

Matrix Problems

Straight Lines

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I. PROBLEM STATEMENT

The base of an equilateral triangle with side $2a$ lies along the y-axis such that the mid-point of the base is at the origin. Find vertices of the triangle.

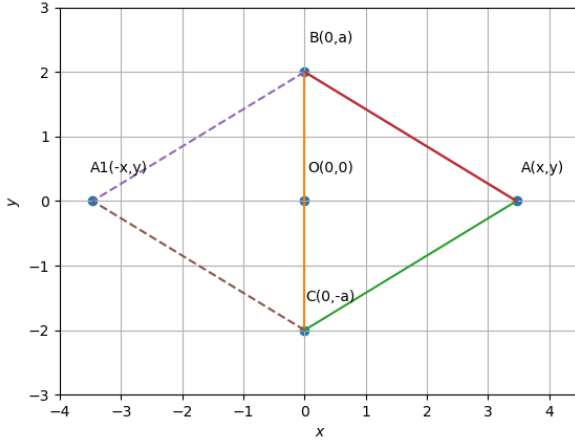


Fig. 1: Equilateral Triangle ABC

II. CONSTRUCTION

B and C are the inputs.

Symbol	Value	Description
B	(0, 2)	Vertex B
C	(0, -2)	Vertex C
A	(x,y)	Vertex A
A1	(x1, y1)	Vertex A

III. SOLUTION

Given the base with $2a$ lies on the y-axis with the mid-point of the base is at origin. The vertices of the two points on y-axis will be

$$\mathbf{B} = \begin{pmatrix} 0 \\ a \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ -a \end{pmatrix} \quad (1)$$

Given $\triangle ABC$ is an equilateral triangle i.e

$$\|\mathbf{A} - \mathbf{B}\| = \|\mathbf{B} - \mathbf{C}\| = \|\mathbf{C} - \mathbf{A}\| = 2a \quad (2)$$

As $AB = AC$, triangle is isosceles and by properties of isosceles triangle, altitude is perpendicular bisector of base.

$$\text{Therefore } \angle AOC = \angle AOB = 90^\circ \text{ and } \|\mathbf{O} - \mathbf{B}\| = \|\mathbf{O} - \mathbf{C}\| = a$$

By Cosine laws,

$$\cos \mathbf{B} = \cos \mathbf{C} = a * \frac{1}{2a} = \frac{1}{2} \quad (3)$$

$$\angle B = \angle C = \arccos \frac{1}{2} = 60^\circ \quad (4)$$

$$\angle A = 180^\circ - (60^\circ * 2) = 60^\circ \quad (5)$$

Therefore, the equilateral triangle have all internal angles equal to 60°

$$(\mathbf{x} - \mathbf{B})^\top (\mathbf{x} - \mathbf{C}) = \|\mathbf{x} - \mathbf{B}\| \cdot \|\mathbf{x} - \mathbf{C}\| \cdot \cos \theta \quad (6)$$

$$(\mathbf{x}^\top \cdot \mathbf{x}) - (\mathbf{x}^\top \cdot \mathbf{C}) - (\mathbf{B}^\top \cdot \mathbf{x}) + (\mathbf{B}^\top \cdot \mathbf{C}) = 2a \cdot 2a \cos 60^\circ \quad (7)$$

$$\|\mathbf{x}\|^2 - \mathbf{x}^\top (\mathbf{B} + \mathbf{C}) - \mathbf{B}^\top \cdot \mathbf{C} = 2a \cdot 2a \cdot \frac{1}{2} \quad (8)$$

$$\|\mathbf{x}\|^2 - \mathbf{x}^\top (0) - \begin{pmatrix} 0 \\ a \end{pmatrix} \begin{pmatrix} 0 & -a \end{pmatrix} = 4a^2 \quad (9)$$

$$\|\mathbf{x}\|^2 + a^2 = 4a^2 \quad (10)$$

$$\|\mathbf{x}\|^2 = 3a^2 \quad (11)$$

Considering, the line equation of AB

$$\|\mathbf{x} - \mathbf{B}\|^2 = 4a^2 \quad (12)$$

$$(\mathbf{x} - \mathbf{B})^\top \cdot (\mathbf{x} - \mathbf{B}) = 4a^2 \quad (13)$$

$$\|\mathbf{x}\|^2 - 2 \cdot \mathbf{x}^\top \mathbf{B} + \|\mathbf{B}\|^2 = 4a^2 \quad (14)$$

$$3a^2 - 2 \cdot \mathbf{x}^\top \mathbf{B} + a^2 = 4a^2 \quad (15)$$

$$\mathbf{x}^\top \mathbf{B} = 0 \quad (16)$$

Since we can write,

$$\mathbf{B} = a \cdot \mathbf{e}_2 \quad (17)$$

$$\mathbf{x}^\top \cdot a \cdot \mathbf{e}_2 = 0 \quad (18)$$

$$\mathbf{x}^\top \cdot \mathbf{e}_2 = 0 \quad (19)$$

$$\mathbf{x} = \lambda \mathbf{e}_1 \quad (20)$$

From this its clearly concluded that third vertex will lie on x-axis. From the equation (11)

$$\mathbf{x} = \sqrt{3}a \quad (21)$$

Hence,the coordinates of the vertices of triangle are

$$\mathbf{A} = \begin{pmatrix} \pm\sqrt{3}a \\ 0 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ a \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ -a \end{pmatrix} \quad (22)$$

Get Python Code for image from

<https://github.com/ManojChavva/FWC/blob/main/Matrix/line/code-py/triangle.py>

Get LaTeX code from

<https://github.com/ManojChavva/FWC/blob/main/Matrix/line/line.tex>