BUAN/OPRE 6398
Prescriptive Analytics

Time Series Forecasting
(Stationary Data)

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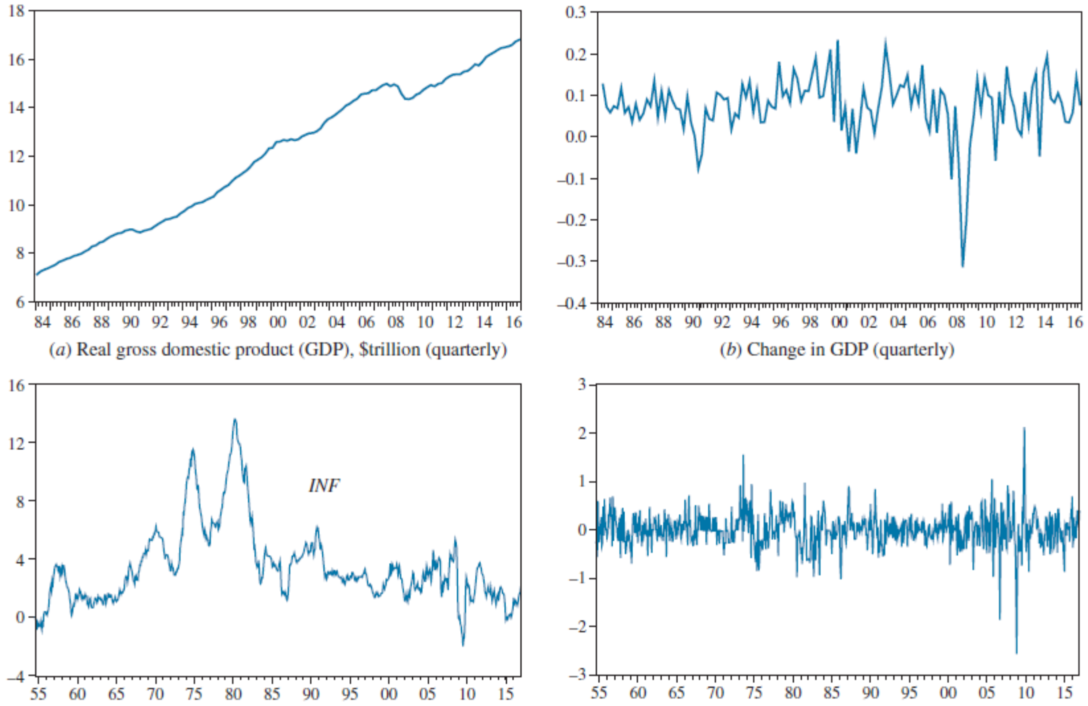
Lecture Outline

- Predicting Models for Stationary Data
  - Moving Average
  - Weighted Moving Average
  - Exponential Smoothing
  - A model for stationary data with additive seasonal effects
  - A model for stationary data with multiplicative seasonal effects

Introduction

- A *time-series* data is a set of observations on a quantitative variable collected over time.
- Examples
  - Dow Jones Industrial Averages
  - Historical data on sales, inventory, customer counts, interest rates, and costs.
  - The inflation, unemployment, and growth rates between 1970 to 2020

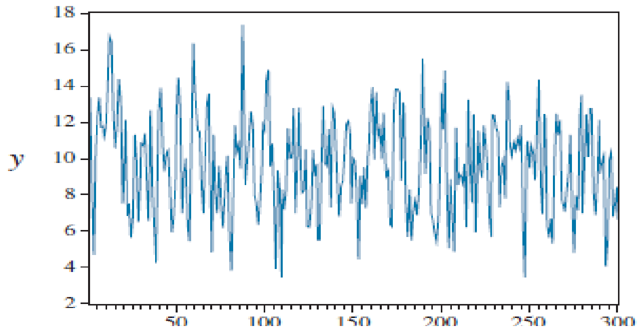


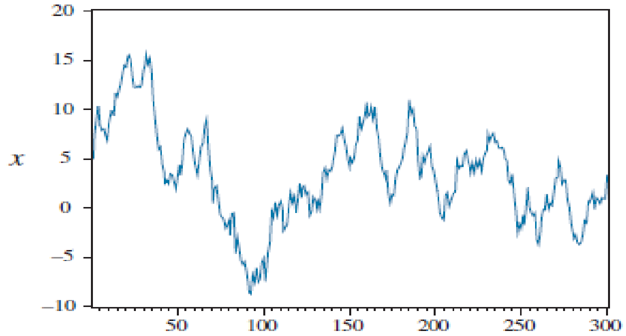
(a) Real gross domestic product (GDP), $trillion (quarterly)
(b) Change in GDP (quarterly)
(c) Inflation, percent (monthly)
(d) Change in inflation (monthly)

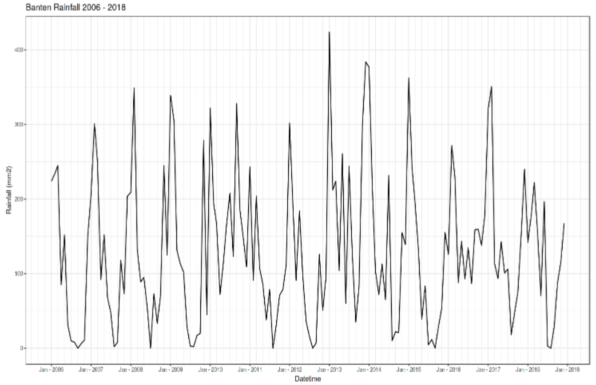
- In time series analysis,
  - we analyze the past behavior of data to predict its future behavior (extrapolation).

Some Time Series Terms

- **Stationary Data:**
  - No significant trend, non-explosive, non-wandering.
- **Nonstationary Data:**
  - Significant upward/ downward trend, wandering up and down over time.
- **Seasonal Data:**
  - Repeating patterns at regular intervals over time.
    - Additive effects
    - Multiplicative effects







Extrapolation Model

- General form of an extrapolation model:

$$\hat{Y}_{t+1} = f(Y_t, Y_{t-1}, Y_{t-2}, \dots)$$

  - $\hat{Y}_{t+1}$ = The *predicted* value for Y at period $t + 1$.
  - Y_{t-j} = The *actual* value of Y at period $t - j$; $j = 0,1,2, \dots$

- Goal of an extrapolation model:
  - Identifying the function $f(\cdot)$ that produces the most accurate forecast of future values of Y.
    - There are many different time series techniques.
    - It is usually impossible to know which technique will be best for a particular data set.
    - Try out several different techniques and select the one that seems to work best.
    - We need a way to compare different time series techniques for a given data set.

Measuring Accuracy

- **Informal Approach**
  - Constructing line graphs that show the actual value versus the predicted values for various modeling techniques.
- **Formal Techniques**
  - Minimizing Mean Absolute Deviation (MAD):

$$MAD = \frac{1}{n}\sum\left|Y - \hat{Y}\right|$$

  - Minimizing Mean Absolute Percent Error (MAPE)

$$MAPE = \frac{1}{n}\sum\frac{\left|Y - \hat{Y}\right|}{Y}$$

  - Minimizing Mean Square Error (MSE)

$$MSE = \frac{1}{n}\sum\left(Y - \hat{Y}\right)^2$$

  - Minimizing Root Mean Square Error (RMSE)

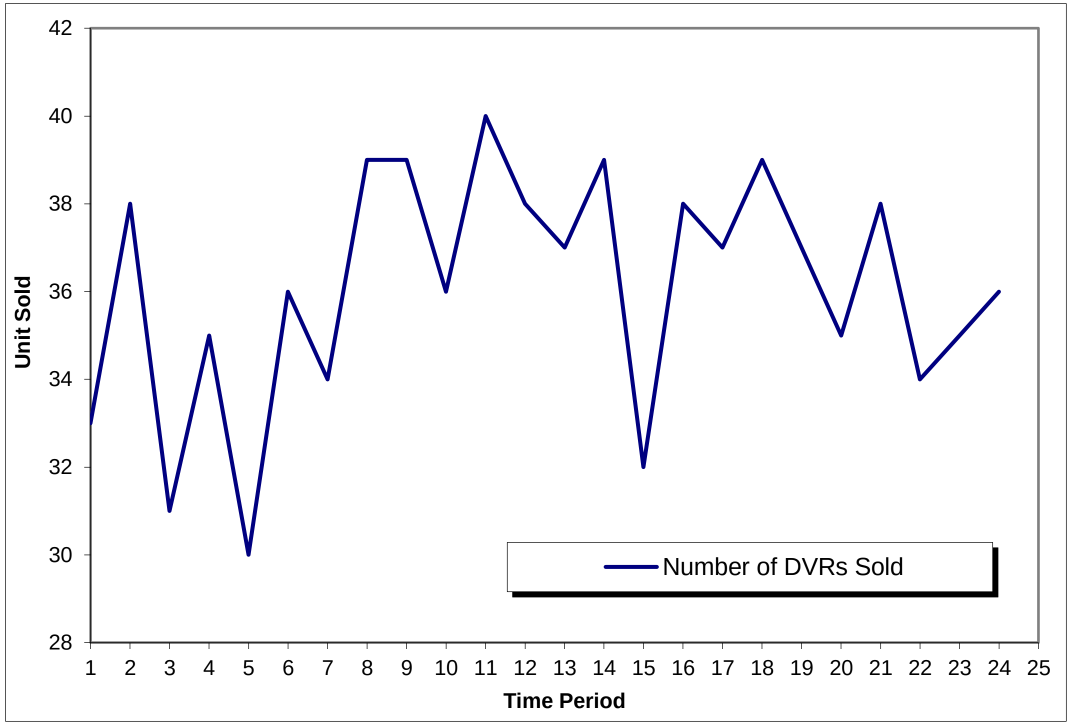
$$RMSE = \sqrt{MSE}$$

Stationary Models: An Example

- Electra-City is a retail store that sells audio and video equipment for homes and cars.
- Each month, the manager of the store must order merchandise from a distant warehouse.

- The manager wants to estimate the number of DVRs the store will sell in the next four months.
- He has collected sales data for the past 24 months.

See file Fig11-1.xlsm



Moving Average (MA)Technique

- The MA technique is the simplest forecasting method.
- MA over the past k periods is:

$$\hat{Y}_{t+1} = \frac{Y_t + Y_{t-1} + \cdots + Y_{t-k+1}}{k}$$

- Note: No general rule for determining the best k exists. We must try out several k values to see what works best.
- In forecasting sale units for Electra-City with MA, let's compare two MAs: 2-period and 4-period

See file Fig11-2.xlsm

  - In excel, **MSE = SUMXMY2(Y range, $\hat{Y}$ range)/count($\hat{Y}$ range)**

$$MSE_{k=2} = 7.2,\ MSE_{k=4} = 7.8$$

  - The 2-period MA is preferred. Thus,

$$\hat{Y}_{25} = \frac{Y_{24} + Y_{23}}{2} = \frac{36+35}{2} = 35.5$$

- Assuming data is stationary and exhibits no trends: $\hat{Y}_{26} = \hat{Y}_{27} = \hat{Y}_{28} = 35.5$

Comments on Comparing MSE Values

- Care should be taken when comparing MSE values of two different forecasting techniques.
- The lowest MSE may result from a technique that fits older values very well but fits recent values poorly.
  - Hence, it is sometimes wise to compute the MSE using only the most recent values.
  - E.g., in the Electra-City example, when MSE is calculated based on the past 10 months, the 4-period MA model is preferred to the 2-period MA model.

See file Fig11-3.xlsm

Weighted Moving Average (WMA) Technique

- The MA technique assigns an equal weight $(1/k)$ to past values:

$$\hat{Y}_{t+1} = \frac{1}{k}Y_t + \frac{1}{k}Y_{t-1} + \cdots + \frac{1}{k}Y_{t-k+1}$$

- But, sometimes, periods might have different importance in forecasting.
- The WMA technique allows different weights to be assigned to past values.

$$\hat{Y}_{t+1} = w_0 Y_t + w_1 Y_{t-1} + \cdots + w_k Y_{t-k}$$

  – Where $0 \le w_i \le 1$ and $\sum w_i = 1$
  – Under WMA, in addition to k, we must determine w_i's.
  See file Fig11-6.xlsm

Min MSE $= \frac{1}{n}\sum\left(Y_t - \hat{Y}_t\right)^2$
ST: $0 \le w_i \le 1$
$\sum w_i = 1$

- Forecasting with WMA (assuming $k = 2$):

$$\hat{Y}_{25} = 0.291(36) + 0.709(35) = 35.29$$

  – Assuming that data is stationary and exhibits no trends,

$$\hat{Y}_{26} = \hat{Y}_{27} = \hat{Y}_{28} = 35.29$$

Exponential Smoothing Technique

- Exponential smoothing is another averaging technique for stationary data.

$$\hat{Y}_{t+1} = \hat{Y}_t + \alpha(Y_t - \hat{Y}_t)$$
$$0 \leq \alpha \leq 1$$

  - $\hat{Y}_{t+1}$ = The predicted value for period $t + 1$
  - $\hat{Y}_t$ = The predicted value for period t
  - Y_t = The actual value at period t
  - $Y_t - \hat{Y}_t$ = prediction error at period t

- Iterating the above equation, we have:

$$\hat{Y}_{t+1} = \alpha Y_t + \alpha(1-\alpha)Y_{t-1} + \alpha(1-\alpha)^2 Y_{t-2} + \cdots + (1-\alpha)^t \hat{Y}_1$$

- As we see, the exponential smoothing technique is a special case of the WMA technique.
  - The weights become smaller and smaller as we move farther away from the current period.

Exponential Smoothing: Optimal α

$$\text{Min MSE} = \frac{1}{n}\sum\left(Y_t - \hat{Y}_t\right)^2$$

$$\text{ST: } 0 \leq \alpha \leq 1$$

- Where, $\hat{Y}_t = \hat{Y}_{t-1} + \alpha(Y_{t-1} - \hat{Y}_{t-1})$
- Note: For the very first period, we assume that $\hat{Y} = Y$.

See file Fig11-10.xlsm

$$\alpha = 0.2604$$

- Forecasted values:

$$\hat{Y}_{25} = \hat{Y}_{24} + \alpha\left(Y_{24} - \hat{Y}_{24}\right)$$

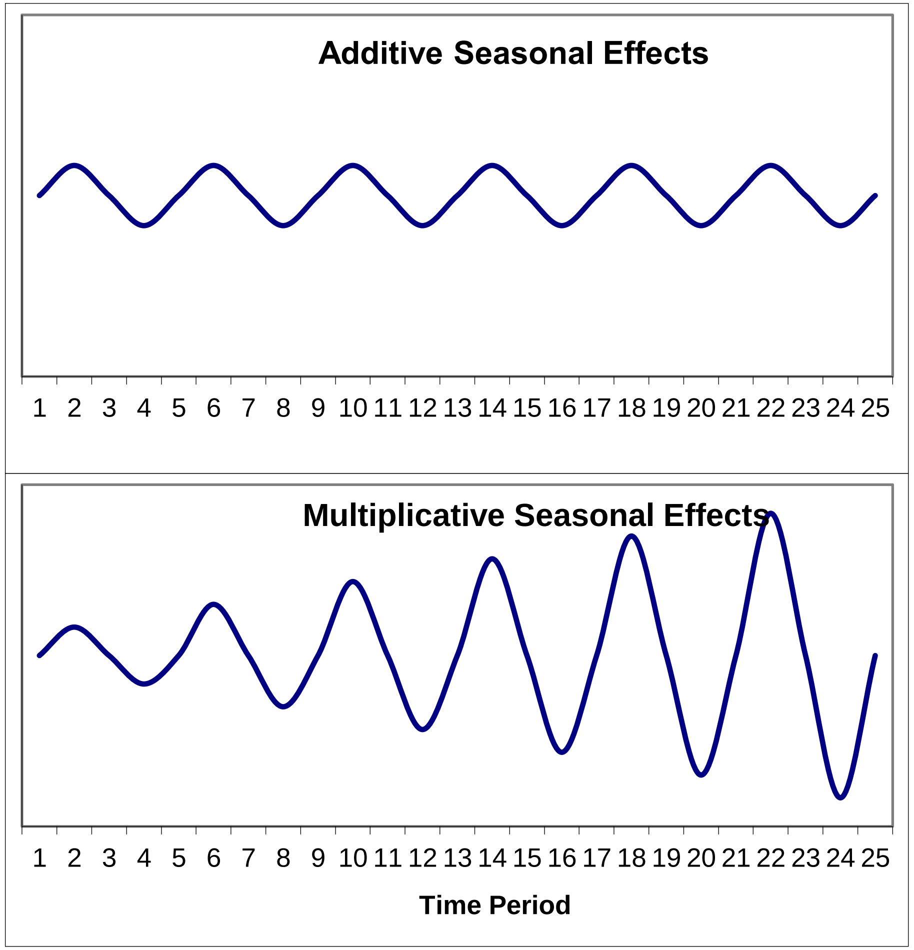
$$= 35.91 + 0.2604(36 - 35.91) = 35.93$$

- Since the data is assumed to be stationary:

$$\hat{Y}_{26} = \hat{Y}_{27} = \hat{Y}_{28} = 35.93$$

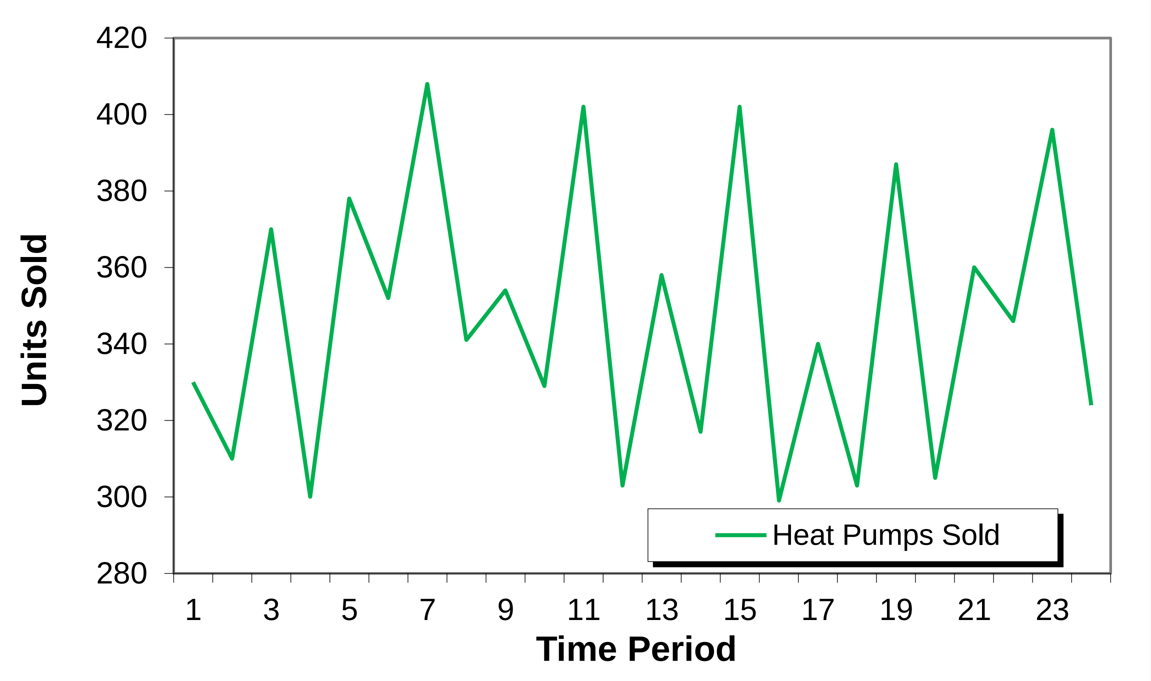
Seasonality

- **Seasonality:** A regular and repeating pattern in time series data.
- E.g.,
  - Fuel oil price jumps up during the winter.
  - Price of the suntan lotion and ice cream increases during the summer.
- Two types of seasonal effects
  - Additive effects
  - Multiplicative effects



Seasonal Effects
An Example

- Savannah Climate Control (SCC) sells residential heat pumps.
  - SCC collected quarterly sales data for the past 6 years.
  - Using past data, SCC wants to predict the sales in next year's quatres.
- Sales of heat pumps tend to be
  - higher than average in the winter and summer quarters
  - lower than average in the spring and fall quarters



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Stationary Data with Additive Seasonal Effects

- Let $p =$ The number of seasonal periods ($p = 4$ in quarterly data).
- The **expected level** at period t is the weighted average of the *de-seasonalized value* at t and the *expected level* at $t - 1$:

$$E_t = \alpha\left(Y_t - S_{t-p}\right) + (1-\alpha)E_{t-1}$$
$$0 \le \alpha \le 1$$

- The **seasonal factor** at period t is the weighted average of the *expected seasonal effect* at period t and the *seasonal factor* at period $t - p$:

$$S_t = \beta(Y_t - E_t) + (1-\beta)S_{t-p}$$
$$0 \le \beta \le 1$$

- The predicted value at period $t + n$ (for $n = 1,2, \dots, p$) is:

$$\boldsymbol{\hat{Y}_{t+n} = E_t + S_{t+n-p}}$$

- Note: E_t and S_t for the first p periods are:

$$E_t = \frac{1}{p}\sum_{i=1}^{p} Y_i$$
$$S_t = Y_t - E_t$$

Forecasting Stationary Data with Additive Seasonal Effects

- Forecasts for time periods 25 to 28 at period 24:

$$\hat{Y}_{t+n} = E_t + S_{t+n-p}$$
$$\hat{Y}_{24+n} = E_{24} + S_{24+n-4} \text{ for } n = 1,2,3,4$$

$$\hat{Y}_{25} = E_{24} + S_{21} = 234.55 + 8.45 = 363.00$$
$$\hat{Y}_{26} = E_{24} + S_{22} = 234.55 - 17.82 = 336.73$$
$$\hat{Y}_{27} = E_{24} + S_{23} = 234.55 + 46.58 = 401.13$$
$$\hat{Y}_{28} = E_{24} + S_{24} = 234.55 - 31.73 = 322.81$$

See file Fig11-15.xlsm

Stationary Data with Multiplicative Seasonal Effects

- Let $p =$ The number of seasonal periods ($p = 4$ in quarterly data).
- The **expected level** at period t is the weighted average of the *de-seasonalized value* at t and the *expected level* at $t - 1$:

$$E_t = \alpha\left(Y_t/S_{t-p}\right) + (1-\alpha)E_{t-1}$$
$$0 \le \alpha \le 1$$

- The **seasonal factor** at period t is the weighted average of the *expected seasonal effect* at period t and the *seasonal factor* at period $t - p$:

$$S_t = \beta(Y_t/E_t) + (1-\beta)S_{t-p}$$
$$0 \le \beta \le 1$$

- The predicted value at period $t + n$ (for $n = 1,2, \dots, p$) is:

$$\boldsymbol{\hat{Y}_{t+n} = E_t \times S_{t+n-p}}$$

- Note: E_t and S_t for the first p periods are:

$$E_t = \frac{1}{p}\sum_{i=1}^{p} Y_i$$
$$S_t = Y_t/E_t$$

Forecasting Stationary Data with Multiplicative Seasonal Effects

- Forecasts for time periods 25 to 28 at period 24:

$$\hat{Y}_{t+n} = E_t \times S_{t+n-p}$$

$$\hat{Y}_{24+n} = E_{24} \times S_{24+n-4}$$

$$\hat{Y}_{25} = E_{24} \times S_{21} = 353.95 \times 1.015 = 359.13$$
$$\hat{Y}_{26} = E_{24} \times S_{22} = 353.95 \times 0.946 = 334.94$$
$$\hat{Y}_{27} = E_{24} \times S_{23} = 353.95 \times 1.133 = 400.99$$
$$\hat{Y}_{28} = E_{24} \times S_{24} = 353.95 \times 0.912 = 322.95$$

See file Fig11-18.xlsm

End of Lecture 9