COMP 409: Homework 2

Sample Solutions

1 Propositional semantics

1. *Implement a truth evaluator* $eval(\phi, \tau)$*, which evaluates whether a formula* ϕ *holds for a truth assignment* τ*. Use the evaluator to answer the question.*

   The code is available upon request. The results are:

$$\alpha(I_1) = \beta(I_1) = \gamma(I_1) = \delta(I_1) = 0 \Rightarrow \psi(I_1) = 1$$

$$\alpha(I_2) = 0; \beta(I_2) = \gamma(I_2) = \delta(I_2) = 1 \Rightarrow \psi(I_2) = 1$$

2 Validity

2.1 Prove Validity

2.1.1 $(((p \to q) \to p) \to p)$ - Peirce's Law

Let $\tau \in 2^{Prop}$.
Case 1: $\tau \models p$ and $\tau \models q \to \tau \models (p \to q)$
$\tau \models p$ and $\tau \models (p \to q) \to \tau \models ((p \to q) \to p)$
$\tau \models p$ and $\tau \models ((p \to q) \to p) \to \tau \models (((p \to q) \to p) \to p)$

Case 2: $\tau \models p$ and $\tau \not\models q \to \tau \not\models (p \to q)$
$\tau \not\models (p \to q) \to \tau \models ((p \to q) \to p)$
$\tau \models p$ and $\tau \models ((p \to q) \to p) \to \tau \models (((p \to q) \to p) \to p)$

Case 3: $\tau \not\models p \to \tau \models (p \to q)$
$\tau \not\models p$ and $\tau \models (p \to q) \to \tau \not\models ((p \to q) \to p)$
$\tau \not\models ((p \to q) \to p) \to \tau \models (((p \to q) \to p) \to p)$

2.1.2 $false \to p$ - Ex Falso Quolibet

Let $\tau \in 2^{Prop}$. $\tau \not\models false \to \tau \models (false \to p)$

2.1.3 $((\neg p) \vee p)$ - Excluded Middle

Let $\tau \in 2^{Prop}$. If $\tau \models p$, then $\tau \models (p \vee (\neg p))$.
If $\tau \not\models p$, then $\tau \models (\neg p) \implies \tau \models (p \vee (\neg p))$.
Therefore $\models (p \vee (\neg p))$.

2.1.4 $((p \to q) \vee p)$ - Weak Excluded Middle

Let $\tau \in 2^{Prop}$. We know that $\tau \models ((p \to q) \vee p) \leftrightarrow \tau \models (p \to q)$ or $\tau \models p$.
Case 1: $\tau \models p$, then $\tau \models ((p \to q) \vee p)$.
Case 2: $\tau \not\models p$ then $\tau \models (p \to q)$. Therefore, $\tau \models ((p \to q) \vee p)$.

2.2 Disjunctional Propositional Completeness

Suppose α and β were both not valid. This would mean that there exist assignments $\tau_1 \in 2^{\mathbf{AP}(\alpha)}$ and $\tau_2 \in 2^{\mathbf{AP}(\beta)}$ s.t $\tau_1 \not\models \alpha$ and $\tau_2 \not\models \beta$. Note that $\tau_1 \cap \tau_2 = \emptyset$.

Take the assignment $\tau \in 2^{\mathbf{AP}(\alpha \vee \beta)}$ where $\tau = \tau_1 \cup \tau_2$. Since, because $\mathbf{AP}(alpha) \cap \mathbf{AP}(beta) = \emptyset$, $\tau|_{\mathbf{AP}(\alpha)} = \tau_1$ and $\tau|_{\mathbf{AP}(\beta)} = \tau_2$.

Since $\models (\alpha \vee \beta)$, $\tau \models (\alpha \vee \beta) \implies$ either $\tau \models \alpha$ or $\tau \models \beta$ since it is sufficient to look at the relevant propositions only, $\implies$ either $\tau|_{\mathbf{AP}(\alpha)} \models \alpha$ or $\tau|_{\mathbf{AP}(\beta)} \models \beta \implies$ either $\tau_1 \models \alpha$ or $\tau_2 \models \beta$. We know that this is not possible. Therefore our assumption must be faulty.

Hence either $\models \alpha$ or $\models \beta$.

3 Logical implication

1. *Analyze the binary connectives $\wedge$, $\vee$, $\to$, $\leftrightarrow$ to see whether they are commutative or associative. Prove your claims.*

   We will show that $\wedge$, $\vee$, and $\leftrightarrow$ are commutative and associative. The basic strategy in all cases is to use the commutativity and the associativity of the set-theoretic operations $\cap$, $\cup$, and set equality $(=)$. An alternative approach is to use truth tables.

   - $\wedge$ **is commutative.** Proof: $models(\theta \wedge \psi) = models(\theta) \bigcap models(\psi)$ $= models(\psi) \bigcap models(\theta) = models(\psi \wedge \theta)$ (since set intersection is commutative). By definition of $\models\dashv$ it follows that
   $$(\theta \wedge \psi) \models\dashv (\psi \wedge \theta).$$
   - $\wedge$ **is associative.** Proof: $models((\phi \wedge \theta) \wedge \psi) = models(\phi \wedge \theta) \bigcap models(\psi)$ $= models(\phi) \bigcap models(\theta) \bigcap models(\psi) = models(\phi) \bigcap models(\theta \wedge \psi)$ $= models(\phi \wedge (\theta \wedge \psi))$. By definition of $\models\dashv$ it follows that
   $$((\phi \wedge \theta) \wedge \psi) \models\dashv (\phi \wedge (\theta \wedge \psi)).$$

- $\vee$ **is commutative.** Proof: $models(\theta \vee \psi) = models(\theta) \bigcup models(\psi) = models(\psi) \bigcup models(\theta) = models(\psi \vee \theta)$ (since set union is commutative). By definition it follows that

$$(\theta \vee \psi) \models\Dashv (\psi \vee \theta).$$

- $\vee$ **is associative.** Proof: $models((\phi \vee \theta) \vee \psi) = models(\phi \vee \theta) \bigcup models(\psi) = models(\phi) \bigcup models(\theta) \bigcup models(\psi) = models(\phi) \bigcup models(\theta \vee \psi) = models(\phi \vee (\theta \vee \psi))$ (since set union is associative). By definition it follows that

$$((\phi \vee \theta) \vee \psi) \models\Dashv (\phi \vee (\theta \vee \psi)).$$

- $\rightarrow$ **is NOT commutative.** Proof by counterexample. Take $Prop = \{p, q, r\}$. Notice that $models(q \rightarrow p) = (2^{AP} - models(q)) \bigcup models(p) = \{\emptyset, \{p\}\} \bigcup \{\{p\}, \{p, q\}\} = \{\emptyset, \{p\}, \{p, q\}\}$. Similarly, $models(p \rightarrow q) = \{\emptyset, \{q\}, \{p, q\}\}$. Since $models(p \rightarrow q) \neq models(q \rightarrow p)$ then $(p \rightarrow q)$ is NOT logically equivalent to $(q \rightarrow p)$.
- $\rightarrow$ **is NOT associative.** Proof by counterexample (different approach than above). Take $Prop = \{p, q, r\}$ and a world $\tau = \emptyset$ (i.e. all propositions are false). From the EE view, we have $((p \rightarrow q) \rightarrow r)(\tau) = 0$ and $(p \rightarrow (q \rightarrow r))(\tau) = 1$. Then, $((p \rightarrow q) \rightarrow r) \leftrightarrow (p \rightarrow (q \rightarrow r))$ is not valid, so by definition of valid $((p \rightarrow q) \rightarrow r)$ is NOT logically equivalent to $(p \rightarrow (q \rightarrow r))$.
- $\leftrightarrow$ **is commutative.** Proof: $models(\theta \leftrightarrow \psi) = ((2^{Prop} - models(\theta)) \bigcap (2^{Prop} - models(\psi))) \bigcup (models(\theta) \bigcap models(\psi)) = ((2^{Prop} - models(\psi)) \bigcap (2^{Prop} - models(\theta))) \bigcup (models(\psi) \bigcap models(\theta)) = models(\psi \leftrightarrow \theta)$ by set intersection and union commutativity. By definition of $\models\Dashv$ it follows that

$$(\theta \leftrightarrow \psi) \models\Dashv (\psi \leftrightarrow \theta).$$

- $\leftrightarrow$ **is associative.** Proof: $models((\phi \leftrightarrow \theta) \leftrightarrow \psi) = models(\phi \leftrightarrow \theta) \cap models(\psi)) \cup ((2^{Prop} - models(\phi \leftrightarrow \theta)) \cap (2^{Prop} - models(\psi)))$. Expanding $models(\phi \leftrightarrow \theta) = (models(\phi) \cap models(\theta)) \cup ((2^{Prop} - models(\phi)) \cap (2^{prop} - models(\theta)))$ and regrouping using associativity from set theory, we get this is equal to $(models(\phi) \cap models(\theta \leftrightarrow \psi)) \cup ((2^{Prop} - models(\phi)) \cap (2^{prop} - models(\theta \leftrightarrow \psi))) = models(\phi \leftrightarrow (\theta \leftrightarrow \psi))$. By definition of $\models\Dashv$ it follows that

$$((\phi \leftrightarrow \theta) \leftrightarrow \psi) \models\Dashv (\phi \leftrightarrow (\theta \leftrightarrow \psi)).$$

2. *Prove de Morgan's Laws*

   - $\models ((\neg(p \wedge q)) \leftrightarrow ((\neg p) \vee (\neg q))$
   - $\models ((\neg(p \vee q)) \leftrightarrow ((\neg p) \wedge (\neg q))$

Let

$$
\begin{aligned}
\phi &= ((\neg(p \wedge q)) \leftrightarrow ((\neg p) \vee (\neg q)) \\
\psi &= ((\neg(p \vee q)) \leftrightarrow ((\neg p) \wedge (\neg q))
\end{aligned}
$$

Proof:
By definition, ϕ is valid iff

$$
\begin{aligned}
& models(\phi) = 2^{Prop} \\
\text{iff} \quad & models(\phi) = 2^{AP(\phi)} \text{ (by Relevance Lemma)} \\
\text{iff} \quad & \phi(\tau) = 1, \forall \tau \in 2^{AP(\phi)} \text{ (by Lemma from class)}
\end{aligned}
$$

Similarly, ψ is valid iff

$$
\begin{aligned}
& models(\psi) = 2^{Prop} \\
\text{iff} \quad & models(\psi) = 2^{AP(\psi)} \text{ (by Relevance Lemma)} \\
\text{iff} \quad & \psi(\tau) = 1, \forall \tau \in 2^{AP(\psi)} \text{ (by Lemma from class)}
\end{aligned}
$$

The following table shows that $models(\phi) = 2^{AP(\phi)}$ and $models(\psi) = 2^{AP(\psi)}$.

p	q	$((\neg(p \wedge q))$	$((\neg p) \vee (\neg q)$	ϕ	$((\neg(p \vee q))$	$((\neg p) \wedge (\neg q))$	ψ
0	0	1	1	1	1	1	1
0	1	1	1	1	0	0	1
1	0	1	1	1	0	0	1
1	1	0	0	1	0	0	1

3. *Show that logical and material equivalence coincide, that is,* $\phi \models\!\!=\!\!| \psi$ *iff* $\models (\phi \leftrightarrow \psi)$.

   By definition,

$$
\begin{aligned}
& \phi \models\!\!\!\dashv \psi \\
\text{iff} \quad & \phi \models \psi \text{ and } \psi \models \phi && \text{(by definition of } \models\!\!\!\dashv \text{)} \\
\text{iff} \quad & \models (\phi \rightarrow \psi) \text{ and } \models (\phi \rightarrow \psi) && \text{(by Lemma 1 in Lecture 8)} \\
\text{iff} \quad & \models (\phi \leftrightarrow \psi) && \text{(by definition of } \leftrightarrow \text{)}
\end{aligned}
$$

4. *Show that* $\phi \models \psi$ *iff* $(\phi \wedge \neg\psi)$ *is not satisfiable.*

   Proof:
   By definition,

$$\begin{array}{ll} & (\phi \wedge \neg\psi) \text{ is not satisfiable} \\ \text{iff} & \neg(\phi \wedge \neg\psi) \text{ is valid} \\ \text{iff} & \models \neg(\phi \wedge \neg\psi) \\ \text{iff} & \models (\neg\phi \vee \psi) \text{ (by de Morgan's Law)} \\ \text{iff} & \models (\phi \rightarrow \psi) \text{ (by definition of } \rightarrow) \\ \text{iff} & \phi \models \psi \text{ (by definition of logical implication)} \end{array}$$

5. *Prove that* $\models$ *is reflexive, transitive, but not symmetric. Prove that* $\models\!\!\dashv$ *is reflexive, transitive, and symmetric.*

   Notice that $\models$ is *not* a relation between $2^{\textsc{Prop}}$ and $\textsc{Form}$, nor between $2^{\textsc{Prop}}$ and $2^{\textsc{Form}}$ (otherwise neither of the properties would make sense). The only option that is left is $\models \subseteq 2^{\textsc{Form}} \times 2^{\textsc{Form}}$.

   Let Φ, $\Psi \in 2^{\textsc{Form}}$

   - $\models$ **is reflexive.** Proof: since $models(\Phi) \subseteq models(\Phi)$ then $\Phi \models \Phi$ by definition.
   - $\models$ **is transitive.** Proof: if $\Phi \models \Psi$ and $\Psi \models \Theta$ then $models(\Phi) \subseteq models(\Psi)$ and $models(\Psi) \subseteq models(\Theta)$. Since subset is a transitive relation, $models(\Psi) \subseteq models(\Theta)$, so $\Phi \models \Theta$.
   - $\models$ **is NOT symmetric.** Proof by counterexample. Let $\Phi = \{(p \wedge q)\}$, $\Psi = \{p\}$, $Prop = \{p, q\}$. Then, $models(\Phi) = \{\{p, q\}\}$ and $models(\Psi) = \{\{p\}, \{p, q\}\}$. Since $models(\Phi) \subseteq models(\Psi)$ but $models(\Psi) \not\subseteq models(\Phi)$, it follows from the definition of $\models$ that $\Phi \models \Psi$ but $\Psi \not\models \Phi$.

   In the case of $\models\!\!\dashv$ we also have sets of formulas on both sides.

   - $\models\!\!\dashv$ **is reflexive.** Proof: Since $models(\Phi) = models(\Phi)$ then $\Phi \models\!= |\Phi$ by definition.
   - $\models\!\!\dashv$ **is transitive.** Proof: if $\Phi \models\!\!\dashv \Psi$ and $\Psi \models\!\!\dashv \Theta$ then $models(\Phi) = models(\Psi)$ and $models(\Psi) = models(\Theta)$, so $models(\Phi) = models(\Theta)$ and by definition $\Phi \models\!\!\dashv \Theta$.
   - $\models\!\!\dashv$ **is symmetric.** Proof: $\Phi \models\!\!\dashv \Psi$ iff $models(\Phi) = models(\Psi)$ iff $models(\Psi) = models(\Phi)$ iff $\Psi \models\!\!\dashv \Phi$.

6. *Show that if* ϕ *and* ψ *are logically equivalent and* ϕ *is valid, then* ψ *is valid.*

   Restating the problem formally, we show that if $\phi \models\!\!\dashv \psi$ and $\models \phi$, then $\models \psi$ (by definition of logical equivalence and logical validity).

Proof: By definition of $\models\mid$, $\phi \models\mid \psi$ iff $models(\phi) = models(\psi)$. By definition of logically valid, $\models \phi$ iff $models(\phi) = 2^{Prop}$. Thus, $models(\psi) = 2^{Prop}$ and hence $\models \psi$.

7. *Which of the formulas implies the other?*

   Let

$$\begin{aligned} \phi &= (p \leftrightarrow (q \leftrightarrow r)) \\ \psi &= ((p \wedge (q \wedge r)) \vee ((\neg p) \wedge (\neg q) \wedge (\neg r))) \end{aligned}$$

   Then, $models(\phi) = \{\{p\}, \{p, q, r\}, \{q\}, \{r\}\}$ and $models(\psi) = \{\{p, q, r\}, \emptyset\}$. Since neither $models(\phi) \subseteq models(\psi)$ nor $models(\psi) \subseteq models(\phi)$ then neither $\phi \models \psi$ nor $\psi \models \phi$.

8. *Show that* $\Sigma \cup \{\alpha\} \models \beta$ *iff* $\Sigma \models (\alpha \rightarrow \beta)$.

   First, we need some help:

   Lemma 1. *If A and B are sets, then $A \subseteq (A \cap B) \cup \overline{B}$.*

   Proof. Suppose that $a \in A$. We must show that $a \in A \subseteq (A \cap B) \cup \overline{B}$. Either $a \in B$ or $a \notin B$. In the first case $a \in (A \cap B)$, in the second, $a \in \overline{B}$. In either case $a \in A \subseteq (A \cap B) \cup \overline{B}$, which is what we had to prove. □

   Proof of the main problem.

$$\begin{aligned}
& \Sigma \cup \{\alpha\} \models \beta \\
\text{iff} \quad & models(\Sigma \cup \{\alpha\}) \subseteq models(\beta) \\
\text{iff} \quad & models(\Sigma) \cap models(\alpha) \subseteq models(\beta) \\
\text{iff} \quad & (models(\Sigma) \cap models(\alpha)) \cup (2^{Prop} - models(\alpha)) \\
& \subseteq (2^{Prop} - models(\alpha)) \cup models(\beta) \qquad (1) \\
& \text{(union the same set to each side)} \\
\text{iff} \quad & models(\Sigma) \subseteq (2^{Prop} - models(\alpha)) \cup models(\beta) \text{ (by Lemma)} \qquad (2) \\
\text{iff} \quad & models(\Sigma) \subseteq models(\alpha \rightarrow \beta) \text{ (by defn of } \rightarrow) \\
\text{iff} \quad & \Sigma \models (\alpha \rightarrow \beta) \text{ (by defn of } \models)
\end{aligned}$$

   Note that the step from (1) to (2) is justified by Lemma 1, but this is not the case when we traverse the proof in the opposite direction. For this we need another lemma, which we state and prove below.

   Lemma 2. *Let Σ, U, A and B be sets. If*

$$\begin{aligned} \Sigma &\subseteq (U - A) \cup B, \text{ then} \\ (\Sigma \cap A) \cup (U - A) &\subseteq (U - A) \cup B. \end{aligned}$$

Proof. Since $(\Sigma \cap A) \subseteq \Sigma$ and $\Sigma \subseteq (U - A) \cup B$, we can conclude that $(\Sigma \cap A) \subseteq (U - A) \cup B$. Clearly $(U - A) \subseteq (U - A) \cup B$, therefore the left-hand side is the union of two sets which are both contained in the right-hand side. □

The proof of Lemma 2 concludes the proof of the main problem. □

9. *Prove or refute the following statements.*

- If either $\Sigma \models \alpha$ or $\Sigma \models \beta$ then $\Sigma \models (\alpha \vee \beta)$. **TRUE.**
  Proof:

$$\begin{array}{rcl} \text{If either } models(\Sigma) & \subseteq & models(\alpha) \text{ or} \\ models(\Sigma) & \subseteq & models(\beta) \text{ then} \\ models(\Sigma) & \subseteq & models(\alpha) \cup models(\beta) \\ & \subseteq & models(\alpha \vee \beta) \text{ by definition.} \end{array}$$

  Then $\Sigma \models (\alpha \vee \beta)$.

- If $\Sigma \models (\alpha \vee \beta)$, then either $\Sigma \models \alpha$ or $\Sigma \models \beta$. **FALSE.**
  Proof by counterexample:
  Take $Prop = \{p, q\}, \Sigma = \{p\}, \alpha = (p \wedge q), \beta = (\neg(p \wedge q))$ Then:

$$\begin{array}{rcl} models(\Sigma) & = & \{\{p\}, \{p, q\}\} \\ models(\alpha) & = & \{\{p, q\}\} \\ models(\beta) & = & \{\emptyset, \{p\}, \{q\}\} \end{array}$$

$$\begin{array}{rcl} \text{Here, } models(\Sigma) & \subseteq & models(\alpha) \cup models(\beta) \text{ iff } \Sigma \models (\alpha \vee \beta) \\ \text{but } models(\Sigma) & \not\subseteq & models(\alpha) \text{ therefore } \Sigma \not\models \alpha \\ \text{and } models(\Sigma) & \not\subseteq & models(\beta) \text{ therefore } \Sigma \not\models \beta \end{array}$$

  Which completes the proof.

10. *A set Σ of formulas is independent if $\Sigma - \{\sigma\} \not\models \sigma$, $\forall \sigma \in \Sigma$.*

Let $\Sigma = \{\sigma_1, \sigma_2, ..., \sigma_n\}$ be a finite set of formulas, if Σ is independent already then we have the final result we wanted.

If Σ is not independent, then it means that there is at least one σ_i, such that $\sigma_1 \wedge \sigma_2 \wedge ... \wedge \sigma_{i-1} \wedge \sigma_{i+1} \wedge ... \wedge \sigma_n \models \sigma_i$. That is σ_i is implied by the conjunction all of the formulas from Σ except σ_i itself.

In order to make Σ an independent set of formulas, we can eliminate σ_i from the set. The set $\Sigma - \{\sigma_i\}$ will still be equivalent to the original set. We can continue to do this until there are no more dependent formulas in the resulting set.

On the other hand, if Σ is an infinite set of formulas ($\Sigma = \{\sigma_1, \sigma_2, ...\}$), such as:

$$\begin{aligned} \sigma_1 &= p_1 \\ \sigma_2 &= p_1 \wedge p_2 \\ &\vdots \\ \sigma_n &= p_1 \wedge p_2 \wedge ... \wedge p_n \\ &\vdots \end{aligned}$$

In this case, the set itself is not independent. If we take a finite subset of the formulas, it's not going to be equivalent, since an infinite number of formulas can not be generated from the finite subset (any formula with atomic propositions with subscripts greater than the greatest subscript present in the subset).

If we take an infinite subset, say by eliminating σ_i from the set, we are still able to generate the missing formula from any $\sigma_j, j > i$. Take σ_{i+1}, for example, if $p_1 \wedge p_2 \wedge ... \wedge p_i \wedge p_{i+1}$ holds then $p_1 \wedge p_2 \wedge ... \wedge p_i$ holds also. This infinite subset is equivalent, but it is not independent since it possesses the same properties as the original set. Any other infinite subset is also equivalent to the original set, due to the same reasoning.

4 Substitutions

If ϕ is valid, then $\phi[p \mapsto \theta]$ is valid.

1. If $\models \varphi$, then $\models \varphi[p \rightarrow \theta]$.

   Proof: Suppose that $\models \varphi$. Let $\tau \in 2^{\text{PROP}}$. We wish to show that $\tau(\varphi[p \rightarrow \theta]) = 1$. Let $\sigma \in 2^{\text{PROP}}$ be identical to τ except that $\sigma(p) = \tau(\theta)$. First we show that $\tau(\varphi[p \rightarrow \theta]) = \sigma(\varphi)$.

   Case: $\varphi = p$. Then $\tau(p[p \rightarrow \theta]) = \tau(\theta) = \sigma(p)$.

   Case $\varphi = q \neq p$. Then $\tau(q[p \rightarrow \theta]) = \tau(q) = \sigma(q)$.

   Case $\varphi = (\neg\psi)$. Then $\tau((\neg\psi)[p \rightarrow \theta]) = \tau(\neg(\psi[p \rightarrow \theta])) = \neg(\tau(\psi[p \rightarrow \theta]))$ $= \neg(\sigma(\psi)) = \sigma(\neg(\psi)) = \sigma(\neg\psi)$.

   Case $\varphi = (\theta \circ \psi)$. Then $\tau((\theta \circ \psi)[p \rightarrow \theta]) = \tau(\theta[p \rightarrow \theta] \circ \psi[p \rightarrow \theta]) = \circ(\tau(\theta[p \rightarrow \theta]), \tau(\psi[p \rightarrow \theta])) = \circ(\sigma(\theta), \sigma(\psi)) = \sigma(\theta \circ \psi)$.

   Now, since every truth assignment makes φ true (because it is valid), σ in particular makes it true, and so τ makes $\varphi[p \rightarrow \theta]$ true.

2. If α is equivalent to β, then $\varphi[p \to \alpha]$ is equivalent to $\varphi[p \to \beta]$.

   Proof: Let τ be an arbitrary truth assignment. We wish to show $\varphi[p \to \alpha]$ is equivalent to $\varphi[p \to \beta]$. We proceed by structural induction.

   Case $\varphi = p$. Then $\tau(p[p \to \alpha]) = \tau(\alpha) = \tau(\beta)$ (because α, β equivalent) $= \tau(p[p \to \beta])$.

   Case $\varphi = q \neq p$. Then $\tau(q[p \to \alpha]) = \tau(q) = \tau(q[p \to \beta])$.

   Case $\varphi = (\neg\theta)$. Then $\tau((\neg\theta)[p \to \alpha]) = \tau(\neg\theta[p \to \alpha]) = \neg(\tau(\theta[p \to \alpha])) = \neg(\tau(\theta[p \to \beta]))$ (by the inductive hypothesis) $= \tau(\neg\theta[p \to \beta]) = \tau((\neg\theta)[p \to \beta])$.

   Case $\varphi = (\theta \circ \psi)$. Then $\tau((\theta \circ \psi)[p \to \alpha]) = \tau(\theta[p \to \alpha] \circ \psi[p \to \alpha]) = \circ(\tau(\theta[p \to \alpha]), \tau(\psi[p \to \alpha])) = \circ(\tau(\theta[p \to \beta]), \tau(\psi[p \to \beta]))$ (I.H.) $= \tau(\theta[p \to \beta] \circ \psi[p \to \beta]) = \tau((\theta \circ \psi)[p \to \beta])$.

5 Puzzle

Let the assertion "Box i has the gold" be represented by p_i and the assertion "The clue on Box i says the truth" be represented by q_i. . The following facts are given:

- Only one box have the gold: $og = ((p_1 \vee p_2 \vee p_3) \wedge (\neg(p_1 \wedge p_2) \wedge \neg(p_1 \wedge p_3) \wedge \neg(p_2 \wedge p_3)))$.
- Box1: $b1 = (q_1 \leftrightarrow \neg p_1)$.
- Box2: $b2 = (q_2 \leftrightarrow \neg p_2)$.
- Box3: $b3 = (q_3 \leftrightarrow p_2)$.
- Only one clue is true: $ot = (q_1 \vee q_2 \vee q_3) \wedge (\neg(q_1 \wedge q_2) \wedge \neg(q_1 \wedge q_3) \wedge \neg(q_2 \wedge q_3))$.

We need a satisfying assignment to $og \wedge b1 \wedge b2 \wedge b3 \wedge ot$ to solve the puzzle. The only satisfying assignment would be $p_1 = 1, p_2 = 0, p_3 = 0, q_1 = 0, q_2 = 1, q_3 = 0$, which means box 1 contains the gold.