Digital Communication

Through Simulations

Manoj Chavva

Contents

Chapter 1

Two Dice

1.1 Sum of Independant Random Variables

Two dice, one red and one grey, are thrown at the same time. The event defined by the sum of the two numbers appearing on the top of the dice can have 11 possible outcomes 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12. A student argues that each of these outcomes has a probability $\frac{1}{11}$. Do you agree with this argument? Justify your answer.

1.1.1 *The Uniform Distribution:* Let $X_i \in \{1,2,3,4,5,6\}, i = 1,2$, be the random variables representing the outcome for each die. Assuming the dice to be fair, the probability mass function (pmf) is expressed as

$$p_{X_i}(n) = \Pr\left(X_i = n\right) = \begin{cases} \frac{1}{6} & 1 \le n \le 6 \\ 0 & otherwise \end{cases} \tag{1.1.1.1}$$

The desired outcome is

$$X = X_1 + X_2, \tag{1.1.1.2}$$

$$\implies X \in \{1,2,\dots,12\} \tag{1.1.1.3}$$

The objective is to show that

$$p_X(n) \neq \frac{1}{11} \tag{1.1.1.4}$$

1.1.2 *Convolution:* From (1.1.1.2),

$$p_X(n) = \Pr\left(X_1 + X_2 = n\right) = \Pr\left(X_1 = n - X_2\right) \tag{1.1.2.1}$$

$$= \sum_k \Pr\left(X_1 = n - k | X_2 = k\right) p_{X_2}(k) \tag{1.1.2.2}$$

after unconditioning. $\because$ X_1 and X_2 are independent,

$$\Pr\left(X_1 = n - k | X_2 = k\right)$$

$$= \Pr\left(X_1 = n - k\right) = p_{X_1}(n-k) \tag{1.1.2.3}$$

From (1.1.2.2) and (1.1.2.3),

$$p_X(n) = \sum_k p_{X_1}(n-k) p_{X_2}(k) = p_{X_1}(n) * p_{X_2}(n) \tag{1.1.2.4}$$

where $*$ denotes the convolution operation. Substituting from (1.1.1.1) in (1.1.2.4),

$$p_X(n) = \frac{1}{6}\sum_{k=1}^{6} p_{X_1}(n-k) = \frac{1}{6}\sum_{k=n-6}^{n-1} p_{X_1}(k) \tag{1.1.2.5}$$

$$\because p_{X_1}(k) = 0, \quad k \le 1, k \ge 6. \tag{1.1.2.6}$$

From (1.1.2.5),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{1}{6}\sum_{k=1}^{n-1} p_{X_1}(k) & 1 \le n-1 \le 6 \\ \frac{1}{6}\sum_{k=n-6}^{6} p_{X_1}(k) & 1 < n-6 \le 6 \\ 0 & n > 12 \end{cases} \tag{1.1.2.7}$$

Substituting from (1.1.1.1) in (1.1.2.7),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{n-1}{36} & 2 \le n \le 7 \\ \frac{13-n}{36} & 7 < n \le 12 \\ 0 & n > 12 \end{cases} \tag{1.1.2.8}$$

satisfying (1.1.1.4).

1.1.3 *The Z-transform:* The Z-transform of $p_X(n)$ is defined as

$$P_X(z) = \sum_{n=-\infty}^{\infty} p_X(n) z^{-n}, \quad z \in \mathbb{C} \tag{1.1.3.1}$$

From (1.1.1.1) and (1.1.3.1),

$$P_{X_1}(z) = P_{X_2}(z) = \frac{1}{6}\sum_{n=1}^{6} z^{-n} \tag{1.1.3.2}$$

$$= \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}, \quad |z| > 1 \tag{1.1.3.3}$$

upon summing up the geometric progression.

$$\because p_X(n) = p_{X_1}(n) * p_{X_2}(n), \tag{1.1.3.4}$$

$$P_X(z) = P_{X_1}(z) P_{X_2}(z) \tag{1.1.3.5}$$

The above property follows from Fourier analysis and is fundamental to signal processing. From (1.1.3.3) and (1.1.3.5),

$$P_X(z) = \left\{\frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}\right\}^2 \tag{1.1.3.6}$$

$$= \frac{1}{36}\frac{z^{-2}\left(1 - 2z^{-6} + z^{-12}\right)}{\left(1 - z^{-1}\right)^2} \tag{1.1.3.7}$$

Using the fact that

$$p_X(n-k) \overset{\mathcal{H}}{\longleftrightarrow} ZP_X(z)z^{-k}, \tag{1.1.3.8}$$

$$nu(n) \overset{\mathcal{H}}{\longleftrightarrow} Z\frac{z^{-1}}{(1-z^{-1})^2} \tag{1.1.3.9}$$

after some algebra, it can be shown that

$$\frac{1}{36}\left[(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \overset{\mathcal{H}}{\longleftrightarrow} Z\frac{1}{36}\frac{z^{-2}\left(1-2z^{-6}+z^{-12}\right)}{(1-z^{-1})^2} \tag{1.1.3.10}$$

where

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{1.1.3.11}$$

From (1.1.3.1), (1.1.3.7) and (1.1.3.10)

$$p_X(n) = \frac{1}{36}\left[(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \tag{1.1.3.12}$$

which is the same as (1.1.2.8). Note that (1.1.2.8) can be obtained from (1.1.3.10) using contour integration as well.

1.1.4 The experiment of rolling the dice was simulated using Python for 10000 samples. These were generated using Python libraries for uniform distribution. The frequencies for each outcome were then used to compute the resulting pmf, which is plotted in Figure 1.1.4.1. The theoretical pmf obtained in (1.1.2.8) is plotted for comparison.

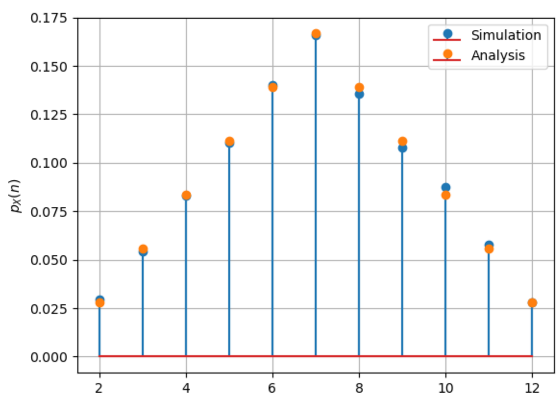

Figure 1.1.4.1: Plot of $p_X(n)$. Simulations are close to the analysis.

1.1.5 The python code is available in

```
chapters/codes/dice.py
```

Chapter 2

Random Numbers

2.1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

2.1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
chapter2/codes/exrand.c
chapter2/codes/coeffs.h
```

2.1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{2.1.2.1}$$

```
chapter2/codes/cdf_plot.py
```

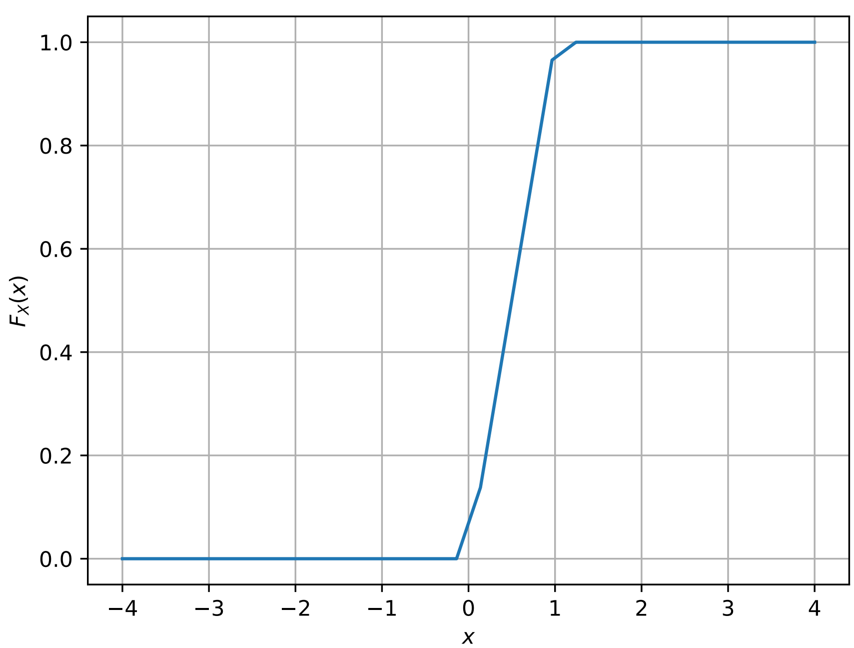

Figure 2.1.2.1: The CDF of U

2.1.3 Find a theoretical expression for $F_U(x)$.
Solution:

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 0 \end{cases} \tag{2.1.3.1}$$

2.1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{2.1.4.1}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{2.1.4.2}$$

Write a C program to find the mean and variance of U.
Solution: The following code prints the mean and variance of U

```
chapter2/codes/mv.c
```

The output of the program is

```
Uniform stats:
Mean: 0.500007
Variance: 0.083301
```

2.1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{2.1.5.1}$$

Solution: For a random variable X, the mean μ_X is given by

$$\mu_X = E[X] = \int_{-\infty}^{\infty} x dF_U(x) \tag{2.1.5.2}$$

$$\sigma_X^2 = E\left[X^2\right] - \mu_X^2 = \int_{-\infty}^{\infty} x^2 dF_U(x) - \mu_X^2 \tag{2.1.5.3}$$

Variance σ_X^2 is given by
Substituting the CDF of U from (2.1.3.3) in (2.1.5.2) and (2.1.5.3), we get

$$\textbf{Mean} = \mu_U \tag{2.1.5.4}$$

$$\text{E}[X] = \int_a^b x \cdot \frac{1}{b-a} dx \tag{2.1.5.5}$$

$$Here\ a = 0, b = 1 \tag{2.1.5.6}$$

$$\mu = \text{E}[X] = \frac{1}{2} = 0.5 \tag{2.1.5.7}$$

$$\text{E}[X^2] = \int_a^b x^2 \cdot \frac{1}{b-a} dx = \frac{b^3 - a^3}{3} \cdot \frac{1}{b-a} \tag{2.1.5.8}$$

$$\text{Variance} \tag{2.1.5.9}$$

$$\sigma^2 = E\left(X^2\right) - [E(X)]^2 = \frac{(a-b)^2}{12} \tag{2.1.5.10}$$

$$\sigma^2 = \frac{1}{12} = 0.08 \tag{2.1.5.11}$$

Hence, the output values of program and theory are same.

2.2 Central Limit Theorem

2.2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.2.1.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Use the following code and it will generate gau.dat file with the required Random variables the C program.

```
chapter2/codes/rv.c
```

2.2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Let X be a random variable (either continuous or discrete), then the CDF of X has the following properties

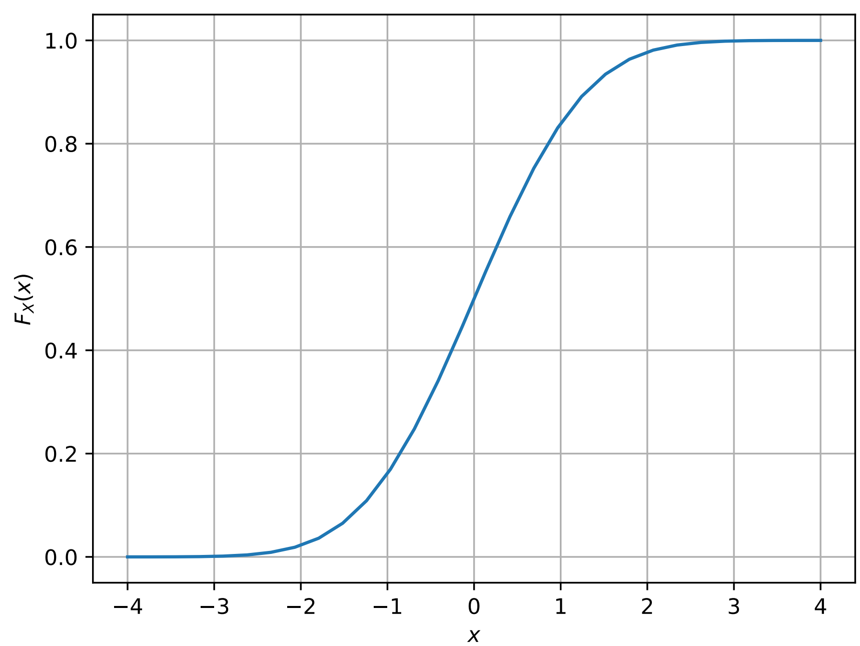

Figure 2.2.2.1: The CDF of X

The CDF is a non-decreasing

The maximum of the CDF is when

$$x = \infty : F_X(\infty) = 1 \tag{2.2.2.1}$$

The minimum of the CDF is when

$$x = -\infty : F_X(-\infty) = 0 \tag{2.2.2.2}$$

If the CDF F_X is continuous at any $a \leq x \leq b$, then

$$\Pr\left[a \leq X \leq b\right] = F_X(b) - F_X(a) \tag{2.2.2.3}$$

For any random variable X (discrete or continuous), $P[X = b]$ is

$$\Pr\left[X \;=\; b\right] = \begin{cases} F_X(b) - F_X(b-) & \text{if } F_X \text{ is discontinuous at x = b} \\ 0 & otherwise \end{cases} \tag{2.2.2.4}$$

2.2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2.3.1}$$

What properties does the PDF have? **Solution:**

```
chapter2/codes/rv.c
```

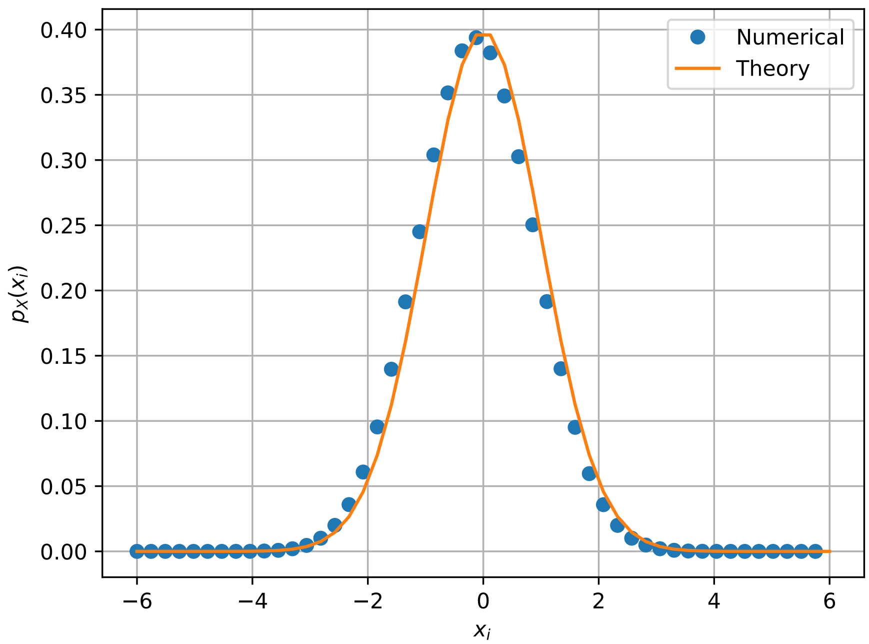

Figure 2.2.3.1: The PDF of X

The properties of PDF are

$$f_X(x) \geq 0 \text{for all} X \in \mathbb{R} \tag{2.2.3.2}$$

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1 \tag{2.2.3.3}$$

2.2.4 Find the mean and variance of X by writing a C program.
Solution: The following code prints the mean and variance of X

```
chapter2/codes/mean.c
```

The output of the program is

```
Gaussian stats:
Mean: 0.000294
Variance: 0.999561
```

2.2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.2.5.1}$$

repeat the above exercise theoretically.
Solution: The mean of gievn PDF is given by $E\left[X\right]$,

$$E\left[X\right] = \mu_X = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.2.5.2}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} xe^{-\frac{x^2}{2}} dx \tag{2.2.5.3}$$

$$= 0 \tag{2.2.5.4}$$

$$\mu_X = 0 \tag{2.2.5.5}$$

Variance is given by

$$\sigma^2 = E\left(X\right)^2 - E^2\left(X\right) \tag{2.2.5.6}$$

Substituting μ_X and the PDF

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} x^2 e^{-\frac{x^2}{2}} dx \tag{2.2.5.7}$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} \sqrt{2u} e^{-u} du \quad \left(Let \frac{x^2}{2} = u\right) \tag{2.2.5.8}$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-u} u^{\frac{3}{2}-1} du \quad \left(Let \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right) \tag{2.2.5.9}$$

$$= \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) \tag{2.2.5.10}$$

$$= \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) \tag{2.2.5.11}$$

$$= 1 \tag{2.2.5.12}$$

2.3 From Uniform to Other

2.3.1 Generate samples of

$$V = -2 \ln\left(1 - U\right) \tag{2.3.1.1}$$

and plot its CDF.
Solution:

```
chapter2/2_3_cdf.py
```

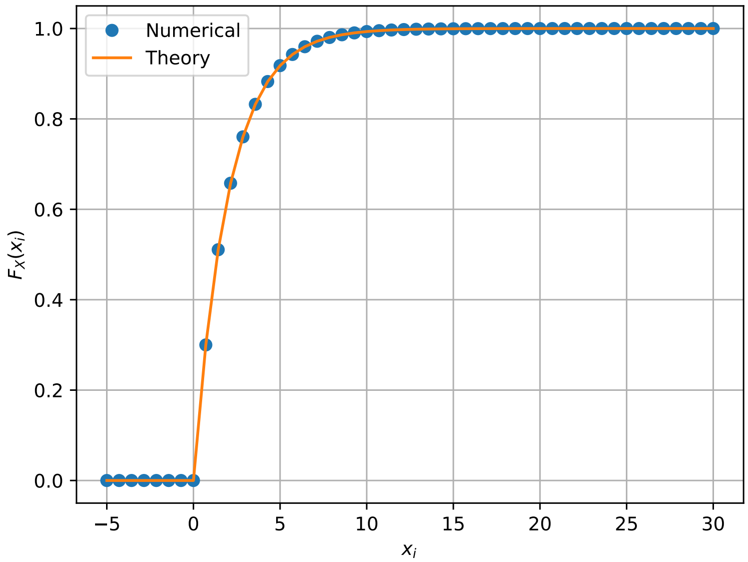

Figure 2.3.1.1: The CDF of V

2.3.2 Find a theoretical expression for $F_V(x)$.

$$F_V(x) = P(V \le x) \tag{2.3.2.1}$$
$$= P(-2\ln(1-U) \le x) \tag{2.3.2.2}$$
$$= P(U \le 1 - e^{\frac{-x}{2}}) \tag{2.3.2.3}$$
$$= F_U(1 - e^{\frac{-x}{2}}) \tag{2.3.2.4}$$

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \tag{2.3.2.5}$$

Substituting the above in (2.3.2.4),

$$F_U\left(1 - e^{-\frac{x}{2}}\right) = \begin{cases} 0 & 1 - e^{-\frac{x}{2}} < 0 \\ 1 - e^{-\frac{x}{2}} & 0 \le 1 - e^{-\frac{x}{2}} \le 1 \\ 1 & 1 - e^{-\frac{x}{2}} > 1 \end{cases} \tag{2.3.2.6}$$

After some algebra, the above conditions yield

$$F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\frac{x}{2}} & x \ge 0 \end{cases} \tag{2.3.2.7}$$

2.4 Triangular Distribution

2.4.1 Generate

$$T = U_1 + U_2 \tag{2.4.1.1}$$

Solution: Download the following files and execute the C program.

```
chapter2/codes/tri.c
```

2.4.2 Find the CDF of T.

Solution:

```
chapter2/codes/tcdf.py
```

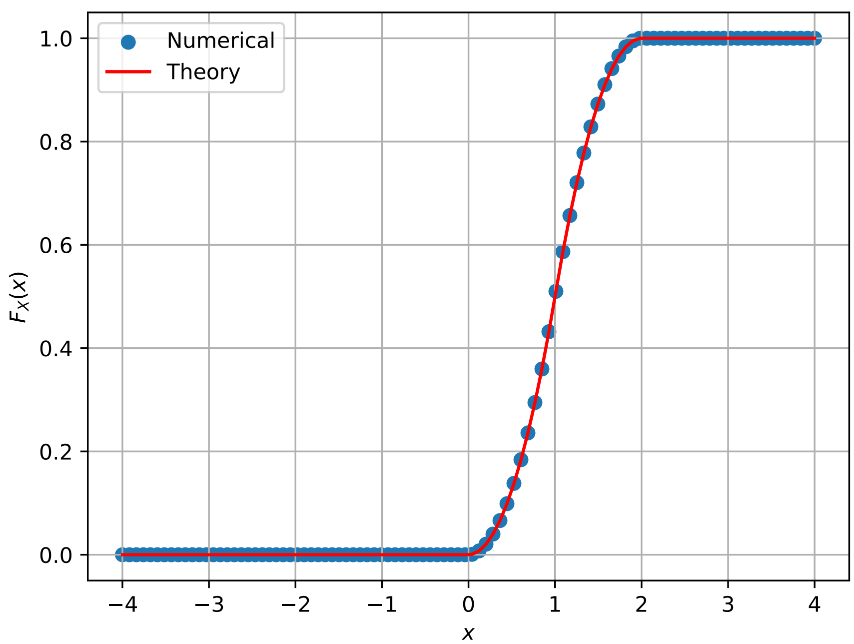

Figure 2.4.2.1: The CDF of T

2.4.3 Find the PDF of T.

```
chapter2/codes/tdpf.py
```

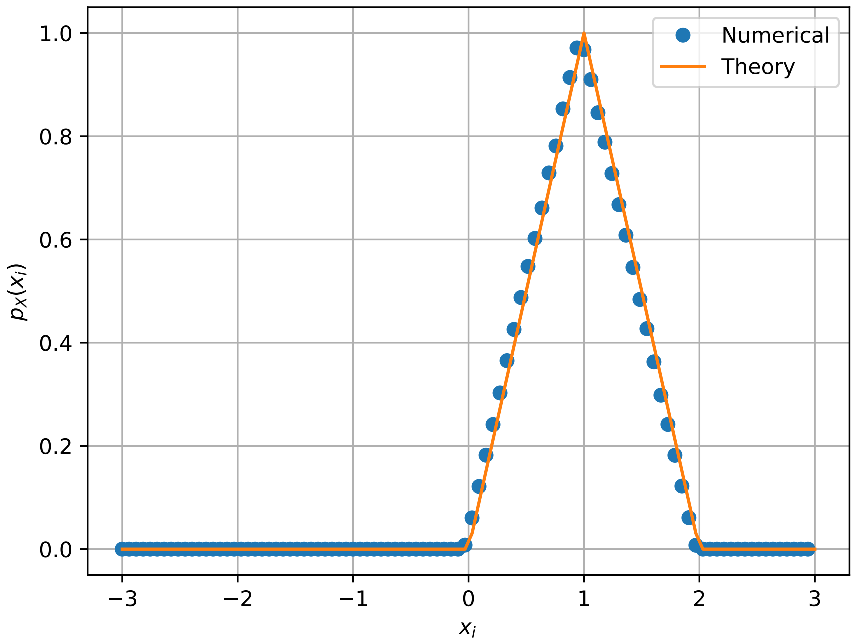

Figure 2.4.3.1: The PDF of T

2.4.4 Find the theoretical expressions for the PDF and CDF of T.
Solution:
CDF $F_U(x)$ is

$$F_T(x) = \begin{cases} 0 & x \le a \\ \frac{(x-a)^2}{(b-a)(c-a)} & a < x \le c \\ 1 - \frac{(b-x)^2}{(b-a)(c-a)} & c < x \le b \\ 1 & x > b \end{cases} \tag{2.4.4.1}$$

PDF $p_T(x)$

$$p_T(x) = \frac{d}{dx} F_T(x) \tag{2.4.4.2}$$

$$p_T(x) = \begin{cases} 0 & x \le a \\ \frac{2(x-a)}{(b-a)(c-a)} & a < x \le c \\ \frac{2(b-x)}{(b-a)(c-a)} & c < x \le b \\ 0 & x > b \end{cases} \tag{2.4.4.3}$$

2.4.5 Verify your results through a plot.
Solution: Refer the Fig. 2.4.2.1 and Fig. 2.4.3.1

Chapter 3

Maximum Likelihood Detection: BPSK

3.1 Maximum Likelihood

3.1.1 Generate equiprobable $X \in \{1, -1\}$.
Solution: Refer the below code section,

```
chapter3/codes/eqi_prob.py
```

3.1.2 Generate

$$Y = AX + N, \tag{3.1.2.1}$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.
Solution: Refer the below code section,

```
chapter3/codes/Y_gau.py
```

3.1.3 Plot Y using a scatter plot.
Solution:

```
chapter3/codes/scatter.py
```

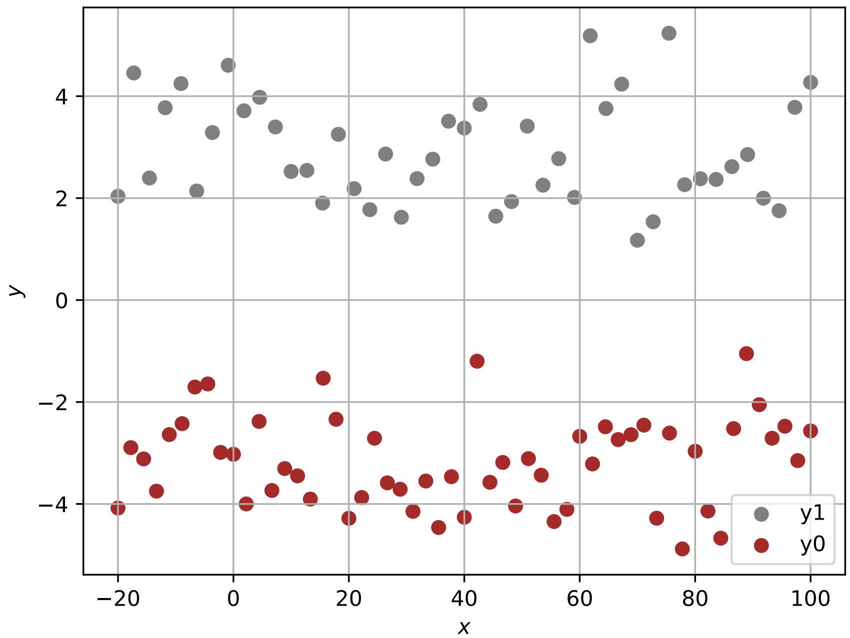

Figure 3.1.3.1: Scatter plot of Y

3.1.4 Guess how to estimate X from Y.

Solution: According to decision rule

$$P(Y > y)y \underset{-1}{\overset{1}{\gtrless}} 0 \tag{3.1.4.1}$$

3.1.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{3.1.5.1}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1|X = -1\right) \tag{3.1.5.2}$$

Solution: using decision rule in

$$\Pr\left(\hat{X} = -1|X = 1\right) = \Pr\left(Y < 0|X = 1\right) \tag{3.1.5.3}$$
$$= \Pr\left(AX + N < 0|X = 1\right) \tag{3.1.5.4}$$
$$= \Pr\left(A + N < 0\right) \tag{3.1.5.5}$$
$$= \Pr\left(N < -A\right) \tag{3.1.5.6}$$
$$= \Pr\left(N > A\right) \tag{3.1.5.7}$$

$$\Pr\left(\hat{X} = 1|X = -1\right) = \Pr\left(Y > 0|X = -1\right) \tag{3.1.5.8}$$
$$= \Pr\left(N > A\right) \tag{3.1.5.9}$$

$$\Pr\left(N > A\right) = \Pr\left(\frac{N-0}{1} > \frac{A-0}{1}\right)$$
$$= Q(\frac{A-0}{1}) = Q(A)$$
$$P_{e|0} = P_{e|1} = Q(A) = \Pr\left(N > A\right) \tag{3.1.5.10}$$

3.1.6 Find P_e assuming that X has equiprobable symbols.
Solution:

$$P_e = \Pr\left(X = 1\right) P_{e|1} + \Pr\left(X = -1\right) P_{e|0} \tag{3.1.6.1}$$

Since X is equiprobable

(3.1.6.2)

$$P_e = \frac{1}{2}P_{e|1} + \frac{1}{2}P_{e|0} \tag{3.1.6.3}$$

Substituting

$$P_e = \Pr\left(N > A\right) \tag{3.1.6.4}$$

Given a random varible $X \sim \mathcal{N}\left(0, 1\right)$ the Q-function is defined as

$$Q(x) = \Pr\left(X > x\right) \tag{3.1.6.5}$$
$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp\left(-\frac{u^2}{2}\right) du. \tag{3.1.6.6}$$

(3.1.6.7)

Using the Q-function, P_e is rewritten as

$$P_e = Q(A) \tag{3.1.6.8}$$

3.1.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.
Solution:

```
chapter3/codes/bpsk_pe.py
```

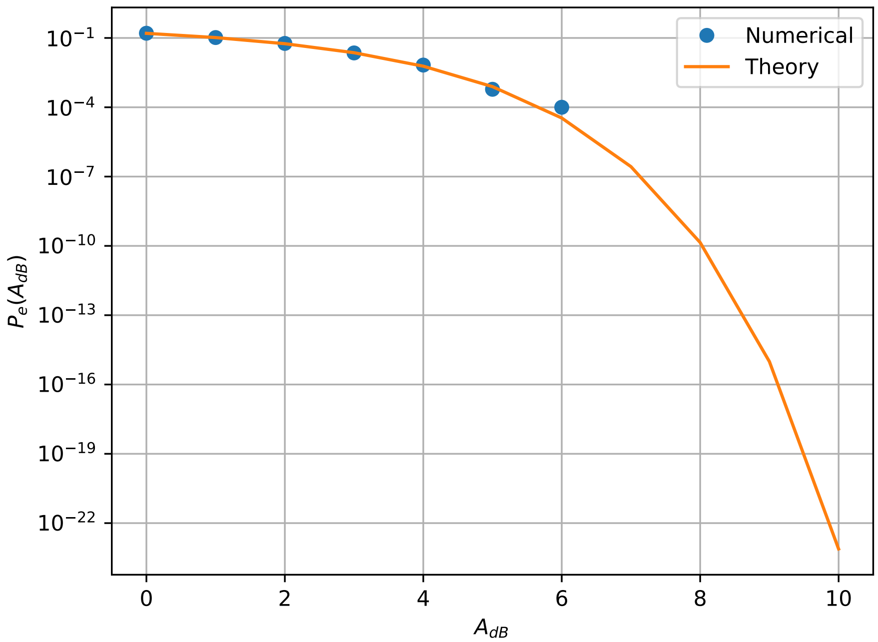

Figure 3.1.7.1: P_e versus A plot

3.1.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.

Solution:

Let $\Pr(0)$, $\Pr(1)$ is a probability of transmitting bit zero and bit one;
$P_{e|0}$,$P_{e|1}$is a probability of error when detecting bit zero and bit one.

$$P_e = P_{e|0}\Pr(0) + P_{e|1}\Pr(1) \tag{3.1.8.1}$$

Let V_0,V_1 be a nominal signal voltage of bit zero and one signal at the transmitter.

$$P(e \mid 0) = \int_{\delta}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\left(\nu - V_0\right)^2 / 2\sigma^2\right) d\nu$$
$$P(e \mid 1) = \int_{-\infty}^{\delta} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\left(\nu - V_1\right)^2 / 2\sigma^2\right) d\nu$$

where δ is a detection threshold. Differentiating $P(e)$ of (3.1.8.1) w.r.t. T, we arrive at

$$-\Pr(0)\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\left(\delta - V_0\right)^2 / 2\sigma^2\right) +$$
$$\Pr(1)\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\left(\delta - V_1\right)^2 / 2\sigma^2\right)$$

To find an optimal threshold, we equate the above expression to zero:

$$\Pr(0)\exp\left(-\frac{\left(\delta - V_0\right)^2}{2\sigma^2}\right) = \Pr(0)\exp\left(-\frac{\left(\delta - V_1\right)^2}{2\sigma^2}\right)$$
$$\delta = \frac{V_0 + V_1}{2} + \sigma^2 \ln\left(\frac{\Pr(1)}{\Pr(0)}\right)$$

$$\implies \Pr(1) = \Pr(0) = \frac{1}{2}$$
$$\implies V_0 = 1, V_1 = -1$$
$$\therefore \delta = 0$$

3.1.9 Repeat the above exercise when

$$p_X(0) = p \tag{3.1.9.1}$$

Solution:

$$P_e = (1-p)P_{e|1} + pP_{e|0} \tag{3.1.9.2}$$

From Above problem we know

$$\delta = \frac{V_0 + V_1}{2} + \sigma^2 \ln\left(\frac{P(1)}{P(0)}\right)$$
$$\delta = \ln\left(\frac{1-p}{p}\right)$$

-

Chapter 4

Transformation of Random Variables

4.1 Gaussian to Other

4.1.1 Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{4.1.1.1}$$

Solution: Refer The CDF and PDF of V plots in Fig. 4.1.1.1 and Fig. 4.1.1.2

```
chapter4/codes/sos.py
chapter4/codes/sos1.py
```

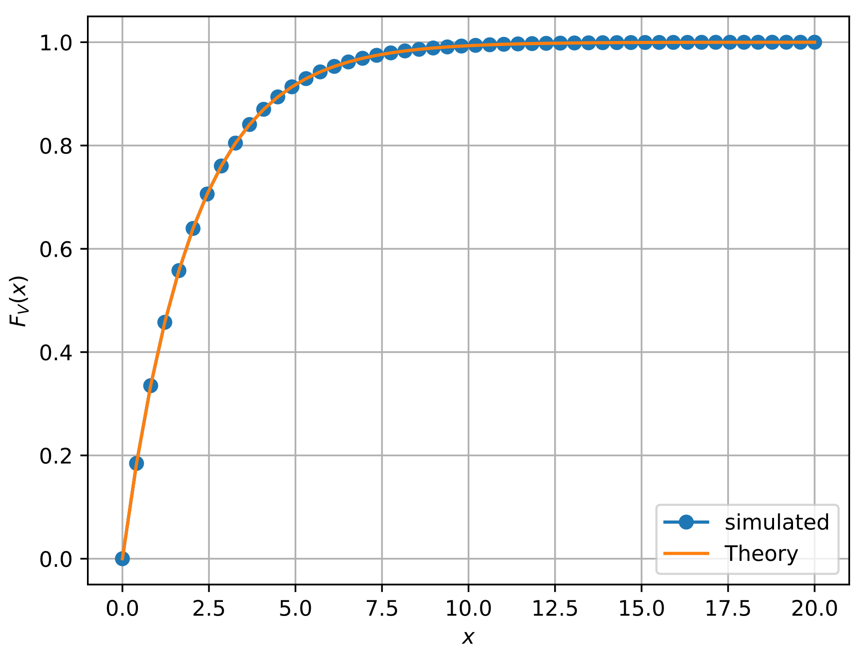

Figure 4.1.1.1: CDF of V

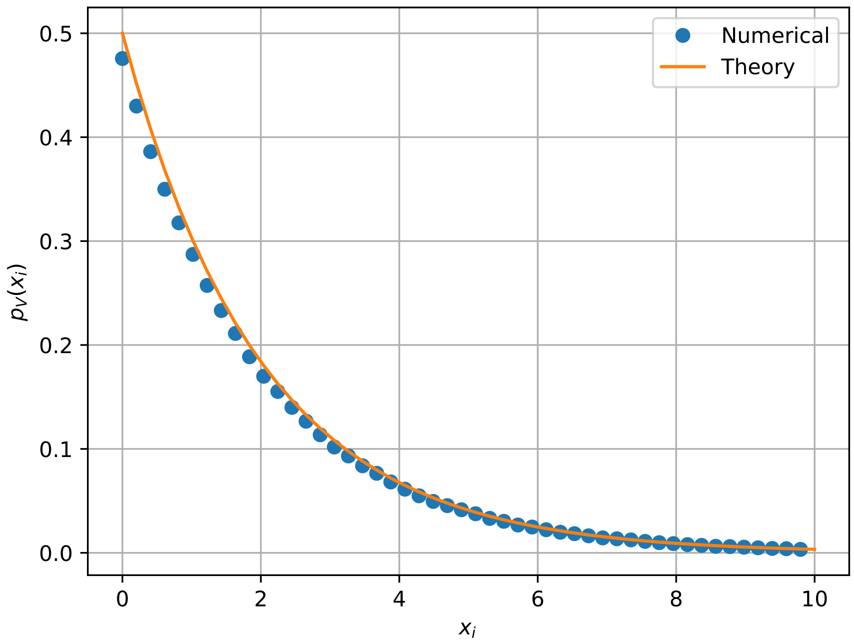

Figure 4.1.1.2: PDF of V

4.1.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{4.1.2.1}$$

find α.
Solution:

```
chapter4/codes/cdf6.py
```

from 4.1.2.1 *alpha*= 0.5

4.1.3 Plot the CDF and PDF of

$$A = \sqrt{V} \tag{4.1.3.1}$$

Solution: The CDF of A is given by,

$$F_A(a) = \Pr(A < a) \tag{4.1.3.2}$$
$$= \Pr\left(\sqrt{V} < a\right) \tag{4.1.3.3}$$
$$= \Pr\left(V < a^2\right) \tag{4.1.3.4}$$
$$= F_V\left(a^2\right) \tag{4.1.3.5}$$
$$= 1 - \exp\left(-\frac{a^2}{2}\right) \tag{4.1.3.6}$$

Using (2.2.3.1), the PDF is found to be

$$p_A(a) = a\exp\left(-\frac{a^2}{2}\right) \tag{4.1.3.7}$$

```
chapter4/codes/cpdf.py
chapter4/codes/cpdf1.py
```

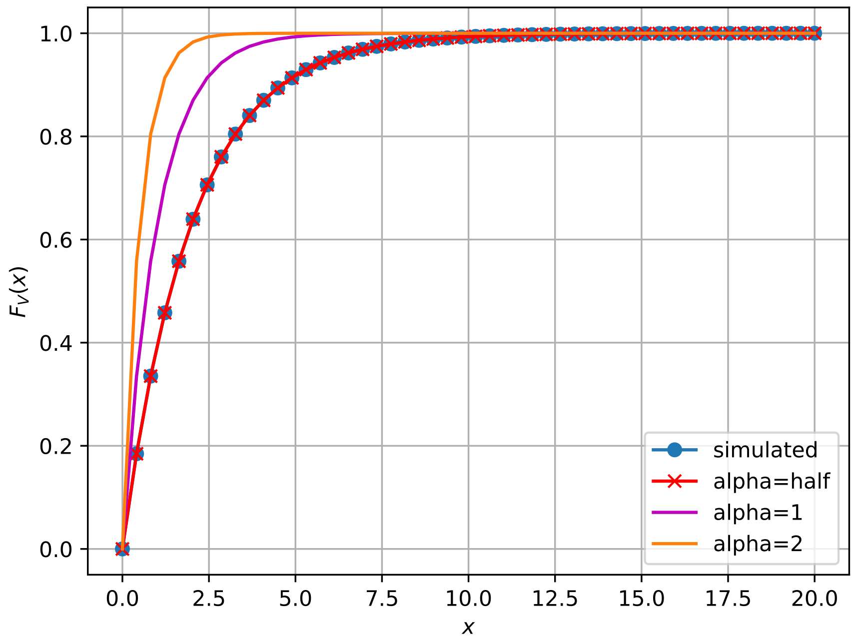

Figure 4.1.2.1: The CDF of V for different alpha

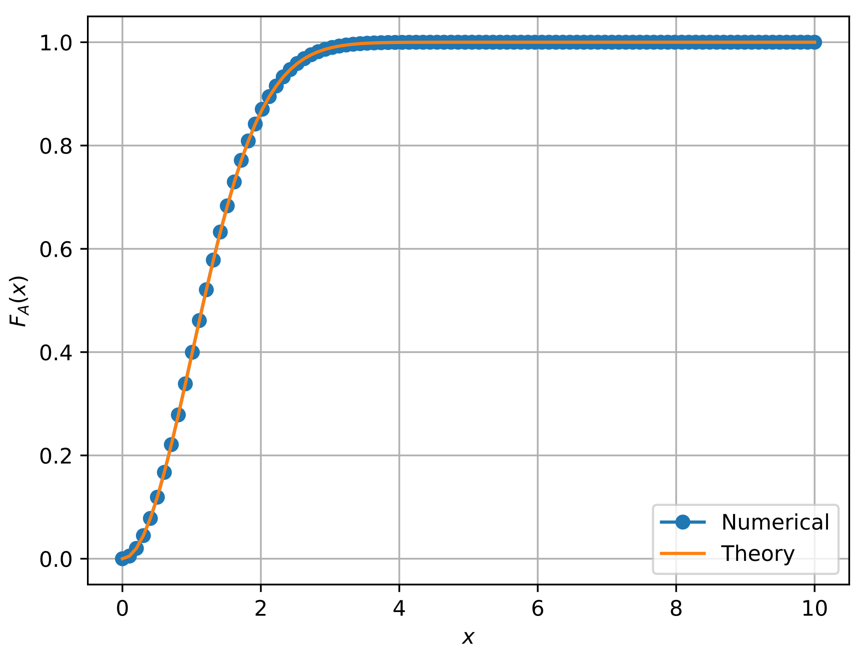

Figure 4.1.3.1: CDF of A

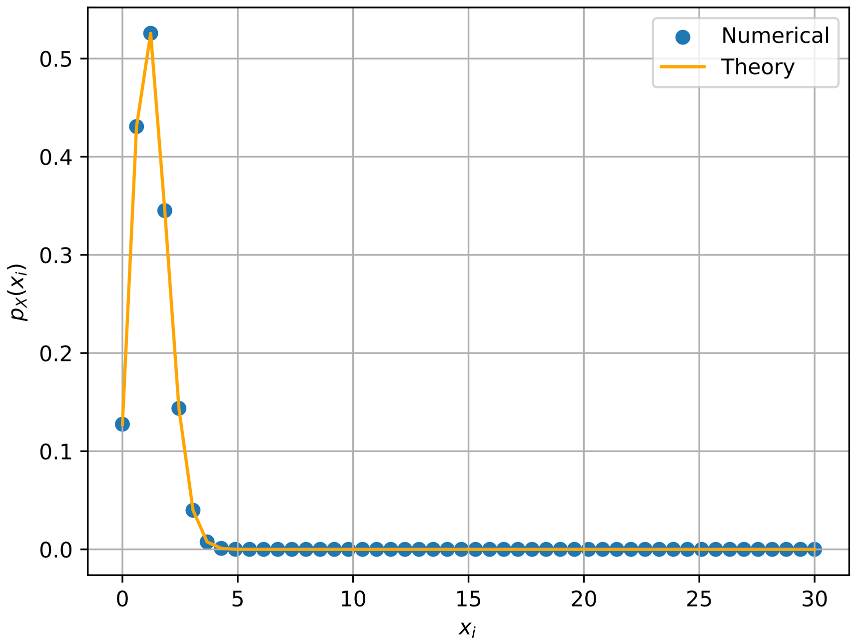

Figure 4.1.3.2: PDF of A

4.2 Conditional Probability

4.2.1 Plot

$$P_e = \Pr\left(\hat{X} = -1|X = 1\right) \tag{4.2.1.1}$$

for

$$Y = AX + N, \tag{4.2.1.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}\left(0, 1\right), X \in \left(-1, 1\right)$ for $0 \le \gamma \le 10$ dB.
Solution: Refer Fig. 4.2.4.1

```
chapter4/codes/cp.py
```

4.2.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$
Solution: Solution: The estimated value $\hat{X}$ is given by

$$\hat{X} = \begin{cases} +1 & Y > 0 \\ -1 & Y < 0 \end{cases} \tag{4.2.2.1}$$

For $X = 1$,

$$Y = A + N \tag{4.2.2.2}$$
$$P_e = \Pr\left(\hat{X} = -1|X = 1\right) \tag{4.2.2.3}$$
$$= \Pr\left(Y < 0|X = 1\right) \tag{4.2.2.4}$$
$$= \Pr\left(A < -N\right) \tag{4.2.2.5}$$
$$= F_A(-N) \tag{4.2.2.6}$$
$$= \int_{-\infty}^{-N} f_A(x)dx \tag{4.2.2.7}$$

By definition

$$f_A(x) = \begin{cases} \frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right) & x \geq 0 \\ 0 & otherwise \end{cases} \tag{4.2.2.8}$$

If $N > 0, f_A(x) = 0$. Then,

$$P_e = 0 \tag{4.2.2.9}$$

If $N < 0$. Then,

$$P_e(N) = \int_{-\infty}^{-N} f_A(x)dx \tag{4.2.2.10}$$

$$= \int_{-\infty}^{0} 0dx + \int_{0}^{-N} f_A(x)dx \tag{4.2.2.11}$$

$$= \int_{0}^{-N} \frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right)dx \tag{4.2.2.12}$$

$$= 1 - \exp\left(-\frac{N^2}{2\sigma^2}\right) \tag{4.2.2.13}$$

Therefore,

$$P_e(N) = \begin{cases} 1 - \exp\left(-\frac{N^2}{2\sigma^2}\right) & N < 0 \\ 0 & otherwise \end{cases} \tag{4.2.2.14}$$

4.2.3 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x)p_X(x)\,dx \tag{4.2.3.1}$$

Find $P_e = E\left[P_e(N)\right]$.
Solution: Since $N \sim \mathcal{N}(0,1)$,

$$p_N(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right) \tag{4.2.3.2}$$

And from (4.2.2.14)

$$P_e(x) = \begin{cases} 1 - \exp\left(-\frac{x^2}{2\sigma^2}\right) & x < 0 \\ 0 & otherwise \end{cases} \tag{4.2.3.3}$$

$$P_e = E\left[P_e(N)\right] = \int_{-\infty}^{\infty} P_e(x)p_N(x)\,dx \tag{4.2.3.4}$$

If $x < 0, P_e(x) = 0$ and using the fact that for an even function

$$\int_{-\infty}^{\infty} f(x) = 2\int_{-\infty}^{0} f(x) \tag{4.2.3.5}$$

we get

$$P_e = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \exp\left(-\frac{x^2}{2}\right)\left(1 - \exp\left(-\frac{x^2}{2\sigma^2}\right)\right) dx \tag{4.2.3.6}$$

$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx - \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(1+\sigma^2)x^2}{2\sigma^2}\right) dx \tag{4.2.3.7}$$

$$= \frac{\sqrt{2\pi} - \sqrt{\frac{\pi(2\sigma^2)}{1+\sigma^2}}}{2\sqrt{2\pi}} \tag{4.2.3.8}$$

$$= \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\sigma^2}{1+\sigma^2}} \tag{4.2.3.9}$$

For a Rayleigh Distribution with scale $= \sigma$,

$$E\left[A^2\right] = 2\sigma^2 \tag{4.2.3.10}$$

$$\gamma = 2\sigma^2 \tag{4.2.3.11}$$

$$\therefore P_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\gamma}{2+\gamma}} \tag{4.2.3.12}$$

4.2.4 Plot P_e in problems 4.2.1 and 4.2.3 on the same graph w.r.t γ. Comment.
Solution: P_e plotted in same graph in Fig. 4.2.4.1. The value of P_e is much higher when the channel gain A is Rayleigh distributed than the case where A is a constant (compare with Fig. 3.1.7.1).

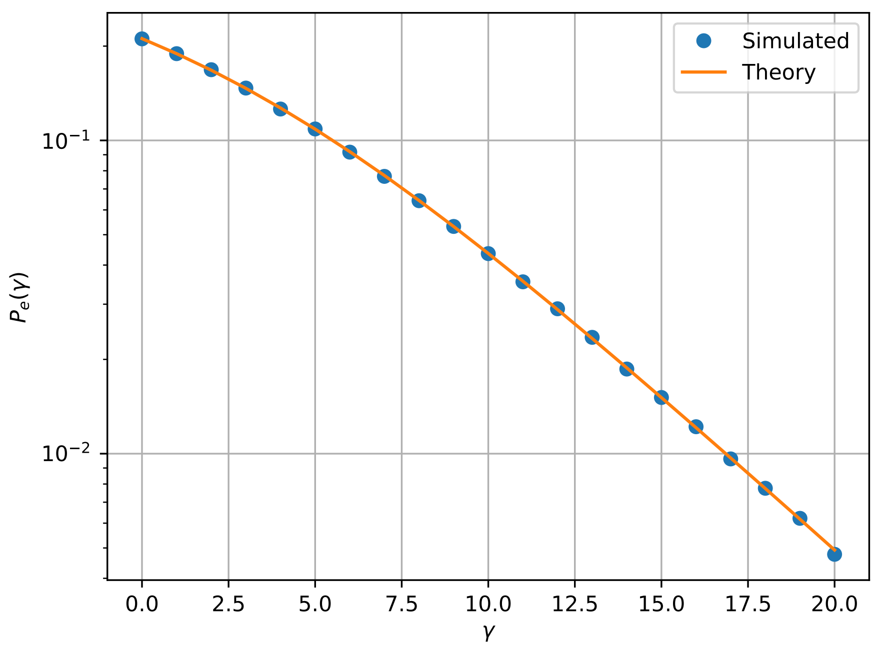

Figure 4.2.4.1: P_e versus γ

Chapter 5

Bivariate Random Variables: FSK

5.1 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{5.1.0.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.1.0.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \tag{5.1.0.3}$$

5.1.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{5.1.1.1}$$

on the same graph using a scatter plot.
Solution: Refer Fig. 5.1.1.1 for plot,

```
chapter5/codes/biv_scatter.py
```

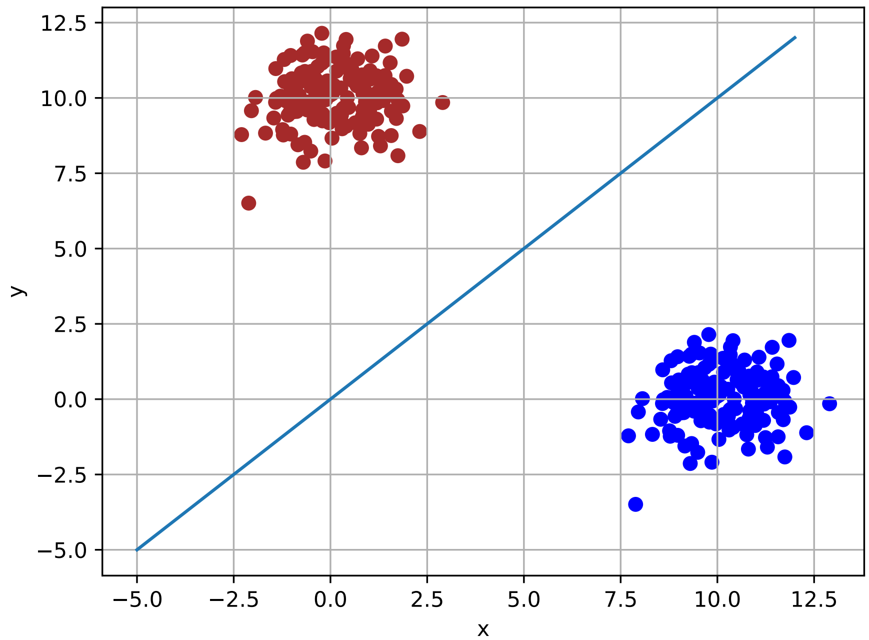

Figure 5.1.1.1: Scatter plot of $\mathbf{y}|\mathbf{s}_0$ and $\mathbf{y}|\mathbf{s}_1$

5.1.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

Solution: The real vector

$$s_0 = \left(s_{0|1}, \ldots, s_{0|n}\right)^\top \tag{5.1.2.1}$$

$$s_1 = \left(s_{1|1}, \ldots, s_{1|n}\right)^\top \tag{5.1.2.2}$$

$$\tag{5.1.2.3}$$

For the given s_0, $n_1, n_2 \sim \mathcal{N}(0,1)$.

$$p\left(\mathbf{y}|s_0\right) = \frac{1}{\sqrt{\left(2\pi\sigma^2\right)^{\frac{n}{2}}}} \exp \sum_{k=1}^{n} \frac{-\left(y - s_0\right)^2}{2\sigma^2} \tag{5.1.2.4}$$

Similarly,

$$p\left(\mathbf{y}|s_1\right) = \frac{1}{\sqrt{\left(2\pi\sigma^2\right)^{\frac{n}{2}}}} \exp \sum_{k=1}^{n} \frac{-\left(y - s_1\right)^2}{2\sigma^2} \tag{5.1.2.5}$$

The likelihood ratio is given by

$$\Lambda(y) = \exp \sum_{k=1}^{n} \frac{\left(y - {s_1}^2\right) - \left(y - s_0\right)^2}{2\sigma^2} \tag{5.1.2.6}$$

$$= \exp\left[\frac{\left(s_0 - s_1\right)^\top y}{\sigma^2} + \frac{s_1^\top s_1 - s_0^\top s_0}{2\sigma^2}\right] \tag{5.1.2.7}$$

$$\Lambda(Y) = \frac{p_\gamma|H\left(\mathbf{y}|1\right)}{p_\gamma|H\left(\mathbf{y}|0\right)} \; \substack{\leq H=1 \\ \geq H=0} \; \frac{P_0}{P_1} = \eta \tag{5.1.2.8}$$

$\Lambda(y)$ is called likelihood ratio and is function of y and $\eta = \frac{P_0}{P_1}$ is called the threshold substituting (5.1.2.7) in (5.1.2.8) and taking the logarthim of both sides,

$$LLR(y) = \frac{(s_0 - s_1)^\top y}{\sigma^2} + \frac{s_1^\top s_1 - s_0^\top s_0}{2\sigma^2} \lessgtr_{H=0}^{H=1} \ln \frac{P_0}{P_1} = \ln \eta \tag{5.1.2.9}$$

rewriting 5.1.2.9 in the form

$$(s_0 - s_1) \lessgtr_{H=0}^{H=1} \sigma^2 \ln \eta + \frac{\left(s_0^\top s_0 - s_1^\top s_1\right)}{2} = \phi \tag{5.1.2.10}$$

In general as seen analytically by 5.1.2.7 points of constant likelihood ratio are points for $(s_0 - s_1)^\top y$ is constat and this is the quation of affine space.
We have seen from 5.1.2.10 that comparing $\Lambda(y)$ to the threshold η is equivalent to comparing $(s_0 - s_1)^\top y$ to the threshold ϕ. Thus the affine space $(s_0 - s_1)^\top y = \phi$ seprates the observation space in tto two regions, Where $H = 1$ for $(s_0 - s_1)^\top y \geq \phi$ and $H = 0$ otherwise.

$$s_0^\top s_0 - s_1^\top s_1 = (s_0 - s_1)^\top (s_0 + s_1). \tag{5.1.2.11}$$

Substituting this in (5.1.2.9), we get

$$LLR(y) = \left[\frac{(s_0 - s_1)^\top}{\sigma^2}\left(y - \frac{s_0 + s_1}{2}\right)\right] \lessgtr_{H=0}^{H=1} \ln \frac{P_0}{P_1} = \ln \eta \tag{5.1.2.12}$$

By equating,$s_0 = s_1$

$$p\left(\mathbf{y}|s_0\right) = p\left(\mathbf{y}|s_1\right) \tag{5.1.2.13}$$

$$\left(\mathbf{y} - \mathbf{s}_0\right)^\top \left(\mathbf{y} - \mathbf{s}_0\right) = \left(\mathbf{y} - \mathbf{s}_1\right)^\top \left(\mathbf{y} - \mathbf{s}_1\right) \tag{5.1.2.14}$$

$$\mathbf{y}^\top \mathbf{y} - 2\mathbf{s}_0^\top \mathbf{y} + \mathbf{s}_0^T \mathbf{s}_0 = \mathbf{y}^\top \mathbf{y} - 2\mathbf{s}_1^\top \mathbf{y} + \mathbf{s}_1^T \mathbf{s}_1 \tag{5.1.2.15}$$

$$2\left(\mathbf{s}_1 - \mathbf{s}_0\right)^\top \mathbf{y} = \|\mathbf{s}_1\|^2 - \|\mathbf{s}_0\|^2 \tag{5.1.2.16}$$

$$\left(\mathbf{s}_1 - \mathbf{s}_0\right)^\top \mathbf{y} = 0 \tag{5.1.2.17}$$

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}^\top \mathbf{y} = 0 \tag{5.1.2.18}$$

This decision regions are separated by the affine space that forms the perpendicular section between s_1 and s_0.
Finally, we use 5.1.2.12 to evaluate $\Pr(e|H = 0)$. $E\left[Y - (s_0 + s_1)/2|H = 0)\right] = \frac{s_1 - s_0}{2}$ So

$$E\left[LLR(Y)|H = 0\right] = \frac{-\left(s_0 - s_1\right)^T \left(s_0 - s_1\right)}{2\sigma^2} \tag{5.1.2.19}$$

Defining γ as

$$\gamma = \frac{\|s_0 - s_1\|}{\sigma} \tag{5.1.2.20}$$

This simplifies to

$$E\left[LLR(Y)|H = 0\right] = -\frac{\gamma^2}{2} \tag{5.1.2.21}$$

Similarly, we see that

$$VAR\left[LLR(Y)|H = 0\right] = \gamma^2 \tag{5.1.2.22}$$

Thus conditional on s_0 with the probabiltiy of error

$$\Pr(e|s_0) = Q\left(\frac{-\ln\eta}{\gamma} + \frac{\gamma}{2}\right) \tag{5.1.2.23}$$

Thus conditional on s_1 with the probabiltiy of error

$$\Pr(e|s_1) = Q\left(\frac{\ln\eta}{\gamma} + \frac{\gamma}{2}\right) \tag{5.1.2.24}$$

5.1.3 Plot

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \tag{5.1.3.1}$$

with respect to the SNR from 0 to 10 dB.
Solution: The blue dots in Fig. 5.1.4.1 are the P_e versus SNR plot. It is generated using the below code,

```
chapter5/codes/biv_pe_snr.py
```

5.1.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.
Solution:

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \tag{5.1.4.1}$$

Given that $\mathbf{s}_0$ was transmitted, the received signal is

$$\mathbf{y}|\mathbf{s}_0 = \begin{pmatrix} A \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \tag{5.1.4.2}$$

From decision rule, the probability of error is given by

$$P_e = \Pr\left(y_1 < y_2 | \mathbf{s}_0\right) = \Pr\left(A + n_1 < n_2\right) \tag{5.1.4.3}$$
$$= \Pr\left(n_2 - n_1 > A\right) \tag{5.1.4.4}$$

Note that $n_2 - n_1 \sim \mathcal{N}\left(0, 2\right)$. Thus,

$$P_e = \Pr\left(\sqrt{2}w > A\right) \tag{5.1.4.5}$$
$$\Pr\left(w > \frac{A}{\sqrt{2}}\right) \tag{5.1.4.6}$$
$$\implies P_e = Q\left(\frac{A}{\sqrt{2}}\right) \tag{5.1.4.7}$$

where $w \sim \mathcal{N}\left(0, 1\right)$.
Fig. 5.1.4.1 compares the theoretical and simulation plots.

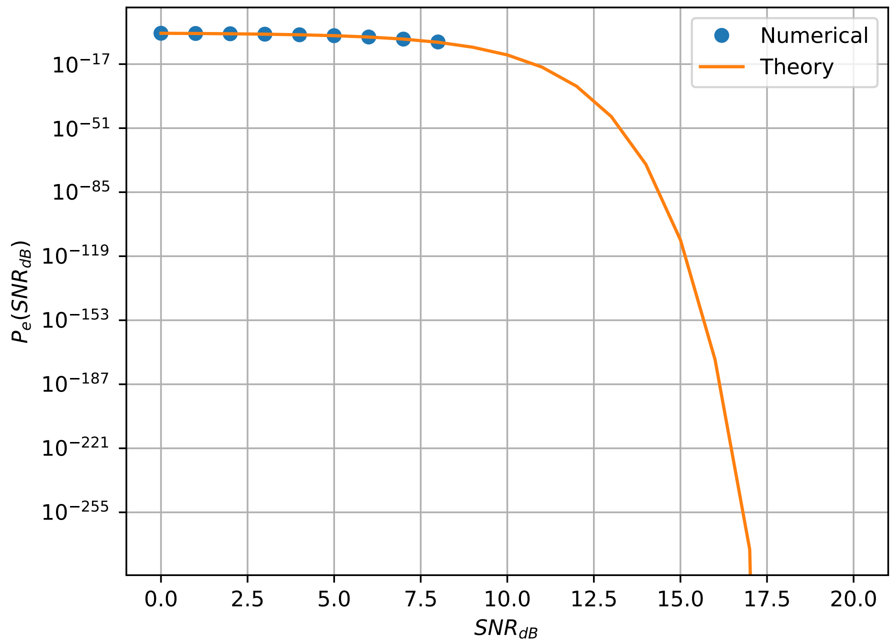

Figure 5.1.4.1: P_e with respect to the SNR from 0 to 10 dB