged atom. This results in a build up of charges on the p and n side. This produces a potential difference between the sides of the p-n junction. This potential difference creates a second drift current across the p-n junction, I_{drift}. A pn junction is shown in figure 3.3

3.8.3 Insulators

Insulators are materials with very low conductivity (about anything below 10^{-5}). The reason materials are insulators is because the valence electrons have typical thermal energies well below the energies required to reach the conduction band (i.e. the outer electrons are tightly bound to the atoms).

3.9 Superconductivity

Superconductivity is a phase which materials enter which is characterized by an infinite conductivity (or 0 resistivity). Many materials can undergo transitions into a superconductive state once the temperature of the material is lowered beyond what is called the critical temperature of the material. The most prominent theory of superconductivity is BSC theory. This theory says that in the superconducting state pairs of electrons condense into a bosonic state. In this way electrons can all be in the same state eliminating collisions between the electrons and hence resistivity. One corollary to superconductivity is known as the Meissner effect. The Meissner effect is the effect that a material expels any magnetic field from itself as it enters the superconducting state.

3.10 Boundary Conditions

The boundary conditions given by Maxwell's equations at an interface are

$$\epsilon_1 E^{\perp} - \epsilon_2 E^{\perp} = \sigma_f \tag{3.42}$$

$$B_1^{\perp} - B_2^{\perp} = 0 \tag{3.43}$$

$$\mathbf{E_1^{\|}} - \mathbf{E_2^{\|}} = 0 \tag{3.44}$$

$$\frac{1}{\mu_1}\mathbf{B_1^{\|}} - \frac{1}{\mu_2}\mathbf{B_1^{\|}} = \mathbf{K_f} \times \hat{n} \tag{3.45}$$

Consider a wave hitting a perfect conductor. The electric and magnetic fields are both perfectly expelled from inside the conductor. The boundary conditions dictate that the tangential components of the electric field is continuous. Thus

$$\mathbf{E}_i^{\|} + \mathbf{E}_r^{\|} = 0 \tag{3.46}$$

Thus the reflected wave electric field must be the negative of the incoming electric field (i.e. out of phase by π). Now using the right hand rule it is easy to show that the reflected magnetic field must not change phase. On a side note the magnetic field doesn't need to be zero since the parallel components of the magnetic field aren't continuous.

3.11 Current

The current density through a material with an electric field is given by

$$\mathbf{J} = \sigma \mathbf{E} \tag{3.47}$$

The drift speed of electrons in a material is given by

$$v = \frac{I}{nAQ} \tag{3.48}$$

Where I is the current, n is the number density of charge carriers (electrons), Q is the charge of the charge carriers, and A is the cross sectional area of the medium. The current given by a current density is given by

$$I = \int \mathbf{J} \cdot \mathbf{dA} \tag{3.49}$$

The current density given by number density of charge carriers n and c change q for each charge carrier is given by

$$\mathbf{J} = nq\mathbf{v_d} \tag{3.50}$$

Chapter 4

Electronics

4.1 General Knowledge

Note that electronic circuits are analogous to the classical mechanic harmonic oscillator. The analogous quantities are in table 4.1

Table 4.1: The Electrical Analogues of the Classic Harmonic Oscillator

Harmonic Oscillator	Series RLC Circuit	Parallel RLC Circuit
Displacement x	Charge Q	Voltage V
$\frac{dx}{dt}$	I	$\frac{dV}{dt}$
Mass M	L	C
Spring Constant K	$1/C$	$1/L$
Friction	R	$1/R$
F	V	$\frac{dI}{dt}$

Def 14. Feedback: Taking some output from the circuit and feeding it back in as input. Positive feedback can be used to amplify signals (hence positive). Negative feedback can control different parameters of the circuit such as stability, distortion, etc.

4.2 Resistors

The voltage (V) across a resister with resistance R and current I is given by

$$V = IR \tag{4.1}$$

The power through a resistor is given by

$$P = IV \tag{4.2}$$

Given an tube of length L, cross sectional area A, and resistivity ρ the resistance is given by

$$R = \rho \frac{L}{A} \tag{4.3}$$

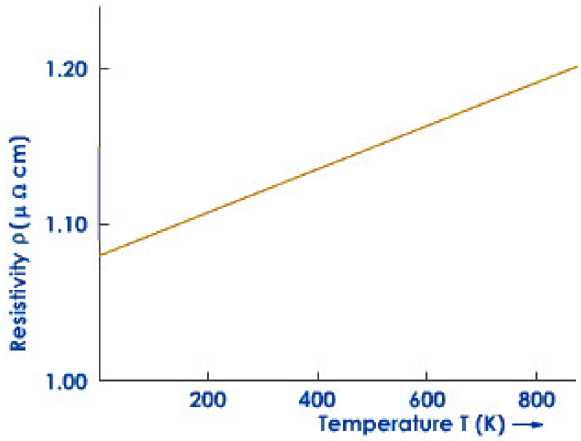


Figure 4.1: Resistivity of a metal

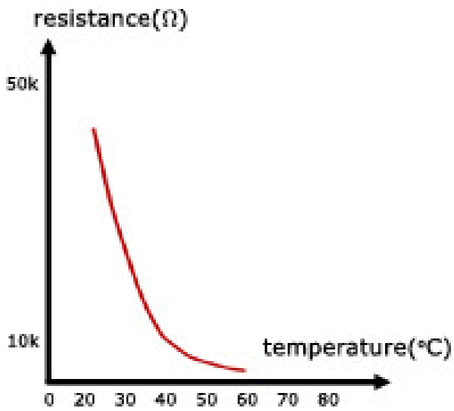


Figure 4.2: Resistivity of a semiconductor as a function of temperature

For metals (especially at high temperature) the resistivity increases approximately linearly with temperature (see figure 4.1. For semiconductors the resistivity decreases with temperature. The temperature dependence on resistivity of a semiconductor is shown in figure 4.2. The resistance of a combination of resistors is given by

$$R_{eq} = \sum_{j=1}^{n} R_j \qquad \text{Resistors in Series} \tag{4.4}$$

$$\frac{1}{R_{eq}} = \sum_{j=1}^{n} \frac{1}{R_j} \qquad \text{Resistors in parallel} \tag{4.5}$$

4.3 Capacitors

The capacitance is a measure for an electrical component of how much electric potential energy is stored for a given electric potential. The capacitance (C) of a capacitor with charge Q and potential V is given by

$$C = \frac{Q}{V} \tag{4.6}$$

The energy stored by a capacitor is given by

$$\int q dV = \int q \frac{dq}{C} = \frac{Q^2}{2C} \tag{4.7}$$

The capacitance of a combination of n capacitors is given by

$$\frac{1}{C_{eq}} = \sum_{j=1}^{n} \frac{1}{C_j} \qquad \text{Capacitors in series} \tag{4.8}$$

$$C_{eq} = \sum_{j=1}^{n} C_j \qquad \text{Capacitors in Parallel} \tag{4.9}$$

Note that capacitors hooked up parallel share the same voltage while capacitors hooked up in series share the same charge.

The typical capacitor consists of two parallel plates. When an uncharged capacitor is added to a simple circuit, charge begins to build up on the plates of the capacitor. This charge creates a repulsive force on the charge on the opposite plate produces a current throughout the entire circuit. However capacitors have a maximum amount of charge they can hold given by equation 4.6. As the capacitor approaches the maximum charge no charge builds on the capacitor plates to push the charge on the opposite plate. In other words the current diminishes over time. Instead charge builds up on the capacitor, charging the capacitor. These ideas are summarized in figure 4.3 The important equation

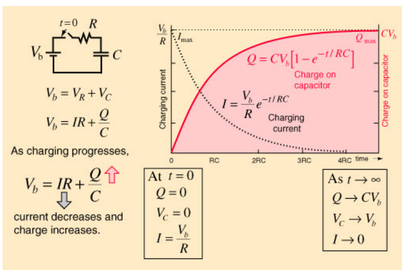


Figure 4.3: Charge and Current Through RC Circuit as a Function of Time

describing an RC circuit is:

$$q = C\mathcal{E}(1 - e^{-t/\tau_C}) \tag{4.10}$$

Where $\tau_C = RC$

In reality capacitors typically have dielectrics material as an intervening medium between the positive and negative plate. The reasons this is done is twofold. For one thing then the charges on the conducting plates do not come into contact with one another. Secondly (and more importantly), the higher permittivity allows a greater charge to be stored at a given voltage. For example consider a parallel plate capacitor:

$$V = \frac{\sigma d}{\epsilon} \tag{4.11}$$

Thus

$$\frac{C}{A} = \frac{\epsilon}{d} \tag{4.12}$$

Hence the greater ϵ the greater the capacitance and greater the charge that can be stored per volt.

4.4 Inductors

The inductance is a measure of how much magnetic flux an inductor produces through it per unit current run through it. For an inductor:

$$L = \frac{N\phi_B}{i} \tag{4.13}$$

For a solenoid (the typical inductor):

$$N\phi_B = N(\mu_0 niA) = nl(\mu_0 niA) \tag{4.14}$$

$$\Rightarrow \frac{L}{l} = \mu_0 n^2 A \tag{4.15}$$

Faraday's law says that for any inductor

$$\mathcal{E} = -\frac{d(N\phi_B)}{dt} \tag{4.16}$$

$$= -L\frac{di}{dt} \tag{4.17}$$

STOP! Common GRE Problem 14. *Solve the current induced in a rotating loop in a magnetic field. For this problem the key point is finding out what to put as the flux going through the loop. The flux is equal to the normal flux (BA) times some sine or cosine function depending on the definition of time t.*

STOP! Common GRE Problem 15. *Given a closed loop moving through a magnetic field determine the emf induced in the loop as a function of time. When doing these types of problems one needs to watch out for confusing the how area changes and how the rate of area changes with time. Since emf is dependent on ΔA not A as a function of time this is a recipe for trouble.*

The energy stored in the magnetic field of an inductor is analogous to the energy stored in the electric field of a capacitor with $L \to 1/C$ and $i \to q$:

$$U = \frac{Li^2}{2} \tag{4.18}$$

Consider an RL circuit. If the current in the circuit is ramped up then the inductor will initially oppose this change of flux by Faraday's law. Thus the current should increase slowly to some final value for the current in the circuit. The same phenomenon occurs when the inductor is initially charged. The curve should also take the exact same form (there is a positive and negative charge symmetry in the circuit). The result is summarized in figure 4.4

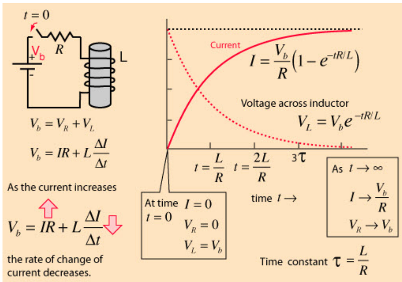


Figure 4.4: Current Through RL Circuit as a Function of Time

STOP! Common GRE Problem 16. *What happens to the current when a circuit has an inductor, resister, and a battery keeping a current running through it and then at some time the battery is taken away? The current in the inductor drops due to the battery being removed but increases in order for the inductor to stop the flux through the inductor from dissipating (Faraday's law), thus there is still a current in the new circuit.*

4.5 LC/RLC/AC Circuits

LC circuits are analogous to the harmonic oscillator in classical mechanics. In this case the angular frequency of oscillation (ω) takes the form:

$$\omega = \frac{1}{\sqrt{LC}} \tag{4.19}$$

At this frequency the current running through the circuit is maximized. The charge and hence the current oscillate sinusoidally:

$$q = Q\cos(\omega t + \phi) \qquad i = \frac{dq}{dt} = -Q\omega\cos(\omega t + \phi) \tag{4.20}$$

Consider an LC circuit that at $t = 0$ contains a charged capacitor. The electrical energy energy oscillates from initially all contained by the capacitor to begin fully contained by the inductor.

The ideal LC circuit does not damp and oscillates on forever. Alternatively consider a more realistic system, the RLC circuit. In this circuit the frequency is

$$\omega' = \sqrt{\omega^2 - \left(\frac{R}{2L}\right)^2} \tag{4.21}$$

While the amplitude decays exponentially with time. In such a circuit the current will not damp out if an emf device is connected to the circuit and makes up for the lost energy due to the resistance. For an AC circuit the root mean squared of the voltage and current are just the max value divided by root 2:

$$I_{rms} = \frac{I_0}{\sqrt{2}} \qquad V_{rms} = \frac{V_0}{\sqrt{2}} \tag{4.22}$$

4.6 Impedance and Reactance

Def 15. Reactance is a measure of the opposition to *change* in the current of voltage of a circuit.

Reactance is analogous to resistance but refers to oscillating current and voltages as in an AC circuit. It has a constant value if the circuit is oscillating a constant frequency. The capacitive reactance is defined as

$$X_C = \frac{1}{\omega_d C} \tag{4.23}$$

Given a circuit with an AC emf source and a single capacitor, the voltage in the circuit oscillates as

$$V = V_C \sin(\omega t) \tag{4.24}$$

By the definition of capacitance:

$$\begin{aligned} q &= CV && (4.25)\\ &= CV_C \sin(\omega t) && (4.26)\\ \Rightarrow i &= CV_C \omega \cos(\omega t) && (4.27)\\ &= \frac{V_C}{X_C} \cos(\omega t) && (4.28) \end{aligned}$$

Hence the current and the voltage are out of phase by 90^0. In this case the voltage lags behind the current.

$$V_C = I_C X_C \tag{4.29}$$

Inductive reactance is defined as:

$$X_L = \omega_d L \tag{4.30}$$

Given a circuit with an AC emf device and an inductor the voltage in the circuit is given by

$$v = V_L \sin(\omega t) \tag{4.31}$$

The current is then given by (see equation 4.17)

$$i = \int dt \frac{V_L}{L} \sin(\omega t) \tag{4.32}$$
$$= -\frac{V_C}{\omega L} \cos(\omega t) \tag{4.33}$$
$$= -\frac{V_L}{X_L} \cos(\omega t) \tag{4.34}$$

Hence the voltage and current are again out of phase by 90^o. However here the current lags the voltage (the current reaches its maximum first)

$$V_L = X_L I_L \tag{4.35}$$

Def 16. Impedance (Z): Measure of both resistance and reactance

For an RLC circuit with an applied emf, Z is given by

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \tag{4.36}$$

With this definition the current is related to the applied emf by

$$\mathcal{E}_0 = I_0 Z \tag{4.37}$$

The resonant frequency for such a circuit is when the impendence is minimized (i.e. $X_L = X_C$) this leads to

$$\omega_{resonant} = \frac{1}{\sqrt{LC}} \tag{4.38}$$

The max voltage through a component is related to the max current through the component through the impedance:

$$V_0 e^{i\omega t} = I_0 e^{i\omega t} Z \tag{4.39}$$

The impedance in different components is given by

Resistor:	$Z = R$
Capacitor:	$Z = \frac{1}{i\omega C}$
Impedance:	$A = i\omega L$

To maximize power transmitted through two different mediums require that the impedance of the source equal the impedance of the output. For a generator and load this says that

$$X_g = -X_l \tag{4.40}$$

Table 4.2: Electronic Components and There Use as Filters

Component	Function
Inductor	Blocks *high* frequency signals and conducts *low* frequency signals
Capacitor	Blocks *low* frequency signals and conducts *high* frequency signals
Resistor	Do not have any frequency dependence (are added to circuits to inductors and capacitors to determine time-constants of the circuits)

4.7 Electronic Filters

Def 17. Passive Filter: Filters that do not depend on an external power supply and/or do not contain active components such as transistors.

Inductors and capacitors each can be used to control the frequencies that pass through a circuit. The way that the common components act for different frequencies is shown in table 4.2 Note that these relations can inferred by looking at the capacitative and inductive reactance:

$$\lim_{\omega\to\infty} X_C = 0 \tag{4.41}$$

$$\lim_{\omega\to\infty} X_L = \infty \tag{4.42}$$

$$\lim_{\omega\to 0} X_C = \infty \tag{4.43}$$

$$\lim_{\omega\to 0} X_L = 0 \tag{4.44}$$

Def 18. High pass filter: A filter which selects out the higher frequency components of a signal.

A high pass filter can be made in two ways. If a circuit is connected to ground through an inductor then the low frequency signals will go to ground and only the high frequencies will pass through the circuit, creating a high-pass filter. Alternatively if the signal goes through a capacitor then only high frequencies will be picked out, creating a high pass filter.

Def 19. Low pass filter: A filter which selects out the lower frequency components of the signal.

A low pass filter can be made in two ways. One way is done by connecting a circuit to ground via a capacitor. Then only the high frequencies will go to ground and the low

frequencies will stay present in the signal. Alternatively a signal can be run through an inductor which will select the low frequency components.
An example of a low pass filter is shown in figure 4.5. In this figure the signal starts on the left of the diagram. The amplitude of the signal is given by the voltage difference of the top and bottom wires before the filter. The amplitude of the signal after the filter is given by the voltage difference between the top and bottoms wires after the filter. At low frequencies the capacitor acts like a resistor and there is a large potential difference after the filter (hence a strong signal). The opposite happens at high frequencies.
See website: `http://www.electronics-tutorials.ws/filter/filter_1.html` for a more thorough explanation.

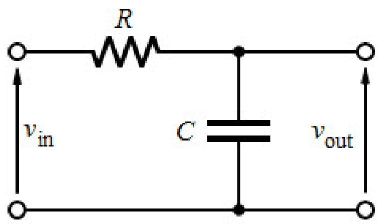


Figure 4.5: A Simple Lowpass Filter

4.8 Circuit Rules

Def 20. Kirchoff's Junction Rule: The sum of the currents entering any junction must be equal to the sum of the currents leaving that junction.

Def 21. Kirchoff's Loop Rule: The algebraic sum of the changes in potential encountered in a complete traversal of any loop of a circuit must be zero.

STOP! Common GRE Problem 17. *What is the resistance going across the resistor in the circuit shown in figure 4.6? At first glance this appears to be a difficult problem but if we just break it down using Kirchoff's loop rule then the voltage going around the loop says that $G - V - IR = 0$. Then we can isolate for the resistance.*

Def 22. Resistance Rule: For a move through a resistance in the direction of the current, the change in the potential is $-iR$: in the opposite direction it is $+iR$.

Def 23. For a move through an ideal emf device in the direction of the emf arrow(negative to positive terminal), the change in the potential is $+\mathcal{E}$: in the opposite direction it is -$\mathcal{E}$.

Def 24. Thevenin Equivalent: For linear electrical networks, any combination of voltage sources, current sources, and resistors with two terminals is electrically equivalent to a single source V and a single series resistor , R.

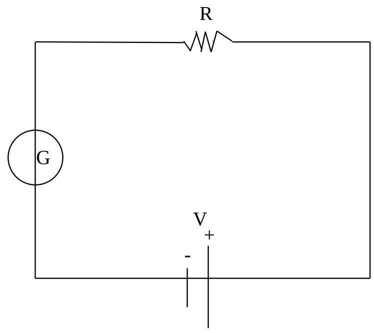


Figure 4.6: A generator in a circuit

4.9 Operation Amplified (Op-amp)

Op-amp's consist of two inputs and one output. The typical op-amp is shown in figure 4.7 The characteristics of an Op-amp are:

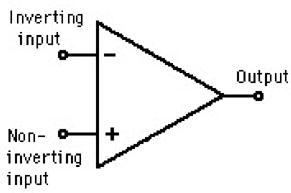


Figure 4.7: Typical Operational Amplifier

- High gain (on the order of a million)
- High input impedance, low output impedance
- Used with split supply
- Used with feedback

4.10 Electrical Energy Transmission Transformers

When electricity is sent from the factory over to the homes of the buildings in a city it is sent over transmission lines. The transmission lines are just like a regular circuit, they work by applying a particular AC voltage to the circuit with the wires acting as a resistor. The power lost due to the transmission of electricity is given by

$$P = I^2 R \tag{4.45}$$

Thus the higher the current the higher the power lost due to resistance in the wires. It is ideal to transmit power at the highest possible voltage and lowest possible current.

Since we want to send electricity at very dangerous voltages we must be able to change the voltage before applying it to our devices. A transformer does just this using Faraday's law of induction. It consists of two coils. The voltage after the transformer (secondary voltage) is given by

$$V_s = V_p \frac{N_s}{N_p} \tag{4.46}$$

By conservation of energy the power transmitted by the first coil is equal the power transmitted by the second coil:

$$I_p V_p = I_s V_s \tag{4.47}$$

Hence

$$I_p = I_s \frac{N_s}{N_p} \tag{4.48}$$

A diagram of a transformer is shown in figure 4.8

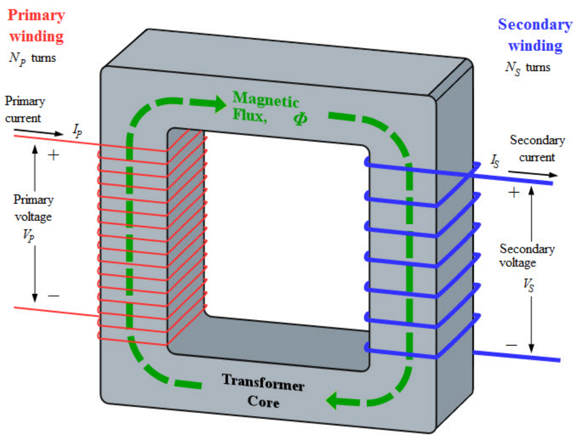


Figure 4.8: A Simple Transformer

4.11 Gates

The different types of logical gates are shown in figure 4.9

4.12 Unrelated Facts

There are a few facts on the PGRE that relate to electronics that don't have much connection to anything else on the test. They are listed below:

- Terminators are used at the end of coaxial cables with matched impedance between the input and output in order to reduce back reflections

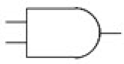
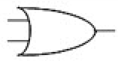
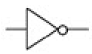
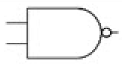
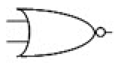
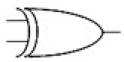
Gate	Symbol	Operator
and		$A \cdot B$
or		$A + B$
not		$\bar{A}$
nand		$\overline{A \cdot B}$
nor		$\overline{A + B}$
xor		$A \oplus B$

Figure 4.9: Logic gates

Chapter 5

Thermodynamics and Statistical Mechanics

5.1 General Knowledge

The average kinetic energy of a material is given by the equipartition theorem:

$$E = N\frac{\nu k_B T}{2} \tag{5.1}$$

Where N is the number of atoms in the sample and ν is the (effective) degrees of freedom. Some important examples are a solid - $\nu = 6$ (3 translational and 3 vibrational degrees of freedom) and a diatomic gas - $\nu = 5$(3 translational 2 rotational degrees of freedom).

It may be useful to know the relation between the gas constant and Boltzmann's constant:

$$nR = Nk_B \tag{5.2}$$

5.2 Ideal Gases

The ideal gas law is

$$pV = nRT = NkT \tag{5.3}$$

The root mean squared velocity of an ideal gas is

$$v_{rms} = \sqrt{\frac{3RT}{M}} \tag{5.4}$$

The mode (most probable speed) is

$$v_{mode} = \sqrt{\frac{2RT}{M}} \tag{5.5}$$

The average speed is

$$v_{avg} = \sqrt{\frac{8RT}{\pi M}} \tag{5.6}$$

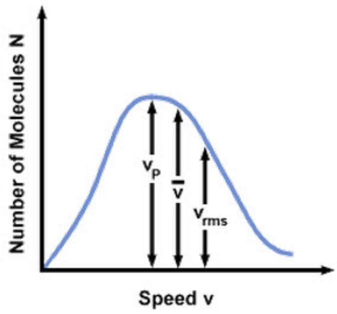


Figure 5.1: Maxwell-Boltzmann distribution

Note that

$$v_{mode} < v_{avg} < v_{rms}$$

A diagram showing the Maxwell-Boltzmann distribution is shown in figure 5.1. The mean free path of an ideal gas is given by

$$\lambda = \frac{1}{\sigma n} = \frac{1}{\sqrt{2}\pi d^2 n} \tag{5.7}$$

Where σ is the cross section, d is the diameter of the molecule and n is the number density of the gas. The average time between collisions is

$$t = \frac{\lambda}{v} \tag{5.8}$$

Using the equipartition theorem this can be rewritten

$$t = \frac{\lambda}{\sqrt{\frac{\nu k_B T}{m}}} \tag{5.9}$$

$$= \frac{\sqrt{m}\lambda}{\sqrt{\nu k_B T}} \tag{5.10}$$

The internal energy of an ideal gas is

$$\Delta U = c_v \Delta T \tag{5.11}$$

Where c_c is the specific heat capacity. Further for a an ideal gas the specific heats at constant volume and constant pressure are related by

$$c_p - c_v = nR \tag{5.12}$$

For a mole of ideal gas the volume heat capacity is

$$c_{Ideal,v} = \frac{3}{2}R \tag{5.13}$$

Thus the pressure heat capacity is

$$c_{Ideal,p} = \frac{5}{2}R \tag{5.14}$$

The reason the heat capacity is greater for an ideal gas under constant pressure is because the gas does work on its environment when its pressure remains constant and temperature is increased. Thus instead of raising the temperature per unit heat input into the system the heat is being used to do work.

The heat capacity of an ideal diatomic gas is approximately constant with temperature. However at very high temperature an ideal gas has 7 degrees of freedom (3 translational, 2 rotational, and 2 vibrational) hence the heat capacity is $\frac{7}{2}Nk_B$. At low temperatures on the other hand an ideal gas has fewer degrees of freedom (no rotational or vibrational) so heat capacity is $\frac{3}{2}Nk_B$.

5.3 Terminology

Def 25. Isothermal Process: Process which occurs at a constant *temperature*

Def 26. isobaric Process: Process which occurs at a constant *pressure*

Def 27. Adiabatic Process: Process which occurs with no energy transferred as heat

Def 28. Isochoric Process: Process which occurs at a constant *volume*

For an adiabatic expansion of a gas:

$$PV^{\gamma} = \text{const} \tag{5.15}$$

From this it is easy enough to derive the work done by expansion of a gas: (defined constant as α and assumed $\gamma \neq 1$)

$$\begin{aligned} W &= \int_{V_1}^{V^2} PdV && (5.16) \\ &= \int_{V_1}^{V^2} \frac{\alpha}{V^{\gamma}} dV && (5.17) \\ &= \alpha \frac{1}{-\gamma+1} \left(\frac{1}{V_2^{\gamma-1}} - \frac{1}{V_1^{\gamma-1}} \right) && (5.18) \\ &= \frac{P_2V_2 - P_1V_1}{-\gamma+1} && (5.19) \end{aligned}$$

Note that for a monotonic gas $\gamma = 5/3$. For an isothermal expansion of a gas $PV = nRT$

which is a constant. The work done in the expansion of a gas is

$$W = \int P dV \tag{5.20}$$
$$= nRT \int \frac{1}{V} dV \tag{5.21}$$
$$= P_i V_i \ln\left(V_2/V_1\right) \qquad \text{(i =1,2)} \tag{5.22}$$
$$\tag{5.23}$$

In general adiabatic processes do less work then isothermal processes.

5.4 Probability Distributions

The partition function is defined as

$$\mathcal{Z} = \sum_s g_s e^{-\beta\epsilon_s} \tag{5.24}$$

Where β is equation to $1/(k_B T)$, the sum over s is over all possible states, ϵ_s is energy of state s and g_s is the degeneracy of state s. The degeneracy is required such that states with more levels become more dominant in the partition function.
Given N subsystems with partition functions, $\zeta_1, \zeta_2, ...$ the partition function of the system is given by

$$\mathcal{Z} = \prod_{i=1}^{N} \zeta_i \tag{5.25}$$

The average energy of a system can be given in terms of the partition function:

$$\langle E \rangle = \frac{1}{\mathcal{Z}} \sum_s \epsilon_s e^{-\beta\epsilon_s} = -\frac{\partial ln(\mathcal{Z})}{\partial\beta} \tag{5.26}$$

The probability of finding a system (both quantum or classical) in state s is

$$P(s) = \frac{e^{-\beta\epsilon_s}}{\mathcal{Z}} \tag{5.27}$$

Thus the number of particles in state s are given by

$$N_s = N g_s \frac{e^{-\beta\epsilon_s}}{\mathcal{Z}} \tag{5.28}$$

Where g_s is the degeneracy of state s. This idea can be extends directly the number of systems s.

In a **classical** system the number of particles in state s is just the Boltzmann distribution

$$n(\epsilon_s)_c = \frac{g_s}{e^{(\epsilon_s - \mu)\beta}} \tag{5.29}$$

In a **Fermionic** quantum system system the number of particles in state s (and hence the occupancy) is given by the fermi function (where g_s is the degeneracy of the state):

$$n(\epsilon_s)_f = \frac{g_s}{e^{(\epsilon_s-\mu)\beta}+1} \tag{5.30}$$

In a **Bosonic** quantum system the occupancy function is

$$n(\epsilon_s)_b = \frac{g_s}{e^{(\epsilon_s-\mu)\beta}-1} \tag{5.31}$$

5.5 Entropy

Entropy is defined as

$$S = k_B ln(\Omega) \tag{5.32}$$

$$\partial S = \frac{\partial Q}{T} \quad \Rightarrow \quad S(T) = \cancelto{0}{S(0)} + \int_0^T \frac{dQ}{T} \tag{5.33}$$

A process is reversible if the change in entropy in the process is zero (and hence no heat is exchanged either)
Temperature can be defined in terms of entropy:

$$\frac{1}{T} \equiv \left(\frac{\partial S}{\partial E}\right)_{V,N} \tag{5.34}$$

Entropy is also given in terms of the partition function by

$$S = \frac{\partial}{\partial T}\left(k_B T \ln \mathcal{Z}\right) \tag{5.35}$$

Thus given by energies of the system we can find the partition function and subsequently find the entropy. At high temperatures we can often simply the partition function to a number (by expanding the exponentials) this gives a constant entropy with temperature.

5.6 Heat Capacity

Heat capacity is a measure of how much the temperature of an object changes with respect to how much heat is supplied to an object. Heat capacities largely depend on the system size. Thus heat capacities are divided by a measure of the system size to give either molar heat capacities (divide by the number of moles in the system) or specific heat capacity (divide by the mass of the system) There are two important types of molar heat capacities; isochoric and isobaric:

$$C_p = \frac{1}{n}\left(\frac{\partial Q}{\partial T}\right)_p \qquad C_V = \left(\frac{1}{n}\frac{\partial Q}{\partial T}\right)_V \tag{5.36}$$

The number of degrees in a system is not a simple constant but a function of temperature. Two methods to calculate the heat capacity as a function of temperature were given by Einstein and Debye.

5.6.1 Einstein

Einstein considered each atom as a harmonic oscillator and the total energy of the system was simply given the product of the energy of the harmonic oscillator (assuming all the harmonic oscillators had the same frequency) and the number of harmonic oscillators in the system. Einstein's model gave qualitatively good results for the heat capacity as $T \to \infty$, however the model did not have the correct temperature dependence. The importance of Einstein's model was the use of importance of using quantum statistics to explain low temperature behavior. Einstein's assumptions leads to the following expression for the energy of a crystal in the high energy limit

$$E = N\epsilon_0 + Mk_BT \tag{5.37}$$

Where M is the molar mass of the molecules. The heat capacity is given by

$$\begin{aligned} C = \frac{\partial E}{\partial T} &= \frac{\partial}{\partial T}\left(\frac{N\nu}{2}k_BT\right) && (5.38) \\ &= 3Nk_B && (5.39) \end{aligned}$$

Note that here $\nu = 6$ since we have a 3 dimensional harmonic oscillator (3 degrees of freedom for kinetic energy and 3 for potential energy). Hence the heat capacity is approximately constant with temperature. This was found experimentally and is called the Dulong-Petit law.

5.6.2 Debye

Debye improved Einstein's model by considering the density of states instead of assuming all the atoms had the same energy (and hence that all oscillators oscillate at the same frequency). However he still assumed 3N independent oscillators. He got the correct T^3 behavior as $T \to 0$.

5.6.3 Practical Use of Specific Heat

The specific heat is most commonly used in order to the find the relationship between inputting heat into an object and the temperature change:

$$\Delta Q = nC_V\Delta T \tag{5.40}$$

Where C_V is the molar heat capacity.
This can easily be derived from equation 5.36:

$$\begin{aligned} \int nC_V\partial T &= \int\left(\frac{\partial Q}{\partial T}\right)\partial T && (5.41) \\ nC_VT &= \Delta Q \qquad \text{Given } \Delta Q \text{ is exact} && (5.42) \end{aligned}$$

The specific heat of water is $\frac{4200J}{kg\cdot k}$.

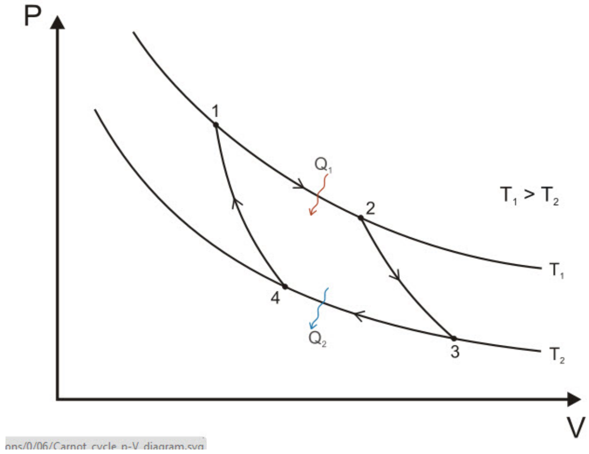


Figure 5.2: The Carnot cycle PV diagram

STOP! Common GRE Problem 18. *An element was heated for a long time with a power P but does not change temperature. How long will it take the element to change a given temperature when it is stopped heating? The element being heated for a long time without changing temperature implies the element changes emits all the power P that it is taking in. Thus the power also gives us the energy dissipation rate when the heater is removed.*

5.7 P-V Diagrams

These diagrams are used to give a visual of a process. There are some subtleties in the diagrams that are rarely explained. First of all any cyclic process has a net positive work output if the cycle is traversed **clockwise direction** and net negative work output if the cycle is traversed **counter-clockwise**. Second of all constant temperature processes within a cycle are often indicated by putting a long black line through a process and indicating a temperature that the process takes place in. Lastly each processes is typically either isothermal (constant temperature) or adiabatic (zero net heat flow). As an example the Carnot cycle PV diagram is shown in figure 5.2. In a PV diagram if the cycle undergone by the system is a closed cycle then the change in internal energy throughout the cycle is zero. Thus the net change in the heat in the cycle is just the net change in the work.

5.8 Engines

A Carnot engine is an ideal engine. All the processes involved are reversible (i.e. no loss of energy due to friction, etc.). It is has the highest possible efficiency for any engine operating between two given temperatures. It has a four step cycle:

1. Isothermal Expansion: Performed at constant temperature T_h, the compressed gas expands doing work on its surroundings, but in the mean time its supplied with heat so as the keep the temperature constant. By moving a piston or some sort of energy generator energy is "collected" from the system.

2. Adiabatic Expansion: No longer supplied with constant heat the gas begins to cool as its expands and continues to do work on its environment.

3. Isothermal Compression: Heat is taken away from the gas. However instead of undergoing a change in temperature the gas compresses.

4. Adiabatic Compression: The gas no longer has heat removed from it. However the gas continues to compress, this produces an increase in the temperature of the gas back to its starting temperature T_H

Efficiency: The efficiency of the engine is defined as (Entropy doesn't change since all processes are reversible)

$$Efficiency = \frac{W}{Q_h} = \frac{Q_h - Q_c}{Q_h} = 1 - \frac{T_c}{T_h} \tag{5.43}$$

5.9 Three Laws of Thermodynamics

0. If two systems are each in thermal, mechanical, and/or diffusive equilibrium with a thrid system, then they are all in equilibrium with each other

1. $dE = dQ - dW + \mu dN = SdT - pdV + \mu dN$

2. $dS \geq 0$ i.e. Entropy is always increases

3. $S(0) = 0$

One interesting consequence of the second law of thermodynamics is that energy flows from hot to cold. Thus a cooler body can never heat a hotter body.

STOP! Common GRE Problem 19. *Can the radiation emitted from a cooler body if focused by a lens heat a hotter body? No, even though the radiation is focused, the cooler body can never heat the hotter body.*

Another interesting consequence of the second law of thermodynamics is that if a system is at the arrangement which maximizes the number of states of the system (i.e. entropy is a maximum) then the system cannot spontaneously change.

Table 5.1: Important Properties of F,H, and G

	Nickname	When is it Useful?	Why?
F	"Work Function"	T is constant	Minimum
H	"Heat Function"	p is constant	Maximum
G	"Gibbs Function"	p and T are constant	Minimum

5.10 Helmholtz Free Energy, Enthalpy, and Gibbs Free Energy

Each of Helmholtz Free Energy (F), Enthalpy (H) and Gibbs Free Energy (G) has units of energy. They are all defined in terms parameters from the first law of thermodynamics. They are defined as follows:

$$F \equiv E - TS = -pV + \mu N \tag{5.44}$$

$$H \equiv E + pV = TS + \mu N \tag{5.45}$$

$$G \equiv E - TS + pV = \mu N \tag{5.46}$$

Alternatively the differential of the definitions can be used:

$$dF = -pdV + \mu dN - SdT \tag{5.47}$$

$$dH = Vdp + TdS + \mu dN \tag{5.48}$$

$$dG = -TdS + Vdp + \mu dN \tag{5.49}$$

F, H, and G have interesting attributes in special cases. These attributes are summarized below: Furthermore the Helmholtz function is closely related the partition function:

$$\mathcal{Z} = e^{-\beta F} \tag{5.50}$$

The chemical potential is related the Helmholtz energy by

$$\mu(T, V, N) = \left(\frac{\partial F}{\partial N}\right) \tag{5.51}$$

Note that this equation can be easily derived from 5.47.

5.11 Blackbodies

The total power emitted by a blackbody is given by:

$$P = \sigma \epsilon A T^4 \tag{5.52}$$

Where σ is a constant, ϵ is the emissivity of the object (measure of how close the object is to an ideal blackbody), A is the surface are of the object and T is the temperature

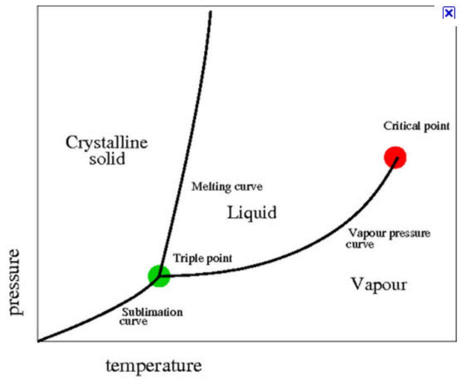


Figure 5.3: The Critical and triple points of a compound

of the object. The wavelength of maximum intensity emitted by an object is given by Wien's law:

$$\lambda_{\text{Max Intensity}} = \frac{2.9 \times 10^{-3} m \cdot K}{T} \tag{5.53}$$

note: online note suggested to memorize this, possibly including the constant
The Plank distribution for the intensity of light emitted by a blackbody is given as a function of frequency by

$$I(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1} \tag{5.54}$$

5.12 Phase Diagrams

Def 29. Critical Point: Point at which there is no distinction between liquid and vapour

Def 30. Triple Point: The one value of T and P at which all three phases coexist.

The different points are shown in figure 5.3.

Def 31. Critical Isotherm: a isothermal line on a PV diagram which just touches a liquid-gas boundary

On a diagram that contains a liquid-gas system the region which shows constant pressure behaviour despite a charge in volume is the region of liquid-vapour equilibrium.

When a material undergoes a phase transition many properties of the system undergo a discontinuity. An example of this is when a material undergoes a first order phase transition to superconductivity. In this case the specific heat capacity is discontinuous.

Chapter 6

Optics

6.1 General Knowledge

The electromagnetic spectrum is show in figure 6.1. The speed of light in a material with

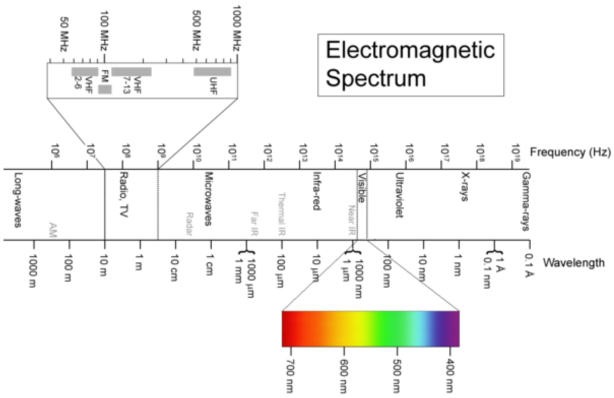


Figure 6.1: The Electromagnetic Spectrum

permittivity (ϵ) and magnetic permeability (μ) is given by:

$$v = \frac{1}{\sqrt{\epsilon\mu}} \tag{6.1}$$

The uncertainty principle of optics is given by

$$\Delta\tau\Delta\nu \approx 1 \tag{6.2}$$

Where $\delta\tau$ is the coherence time of a light pulse and $\delta\nu$ is the bandwidth of the signal. Phase and group velocity of a wave are given by:

$$v_{phase} = \frac{\omega}{k} \qquad v_{group} = \frac{d\omega}{dk} \tag{6.3}$$

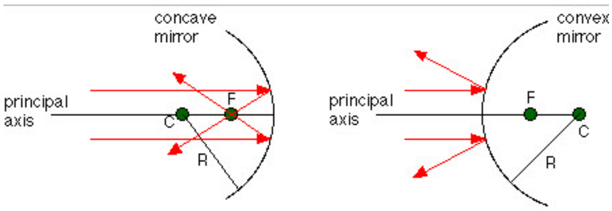


Figure 6.2: Important points on spherical mirrors

The index of refraction is a ratio of the speed of light in a vacuum to the speed in a medium. The index of refraction is equal to

$$n = \sqrt{\frac{\epsilon\mu}{\epsilon_o\mu_o}} \tag{6.4}$$

Typically the material is nonmagnetic and thus n is just the square root of the relative permittivity ($\sqrt{\frac{\epsilon}{\epsilon_o}}$)

As a frame of reference for human vision: human eyes can see motion up to about 10 Hz.

6.2 Images

Choose the convention that object distances are positive quantities and image distances are negative quantities. Due to this convention for a plane mirror the object distance (p) is equal to the negative of the image distance (i):

$$p = -i \tag{6.5}$$

Next consider spherical mirrors. To move forward several definitions are required:

Def 32. Center of Curvature: The center of the sphere making up the spherical mirror. The distance to the center of curvature is called the radius of curvature.

Def 33. Focal Point: When parallel rays reach a concave mirror the point at which these rays converge to is called the focal point (the distance to this point is called the focal length).

These two points are shown in figure 6.2.

Def 34. Magnification: The ratio of the image height (h') by the object height (h).

For a spherical mirror the following relation between the object and image distances holds true:

$$\frac{1}{p} + \frac{1}{i} = \frac{1}{f} \tag{6.6}$$

Furthermore the magnification of the image is equal to

$$m = -\frac{i}{p} \tag{6.7}$$

For a spherically refracting surface the following holds true, where r is the radius of curvature of the surface:

$$\frac{n_1}{p} + \frac{n_2}{i} = \frac{n_2 - n_1}{r} \tag{6.8}$$

For a thin lens the following two equations hold true:

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{i} \tag{6.9}$$

$$\frac{1}{f} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \tag{6.10}$$

Where here r_1 is a radius of curvature at the side close to the object and r_2 is the radius of curvature at the side far from the object.

6.3 Telescopes and Microscopes

When looking at an object through a microscope you can placing the object next to your lens. Thus the lens has rays coming in from different directions. In order to magnify the image a compound microscope can be used. Such a microscope consists of two lenses separated by tube length s as well as the sum of the two focal lengths. This produces parallel rays (hence coherent) into your eye.
On the other hand telescopes are used for incoming rays that are already parallel. Telescopes are available in a variety of forms. One form, called the refracting telescope (see figure 6.3) uses two lenses. The lenses are separated by a distance equal to the sum of the focal lengths of the two lenses.

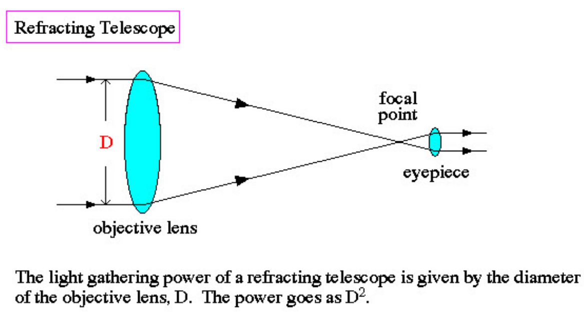


Figure 6.3: A Refracting Telescope

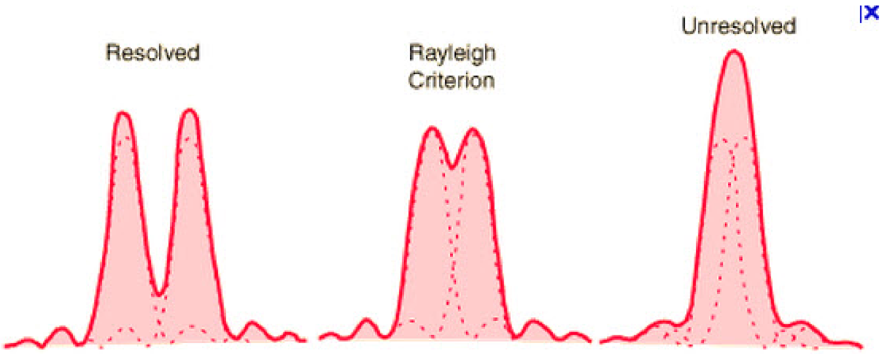


Figure 6.4: The Rayleigh criterion

6.4 Rayleigh Criterion

Images that travel through circular apertures (such as your pupils) are diffraction patterns. Suppose we want to quantify whether two images are resolvable. One useful condition is the angular separation between the two images such that the central maximum of one image is coincides with the first minimum of the second image. This is given by the Rayleigh criterion:

$$\theta_R = 1.22\frac{\lambda}{d} \tag{6.11}$$

Where d is the diameter of the aperture. The Rayleigh condition is shown in figure 6.4 The idea behind the Rayleigh criterion is that as the opening of the aperture is increased it is easier to detect different signals (and hence the minimum separation of two signals that can be distinguished is smaller)

6.5 Non-Relativistic Doppler Shift

The non relativistic Doppler shift is dependent on whether the source (s) or the detector (D) is moving. The equation is

$$f_s\left(v \pm v_D\right) = f_D\left(v \pm v_s\right) \tag{6.12}$$

Where the plus or minus is given by the following rule:

If the motion of the source or detector is toward the other the sign on its speed must such that it gives an increase in the frequency.

Note it is more common in the GRE for them to ask for wavelength changes. In this case since the speed is unchanged by the Doppler shift:

$$\frac{1}{\lambda_s}\left(v \pm v_s\right) = \frac{1}{\lambda_D}\left(v \pm v_D\right) \tag{6.13}$$

6.6 Compound Lenses

STOP! Common GRE Problem 20. *Find the point where an image is formed after a ray transverses through an optical system of two lenses (i.e. a compound lens). This is done by working through each lens separately and remembering the correct sign convention. Where this sign convention says that if an image is formed past the second lens then the distance is negative.*

6.7 Young's Double Slit Experiment

The maximum positions in Young's double slit experiment are given by

$$d\sin(\theta) = m\lambda \tag{6.14}$$

While the minima are given by

$$d\sin(\theta) = \left(m + \frac{1}{2}\right)\lambda \tag{6.15}$$

Alternatively using small angles its easy enough to show that the positions of the peaks from the central maxima are

$$y_m = \frac{m\lambda L}{d} \tag{6.16}$$

Where m is an integer, L is the distance from the slits and d is the distance between the slits.

6.8 Single Slit Diffraction

For a single slit (of width a) diffraction the result is analogous to Young's double slit experiment. The angle at which maxima occur is given by

$$a\sin\theta = \lambda\left(m + \frac{1}{2}\right) \tag{6.17}$$

The angle in which minima occur is given by

$$a\sin\theta = m\lambda \tag{6.18}$$

Notice that this is the same as Young's formula only the points of maxima and minima are reversed.

STOP! Common GRE Problem 21. *Find the values of the diameter of a camera pinhole such that the image is the sharpest. Starting off with the equation for where we have minima $d\sin\theta = m\lambda$. We can say that a sharp image means that the main maximum "fits" inside the width of the slit. In other words $a \approx y$ where y is the distance from the axis of symmetry. Making the small angle approximation and inputting this conditions gives an optimum pinhole size of $\sqrt{\lambda D}$ (where D is the distance to the image)*

Young's double slit and single slit diffraction can be combined. They each yield values of m for which the intensity profile disappears. However its important to consider two different m values. In other words if we are looking for a diffraction minima and an interference maxima:

$$a \sin\theta = m_1\lambda; \qquad d \sin\theta = m_2\lambda \tag{6.19}$$

Dividing these equations gives expression for width of the slits in terms of the slit separation:

$$\frac{a}{d} = \frac{m_1}{m_2} \tag{6.20}$$

Note going a step further we can require $d > a$ which implies that $m_2 > m_1$.

6.9 Diffraction Grating

For light transmitted through a diffraction grating the positions of the maxima is the same as for Young's double slit experiment. The light diffracted is found by summing light diffracted from each of the elements, and is essentially a convolution of diffraction and interference patterns. As the number of slits goes to infinity and the thickness of the slits goes to zero the followed equation for the positions of the maxima is obeyed:

$$d \sin\theta = m\lambda \tag{6.21}$$

Where d is the spacing between the slits. If instead the question gives the number of slits (N)per unit length (L)then

$$\frac{L}{N} \sin\theta = m\lambda \tag{6.22}$$

6.10 Bragg Diffraction

The Bragg law is the condition for constructive interference for light entering a crystal. The condition is the same as if the crystal were made up of two planes:

$$2d \sin(\theta) = m\lambda \tag{6.23}$$

6.11 Thin Films

Light reflecting off a thin film suspended in air will interfere to produce a maxima if

$$2d = \left(m + \frac{1}{2}\right) \frac{\lambda}{n_2} \text{for m =0,1,2,..} \tag{6.24}$$

Where d is the thickness of the film and n is the index of refraction of the film. A thin film setup is shown in figure 6.5 Note that the results will be modified depending on the substrate on the other side of the thin film.

In a soap film there are often large spots in which there isn't any visual refraction. This is because at these points the thickness of the films became greater then the thickness for visible light.

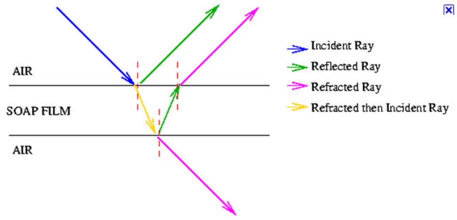


Figure 6.5: Thin film setup

6.12 Michelson's Interferometer

The outline of the Michelson Interferometer is shown in figure 6.6 The Michelson Inter-

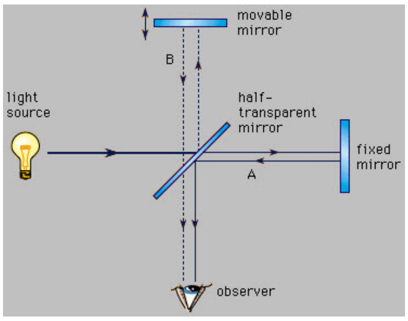


Figure 6.6: The Michelson Interferometer

ferometer can be used to measure very small changes in distance. If one of the mirrors is moved by $\lambda/2$, the path difference of the two rays is changed by one λ and hence the number of fringes will change by one. If a cylinder with index of refraction n is placed along one ray then that ray will experience a change in the number of wavelengths passing through the space occupied by the cylinder. Originally if N is the number of wavelengths passed through the cylinder volume:

$$N = \frac{2L}{\lambda} \tag{6.25}$$

However after the cylinder is inserted

$$N' = \frac{2Ln}{\lambda} \tag{6.26}$$

Thus the change in the number of wavelengths is

$$N' - N = \frac{2L}{\lambda}(n-1) \tag{6.27}$$

If a single mirror is moved by a distance L then the number of fringes that pass by (check this) is given by

$$\Delta N = \frac{2L}{\lambda} \tag{6.28}$$

6.13 Newton's Rings

Newton's rings are due to the interference of light in the apparatus shown in figure 6.7 Its simple enough to derive that the radii of the rings are:

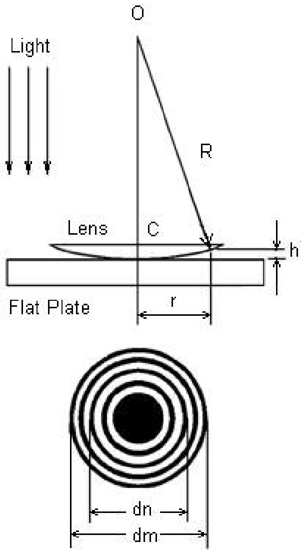


Figure 6.7: Newton's Rings

$$r = \sqrt{R(m + \frac{1}{2})\lambda} \tag{6.29}$$

Where m is an integer

6.14 Polarization

Unpolarized light that went through a linear polarizer has its intensity reduced by a factor of two:

$$I = \frac{I_0}{2} \tag{6.30}$$

Linearly polarized light entering a second linear polarizer has an intensity distribution given by Malus's law:

$$I = I_0 cos^2(\theta) \tag{6.31}$$

Where θ is the angle between the initial polarization axis and the polarization axis of the second linear polarizer.

Note that light with opposite polarizations does not interfere. Thus if the electric field is in two polarizations that are out of phase then the intensity is still proportional the sum of the square of the electric field components separately.

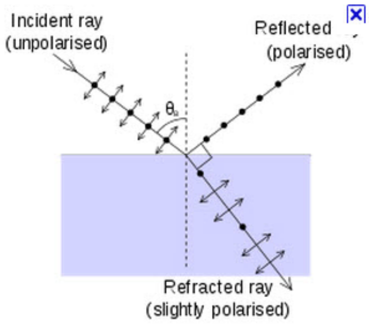


Figure 6.8: Unpolarized light passing through interface at Brewster's angle

6.15 Common Laws

Snell's law is given by:

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2) \tag{6.32}$$

Total internal reflection occurs if $\theta_2 = 90^o$. This will be the case if:

$$\theta_1 = \sin^{-1}\left(\frac{n_2}{n_1}\right) \tag{6.33}$$

Brewster's law says that if input the reflected ray and the transmitted ray are perpendicular to one another then the reflected ray is perfectly linearly polarized. This occurs if:

$$\theta_B = \tan^{-1}\left(\frac{n_2}{n_1}\right) \tag{6.34}$$

A diagram of light entering a medium at Brewster's angle is shown in figure 6.8

6.16 Beats

Bad Source:
When two waves are combined with angular frequencies ω_1 and ω_2 the resulting beat has angular frequency $\omega_1 - \omega_2$ (easy to derive):

$$\sin(\omega_1 t) + \sin(\omega_2 t) = 2cos\left(\frac{\omega_1 - \omega_2}{2}t\right)\sin\left(\frac{\omega_1 + \omega_2}{2}t\right) \tag{6.35}$$

6.17 Holograms

A photograph represents a single fixed image of a point. A hologram records light from a range of directions, not only one. Hence it records the light as well as where it can from. The recording process of a hologram is shown in figure 6.9. Note that holograms use interference of laser beams to record the data. The types of data recorded are amplitude and phase of the incoming waves.

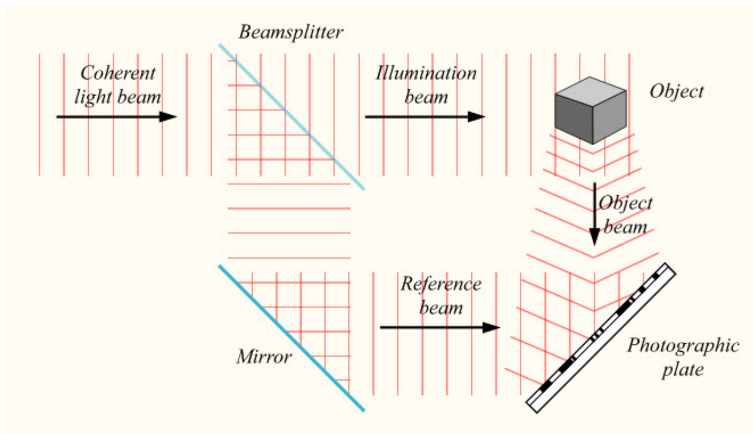


Figure 6.9: The Recording Process of a Hologram

Chapter 7

Astronomy

7.1 General Knowledge

Def 35. Parsec: Unit of length equal to about 3.26 lightyears.

Def 36. Arcsecond: Unit of angular measurement. Equal to 1/3600 of a degree

Def 37. Cosmic Background Radiation: Thermal radiation that is in the entire universe almost uniformly. The radiation is is remnants of the Big Bang.

The temperature of the universe is inversely proportional the radius of the universe. In other words

$$T \propto \frac{1}{r} \tag{7.1}$$

Thus as time progresses and the universe expands the temperature of the universe is reduced (as the universe expands the temperature is decreasing).

STOP! Common GRE Problem 22. *What can be inferred from the maximum and minimum positions of planets in orbit? For this it may be useful to remember that all objects fall at the same speed (hence you can't measure the mass of a moon orbiting a planet using typical methods)*

7.2 Redshift and Blueshift

Def 38. Redshift: Occurs when an a electromagnetic wave increases in wavelength due to the doppler shift (becomes "redder")

Def 39. Blueshift: Occurs when an electromagnetic wave decreases in wavelength due to the doppler shift (becomes "bluer")

The redshift is often refers to a dimensionless quantity z given by

$$z = \frac{\lambda_{obsv} - \lambda_{emit}}{\lambda_{emit}} = \frac{f_{emit} - f_{obsv}}{f_{obsv}} \tag{7.2}$$

Note that if the light was redshifted (i.e. increased in wavelength) then $z > 0$ as one would expect. For small speeds the redshift is approximately given by

$$z_{v<<c} = \frac{v}{c} \tag{7.3}$$

7.3 Hubble's Law

Hubble's law says that all objects around Earth have relative velocity (found using the doppler shift) to the Earth around each other. Further the law says that the velocity of various galaxies galaxies receding from Earth is proportional to their distance to Earth and all other interstellar bodies by Hubble's constant:

$$v = H_0 D \tag{7.4}$$

Where v is the velocity, H_0 is the Hubble constant and D is the proper distance to a galaxy. The reciprical of the Hubbard constant is the Hubble time. It gives a measure of the expansion timescale of the Universe.

7.4 Black Holes

The Schwarzschild Radius of a black hole is

$$r = \frac{2GM}{c^2} \tag{7.5}$$

Where G is the gravitational constant, c is the speed of light, and M is the mass of the potential black hole.

Chapter 8

Relativity

8.1 General Knowledge

The Lorentz factor (γ) is defined as

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \qquad \Longleftrightarrow \qquad v = \frac{c\sqrt{\gamma^2 - 1}}{\gamma} \tag{8.1}$$

Since no calculators are allowed on the GRE's it may be more useful to use the following:

$$\gamma \approx 1 + \frac{1}{2}\frac{v^2}{c^2}, \qquad \frac{1}{\gamma} \approx 1 - \frac{1}{2}\frac{v^2}{c^2} \tag{8.2}$$

Note that this approximation gives strictly incorrect answers for $\gamma \geq 1.5$ since it says that $v > c$

8.2 Gamma Table

If there is extra time to study for the test consider studying the gamma table:

γ	$v(\gamma)$
1	0
1.005	$0.1c$
1.048	$0.3c$
1.155	$0.5c$
1.25	$0.6c$
1.5	$0.75c$
1.75	$0.82c$
2	$\frac{\sqrt{3}}{2}c$
2.5	$0.92c$
3	$\frac{2\sqrt{2}}{3}c$

8.3 Lorentz Transformation

The Lorentz transformation in the convention used above takes the following form for a boost:

$$\begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{8.3}$$

Note when expanding the 4-vector notation it's important to keep in mind minus signs (in other words $x_o = -ct$). Alternatively one can memorize the expanded form for these equations:

$$x' = v\gamma t + \gamma x \tag{8.4}$$

$$t' = \gamma t + \frac{v}{\beta^2}\gamma x \tag{8.5}$$

8.4 Primary Consequences

The two most commonly used consequences of relatively are lorentz contraction and time dilation. Let $'$ indicate the rest frame

$$t' = \frac{t}{\gamma} \tag{8.6}$$

$$l' = \gamma l \tag{8.7}$$

The energy (E) of a particle with rest mass m and momentum p is given by

$$E^2 = p^2c^2 + m^2c^4 \tag{8.8}$$

where the relativistic momentum is given by

$$p = \gamma m v \tag{8.9}$$

Alternatively the total energy can be given as

$$E = \gamma mc^2 = \sqrt{p^2c^2 + m^2c^4} \tag{8.10}$$

Furthermore the kinetic energy is

$$E_{kin} = E - mc^2 = (\gamma - 1)mc^2 \tag{8.11}$$

The momentum of a photon is

$$p_\gamma = \frac{E}{c} = \frac{h\nu}{c} = \frac{h}{\lambda} \tag{8.12}$$

If you are in an inertial frame and you see two objects moving with speeds v_1 and v_2 relative to you then you cannot add them directly to get the speeds relative to one another. You must the Einstein velocity addition rule. Consider an object moving relative to another with speed v in the x direction.

$$u'_x = \frac{u_x + v}{1 + u_x v/c^2} \tag{8.13}$$

$$u'_y = \frac{u_y}{\gamma(1 + u_x v/c^2)} \tag{8.14}$$

$$u'_z = \frac{u_z}{\gamma(1 + u_x v/c^2)} \tag{8.15}$$

Note it is essential to memorize the x direction formula though the other two directions do not seem very important. Further note that the x direction velocity equation can be easily remember be knowing that the speed must converge to c if both speeds being added are c.

8.5 Spacetime Intervals

Def 40. Spacetime Interval: The scalar product of the difference in spacetime between two events with itself. Defining time to be the quantity that changes sign from when going from a covariant to a contravariant vector gives the spacetime interval

$$I \equiv -c^2t^2 + d^2 \tag{8.16}$$

Note that using the spacetime interval can be a big time-saver on some problems which can be solved with lorentz transformations or length contraction and time dealation.

Def 41. Timelike: Two points in spacetime have a timelike spacetime interval if $I < 0$. The reason for this is if two events are separated only in time (i.e. d=0) then $I < 0$. If two events are separated by a timelike spacetime interval then there exists some inertial frame (accessible by a Lorentz transformation) in which they occur at the same point in *space*. In other words if there exist two points A and B in spacetime that are separated by a timelike spacetime interval then if some is at the point where A occurs and leaves toward B with speed $v = d/t$ then they would make it to the B just in time so that B also occurs on their position

Def 42. Spacelike: Two points in spacetime have a spacelike seperation if $I > 0$. The reason for this is if two events are separated only in space (t=0) then $I > 0$. If two events are separated by a spacelike spacetime interval then there exists some inertial frame (accessible by a Lorentz transformation) at which they occur at the same time. In other words if two events A and B will occur and they separated by distance d and time t then a person moving at speed c^2t/d will see these events as simultaneous events.

Def 43. Lightlike: Two points are lightlike if two events are connected by a signal traveling at the speed of light (I=0)

These ideas are illustrated in figure 8.1

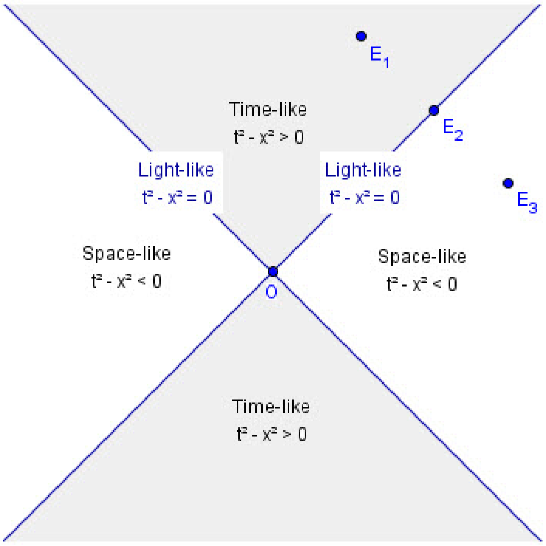


Figure 8.1: Spacelike and Timelike Events

STOP! Common GRE Problem 23. *Can two events separated by Δx and Δt and occur at the same point (either spatially or temporally)? This can be solved by equating spacetime intervals of the two cases (since the spacetime interval is conserved for all non accelerating reference frames)*

8.6 Doppler Effect

The relativistic doppler shift for motion away from the source takes the form:

$$f_s = f_o\sqrt{\frac{1+\beta}{1-\beta}} \tag{8.17}$$

Note: This equation is different depending on whether the source and observer are moving toward or away from one another. However the form is always the same and the general rule is the frequency must increase if the source and observer are must toward one another. The relativistic doppler shift for motions transverse (at exactly 90^o) to the direction of the source is

$$f = \frac{f_s}{\gamma} \tag{8.18}$$

8.7 Lorentz Transformation of Electric and Magnetic Field

Given motion along the x axis:

$$E'_x = E_x \qquad E'_y = \gamma(E_y - vB_z) \qquad E'_z = \gamma(E_z + vB_y) \tag{8.19}$$

$$B'_x = B_x \qquad B'_y = \gamma(B_y + \frac{v}{c^2}E_z) \qquad B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right) \tag{8.20}$$

One thing to notice is that if the the magnetic field is zero then the electric field is strictly dependent on the electric field by a factor gamma for the fields perpendicular to the motion and is identical for the field parallel to the motion.

Note that for simple motions the solutions for a transforming magnetic and electric field can be calculated using Gauss's law and Ampere's law in the moving frame.

Chapter 9

Quantum Mechanics

9.1 General Knowledge

The commutator is defined as

$$[A, B] = AB - BA \tag{9.1}$$

A Hermitian matrix has real eigenvalues; observables are represented by Hermitian operators. Common commutator, Hermitian conjugate, and Dirac Braket identities are

- $(AB)^\dagger = B^\dagger A^\dagger$
- $[AB, C] = A[B, C] + [A, C]B$
- $[A, B] = -[B, A]$
- $\langle\alpha|\beta\rangle^* = \langle\beta|\alpha\rangle$

The de Broglie wavelength of a particle is given by

$$\lambda = \frac{h}{p} \tag{9.2}$$

The Pauli spin matrices are:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9.3}$$

The spin operator is given by

$$\mathbf{S} = \frac{\hbar}{2}\sigma \tag{9.4}$$

The general Hamiltonian is:

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V \tag{9.5}$$

Time-dependent Schrodinger Equation is

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi, \qquad \Psi(t) = e^{-iHt/\hbar}\Psi(0) \tag{9.6}$$

Time-independent Schrodinger equation is

$$H\psi_n = E_n\psi_n, \qquad \psi_n(t) = e^{-iE_nt/\hbar}\psi_n(0) \tag{9.7}$$

The expectation value of operator $\mathcal{O}$ is given by

$$\langle\mathcal{O}\rangle = \int_{\text{All space}} \Psi(\mathbf{r},t)^*\mathcal{O}\Psi(\mathbf{r},t)d^3r \tag{9.8}$$

$$\hat{p} = -i\hbar\nabla \tag{9.9}$$

$$[\hat{x},\hat{p}] = i\hbar \tag{9.10}$$

$$[f(x),\hat{p}] = i\hbar\frac{df}{dx} \tag{9.11}$$

The spherical harmonics take the form

$$Y_l^m = \Theta(\theta)e^{im\phi} \tag{9.12}$$

Where $\Theta(\theta)$ is some function of θ. An important spherical harmonic is that of the s orbital (independent of θ, ϕ):

$$Y_0^0 = \frac{1}{2}\sqrt{\frac{1}{\pi}} \tag{9.13}$$

STOP! Common GRE Problem 24. *What is the possible eigenvalues of the state that is proportional to* $\cos m\phi$*? Since a cosine is a linear combination of exponentials as* $\frac{e^{im\phi}+e^{-im\phi}}{2}$*, we know that this state is a linear combination of the* $-m$ *and* m *state. Thus the possible eigenvalues are* $m\hbar$ *and* $-m\hbar$*.*

The two important boundary conditions for the wavefunction in one dimension are:

1. $\psi(x)$ is always continuous
2. $\frac{d\psi(x)}{dx}$ is continuous except at points where the potential is infinite (e.g. crossing a delta function potential)

Tunneling shows an exponential decay for the probability of getting past the potential barrier.

The Planck length is a measure of what length scales in which many high energy theoretical theories can be probed. An example of this is string theory. The Planck length is given by (this can be found by dimensional analysis)

$$\ell_p = \sqrt{\frac{\hbar G}{c^3}} \approx 10^{-35}m \tag{9.14}$$

Probability current density in non-relativistic quantum mechanics is given by by

$$\mathbf{j} = \frac{\hbar}{2mi}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) \tag{9.15}$$

9.2 Singlet and Triplet States

Consider a system of two spin half particles. The system can be in either the singlet state (antisymmetric state):

$$|00\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - \langle\downarrow\uparrow\rangle\right) \tag{9.16}$$

or in one of the triplet states (symmetric states)

$$\langle 10\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \tag{9.17}$$

$$\langle 1-1\rangle = |\downarrow\downarrow\rangle \tag{9.18}$$

$$\langle 11\rangle = |\uparrow\uparrow\rangle \tag{9.19}$$

The ground state of Helium is an example of a spin singlet (otherwise the wavefunctions would be symmetric!)

9.3 Infinite Potential Well

$$E_n \propto \frac{n^2}{ma^2} \tag{9.20}$$

The important points are

- $E \propto n^2$
- $E \propto \frac{1}{a^2}$
- $E \propto \frac{1}{m}$

$$\psi = \sqrt{\frac{1}{a}}\sin\left(\frac{n\pi x}{a}\right) \tag{9.21}$$

9.4 Quantum Harmonic Oscillator

The solutions take the form of an exponential. The harmonic oscillator wave functions are shown in figure 9.1. The energies of the harmonic oscillator are

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \tag{9.22}$$

The ground state of the harmonic oscillator is

$$\psi_o = \left(\frac{\beta^2}{\pi}\right)^{1/4} e^{\beta^2 x^2/2} \tag{9.23}$$

Where $\beta = \sqrt{\frac{m\omega}{\hbar}}$.

STOP! Common GRE Problem 25. *If a wall was inserted at the center of the well what would happen to the energies? In this case all even energies eigenfunctions would disappear and the energies would just be the odd energies.*

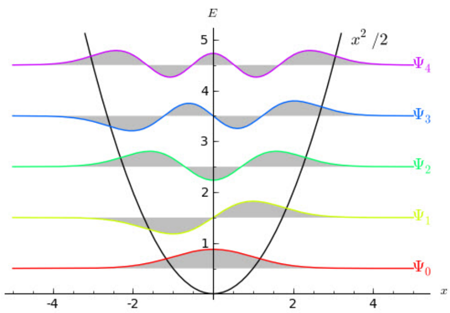


Figure 9.1: The harmonic oscillator wave functions

9.5 Delta Function Potential

Consider the potential well:

$$V = -\alpha\delta(x) \tag{9.24}$$

There is only a single bound state in this system given by

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2}; \qquad E = -\frac{m\alpha^2}{2\hbar^2} \tag{9.25}$$

9.6 Time Independent Non-Degenerate Perturbation Theory and the Variational Principle

The first order energy perturbation by H' (the perturbing energy operator) on state ψ_n is given by

$$E_n^{(1)} = \left\langle \psi_n^{(0)} | H' | \psi_n^{(0)} \right\rangle \tag{9.26}$$

The perturbed wave function (first order) is given by

$$\psi_n^{(1)} = \sum_{k \neq n} \frac{\left\langle \psi_k^{(0)} | H' | \psi_n^{(0)} \right\rangle}{E_n^{(0)} - E_k^{(}0)} \psi_k^{(0)} \tag{9.27}$$

The second order perturbation is given by

$$E_n^{(2)} = \sum_{k \neq n} \frac{| \left\langle \psi_k^{(0)} | H' | \psi_n^{(0)} \right\rangle |^2}{E_n^{(0)} - E_k^{(0)}} \tag{9.28}$$

The variational principle says that the true ground state energy, E_g, can be estimated by

$$\langle \psi | H | \psi \rangle \geq E_g \tag{9.29}$$

Where H is the full Hamiltonian of the system and ψ is an educated guess of the wave function of the ground state.

9.7 Heisenberg Uncertainty Principle

$$\sigma_A \sigma_B \geq \left| \frac{1}{2} \langle [A, B] \rangle \right| \tag{9.30}$$

$$\sigma_p \sigma_x \geq \frac{\hbar}{2} \tag{9.31}$$

$$\sigma_E \sigma_t \geq \frac{\hbar}{2} \tag{9.32}$$

9.8 Angular Momentum

The z component of angular momentum operator is given by:

$$L_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \tag{9.33}$$

The L^2 and $\mathbf{L}$ acting on the eigenstates of the Hydrogen atom Hamiltonian (ℓ) results in the following

$$L^2 \psi = \hbar^2 \ell(\ell + 1)\psi \tag{9.34}$$

$$L_z \psi = \hbar m_\ell \psi \tag{9.35}$$

Since these results are for true for all angular momenta (as long as you are acting on eigenstates of the angular momenta operator) the same results hold true for spin.
The commutators for angular momenta (both spin and orbital) are:

$$[J_x, J_y] = i\hbar J_z, \qquad [J_y, J_z] = i\hbar J_x, \qquad [J_z, J_x] = i\hbar J_y \tag{9.36}$$

Furthermore the raising and lowering operators are defined as:

$$J_\pm = J_x \pm i J_y \tag{9.37}$$

The eigenvalues of the raising or lowering operators are given by

$$J_\pm \psi = \hbar \sqrt{j(j+1) - m(m \pm 1)} \tag{9.38}$$

The expectation values of the spin operators squared are all equal:

$$\langle S_x^2 \rangle = \langle S_y^2 \rangle = \langle S_z^2 \rangle = \frac{\hbar^2}{4} \tag{9.39}$$

9.9 Hydrogen Atom

The Energy of the Hydrogen-like atom with reduced mass μ is given by

$$E_n = -E_1 \left(\frac{Z^2}{n^2} \cdot \frac{\mu}{\mu_e} \right) \tag{9.40}$$

Where $E_1 = -\frac{q^4}{8\hbar^2 \epsilon_0^2} = 13.6eV$, and $\mu_e \approx m_e$ is the reduced mass of the electron.
The important points of this equation are

- The energy constant is proportional to the reduced mass of the system
- The energy constant is proportional to the Z^2
- The energy is proportional to $1/n^2$

As an example the energy of a Helium ionized once is

$$E_{He^+} = \frac{4}{1^2} 13.6eV \approx 55eV \tag{9.41}$$

For positronium the big difference is in the reduced mass:

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{1}{2} m_e \tag{9.42}$$

The common spectral series are: Lyman (transitions to the ground state - UV) and Balmer (transitions to the first excited state - Visible Light) Another important property of the Hydrogen atom is its radius. From the Bohr model the radius of the Hydrogen atom is given as:

$$r_n \approx (a_0) \left(\frac{m_e}{\mu} \frac{n^2}{Z} \right) \tag{9.43}$$

Where a_0 is the Bohr radius and is roughly equal to $0.53\mathring{A}$. Note that for The important points are

- The radius is proportional to the $1/\mu$
- The radius is proportional to n^2
- The radius is proportional to $1/Z$

Note that other the Z instead of Z^2 dependence the energy dependence is the inverse of all these dependencies.

The width spectral line of a gas has an inverse relationship to the density of the gas. Thus the denser the gas used for spectroscopy the broader the spectral line will be.

9.10 Magnetic Moment and Gyromagnetic Ratio

For an electron orbit the magnetic moment (μ) is given by (easy to derive by assuming a circular orbit and the definition of the magnetic moment ($\mu = IA$)

$$\mu = \frac{-e}{2m_e} \mathbf{L} \tag{9.44}$$

For intrinsic angular momentum (spin) QED gives a correction to this formula called the Landé g-factor. This correction gives the spin as:

$$\mu = \frac{-ge}{2m_e} \mathbf{S} \tag{9.45}$$

For an electron $g \approx 2$. The gyromagnetic ratio is defined as the magnetic moment divided by the spin:

$$\gamma = \frac{\mu}{S} = \frac{-eg}{2m_e} \tag{9.46}$$

9.11 Photoelectric Effect

When light is shown in a piece of metal with *short* enough wavelength (i.e. photons with high enough energy) then electrons will be ejected from the metal. This is referred to as the photoelectric effect. In order to detect the ejected electrons a collector plate is placed near the piece of metal. Suppose a potential difference is placed between the collector plate and the piece metal. The potential on the collector plate is negative relative to the metal. The potential then acts as a method of slowing down the incoming electrons. If the potential is ramped up eventually the no electrons will be able to cross the potential barrier. This occurs at the stopping potential V_0. Thus

$$T_{max} = eV_0 \tag{9.47}$$

Where T_{max} is the maximum kinetic energy of the electrons. Employing conservation of energy

$$hf = T_{max} + \Phi \tag{9.48}$$

Where f is the frequency of the light and Φ is the work function of the metal. The work function is the energy required to remove an electron from an atom (also called the ionization energy). Thus if you shine light onto a metal with a low enough frequency (high enough wavelength) then no electrons will be ejected from the metal since $hf < \Phi$. Alternatively one can define a threshold frequency:

$$f_{\text{threshold}} = \frac{\Phi}{h} \tag{9.49}$$

The kinetic energy of the electrons as a function of frequency is shown in figure 9.2.

9.12 Stern-Gerlach Experiment

The Stern Gerlach experiment consisted of silver atoms passed through an inhomogeneous magnetic field. Silver has atomic number 47. With the corresponding configuration:

$$Ag = [Kr]5s^1 4d^{10} \tag{9.50}$$

Hence there is one electron on the $5s$ orbital that is in a linear combination of the spin up and spin down state:

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle - |\downarrow\rangle\right) \tag{9.51}$$

As silver is sent through the magnetic field the system collapses to one of the two states and the beam of silver atoms is split into two beams one for spin up and one for spin down. The Stern Gerlach experiment was monumental in proving the existence of spin. The Stern Gerlach experiment is shown in figure 9.3.

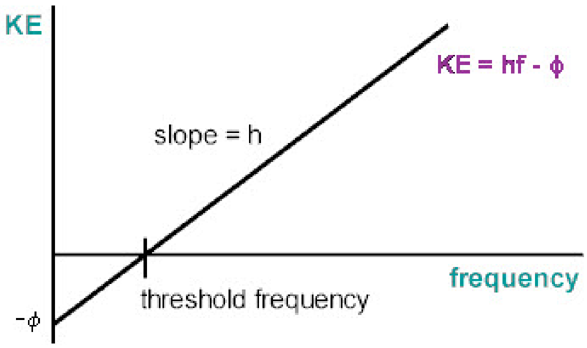


Figure 9.2: Kinetic energy of electrons emitted in the photoelectric effect as a function of frequency of incoming radiation

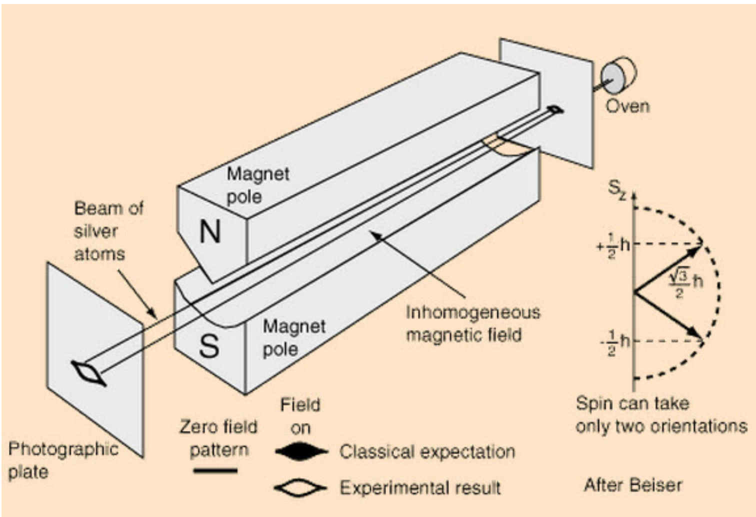


Figure 9.3: The Stern Gerlach Experiment

9.13 Franck-Hertz Experiment

The Franck-Hertz experiment consisted of electrons accelerated by a voltage toward a positively charged grid surrounded by mercury vapour. Past the grid there was a collection plate held at a small negative voltage with respect to the grid. As the electron energy was ramped up mercury was either excited or not depending on whether the energy was close to the correct excitation energy of the mercury atom. The Franck-Hertz experiment showed that the energy levels of mercury are quantized confirming quantum theory. The current of the Franck Hertz experiment as a function of accelerating voltage of the electrons (and hence kinetic energy of the electrons is shown in figure 9.4.

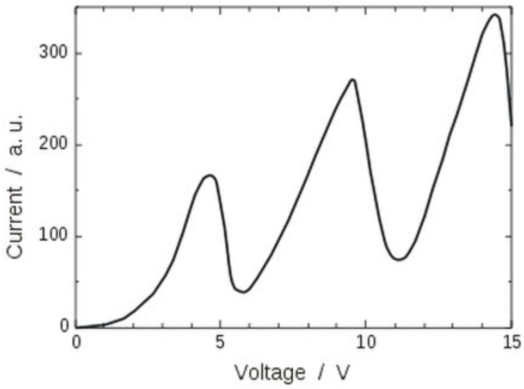


Figure 9.4: Typical results of Franck-Hertz experiment

9.14 Compton Shift

The Compton is the shift in the wavelength of an electron scattered by an incoming photon. The shift can be calculated using energy and momentum conservation. The shift is given by

$$\Delta\lambda = \frac{h}{mc}(1 - \cos\phi) \tag{9.52}$$

Where ϕ is the scattering angle from the incoming direction of the photon. The importance of the Compton effect was that it clearly showed the particle nature of the photon. If light were simply a wave then it would be an incoming wave with a certain frequency which would cause the electron to oscillate with that same frequency.

9.15 Perturbations

The fine structure of Hydrogen is composed of two contributions: relativity and spin-orbit coupling between the electron spin and the electron orbital angular momentum. The effect of fine structure is to split states with different values of total angular momentum, j. Thus states with the same n value are no longer degenerate. The lamb shift is a screening effect of the nucleus due to spontaneous electron-positron creation and annihilation. The hyperfine structure of Hydrogen is due to coupling between the proton spin and electron spin. The effect is to split the triplet and singlet states. The hyperfine splitting of the ground state of Hydrogen produces the famous 21 cm line which is often observed in astronomy. The fine structure effect is shown in figure 9.15

9.16 Cross Sections

The differential cross section is given by

$$D(\theta) = \frac{d\sigma}{d\Omega} \tag{9.53}$$

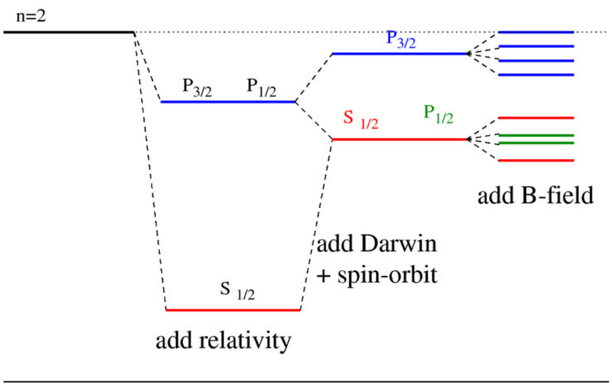


Figure 9.5: The Effect of fine structure on hydrogen

The total cross section is given by

$$\sigma = \int D(\theta) d\Omega \tag{9.54}$$

In spherical coordinates the differential solid angle is

$$d\Omega = \sin\theta d\theta d\phi \tag{9.55}$$

9.17 Possible Useful Theorems

Def 44. Ehrnesfest's Theorem: Expectations values of quantum operators obey the classical rules.

An example of this theorem is

$$m \frac{d^2 \langle x \rangle}{dt^2} = \frac{d \langle p \rangle}{dt} = -\frac{d \langle V \rangle}{dx} \tag{9.56}$$

Def 45. Viral Theorem: $2 \langle T \rangle = \left\langle x \frac{dV}{dx} \right\rangle$

Bad Source:

- If the potential energy, $V(x)$, is even, then the wavefunction can be taken to be either even or odd.
- The ground state of an even potential is even and has no nodes.

Chapter 10

Solid State Physics

10.1 General Knowledge

The unit vectors of the lattice ($\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$) are any three vectors such that

$$\mathbf{T} = u_1\mathbf{a_1} + u_2\mathbf{a_2} + u_3\mathbf{a_3} \tag{10.1}$$

Where $(u_1, u_2, u_3) \in \mathbb{Z}$
can make up any point in the lattice.

Def 46. Point Operations: The symmetry operations of a crystal that carry a crystal onto itself

The unit cell is the parallelepiped defined by the unit vectors. The volume of the unit cell is given by

$$V = |\mathbf{a_1} \cdot \mathbf{a_2} \times \mathbf{a_3}| \tag{10.2}$$

The number of lattice points in a unit cell are given as follows. If a lattice point is shared among n nearby unit cells then the amount of lattice point belonging to a single unit cell is $1/n$. The number of lattice points belonging to a unit cell is the sum of all these fractions.

Def 47. Bravais Lattice: A type of lattice defined by the particular point group operations that can be done on the lattice such that the lattice stays invariant. In other words its a lattice that looks the same from every point.

Def 48. Primitive Unit Cell: The smallest possible unit cell

The volume of the primitive unit cell can either be found by knowing the lattice vectors for the primitive cell and finding the volume. However it may be non-trivial to find the primitive lattice vectors. If the number of lattice points in some unit cell of volume V is N then the volume of the primitive cell is

$$V_{prim} = \frac{V}{N} \tag{10.3}$$

STOP! Common GRE Problem 26. *Find the volume of the primitive unit cell. It may be useful to know offhand that that a BCC lattice has 2 lattice points and a FCC lattice has 4 lattice points.*

10.2 Reciprocal Lattice

The reciprocal lattice vectors make up the entire momentum space of electrons in the lattice. The reciprocal lattice vectors are given by

$$\mathbf{b_1} = 2\pi \frac{\mathbf{a_2} \times \mathbf{a_3}}{\mathbf{a_1} \cdot \mathbf{a_2} \times \mathbf{a_3}} \tag{10.4}$$

and cyclic permutation of 1,2, 3 ($2 \to 3, 3 \to 1, 1 \to 2$)to obtain vectors 2 and 3. The Brillioun Zone is the collection of points in k space that can be obtained from the unit vectors.

10.3 Free Electron Gas

The free electron model assumes that the electrons do not interact with one another but only behave kinetically and fulfill the Pauli Exclusion Principle. This is a common model for the free electrons in a metal. The S.E. says that the solutions are sinusoidal. The energies are

$$E = \frac{\hbar^2 k^2}{2m} \tag{10.5}$$

The Fermi energy is

$$E_F = \frac{\hbar^2 k_F^2}{2m} \tag{10.6}$$

Where k_F is the Fermi momentum which is equal to

$$k_F = (3\rho\pi^2)^{1/3} \tag{10.7}$$

Where ρ is the free electron number density.

10.4 Effective Mass

In a lattice there are forces that change the way electrons move in a crystal. Without these forces the energy would be given by the free electron (equation 10.5). One way to approximate the motion of the electrons (now modified due to the potential of the lattice) is by simply using a new mass as the mass of the electron called an effective mass. The effective mass is given by

$$m^* = \frac{\hbar^2}{\left(\frac{d^2E}{dk^2}\right)} \tag{10.8}$$

Using this equation the energy of the electrons can be approximated as

$$E \approx= \frac{\hbar^2 k^2}{2m} \tag{10.9}$$

The effective mass of the electrons depends on the second derivative of energy with respect to momentum at a given k point in the crystal.

10.5 Unrelated Facts

- Diamond forms an face centered cubic (FCC) lattice. Thus each carbon atom and its nearest neighbors form a tetrahedron (see figure 10.1).

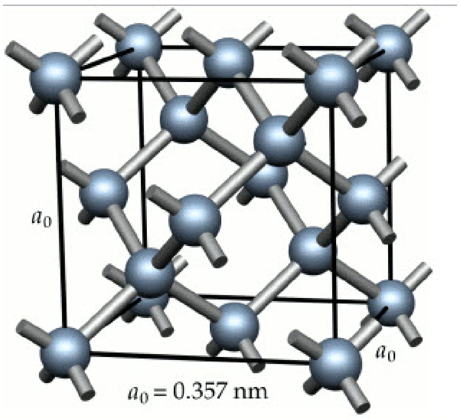


Figure 10.1: The Diamond crystal structure

Chapter 11

Particle Physics

11.1 General Knowledge

The GRE seems to emphasize some seemingly random topics in particle physics the follows are some facts that need to be memorized:

- The J/ψ meson is composed of a charm quark and anti charm quark. It's discovery provided one of the earliest pieces of evidence for quarks (in particular the charm quark)
- The Deutron is a composite particle of a proton and a neutron
- An alpha particle is a composite particle of two protons and two neutrons

One needs to know the spins of common particles. These are shown in table 11.1 The photon has spin 1, electrons spin half, proton have spin half.

STOP! Common GRE Problem 27. *Why can't a photon undergo pair production in free space? Working out the equations shows that linear momentum and energy wouldn't be conserved since if they were the electron and positron would travel at the speed of light (yet angular momentum, lepton number, parity, etc. don't have any problems with this reaction)*

11.2 Hadrons

Def 49. Hadrons: Particles on which the strong force acts

Particle	Spin
Photon	1
Electron	1/2
Proton	1/2

Table 11.1: The spins of common particles

Family	Particle	Approximate Mass (MeV)
Electron	Electron(e^-)	0.511
	Electron neutrino(ν_e)	10^{-7}
Muon	Muon(μ)	106
	Muon neutrino(ν_μ)	10^{-7}
Tau	Tau(τ)	1800
	Tau neutrino(ν_τ)	10^-7

Def 50. Mesons: Hadrons that are bosons. Mesons are composed of a quark and antiquark.

Def 51. Baryons: Hadrons that are fermions. Baryons are composed of 3 quarks.

Every baryon carries a new quantum number called the a baryon number (B). Each baryon has a baryon number $B = +1$ while each antibaryon has a baryon number $B = -1$. Each particle that does not interact with the strong force carry $B = 0$. Each quark carries Baryon number, $B = 1/3$

11.3 Leptons

Def 52. Lepton: Particle which does not interact via the strong force, leaving the weak force as the dominant force

Each lepton has a quantum number called the lepton number (L). Each particle in table 11.3 has a lepton number, $L = 1$. Each of their corresponding antiparticles have a $L = -1$. All other particles which are not leptons have $L = 0$. Experimentally it was found that lepton number is conserved in any particle interaction. This is called *conservation of lepton number*.

11.4 Alpha and Beta Decays

Def 53. Alpha Decay: The spontaneous decay of a nucleus into a new nucleus by emission of an alpha particle (a ^4He) nucleus.

An Alpha decay can be written schematically as:

$$^{A}\mathcal{N} \longrightarrow {}^{A-4}\mathcal{L} + {}^{4}He \tag{11.1}$$

Where $\mathcal{N}$ is some nucleus and $\mathcal{L}$ is the original nucleus with a charge $Z_N - 2e$.

Def 54. Beta Decay:A spontaneous decay of a nucleus that emits an electron or positron together with the corresponding neutrino

There are two types of beta decays, beta minus and beta plus. A beta minus is a decay that emits an electron while a beta plus is a decay that emits a positron. i.e.

$$\beta^- : \mathcal{N} \longrightarrow \mathcal{O} + e^- + \bar{\nu}_e \tag{11.2}$$

$$\beta^+ : \mathcal{N} \longrightarrow \mathcal{M} + e^+ + \nu_e \tag{11.3}$$

Where $\mathcal{M}$ is a new nucleus with a charge $Z_N - e$ and $\mathcal{O}$ is a new nucleus with a charge $Z_N + e$. Note the reaction requires the neutrinos in order to conserve lepton number (for explanation of conservation of Lepton number see section 11.3). It is important to note that the atomic mass of the nucleon does not change in beta decay but only the charge. Alternatively beta decay can be expressed as:

$$\beta^- : n \longrightarrow p + e^- + \bar{\nu}_e \tag{11.4}$$

$$\beta^+ : p \longrightarrow n + e^+ + \nu_e \tag{11.5}$$

Beta decay should be differentiated from the process called internal conversion in which an excited nucleus interacts with an electron in one of the lower atomic orbitals causing the electron to be emitted from the radioactive atom.

11.5 Detectors

In order to resolve particles moving at a certain speed past a detector it is necessary to have a resolving time that is smaller then the time it takes the particle to pass through the detector. In other words if the detector is of length L and the particles have speed v, then the resolving time must be

$$t_{res} \leq \frac{L}{v} \tag{11.6}$$

Chapter 12

Atomic Physics

12.1 General Knowledge

The common notation for a particular isotope of an element is:

$$^{A}E \tag{12.1}$$

Where E represents the element abbreviation and A is the atomic mass in atomic mass units. Since the mass of the electrons is so small A is typically given by:

$$A = Z + N \tag{12.2}$$

Where Z is the number of protons and N is the number of neutrons.

12.2 Spectroscopic Notation

A particular subshell is often designated by the following notation:

$$^{2S+1}L_J \tag{12.3}$$

Where $L = S, P, D, F, ...$ refers to orbital angular momentum, J is the total angular momentum quantum number and S is the spin (typically $s = 1/2$)

12.3 Hydrogen Quantum Numbers

- $n = 1, 2, 3, ..$ is the principle quantum number and controls the radial wavefunction as well as the energy of an orbital.
- $\ell = 0, 1, ...n - 1$ is the orbital quantum number which controls the radial wavefunction and angular wavefunction.
- m $= -\ell, ...\ell$ is the magnetic quantum number and controls the angular wavefunction

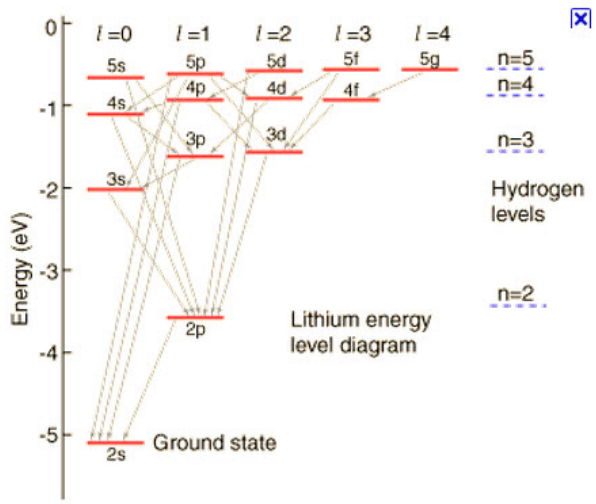


Figure 12.1: The Allowed transitions for Lithium

12.4 Selection Rules

The selections rules for transitions between states designated by n,l,m are given by:

$$\Delta m_\ell = \pm 1 \text{ or } 0$$
$$\Delta \ell = \pm 1 (\neq 0)$$
$$\Delta j = 0, \pm 1$$
$$\Delta m_s = 0$$

These selection rules are illustrated for Lithium in figure 12.4. Notice that there is no allowed transitions in which l doesn't change.

12.5 Hund's Rules

Hund's rules provide a guide for how to place electrons in different energy states by giving guidelines as to which states have higher energies

Def 55. Hund's First Rule : All other things being equal, the state with the highest total spin will have lowest energy

Def 56. Hund's Second Rule: For a given spin, the state with the highest total orbital (consistent with antisymmetrization) angular momentum will have the lowest energy

Def 57. Hund's Third Rule: If a subshell is no more than half filled then the lowest energy level has $J = |L - S|$; if it is more then half filled then $J = L + S$ has the lowest energy.

12.6 Zeeman Effect

The Zeeman effect is the splitting of a spectral line into several components in the presence of a magnetic field. The energy change is given from electromagnetism

$$U = -\boldsymbol{\mu} \cdot \mathbf{B} \tag{12.4}$$

The perturbing Hamiltonian is thus

$$\begin{aligned} H' &= -\boldsymbol{\mu} \cdot \mathbf{B} && (12.5) \\ &= -\frac{ge}{2m}\mathbf{J} \cdot \mathbf{B} && (12.6) \end{aligned}$$

In the absence of a magnetic field all the quantum states with the same orbital quantum number ℓ are degenerate (there is no energetic preference toward particular directions of the orbitals). However, if a magnetic field is applied then the electrons tend to line up with the magnetic field, this makes states with different m_ℓ values to have different energies and hence emit different frequencies of light. In theory this suggests that each line should split into ℓ different lines. However m_ℓ can only change by at most 1 in an atomic transition. Thus each transition energy (and hence frequency of transition line) is split into three (equally spaced) lines due to whether m_ℓ increases by 1, stays the same, or decreases by 1 in the transition.

However this still isn't the whole story. The total angular momentum should include the electron spin. Thus the Zeeman effect can produce more then three lines and not equally spaced.

12.7 Stark Effect

Start effect is the effect of a static electric field on the energy levels of an atom. The energy perturbation is given by electromagnetism:

$$U = \mathbf{p} \cdot \mathbf{E} \tag{12.7}$$

The perturbing Hamiltonian is

$$H' = -qEz = -qEr\cos\theta \tag{12.8}$$

Note that this perturbation is odd. Thus the first order effect on any even state (such as the ground state of Hydrogen) is zero.

12.8 X-ray Spectrum

One way to order the elements in the periodic table is by their X-ray Spectrum. X-rays are produced by bombarding an element with an electron in the kiloelectron-volt range

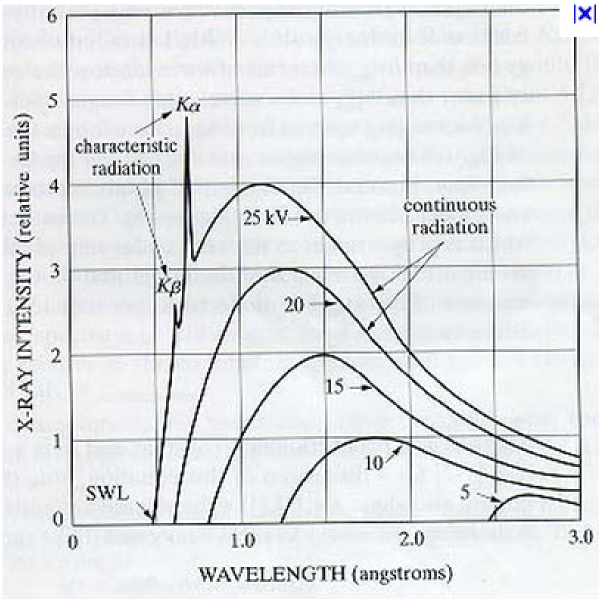


Figure 12.2: The X-ray spectrum of an atom

(i.e. $\lambda \approx 10^{-9}$). These electrons have enough energy to excite the inner electrons of the element near the nucleus. These electrons are not affected much by screening and their energies are highly dependent on the charge of the nucleus. After the inner electrons are excited, higher energy electrons drop down emitted X-rays. These are called "Auger Transitions".

If electrons in the KeV range are bombarded at an element and the emitted radiation is plotted as a function of wavelength it is found that the radiation goes to zero below some minimum wavelength, λ_{min}. This is because this wavelength corresponds to an energy $E = \frac{hc}{\lambda_{min}}$. This is the kinetic energy of the bombarding electron. λ_{min} is only dependent on the incoming energy of the electron. Moseley first used X-Rays to arrange elements in the periodic table. A diagram of the results of an X-ray emission experiment a function of wavelength is shown in figure 12.2 Note: When characterizing the X-Ray spectrum the notation K,L, M, N, ... is used to characterize the $n = 1, 2, 3, 4, ..$ states.

STOP! Common GRE Problem 28. *What type of photon would be emitted from electrons begin ionized near the nucleus? As discussed above, the answer is X-rays. On a side note the photons emitted from nuclear transitions are gamma rays (See Nuclear Physics section).*

The energy of emitted electrons follow a modified formula that takes into account shielding:

$$E = 13.6eV \left(\frac{3}{4}\right)(Z-1)^2 \tag{12.9}$$

Where Z is the charge of the atom. The key point here is the reduction of Z by 1. The energies are almost identical to the Hydrogen-like atom energies.

12.9 Periodic Table

Def 58. Subshell: a set of states determined by n and l quantum numbers. The maximum number of electrons that can fit in a subshell is $2(2l+1)$. In the absence of a magnetic field all states in a given subshell are degenerate.

A closed subshell has no angular moment/magnetic moment and its probability density is spherical symmetric.

12.10 Stimulated and Spontaneous Emission

The frequency of the emitted photon from an atom dropping down to lower energy is given by

$$\nu = \frac{1}{h}(E_2 - E_1) \tag{12.10}$$

12.10.1 Spontaneous Emission

Def 59. Spontaneous Emission: Process by which is spontaneously emitted from an atom due to an electron dropping down from an excited state

If the number of light sources in the excited state are N then

$$\frac{\partial}{\partial t} N = -A_{21} N \tag{12.11}$$

Where A_{21} is called the Einstein A coefficient and is the rate of spontaneous emission (equal to the radiative decay rate which is inversely proportional to the lifetime of the state). Solving the above differential equation gives

$$N(t) = N(0) e^{-A_{21} t} \tag{12.12}$$

12.10.2 Stimulated Emission

Def 60. Stimulated Emission: Process by which an electron in an excited state drops to a state of lower energy by begin bombarded by a photoon of certain frequency.

The key feature of stimulated emission is that the emitted photon has the same frequency, phase, polarization, and direction of travel as the incident photon. This is the basis of the laser.

12.11 Lasers

Laser stands for "light amplification for stimulated emission of radiation". The idea of a laser is to excite the atoms of the laser to a metastable state and keep them in this excited state (creating a "population inversion"). This process of constantly maintaining a population inversion is called optical pumping. Once a population inversion is achieved light is shown on one of the atoms causing stimulated emission. The atom then emits light exciting the other atoms all with a single wavelength. In order to keep a laser running a population inversion must be maintained. The properties of a laser are

1. Light is coherent
2. Light is monochromatic
3. Light has minimal divergence
4. Light has a high intensity

In order to create a laser we need to pump one state and let the atoms fall to a metastable state below the pumped state. This is shown in figure 12.3. There few important types

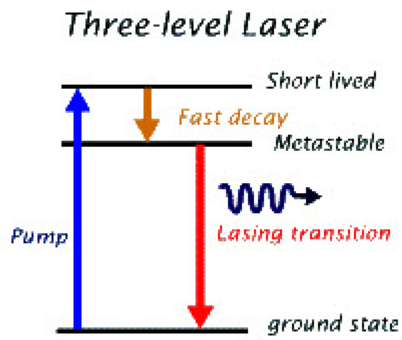


Figure 12.3: Three level laser

of lasers.

Def 61. Diode Laser: A laser formed with a semiconducting active medium. The semiconductor is typically a p-n junction that is injected with electric current.

Def 62. Gas Laser: A laser where a free gas is the active medium. An electric current is run through the gas to excite the atoms.

Chapter 13

Nuclear Physics

13.1 General Knowledge

The notation used for nuclei is as follows. Define Z to be the charge of the nucleus and N to be the number of neutrons. Then the mass of the nuclei is

$$A = Z + N \tag{13.1}$$

The nucleus $\mathcal{X}$ is written as

$${}^{A}_{Z}\mathcal{X} \tag{13.2}$$

13.2 Symmetries

Breaking of different symmetries implies that certain conservation laws do not hold. The different symmetries and corresponding conservation laws are shown in table 13.1 In the

Symmetry	Conservation law
Global gauge invariance of the electromagnetic field	Conservation of charge
Time invariance	Conservation of energy
Translational invariance	Conservation of momentum
Rotation invariance	Conservation of angular momentum

Table 13.1: Symmetries and their conservation laws

case of an isolated nucleus, an electron is equally likely to emit electrons in all directions. However in the case of a magnetic field, the nucleus is most likely to emit electrons in the direction opposite to the magnetic field.

STOP! Common GRE Problem 29. *Suppose a nucleus in an external magnetic field decays and emits an electron in a direction opposite to the magnetic field (and hence the spin of the nucleus) what symmetry does this break? The only symmetry this breaks is parity or reflection symmetry.*

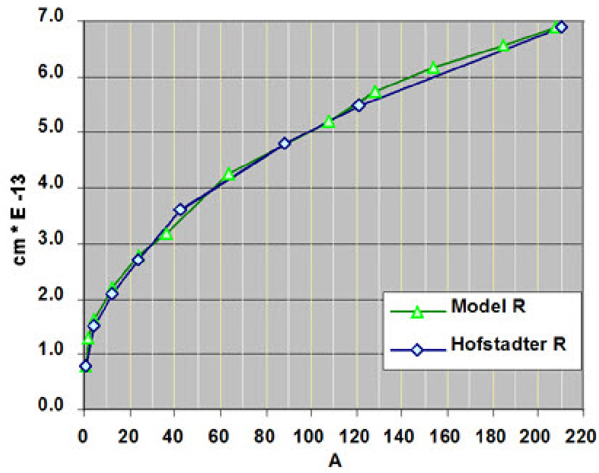


Figure 13.1: Nuclear radii as a function of atomic number

13.3 Properties of Nuclei

In nature heavier elements tend to have more neutrons then protons ($N > Z$) while for light elements the two quantities are about equal ($Z = N$).
The radius (r)of the nucleus scales with the cube root of the atomic mass(A):

$$r \approx r_0 A^{1/3} \tag{13.3}$$

Where $r_0 \approx 1.2 fm$. This relation is shown in figure 13.1

Def 63. Binding Energy: A measure of the energy holding the nucleus together. The binding energy is equal to the difference between the mass of the constituents of the nucleus and the nucleus itself.

Quantitatively the binding energy (B.E.) is given by

$$B.E. = \sum_i \left(m_i c^2\right) - Mc^2 \tag{13.4}$$

Where the sum over i is a sum over all nucleons. The binding energy is much larger then the energy holding together electrons in atoms, thus when nuclei make an atomic transition the energy emitted is greater. Unlike electron transitions this process typically produces gamma rays.

STOP! Common GRE Problem 30. *Find the energy per nucleon of products of a reaction. This problem is solved by first noting that the binding energy is a negative quantity and kinetic energies are positive. From there using conservation of energy the problem can be solved.*

13.4 Half-Life

Radioactive element decay randomly and their probability of decaying at a certain instant is independent of how long it has been around for. This means that they follow the poisson distribution. This can be described with three equivalent equations. Note that all the following section on radioactivity can be derived from the equation for $N(t)$.

$$N(t) = N(0)e^{-\lambda t} \tag{13.5}$$

$$R(t) \equiv \frac{-dN}{dt} = R(0)e^{-\lambda t} \tag{13.6}$$

$$\Rightarrow R(t) = \lambda N(t) \tag{13.7}$$

Where $N(t)$ is the number of nuclei, $R(t)$ is the decay rate(also referred to as the "activity"), and λ is a constant. λ is directly related to the half life by

$$\frac{ln(2)}{\lambda} = t_{1/2} \tag{13.8}$$

Another less common quantity is the mean life(τ) of an atom defined the time it takes the population to reach $1/e$ of it's original population.

$$\tau = \frac{1}{\lambda} \tag{13.9}$$

Consider a decay that can occur through two processes:

$$\frac{dN}{dt} = -\left(\lambda_1 + \lambda_2\right) N \tag{13.10}$$

$$N = e^{-(\lambda_1+\lambda_2)t} \tag{13.11}$$

$$\tag{13.12}$$

The half life is then

$$t_{1/2} = \frac{\ln 2}{\lambda_A + \lambda_B} \tag{13.13}$$

$$\frac{1}{t_{1/2}} = ln2\left(\lambda_A + \lambda_B\right) \tag{13.14}$$

$$\frac{1}{t_{1/2}} = \frac{1}{t^A_{1/2}} + \frac{1}{t^B_{1/2}} \tag{13.15}$$

Thus the reciprocals of the half lives are additive. This makes sense since the half life should decrease with more possible decay routes.

13.5 Types of Radiation

Def 64. Gray: $\frac{\text{Energy}}{\text{Unit Mass}}$ absorbed an object. $1Gy = \frac{1J}{1kg}$

Point of reference: A whole-body short term γ ray dose of $3Gy$ will cause death in 50% of the population(actual typical values are about 2 mGy/year).

There are several types of radiations two important examples are mentioned here:

Def 65. Cherenkov Radiation: Radiation produced when a charged particle passes through a dielectric medium at a speed greater then the *phase* velocity of light in that medium (c/n)

The origin of Cherenkov radiation is that the charged particle polarizes (excites) the molecules surrounding it. Subsequently these molecules drop down to a lower energy state and hence radiates out energy. Unlike fluorescence and emission spectra that have distance spectral peaks, Cherenkov radiation is continuous. In the visible spectrum the intensity per unit frequency is approximately proportional to frequency (i.e. the higher the frequency the more intense the radiation). That is why Cherenkov radiation is blue. There is a cutoff frequency so very high frequency waves such as X-rays are not produced.

Def 66. Bremsstrahlung Radiation: Radiation produced by a decelerating charge.

The Larmor Formula (equation 13.16) gives the power radiated by an accelerating charge as long as the speed of the charged particle is small relative to the speed of light.

$$P = \frac{q^2 a^2}{6\pi\epsilon_0 c^3} \tag{13.16}$$

Where q is the charge of the particle and a is the acceleration of the particle. The key points for the Larmor formula are

- $P \propto q^2$
- $P \propto a^2$.

An electric dipole that oscillates vertically has a radiation distribution as shown in figure 13.2 The radiation goes as $\sin^2(\theta)$. A magnetic dipole gives off radiation that is a smaller in magnitude by a factor of $(c/\omega)^2$. The power of dipole radiation is dependent on ω^4. Hence it is highly dependent on the frequency and the powers of higher frequencies are much greater. The radiation given off by an electric dipole of dipole moment p is

$$P = \frac{\omega^4}{12\pi\epsilon_0 c^3} p^2 \tag{13.17}$$

The important point is that the power is proportional to the frequency to the fourth power.

STOP! Common GRE Problem 31. *How much radiation is given off by an objects oscillating between two spherically symmetrical configurations (For example a charged shell oscillating from one radii to the other). By applying Gauss's law to this problem it is easy to see that the electric field is constant outside the shell and hence no radiation is emitted.*

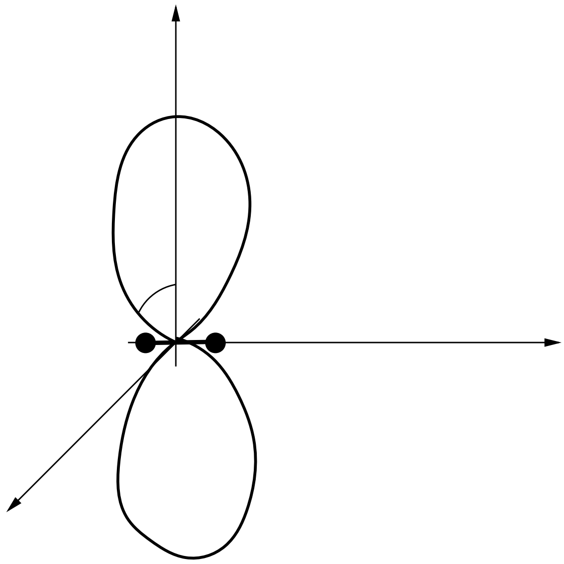


Figure 13.2: Electric Dipole Radiation Spectrum From a Vertically Oscillating Charge

13.6 Fission

$$n + {}^{A}\mathcal{N} \to {}^{A+1}\mathcal{N} \to \mathcal{L} + \mathcal{M} \tag{13.18}$$

A particularly important example of fission occurs in Uranium:

$$n + {}^{235}U \to {}^{236}U \to {}^{140}Xe + {}^{94}Sr + 2n \tag{13.19}$$

Where the pair of neutrons emitted can go on to create other unstable Uranium atoms. In order for fission to occur spontaneously it is required the mass of the reactants be greater then the mass of the products.

13.7 Fusion

It is energetically favourable for elements with a small atomic mass to combine and form larger atoms through a process called fusion. The Sun undergoes fusion in a 4 step:

$$2\left({}^{1}H + {}^{1}H \to {}^{2}H + e^{+} + \nu\right) \tag{13.20}$$
$$2\left(e^{+} + e^{-} \to 2\gamma\right) \tag{13.21}$$
$$2\left({}^{2}H + {}^{1}H \to {}^{3}He + \gamma\right) \tag{13.22}$$
$${}^{3}He + {}^{3}He \to {}^{4}He + {}^{1}H + {}^{1}H \tag{13.23}$$

Notice that the net release in energy is

$$\Delta E = m_{He}c^2 + 2m_Hc^2 - 6m_Hc^2 = m_{He}c^2 - 4m_{He} \tag{13.24}$$

Note that to first order this release is zero (Helium has 2 neutrons and 2 protons). This must be the case because otherwise the energy released from fusion reaction would be enormous. In order for fusion to occur spontaneously it is required the mass of the reactants be greater then the mass of the products (in the case of the Sun $4m_H > m_{He}$).

13.8 Rutherford Scattering

Rutherford scattering refers to bombarding nuclei (generally thin metal foil) with particles (generally alpha particles) and measure the scattered particles. This experiment gave the first evidence that the positive charge in atoms was located in the nucleus. The penetration depth of the bombarding particles can be calculated by setting the kinetic energy of the incoming particle exactly equal to the Coloumb potential encountered:

$$\frac{1}{2}mv^2 = k\frac{Ze^2}{r} \tag{13.25}$$

Where r is the penetration depth.

Chapter 14

Mathematics

14.1 General Knowledge

Euler's formula for expansion of an exponential is:

$$e^{i\phi} = \cos\theta + i\sin\theta \tag{14.1}$$

It can be used to find cosine and sin as exponentials:

$$\cos\theta = \frac{e^{i\phi} + e^{-i\phi}}{2}; \qquad \sin\theta = \frac{e^{i\phi} - e^{-i\phi}}{2i} \tag{14.2}$$

As a way to save time can often use a Taylor expansion to deal with extreme cases. A common example is

$$e^x \approx 1 + x \qquad (\text{For } x \ll 1) \tag{14.3}$$

$$(1+x)^n \approx 1 + nx \qquad (\text{For } x \ll) \tag{14.4}$$

The rotation matrix in two dimensions is

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{14.5}$$

Note this rotation is **counterclockwise**

On an unrelated note one should remember that the curl of a gradient is always zero:

$$\nabla \times (\nabla U) = 0 \tag{14.6}$$

Furthermore the divergence of a curl is also equal to zero:

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0 \tag{14.7}$$

This makes sense since a curl measures how much as field is curling around and divergence measures how much of the field extends outwards.

14.2 Line and Surface Integrals

There are two types of line integrals: Scalar and Vector Integrals. First consider a scalar field ϕ. In cartesian coordinates an integral over some curve C is given by

$$\int_c \phi d\mathbf{r} = \int_c \phi dx\ \hat{x} + \int_c \phi dy\ \hat{y} + \int_c \phi dz\ \hat{z} \tag{14.8}$$

Alternatively the integral can be done in curvilinear coordinates. In spherical and cylindrical coordinates the differentials takes the form (where θ represents the azimuthal angle):

$$d\mathbf{r} = dr\ \hat{r} + rd\phi\ \hat{\phi} + r\sin(\theta)d\theta\ \hat{\theta} \tag{14.9}$$
$$d\mathbf{r} = ds\ \hat{s} + sd\phi\ \hat{\phi} + dz\ \hat{z} \tag{14.10}$$
$$\tag{14.11}$$

For a vector field $\mathbf{F}(x, y, z)$ a line integral is given by

$$\int \mathbf{F} \cdot d\mathbf{r} \tag{14.12}$$

Some possible $d\mathbf{r}$ values are shown above. Lastly it may be useful to do line integrals using general parametrization. If the path of integration is $\mathbf{r}(t) = (r_x(t), r_y(t), r_z(t))$, then the line integral of a vector field is

$$\int_c \mathbf{F} \cdot \mathbf{r}'(t)dt \tag{14.13}$$

14.3 Common Converging Infinite Series

The geometric series:

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ... = \frac{a}{1-r} \tag{14.14}$$

Other common series:

$$\sum_{n=1}^{N} n = 1 + 2 + ... + n = \frac{n(n+1)}{2} \tag{14.15}$$
$$\sum_{n=1}^{N} n^2 = 1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{14.16}$$

14.4 Dirac Delta Function

Some important properties of the Dirac Delta function are:

$$\int \nabla \cdot (\frac{\hat{r}}{r^2}) d\tau = \begin{cases} 4\pi \\ 0 \end{cases} \quad \text{Depending on whether the integration contains } \mathbf{r} = 0 \tag{14.17}$$

$$\int f(x)\delta(x)dx = f(0) \tag{14.18}$$

$$\int \delta(x)dx = 1 \tag{14.19}$$

$$\frac{1}{2\pi} \int_{-a}^{a} e^{i(\mathbf{r}-\mathbf{r}')t} dt = \delta(\mathbf{r} - \mathbf{r}') \tag{14.20}$$

14.5 Fourier Series

The Fourier series of a periodic function f (x) is given by

$$f(x) = \frac{a_n}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \tag{14.21}$$

Where

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{nt\pi}{L}\right) dt \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{n\pi t}{L}\right) \tag{14.22}$$

14.6 Fourier Transform

The Fourier transform of a function $f(x)$ is given by

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-\omega i t} dt \tag{14.23}$$

The magnitude of the Fourier transform evaluated at a certain frequency k gives the amount that that particular frequency is in a function. Using Fourier transforms it can be shown that any periodic function can be written as a linear combination of sines and cosines.

For a wave packet (i.e. Fourier transform) the width of the packet in k space is equal to inverse of the width in real space.

14.7 Trig Identities

It may be useful to memorize a few of the most common trig identities:

$$\sin 2\theta = \frac{1}{2}\sin\theta\cos\theta \tag{14.24}$$

$$\sin^2\theta + \cos^2\theta = 1 \tag{14.25}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \tag{14.26}$$

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \sin\phi\cos\theta \tag{14.27}$$

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \cos\phi\cos\theta \tag{14.28}$$

14.8 Geometry

STOP! Common GRE Problem 32. *What is the radius of a sphere whose surface drops a vertical distance y for every tangent distance t? In order to solve this problem one needs to refer to figure 14.1 and approximate the t is approximately as shown. From then on its simple to use pythagorean Theorem to solve for the radius.*

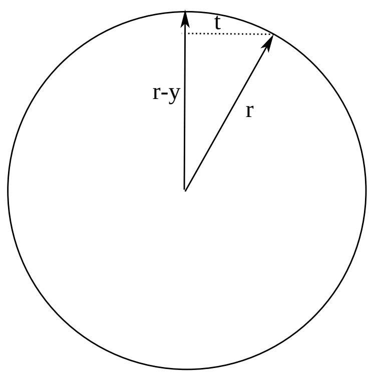


Figure 14.1: Finding the radius of a sphere given the verticle drop per tangent distance

14.9 Logarithmic Graphs

Consider some function $y = Ax^b$

$$\log y = \log Ax^b \tag{14.29}$$

$$= \log A + b\log x \tag{14.30}$$

Hence by plotting the logarithm of y we can extract each of the variables using slopes. If one wanted to plot a semi-log graph then the vertical axis would be $\log A + b\log x$ and the horizontal axis would be x. Alternatively one could plot a log-log graph so that the vertical axis will still be $\log A + b \log x$ but the horizontal axis would be $\log x$. Both of these approaches are best suited for exponential relationships. The three different approaches for $y = 2x^2$ are shown in figure 14.2 The slope of a log log plot (easy to

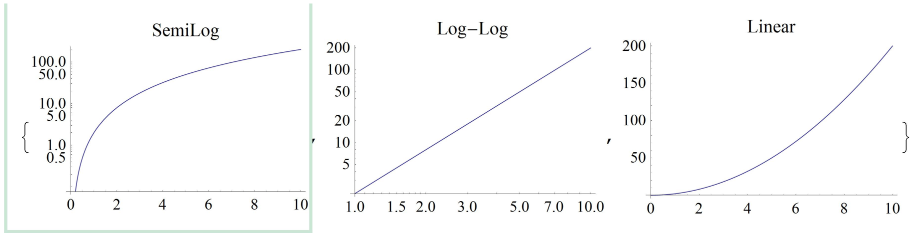


Figure 14.2: Different types of plots for $y = 2x^2$

derive) if y and x are the raw values:

$$slope = \frac{\log(y_2/y_1)}{\log(x_2/x_1)} \tag{14.31}$$

It helps to have some physical intuition about what the slope is on log log plot. If a log log plot is drawn to scale then if we have a relationship of the form $y = Ax^n$ then the line representing the trend will be less then 45^o from the horizontal if $n < 1$ and it would be greater then 45^o from the horizontal if $n > 1$. A loglog plot for a few values of n is shown in figure 14.3

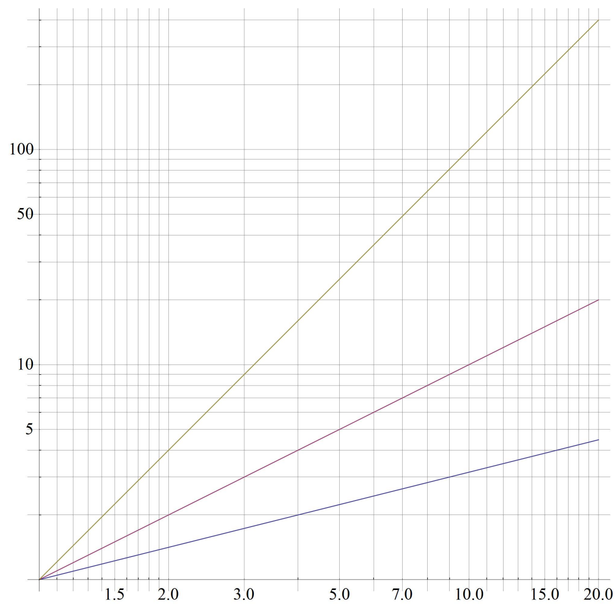


Figure 14.3: Loglog plots of $y = Ax^n$ for different values of n

Chapter 15

Error Analysis

15.1 General Knowledge

Even though accuracy and precision are often used interchangeably, in the scientific method they are distinct:

Def 67. Accuracy : How far off the result is from the true value

Def 68. Precision: How reproducible the result is.

The difference is summarized in figure 15.1 Systematic and statistical uncertainties

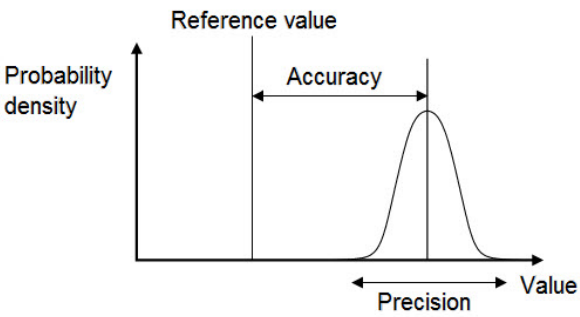


Figure 15.1: Error Analysis

can be differentiated by:

Def 69. Systematic: Uncertainty that cannot be reduced by repeating the experiment over and over again

Def 70. Statistical: Uncertainty that can be reduced by repeating the experiment over and over

15.2 Poisson Distribution

The Poisson distribution describes the results of experiments in which we count events that occur at random but at a definite average rate. The probability of getting the number of counts ν in any definite time interval (μ is just the average number of counts in the definite time interval)

$$P_\mu(\nu) = e^{-\mu}\frac{\mu^\nu}{\nu!} \tag{15.1}$$

The standard deviation from the mean value μ is

$$\sigma_\nu = \sqrt{\mu} \tag{15.2}$$

If this experiment is repeated N times then the standard deviation is

$$\sigma_N = \frac{\sigma}{\sqrt{N}} \tag{15.3}$$

15.3 Propagating Uncertainties

The sum of two or more uncertainties is given by

$$(dx)^2 = (da)^2 + (db)^2 + ... \tag{15.4}$$

The uncertainty of a product of two or more figures with uncertainties is given by

$$\left(\frac{dx}{x}\right)^2 = \left(\frac{da}{a}\right)^2 + \left(\frac{db}{b}\right)^2 + ... \tag{15.5}$$

Given N measurements of the form $x_i \pm \sigma_i$, the weight of measurement i is defined as

$$w_i = \frac{1}{\sigma_i^2} \tag{15.6}$$

With this definition we can define the weighted average as

$$x_{wav} = \frac{\sum_i w_i x_i}{\sum_i w_i} \tag{15.7}$$

The uncertainty in this value is

$$\sigma_{wav} = \frac{1}{\sqrt{\sum_i w_i}} \tag{15.8}$$

Now suppose $w_i = w \; \forall \; i$. In this case

$$x_{wav} = \frac{w\sum_i x_i}{Nw} \tag{15.9}$$

$$= \frac{\sum_i x_i}{N} \tag{15.10}$$

So in the case that the weights are the same the formula reduce to the equation of the mean. Doing the same for the uncertainty gives

$$\sigma_{wav} = \frac{1}{\sqrt{Nw}} \tag{15.11}$$

$$= \frac{1}{1/\sigma\sqrt{N}} \tag{15.12}$$

$$= \frac{\sigma}{N} \tag{15.13}$$

As expected for a uncertainty in N measurements.

Chapter 16

Miscellaneous

16.1 Units

In order to be able to quickly remember formulas it is useful to remember the different units in terms of kg,m,s, and C.

$$N = kgm/s^2$$
$$V = \frac{kgm^2}{Cs}$$
$$J = kgm^2/s^2$$
$$T = \frac{kg}{Cs}$$
$$\Omega = \frac{m^2kg}{sC^2}$$
$$F = C/V = \frac{s^2C^2}{m^2kg}$$
$$H = \frac{m^2kg}{C^2}$$

Further it may be useful to know the units of some common constants:

$$[\epsilon_0] = F/m = \frac{C}{Vm}$$
$$[\mu_0] = \frac{Ns^2}{C^2}$$

16.2 Useful Constants and Formulas

Since the GRE test is a speed test it may be useful to memorize some shortcut formulas and constants. When solving relativistic problems using energy and momentum conservation the product γv is often encountered with the desire to isolate for v or γ. It is straightforward to go ahead isolate for the desired variables but its faster (and safer to use the following formula). Suppose

$$\gamma v = A \tag{16.1}$$

then

$$v = \frac{A}{\sqrt{1 + A^2/c^2}} \qquad \text{and hence} \qquad \gamma = \sqrt{1 + A^2/c^2} \tag{16.2}$$

Some useful constants are listed below. Even though some of these may be given on the GRE its probably not worth the time going to find them.

$$
\begin{aligned}
&h = 4 \times 10^{-15} eVs = 6.6 \times 10^{-34} Js \\
&hc = 2 \times 10^{-25} J \cdot m = 1.24 \times 10^{-6} eVm = 1240 eVnm \\
&e = 1.6 \times 10^{-19} C \\
&N_A = 6 \times 10^{23} \\
&k = 9 \times 10^9 N \cdot m^2/C^2 \\
&\sqrt{2} \approx 1.4 \\
&\tfrac{1}{\sqrt{2}} \approx 0.7 \\
&\sqrt{3} \approx 1.7 \\
&\tfrac{1}{\sqrt{3}} \approx 0.6 \\
&\sin 30^o = 1/2 \\
&\sin 60^o = \sqrt{3}/2 \\
&\cos 30^o = \sqrt{3}/2 \\
&\cos 60^o = 1/2 \\
&\tfrac{1}{6} = 0.17
\end{aligned}
$$

The GRE may require some "common knowledge" regarding several quantities. Some of these quantities are given below

Quantity	*Value*
Nuclear cross section	$\pi(10^{-14})^2 \approx 10^{-28} m$
Air cross section	$10^{-18} m$
Number Density	$10^{25} m^{-3}$
Mean free path of air	10^{-7}

On the GRE it is often required to find the square root of some number that does not have a simple square root. In order to approximate this value quickly one can use binomial theorem. As example suppose we need for take the square root of 10:

$$
\begin{aligned}
\sqrt{10} &= \sqrt{1+9} && (16.3) \\
&= 3\sqrt{1+\frac{1}{9}} && (16.4) \\
&\approx 3\left(1+\frac{1}{18}\right) && (16.5) \\
&\approx 3.167 && (16.6)
\end{aligned}
$$

In this case this method is off by 0.004. Note that this method may be time consuming so its better to estimate square roots without this technique, but if one is stuck on a square root this may be the way to go.

Chapter 17

Reminders

- Insert Jacobians
- In the hydrogen atom, the probability density of being in a position r contains the jacobian, $r^2 \sin(\theta)$
- When expanding a braket don't forget to take the adjoint of the bra vector