a projective invariant

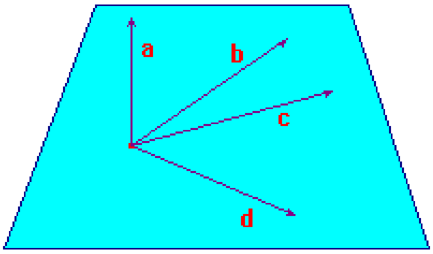
We introduce a quantity which is invariant under the projective group P(2).
This quantity is very significant geometrically, and admits several interesting interpretations. The study of this invariant goes back to Greek geometry, but it looks much more natural in a projective setting.

Suppose that A is a p-point. This is represented by a line through the origin in $\mathbf{R}^3$. If **a** is a direction vector for this line, then, as an element of $\mathbf{RP}^2$, A = [**a**] . Of course, we could replace **a** by **a'**, where **a'** = λ**a** for any non-zero real number λ.

Also, a p-line **L** is represented by a plane Π through the origin in $\mathbf{R}^3$. A p-point A = [**a**] lies on **L** if and only if **a** lies on Π. If B = [**b**] and C = [**c**] also lie on **L**, *i.e.* A,B,C are collinear p-points, then **a**, **b** and **c** lie on Π. Provided that A ≠ B, {**a**, **b**} is a basis for Π, so that there exist *unique* real numbers α and β with **c** = α**a** + β**b**. Further, if C ≠ A,B, then α and β will be *non-zero*.
Of course, α, β, and even their ratio, will depend upon the choice of the vectors **a**, **b** and **c** on their respective lines in $\mathbf{R}^3$.

If we introduce a fourth p-point on **L**, then we do have an invariant.

The cross-ratio porism
Suppose that A,B,C,D are distinct collinear p-points, and that **a,b,c,d** are *chosen* so that A=[**a**], B = [**b**], C = [**c**] and D = [**d**]. Then we have
(1) There exist unique non-zero real numbers α,β,γ and δ such that
c = α**a** + β**b**, and **d** = γ**a** + δ**b**.
(2) The ratio βγ/αδ does *not* depend on the choice of **a,b,c,d**.



Of course, part (1) follows easily from the earlier remarks about *collinear* p-points.

proof of part(2)

Since the final ratio depends only upon the p-points and *not* on the choice of vectors, we can make the

Definition
If A,B,C,D are *distinct, collinear* p-points then the cross-ratio (A,B,C,D) is the value of the ratio described in the cross-ratio porism.

Remarks
(1) The value of the cross-ratio *will* depend on the *order* of the points.
In the notation of the porism, in calculating (B,A,C,D), we have
c = β**b** + α**a** , and **d** = δ**b** + γ**a**, so that (B,A,C,D) = αδ/βγ = 1/(A,B,C,D).
The reader may care to verify that we also have (A,B,D,C) = 1/(A,B,C,D).

(2) Since we require that A,B,C,D are distinct, α,β,γ,δ are non-zero.
Thus (A,B,C,D) ≠ 0.

(3) Since C and D are distinct p-points, (α,β) ≠ λ(γ,δ) for any λ
Thus (A,B,C,D) ≠ 1.

(4) A cross-ratio can take *any* value other than 0 and 1.
To see this, *choose* **c** = **a**+**b**, and **d** = λ**a**+**b**, where λ ≠ 0,1.
The conditions on λ ensure that the p-points A,B,C,D are distinct.
Then (A,B,C,D) = 1.λ/1.1 = λ.

Suppose that **t** is a projective transformation. Then **t** maps p-lines to p-lines.
Thus, if A,B,C,D are collinear p-points, then **t**(A),**t**(B),**t**(C),**t**(D) are also collinear, so their cross-ratio is defined. We shall write **t**(A,B,C,D) for (**t**(A),**t**(B),**t**(C),**t**(D)).

The projective cross-ratio theorem
Cross-ratio is invariant under the projective group P(2).

proof

It is often useful to note that, if A,B,C lie on a p-line **L**, then there is a *unique* p-point D on **L** with a given value of (A,B,C,D). The uniqueness is guaranteed by

Theorem PI1
If A,B,C,D,E are collinear p-points such that (A,B,C,D) = (A,B,C,E), then D = E.

proof of theorem PI1

To use this invariant to obtain euclidean results, we need to consider **embeddings** of RP^2.

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