Unit Vector Perpendicular

1 12^{th} Maths - Chapter 10

This is Problem-2 from Exercise 10.4

1. Find a unit vector perpendicular to each of a vector $\bar{a}+\bar{b}$ and $\bar{a}-\bar{b}$ where $\overrightarrow{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\overrightarrow{b}=\hat{i}+2\hat{j}-2\hat{k}$

2 Solution

A unit vector perpendicular

$$(\mathbf{a}+\mathbf{b})^\top \mathbf{x}=0 \tag{1}$$

$$(\mathbf{a}-\mathbf{b})^\top \mathbf{x}=0 \tag{2}$$

$$\begin{pmatrix}(\mathbf{a}+\mathbf{b})^\top \\ (\mathbf{a}-\mathbf{b})^\top\end{pmatrix}\mathbf{x}=0 \tag{3}$$

$$\begin{pmatrix}\mathbf{a}+\mathbf{b} & \mathbf{a}-\mathbf{b}\end{pmatrix}^\top \mathbf{x}=0 \tag{4}$$

Here.

$$\mathbf{a}=\begin{pmatrix}3\\2\\2\end{pmatrix},\mathbf{b}=\begin{pmatrix}1\\2\\-2\end{pmatrix} \tag{5}$$

$$\mathbf{a}+\mathbf{b}=\begin{pmatrix}3\\2\\2\end{pmatrix}+\begin{pmatrix}1\\2\\-2\end{pmatrix}=\begin{pmatrix}4\\4\\0\end{pmatrix} \tag{6}$$

$$\mathbf{a}-\mathbf{b}=\begin{pmatrix}3\\2\\2\end{pmatrix}-\begin{pmatrix}1\\2\\-2\end{pmatrix}=\begin{pmatrix}2\\0\\4\end{pmatrix} \tag{7}$$

Now using the formula substituing (7) and (8) in (4) and equating,

$$\begin{pmatrix}\mathbf{a}+\mathbf{b} & \mathbf{a}-\mathbf{b}\end{pmatrix}^\top \mathbf{x} = 0 \tag{8}$$

$$\begin{pmatrix}4 & 2\\ 4 & 0\\ 0 & 4\end{pmatrix}^\top \mathbf{x} = 0 \tag{9}$$

$$\begin{pmatrix}4 & 4 & 0\\ 2 & 0 & 4\end{pmatrix} \xleftrightarrow{R_1=\frac{R_1}{4}} \begin{pmatrix}1 & 1 & 0\\ 2 & 0 & 4\end{pmatrix}\mathbf{x} = 0 \tag{10}$$

$$\xleftrightarrow{R_2=\frac{R_2}{2}} \begin{pmatrix}1 & 1 & 0\\ 1 & 0 & 2\end{pmatrix}\mathbf{x} = 0 \tag{11}$$

$$\xleftrightarrow{R_2=R_1-R_2} \begin{pmatrix}1 & 1 & 0\\ 0 & -1 & 2\end{pmatrix}\mathbf{x} = 0 \tag{12}$$

$$\xleftrightarrow{R_2=\frac{R_2}{-1}} \begin{pmatrix}1 & 1 & 0\\ 0 & 1 & -2\end{pmatrix}\mathbf{x} = 0 \tag{13}$$

$$\xleftrightarrow{R_1=R_1-R_2} \begin{pmatrix}1 & 0 & 2\\ 0 & 1 & -2\end{pmatrix}\mathbf{x} = 0 \tag{14}$$

From (15), we get two equations which is $x_1+2x_3=0$ and $x_2-2x_3=0$

$$\mathbf{x}_1 + 2\mathbf{x}_3 = 0 \tag{15}$$

$$\mathbf{x}_1 - 2\mathbf{x}_3 = 0 \tag{16}$$

$$x_1 = \begin{pmatrix}-2x_3\\ 2x_3\\ x_3\end{pmatrix} \tag{17}$$

$$= x_3\begin{pmatrix}-2\\ 2\\ 1\end{pmatrix} \tag{18}$$

$$\overrightarrow{\mathbf{c}} = \begin{pmatrix}-2\\ 2\\ 1\end{pmatrix} \tag{19}$$

The unit vector

$$
\begin{aligned}
\overrightarrow{\mathbf{c}} &= \frac{1}{\|c\|}\mathbf{c} && (20)\\
\|c\| &= \sqrt{c^\top c} && (21)\\
&= \sqrt{\begin{pmatrix} -2 & 2 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}} && (22)\\
&= \sqrt{4+4+1} && (23)\\
&= 3 && (24)\\
\overrightarrow{\mathbf{c}} &= \frac{1}{\|c\|}\mathbf{c} && (25)\\
&= \frac{1}{3}\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} && (26)\\
\overrightarrow{\mathbf{c}} &= \begin{pmatrix} \frac{-2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} && (27)
\end{aligned}
$$