Matrix Problems
Straight Lines

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I. Problem Statement

The base of an equilateral triangle with side 2a lies along the y-axis such that the mid-point of the base is at the origin. Find vertices of the triangle.

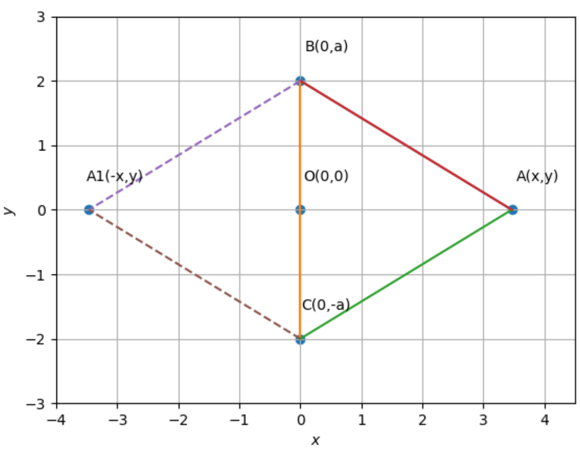

Fig. 1: Equilateral Triangle ABC

II. Construction

B and C are the inputs.

Symbol	Value	Description
B	(0, 2)	Vertex B
C	(0, -2)	Vertex C
A	(x,y)	Vertex A
A1	(x1, y1)	Vertex A

III. Solution

Given the base with 2a is lies on the y-axis with the mid-point of the base is at origin. The vertices of the two points on y-axis will be

$$\mathbf{B} = \begin{pmatrix} 0 \\ a \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ -a \end{pmatrix} \tag{1}$$

Given ΔABC is an equilateral triangle i.e

$$\|\mathbf{A}-\mathbf{B}\| = \|\mathbf{B}-\mathbf{C}\| = \|\mathbf{C}-\mathbf{A}\| = 2a \tag{2}$$

As AB = AC, triangle is isoceles and by properties of isoceles triangle, altitude is perpendicular bisector of base.

Therefore $\angle$AOC = $\angle$AOB = 90^0 and $\|\mathbf{O}-\mathbf{B}\| = \|\mathbf{O}-\mathbf{C}\| = \frac{a}{2}$

By Cosine laws,

$$\cos\mathbf{B} = \cos\mathbf{C} = \frac{a}{2} * \frac{1}{a} = \frac{1}{2} \tag{3}$$

$$\angle B = \angle C = \arccos\frac{1}{2} = 60^0 \tag{4}$$

$$\angle A = 180^0 - (60^0 * 2) = 60^0 \tag{5}$$

Therefore, the equilaterial triangle have all internal angles eaqual to 60^0

$$(\mathbf{x}-\mathbf{B})^\top(\mathbf{x}-\mathbf{C}) = \|\mathbf{x}-\mathbf{B}\|\cdot\|\mathbf{x}-\mathbf{C}\|\cdot\cos\theta \tag{6}$$

$$\left(\mathbf{x}^\top\cdot\mathbf{x}\right) - \left(\mathbf{x}^\top\cdot\mathbf{C}\right) - \left(\mathbf{B}^\top\cdot\mathbf{x}\right) - \left(\mathbf{B}^\top\cdot\mathbf{C}\right) = 2a\cdot 2a\cos 60^0 \tag{7}$$

$$\|\mathbf{x}\|^2 - \mathbf{x}^\top(\mathbf{B}+\mathbf{C}) - \mathbf{B}^\top\cdot\mathbf{C} = 2a\cdot 2a\cdot\frac{1}{2} \tag{8}$$

$$\|\mathbf{x}\|^2 - \mathbf{x}^\top(0) - \begin{pmatrix} 0 \\ a \end{pmatrix}\begin{pmatrix} 0 & -a \end{pmatrix} = 4a^2 \tag{9}$$

$$\|\mathbf{x}\|^2 + a^2 = 4a^2 \tag{10}$$

$$\|\mathbf{x}\|^2 = 3a^2 \tag{11}$$

Considering, the line equation of $\mathbf{AB}$

$$\|\mathbf{x}-\mathbf{B}\|^2 = 4a^2 \tag{12}$$

$$(\mathbf{x}-\mathbf{B})^{\top}\cdot(\mathbf{x}-\mathbf{B})=4a^2 \tag{13}$$

$$\|\mathbf{x}\|^2-2\cdot\mathbf{x}^{\top}\mathbf{B}+\|\mathbf{B}\|^2=4a^2 \tag{14}$$

$$3a^2-2\cdot\mathbf{x}^{\top}\mathbf{B}+a^2=4a^2 \tag{15}$$

$$\mathbf{x}^{\top}\mathbf{B}=0 \tag{16}$$

Since we can write,

$$\mathbf{B}=a\cdot\mathbf{e}_2 \tag{17}$$

$$\mathbf{x}^{\top}\cdot a\cdot\mathbf{e}_2=0 \tag{18}$$

$$\mathbf{x}^{\top}\cdot\mathbf{e}_2=0 \tag{19}$$

$$\mathbf{x}=\lambda\mathbf{e}_1 \tag{20}$$

From this its clearly concluded that third vertex will lie on x-axis. From the equation (11)

$$\mathbf{x}=\sqrt{3}a \tag{21}$$

Hence,the coordinates of the vertices of triangle are

$$\mathbf{A}=\begin{pmatrix}\pm\sqrt{3}a\\0\end{pmatrix}$$

$$\mathbf{B}=\begin{pmatrix}0\\a\end{pmatrix},\mathbf{C}=\begin{pmatrix}0\\-a\end{pmatrix} \tag{22}$$

Get Python Code for image from

https://github.com/ManojChavva/FWC/blob/main/Matrix/line/code-py/triangle.py

Get LaTex code from

https://github.com/ManojChavva/FWC/blob/main/Matrix/line/line.tex