Unit Vector Perpendicular

1 12th Maths - Chapter 10

This is Problem-2 from Exercise 10.4

1. Find a unit vector perpendicular to each of a vector $\bar{a} + \bar{b}$ and $\bar{a} - \bar{b}$ where $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$

2 Solution

A unit vector perpendicular

$$(\mathbf{a} + \mathbf{b})^\top \mathbf{x} = 0 \tag{1}$$

$$(\mathbf{a} - \mathbf{b})^\top \mathbf{x} = 0 \tag{2}$$

$$\begin{pmatrix} (\mathbf{a} + \mathbf{b})^\top \\ (\mathbf{a} - \mathbf{b})^\top \end{pmatrix} \mathbf{x} = 0 \tag{3}$$

$$\begin{pmatrix} \mathbf{a} + \mathbf{b} & \mathbf{a} - \mathbf{b} \end{pmatrix}^\top \mathbf{x} = 0 \tag{4}$$

Here.

$$\mathbf{a} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \tag{5}$$

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} \tag{6}$$

$$\mathbf{a} - \mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \tag{7}$$

Now using the formula substituing (7) and (8) in (4) and equating,

$$\begin{pmatrix} \mathbf{a+b} & \mathbf{a-b} \end{pmatrix}^\top \mathbf{x} = 0 \tag{8}$$

$$\begin{pmatrix} 4 & 2 \\ 4 & 0 \\ 0 & 4 \end{pmatrix}^\top \mathbf{x} = 0 \tag{9}$$

$$\begin{pmatrix} 4 & 4 & 0 \\ 2 & 0 & 4 \end{pmatrix} \xrightarrow{R_1 = \frac{R_1}{4}} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 4 \end{pmatrix} \mathbf{x} = 0 \tag{10}$$

$$\xrightarrow{R_2 = \frac{R_2}{2}} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \mathbf{x} = 0 \tag{11}$$

$$\xrightarrow{R_2 = R_1 - R_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \end{pmatrix} \mathbf{x} = 0 \tag{12}$$

$$\xrightarrow{R_2 = \frac{R_2}{-1}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \end{pmatrix} \mathbf{x} = 0 \tag{13}$$

$$\xrightarrow{R_1 = R_1 - R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \end{pmatrix} \mathbf{x} = 0 \tag{14}$$

From (14), we get two equations which is

$$x_1 + 2x_3 = 0 \tag{15}$$
$$x_2 - 2x_3 = 0 \tag{16}$$

$$x_1 = \begin{pmatrix} -2x_3 \\ 2x_3 \\ x_3 \end{pmatrix} \tag{17}$$

$$= x_3 \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \tag{18}$$

$$\vec{\mathbf{c}} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \tag{19}$$

The unit vector

$$\vec{\mathbf{c}} = \frac{1}{\|\mathbf{c}\|}\mathbf{c} \tag{20}$$

$$\|\mathbf{c}\| = \sqrt{c^\top c} \tag{21}$$

$$= \sqrt{\begin{pmatrix} -2 & 2 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}} \tag{22}$$

$$= \sqrt{4+4+1} \tag{23}$$

$$= 3 \tag{24}$$

$$\vec{\mathbf{c}} = \frac{1}{\|\mathbf{c}\|}\mathbf{c} \tag{25}$$

$$= \frac{1}{3}\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \tag{26}$$

$$\vec{\mathbf{c}} = \begin{pmatrix} \frac{-2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} \tag{27}$$