EMBEDDING CONSTRAINT RELATIONSHIPS INTO C-SEMIRINGS

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INTRODUCTION

These notes provide technical details that are required to embed constraint relationships into the c-semiring framework presented in terms of category theory. It contains all steps required to map a dag to a partial order (Section 1), construct the *free* meet monoid from this partial order (Section 3) as well as the *free* c-semiring (Section 4). A constraint solving algorithm based on branch-and-bound search is presented in §5 for c-semirings and in §7 for meet monoids. A concrete instantiation for constraint relationships along with an example soft constraint problem concludes the report in Section 7.

1. PARTIAL ORDERS AND DIRECTED ACYCLIC GRAPHS

1. A *partial order* $(X, \leq)$ is given by a set X and a binary relation $\leq \, \subseteq X \times X$ such that $\leq$ is reflexive, transitive, and anti-symmetric on X. For $x, y \in X$ we write $x < y$ if $x \leq y$ and $x \neq y$, and $x \geq y$ resp. $x > y$ if $y \leq x$ resp. $y < x$, and $x \parallel y$ if neither $x \leq y$ nor $x \geq y$.
A *partial order homomorphism* $\varphi : P \to Q$ from a partial order $P = (|P|, \leq_P)$ to a partial order $Q = (|Q|, \leq_Q)$ is given by a map $\varphi : |P| \to |Q|$ such that $\varphi(p) \leq_Q \varphi(p')$ if $p \leq_P p'$ for all $p, p' \in |P|$.
The category PO of partial orders has the partial orders as objects and the partial order homomorphisms as morphisms.

2. A *directed acyclic graph*, or *dag*, $(X, \to)$ is given by a set X and a binary relation $\to \subseteq X \times X$ such that $\to^+$ is irreflexive. If $x \to y$, then x is a *predecessor* of y, and y is a *successor* of x.
A *dag homomorphism* $\varphi : G \to H$ from a dag $G = (|G|, \to_G)$ to a dag $H = (|H|, \to_H)$ is given by a map $\varphi : |G| \to |H|$ such that $\varphi(g) \to_H \varphi(g')$ if $g \to_G g'$ for all $g, g' \in |G|$.
The category DAG of dags has the dags as objects and the dag homomorphisms as morphisms.

3. Define the functor $PO\langle - \rangle : \mathrm{DAG} \to \mathrm{PO}$ by

$$PO\langle G \rangle = (|G|, \to_G^*) ,$$
$$PO\langle \varphi : G \to H \rangle = \varphi .$$

Define the functor $DAG : \mathrm{PO} \to \mathrm{DAG}$ by

$$DAG(P) = (|P|, <_P) ,$$
$$DAG(\varphi : P \to Q) = \varphi .$$

For each $G \in |\mathrm{DAG}|$, define $\eta_G^{\mathrm{PO}} : G \to DAG(PO\langle G \rangle)$ by $\eta_G^{\mathrm{PO}}(g) = g$. Then $\eta^{\mathrm{PO}} = (\eta_G^{\mathrm{PO}})_{G \in |\mathrm{DAG}|}$ is a natural transformation from 1_{DAG} to $DAG \circ PO\langle - \rangle$.
Let $G \in |\mathrm{DAG}|$, $P \in |\mathrm{PO}|$, and $\varphi : G \to DAG(P)$. Define $\varphi^{\sharp_{\mathrm{PO}}} : PO\langle G \rangle \to P$ by

$$\varphi^{\sharp_{\mathrm{PO}}}(g) = \varphi(g) .$$

Then $DAG(\varphi^{\sharp_{\mathrm{PO}}})(\eta_G^{\mathrm{PO}}(g)) = \varphi(g)$ and $\varphi^{\sharp_{\mathrm{PO}}}$ is unique with this property.

LEMMA. *$PO\langle G \rangle$ is the free partial order over the dag G.* □

2. UPPER SEMI-LATTICES

1. A (*bounded*) *upper semi-lattice* $(X, \sqcup, \bot)$ is given by a set X, a binary operation $\sqcup : X \times X \to X$, and a constant $\bot \in X$ such that the following axioms are satisfied for all $x, y, z \in X$:
(1) $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
(2) $x \sqcup y = y \sqcup x$
(3) $x \sqcup x = x$
(4) $x \sqcup \bot = x$
In words, $\sqcup$ is associative, commutative, and idempotent, and has $\bot$ as neutral element.
A (*bounded*) *upper semi-lattice homomorphism* $\varphi : U \to V$ from an upper semi-lattice $U = (|U|, \sqcup_U, \bot_U)$ to an upper semi-lattice $V = (|V|, \sqcup_V, \bot_V)$ is given by a map $\varphi : |U| \to |V|$ such that for all $u_1, u_2 \in |U|$:
(1) $\varphi(u_1 \sqcup_U u_2) = \varphi(u_1) \sqcup_V \varphi(u_2)$
(2) $\varphi(\bot_U) = \bot_V$
The category uSL of upper semi-lattices has the upper semi-lattices as objects and the upper semi-lattice homomorphisms as morphisms.

2. Let P be a partial order. Let $\mathcal{I}_{\mathrm{fin}}(P)$ denote the set of finite subsets of $|P|$ which only contain pairwise incomparable elements w.r.t $\le_P$. For a subset $S \subseteq |P|$, let $\mathrm{Max}^{\le_P}(S)$ denote the set of maximal elements of S w.r.t. $\le_P$ (in particular, if S is finite, $\mathrm{Max}^{\le_P}(S) \in \mathcal{I}_{\mathrm{fin}}(P)$).
Define the binary operation $\cup_P : \mathcal{I}_{\mathrm{fin}}(P) \times \mathcal{I}_{\mathrm{fin}}(P) \to \mathcal{I}_{\mathrm{fin}}(P)$ by

$$I \cup_P J = \mathrm{Max}^{\le_P}(I \cup J) .$$

LEMMA. *$(\mathcal{I}_{\mathrm{fin}}(P), \cup_P, \emptyset)$ is an upper semi-lattice.*

Proof. Let $I, J, K \in \mathcal{I}_{\text{fin}}(P)$. For the associativity of $\cup_P$ we have

$$I \cup_P (J \cup_P K) = \mathrm{Max}^{\leq_P}(I \cup \mathrm{Max}^{\leq}(J \cup K)) = \mathrm{Max}^{\leq_P}(I \cup J \cup K) = \\ \mathrm{Max}^{\leq_P}(\mathrm{Max}^{\leq_P}(I \cup J) \cup K) = (I \cup_P J) \cup_P K \ ,$$

since $\mathrm{Max}^{\leq_P}(I \cup \mathrm{Max}^{\leq_P} X) = \mathrm{Max}^{\leq_P}(I \cup X)$ for all $X \in \mathcal{P}_{\text{fin}}\,|P|$. $\cup_P$ inherits commutativity from $\cup$. For the idempotency of $\cup_P$ we have

$$I \cup_P I = \mathrm{Max}^{\leq_P}(I \cup I) = \mathrm{Max}^{\leq_P} I = I \ ,$$

since $I \in \mathcal{I}_{\text{fin}}(P)$. Finally, we have $I \cup_P \emptyset = I$. $\square$

Define the functor $uSL\langle - \rangle : \mathrm{PO} \to \mathrm{uSL}$ by

$$uSL\langle P \rangle = (\mathcal{I}_{\text{fin}}(P), \cup_P, \emptyset) \ ,$$
$$uSL\langle \varphi : P \to Q \rangle = \lambda\{p_1, \dots, p_n\} \in \mathcal{I}_{\text{fin}}(P) \, . \, \mathrm{Max}^{\leq_Q}\{\varphi(p_1), \dots, \varphi(p_n)\} \ .$$

3. Each upper semi-lattice U induces a partial ordering $\leq_U \subseteq |U| \times |U|$ on $|U|$ given by

$$u_1 \leq_U u_2 \iff u_1 \sqcup_U u_2 = u_2 \ .$$

Indeed, $\leq_U$ is reflexive on $|U|$ by the idempotency of $\sqcup_U$, $\leq_U$ is transitive by the associativity of $\sqcup_U$, and $\leq_U$ is anti-symmetric by the commutativity of $\sqcup_U$. Furthermore, $\perp_U$ is the smallest element w.r.t. $\leq_U$, i.e., $\perp_U \leq_U u$ for all $u \in |U|$, by the neutrality of $\perp_U$.
Define the functor $PO : \mathrm{uSL} \to \mathrm{PO}$ by

$$PO(U) = (|U|, \leq_U) \ ,$$
$$PO(\varphi : U \to V) = \varphi \ ,$$

which is well-defined on objects by the remarks above and also morphisms since if $u_1 \leq_U u_2$, i.e., $u_1 \sqcup_U u_2 = u_2$, then $\varphi(u_1) \sqcup_V \varphi(u_2) = \varphi(u_1 \sqcup_U u_2) = \varphi(u_2)$, i.e., $\varphi(u_1) \sqcup_V \varphi(u_2)$.
For each $P \in |\mathrm{PO}|$, define $\eta_P^{\mathrm{uSL}} : P \to PO(uSL\langle P \rangle)$ by $\eta_P^{\mathrm{uSL}}(p) = \{p\}$. Then $\eta^{\mathrm{uSL}} = (\eta_P^{\mathrm{uSL}})_{P \in |\mathrm{PO}|}$ is a natural transformation from 1_{PO} to $PO \circ uSL\langle - \rangle$.
Let $P \in |\mathrm{PO}|$, $U \in |\mathrm{uSL}|$, and $\varphi : P \to PO(U)$. Define $\varphi^{\sharp_{\mathrm{uSL}}} : uSL\langle P \rangle \to U$ by

$$\varphi^{\sharp_{\mathrm{uSL}}}(\{p_1, \dots, p_n\}) = \varphi(p_1) \sqcup_U \dots \sqcup_U \varphi(p_n)$$

for all $\{p_1, \dots, p_n\} \in \mathcal{I}_{\text{fin}}(P)$, where, if $n = 0$, the right hand side is to be understood as $\perp_U$; $\varphi^{\sharp_{\mathrm{uSL}}}$ is indeed an upper semi-lattice homomorphism, since for each $\{p'_1, \dots, p'_n\} \in \mathcal{P}_{\text{fin}}\,|P|$ we have $\varphi^{\sharp_{\mathrm{uSL}}}(\mathrm{Max}^{\leq_P}\{p'_1, \dots, p'_n\}) = \varphi(p'_1) \sqcup_U \dots \sqcup_U \varphi(p'_n)$: if $p'_i \leq_P p'_j$, then $\varphi(p'_i) \leq_{PO(U)} \varphi(p'_j)$, i.e., $\varphi(p'_i) \sqcup_U \varphi(p'_j) = \varphi(p'_j)$.
Then $PO(\varphi^{\sharp_{\mathrm{uSL}}})(\eta_P^{\mathrm{uSL}}(p)) = \varphi(p)$ and $\varphi^{\sharp_{\mathrm{uSL}}}$ is unique with this property.

LEMMA. *$uSL\langle P \rangle$ is the free upper semi-lattice over the partial order P.* $\square$

4. The partial ordering of $PO(uSL\langle P \rangle)$ on $\mathcal{I}_{\text{fin}}(P)$ for a partial order P is called the *lower* or *Hoare* ordering on $\mathcal{I}_{\text{fin}}(P)$ which we denote by $\subseteq_P$; it is explicitly given by

$$\begin{aligned} I \subseteq_P J &\iff I \cup_P J = J \\ &\iff \mathrm{Max}^{\leq_P}(I \cup J) = J \\ &\iff \forall p \in I \, . \, \exists q \in J \, . \, p \leq_P q \end{aligned}$$

for $I, J \in \mathcal{I}_{\text{fin}}(P)$. It is $\emptyset \subseteq_P I$ for all $I \in \mathcal{I}_{\text{fin}}(P)$.

The dual of the Hoare ordering is the *upper* or *Smyth ordering* $\subseteq^P$ on $\mathcal{I}_{\text{fin}}(P)$ defined by $I \subseteq^P J$ if, and only if, $J \subseteq_{P^{-1}} I$, where $P^{-1} = (|P|, \geq_P)$. Explicitly, the Smyth ordering is given by

$$\begin{aligned} I \subseteq^P J &\iff \mathrm{Min}^{\leq_P}(I \cup J) = I \\ &\iff \forall q \in J \,.\, \exists p \in I \,.\, p \leq_P q \end{aligned}$$

where $\mathrm{Min}^{\leq_P}(S)$ is the set of minimal elements of $S \subseteq |P|$. In particular, the Smyth ordering also induces a binary operation $\cup^P : \mathcal{I}_{\text{fin}}(P) \times \mathcal{I}_{\text{fin}}(P) \to \mathcal{I}_{\text{fin}}(P)$ given by

$$I \cup^P J = \mathrm{Min}^{\leq_P}(I \cup J) \ ,$$

which is also associative, commutative, and idempotent. Here, $I \cup^P \emptyset = I$, i.e., $\emptyset$ is again a neutral element for $\cup^P$, but $I \subseteq^P \emptyset$ for all $I \in \mathcal{I}_{\text{fin}}(P)$, i.e., $\emptyset$ is the greatest element of $\mathcal{I}_{\text{fin}}(P)$ w.r.t. $\subseteq^P$.
The *convex* or *Plotkin ordering* on $\mathcal{I}_{\text{fin}}(P)$ is defined by the intersection of $\subseteq_P$ and $\subseteq^P$, which means

$$I \ (\subseteq_P \cap \subseteq^P) \ J \iff (\forall p \in I \,.\, \exists q \in J \,.\, p \leq_P q) \wedge (\forall q \in J \,.\, \exists p \in I \,.\, p \leq_P q)$$

for $I, J \in \mathcal{I}_{\text{fin}}(P)$.
Finally, $\cup_P$ is monotonic w.r.t. $\subseteq_P$, and $\cup^P$ is monotonic w.r.t. $\subseteq^P$, i.e., for all $I, J, K \in \mathcal{I}_{\text{fin}}(P)$,

$$\begin{aligned} &I \subseteq_P J \text{ implies } I \cup_P K \subseteq_P J \cup_P K \ , \\ &I \subseteq^P J \text{ implies } I \cup^P K \subseteq^P J \cup^P K \ . \end{aligned}$$

3. Partially Ordered Commutative Monoids

1. A *partially ordered commutative monoid* $(X, \cdot, \varepsilon, \leq)$ is given by a set X, a binary operation $\cdot : X \times X \to X$, a constant $\varepsilon \in X$, and a partial order relation $\leq \subseteq X \times X$ such that the following axioms are satisfied for $x, x', y, y', z \in X$:

(1) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
(2) $x \cdot y = y \cdot x$
(3) $x \cdot \varepsilon = x$
(4) if $x \leq x'$ and $y \leq y'$, then $x \cdot y \leq x' \cdot y'$

In words, $(X, \cdot, \varepsilon)$ is a commutative monoid with unity ε and $\leq$ is monotone w.r.t. $\cdot$. (By commutativity, it is enough to require that $x \leq x'$ implies $x \cdot y \leq x' \cdot y$ to achieve monotonicity of $\leq$ w.r.t. $\cdot$.)[1]
A *partially ordered commutative monoid homomorphism* $\varphi : M \to N$ from a partially ordered commutative monoid $M = (|M|, \cdot_M, \varepsilon_M, \leq_M)$ to a partially ordered commutative monoid $N = (|N|, \cdot_N, \varepsilon_N, \leq_N)$ is given by a map $\varphi : |M| \to |N|$ such that for all $m, n \in |M|$:

(1) $\varphi(m \cdot_M n) = \varphi(m) \cdot_N \varphi(n)$
(2) $\varphi(\varepsilon_M) = \varepsilon_N$
(3) if $m \leq_M n$, then $\varphi(m) \leq_N \varphi(n)$

The category pocMon of partially ordered commutative monoids has the partially ordered commutative monoids as objects and the partially ordered commutative monoids homomorphisms as morphisms.

[1]Partially ordered commutative monoids have also been called *preference degree structures* [**fargier-rollon-wilson:cj:2010**].

2. A partially ordered commutative monoid M is a *join monoid* if for all $m, n \in |M|$

$$m \leq_M m \cdot_M n \ .$$

This requirement is equivalent to requiring that ε_M is the smallest element w.r.t. $\leq_M$. Indeed, if $m \leq_M m \cdot_M n$ holds for all $m, n \in |M|$, then $\varepsilon_M \leq_M \varepsilon_M \cdot_M n = n$ for all $n \in |M|$. Conversely, if $\varepsilon_M \leq_M n$ for all $n \in |M|$, then $m = m \cdot_M \varepsilon_M \leq_M m \cdot_M n$ for all $m, n \in |M|$ by the monotonicity of $\leq_M$.
Dually, a partially ordered commutative monoid M is a *meet monoid* if for all $m, n \in |M|$

$$m \cdot_M n \leq_M m \ ,$$

and this requirement is equivalent to requiring that ε_M is the greatest element w.r.t. $\leq_M$.[2]
The full sub-categories of pocMon having all join monoids respectively meet monoids as objects are denoted by jMon and mMon, respectively.
There are functors

$$\begin{aligned} &\mathit{jMon} : \mathrm{mMon} \to \mathrm{jMon} && \mathit{mMon} : \mathrm{jMon} \to \mathrm{mMon} \\ &\mathit{jMon}(M) = (|M|, \cdot_M, \varepsilon_M, \leq_M^{-1}) && \mathit{mMon}(M) = (|M|, \cdot_M, \varepsilon_M, \leq_M^{-1}) \\ &\mathit{jMon}(\varphi : M \to N) = \varphi && \mathit{mMon}(\varphi : M \to N) = \varphi \end{aligned}$$

such that $\mathit{jMon} \circ \mathit{mMon} = 1_{\mathrm{jMon}}$ and $\mathit{mMon} \circ \mathit{jMon} = 1_{\mathrm{mMon}}$.

3. For a set X let $\mathcal{M}_{\text{fin}}(X)$ be the set of finite multisets over X. We write $\wr x_1, \dots, x_m \int$ with $x_i \in X$ for $1 \leq i \leq m$ or $\wr l_1 x_1, \dots, l_n x_n \int$ with $x_i \in X$ and $l_i \in \mathbb{N}$ for $1 \leq i \leq n$ for an element of $\mathcal{M}_{\text{fin}}(X)$, $T \uplus U$ for the multiset union of the multisets T and U, and $T \Subset U$ for the sub-multiset relation, which is a partial ordering relation on $\mathcal{M}_{\text{fin}}(X)$.
For a partial order P, the *lower* or *Hoare ordering* on $\mathcal{M}_{\text{fin}}\,|P|$ is the binary relation $\Subset_P \subseteq (\mathcal{M}_{\text{fin}}\,|P|) \times (\mathcal{M}_{\text{fin}}\,|P|)$ given by the transitive closure of

$$\begin{aligned} &T \Subset U \text{ implies } T \Subset_P U \ , \\ &p \leq_P q \text{ implies } T \uplus \wr p \int \Subset_P T \uplus \wr q \int \ . \end{aligned}$$

If $T \Subset_P U$, then $T \uplus \wr r \int \Subset_P U \uplus \wr r \int$ for all $r \in X$, since this holds for both defining clauses of the ordering.
For an element $T = \wr l_1 x_1, \dots, l_n x_n \int \in \mathcal{M}_{\text{fin}}(X)$ with $l_1, \dots, l_n > 0$, $x_i \neq x_j$ for all $1 \leq i \neq j \leq n$, and $n \geq 0$ let $\mathcal{S}(T) = \bigcup_{1 \leq i \leq n} \{(j, x_i) \mid 1 \leq j \leq l_i\}$.

LEMMA. *$T \Subset_P U$ if, and only if, there is an injective mapping $f : \mathcal{S}(T) \to \mathcal{S}(U)$ with $p \leq_P q$ if $f(j, p) = (k, q)$ for all $(j, p) \in \mathcal{S}(T)$.*

Proof. Let first $T \Subset_P U$ hold. Then there are an $n > 1$ and $T_1, \dots, T_n \in \mathcal{M}_{\text{fin}}(X)$ such that $T_1 = T$, $T_n = U$, and either $T_i \Subset T_{i+1}$ or $T_i = T_i' \uplus \wr p \int$ and $T_{i+1} = T_i' \uplus \wr q \int$ with $p \leq_P q$ for all $1 \leq i < n$. For each $1 \leq i < n$ there is a map $f_i : \mathcal{S}(T_i) \to \mathcal{S}(T_{i+1})$ as required in the claim as follows: If $T_{n-1} \Subset T_n$, then we choose $f_i = 1_{\mathcal{S}(T_i)}$. If $T_i = T_i' \uplus \wr p \int$ and $T_{i+1} = T_i' \uplus \wr q \int$ with $p \leq_P q$, then we choose $f_i = 1_{\mathcal{S}(T_i')} \cup \{(j, p) \mapsto (k, q)\}$ where $j = |\{l \mid (l, p) \in \mathcal{S}(T_i')\}| + 1$ and $k = |\{l \mid (l, q) \in \mathcal{S}(T_i')\}| + 1$. Then $f_n \circ \ldots \circ f_1 : \mathcal{S}(T) \to \mathcal{S}(U)$ as required in the claim.

[2] Meet monoids have also been referred to as *partial valuation structures* [**Gadducci2013**] and *ic-monoids* [**Holzl2009**].
For soft constraint applications, $m \leq_M n$ will represent the fact that value m is "worse than" n, so ε will be the top (and best) element of the ordering. Think of m and n as abstract weights where more is worse ("bad points", "penalty points" or heavy items). To illustrate the meaning of "$\leq$" think of a teeter-totter (or a weighing scale) where on the left side the heavier item (or kid) sits and the lightest item (or kid) is the best. The position of the weighing scale then resembles $<$ (precisely $\diagup$). Similarly, a solution violating many constraints will carry a heavy weight and we search for the lightest solution.

For the converse, we prove that if $f : \mathcal{S}(T) \to \mathcal{S}(U)$ is a mapping as required in the claim, then $T \sqsubseteq_P U$ by induction on the cardinality of $\mathcal{S}(T)$. Let $f : \mathcal{S}(T) \to \mathcal{S}(U)$ be given. If $|\mathcal{S}(T)| = 0$, then $\wr\int = T \sqsubseteq U$. Now let $|\mathcal{S}(T)| > 0$ and let $(j, p) \in \mathcal{S}(T)$ such that j is maximal. Then $f(j, p) = (k, q)$ with $p \leq_P q$. Define $g : \mathcal{S}(U) \to \mathcal{S}(U) \setminus \{(k, q)\}$ by $g(l, r) = (l, r)$ if $r \neq q$ or $l < k$, and $g(l, q) = (l - 1, q)$ if $l > k$. Let $T', U' \in \mathcal{M}_{\text{fin}}(X)$ be defined by $T = T' \uplus \wr p \int$ and $U = U' \uplus \wr q \int$. Then $\mathcal{S}(T') = \mathcal{S}(T) \setminus \{(j, p)\}$ and $f' : \mathcal{S}(T') \to \mathcal{S}(U')$ defined by $f'(l, r) = g(f(l, r))$ for all $(l, r) \in \mathcal{S}(T')$ is an injective mapping as required in the claim. By induction hypothesis $T' \sqsubseteq_P U'$ and thus, by the remark above, $T = T' \uplus \wr p \int \sqsubseteq_P U' \uplus \wr p \int \sqsubseteq_P U' \uplus \wr q \int = U$. $\square$

We call such a map a *witness* for $T \sqsubseteq_P U$.
The relation $\sqsubseteq_P$ is obviously transitive and reflexive on $\mathcal{M}_{\text{fin}} |P|$. It is also antisymmetric: Assume for a contradiction that there are T and U with $T \sqsubseteq_P U$ and $U \sqsubseteq_P T$, but $T \neq U$ and choose an T with minimal cardinality satisfying this property. Then $T \neq \wr\int$. Let $f : \mathcal{S}(T) \to \mathcal{S}(U)$ and $g : \mathcal{S}(U) \to \mathcal{S}(T)$ be witnessing maps for $T \sqsubseteq_P U$ and $U \sqsubseteq_P T$. Choose an element $(j, p) \in \mathcal{S}(T)$ such that p is maximal w.r.t. $\leq_P$ in T. Let $f(j, p) = (k, q) \in \mathcal{S}(U)$. Then $p \leq_P q$. If $p \neq q$, there would be a $(j', p') \in \mathcal{S}(T)$ with $p \leq_P q \leq_P p'$ but $p \neq p'$ contradicting the maximality of p in T; thus $f(j, p) = (k, p)$. Assume, without loss of generality, that j and k are maximal. Remove the occurrence of p from T, obtaining T', and from U, obtaining U'. Then $T' \sqsubseteq_P U'$ and $U' \sqsubseteq_P T'$, since $f' : \mathcal{S}(T') \to \mathcal{S}(U')$ with $f'(x) = f(x)$ if $x \neq (j, p)$ and $g' : \mathcal{S}(U') \to \mathcal{S}(T')$ with $g'(y) = g(y)$ if $y \neq (k, p)$ are witnessing maps, contradicting the minimality of T.

4. For a partial order P, multiset union $\uplus$ is monotonic w.r.t. $\sqsubseteq_P$, i.e., for all $T, U, V \in \mathcal{M}_{\text{fin}} |P|$,

$$T \sqsubseteq_P U \text{ implies } T \uplus V \sqsubseteq_P U \uplus V \ ;$$

and $\wr\int \sqsubseteq_P T$ for all $T \in \mathcal{M}_{\text{fin}} |P|$. Since $\uplus$ also is associative and commutative, we have

LEMMA. *$(\mathcal{M}_{\text{fin}} |P|, \uplus, \wr\int, \sqsubseteq_P)$ is a join monoid.* $\square$

Define the functor $jMon\langle - \rangle : \mathrm{PO} \to \mathrm{jMon}$ by

$$jMon\langle P \rangle = (\mathcal{M}_{\text{fin}} |P|, \uplus, \wr\int, \sqsubseteq_P) \ ,$$
$$jMon\langle \varphi : P \to Q \rangle = \lambda \wr p_1, \ldots, p_n \int \in \mathcal{M}_{\text{fin}} |P| \,.\, \wr \varphi(p_1), \ldots, \varphi(p_n) \int \ .$$

5. Dually, the *upper* or *Smyth ordering* on $\mathcal{M}_{\text{fin}} |P|$ is the binary relation $\sqsubseteq^P \subseteq (\mathcal{M}_{\text{fin}} |P|) \times (\mathcal{M}_{\text{fin}} |P|)$, defined by $T \sqsubseteq^P U$ if, and only if, $U \sqsubseteq_{P^{-1}} T$; more explicitly, the Smyth ordering on $\mathcal{M}_{\text{fin}} |P|$ is given by the transitive closure of

$$T \sqsupseteq U \text{ implies } T \sqsubseteq^P U \ ,$$
$$p \leq_P q \text{ implies } T \uplus \wr p \int \sqsubseteq^P T \uplus \wr q \int \ ,$$

i.e., $T \sqsubseteq^P U$ if, and only if, there is an injective mapping $g : \mathcal{S}(U) \to \mathcal{S}(T)$ with $p \leq_P q$ if $g(k, q) = (j, p)$ for all $(k, q) \in \mathcal{S}(U)$; we call such a map a *witness* for $T \sqsubseteq^P U$. The relation $\sqsubseteq^P$ is also a partial ordering on $\mathcal{M}_{\text{fin}} |P|$, and, again, $\uplus$ is monotonic w.r.t. $\sqsubseteq^P$, i.e.,

$$T \sqsubseteq^P U \text{ implies } T \uplus V \sqsubseteq^P U \uplus V \ ;$$

and $T \sqsubseteq^P \wr\int$ for all $T \in \mathcal{M}_{\text{fin}} |P|$. Thus,

LEMMA. *$(\mathcal{M}_{\text{fin}} |P|, \uplus, \wr\int, \sqsubseteq^P)$ is a meet monoid.* $\square$

This meet monoid over a partial order P does not show suprema of finite sets, in general: Consider the partial order $\mathrm{P} = (\{\mathrm{a}, \mathrm{b}, \mathrm{c}\}, \{\mathrm{a} < \mathrm{c}, \mathrm{b} < \mathrm{c}\})$ (which does show suprema). In the meet monoid $(\mathcal{M}_{\text{fin}} |\mathrm{P}|, \uplus, \wr\int, \sqsubseteq^{\mathrm{P}})$, we have

$$\wr \mathrm{c} \int \sqsubset^{\mathrm{P}} \wr \mathrm{a} \int, \wr \mathrm{b} \int \ ,$$

$$\wr \mathrm{a}, \mathrm{b} \int \mathbb{C}^{\mathrm{P}} \wr \mathrm{a} \int, \wr \mathrm{b} \int$$

and no $T \in \mathcal{M}_{\mathrm{fin}}\,|\mathrm{P}|$ exists with $\wr \mathrm{a}, \mathrm{b} \int, \wr \mathrm{c} \int \mathbb{C}^{\mathrm{P}} T \mathbb{C}^{\mathrm{P}} \wr \mathrm{a} \int, \wr \mathrm{b} \int$ since, e.g., for $\wr \mathrm{c} \int \mathbb{C}^{\mathrm{P}} T$, T can only be $\wr \int$ by the first rule (with $\wr \mathrm{a}, \mathrm{b} \int, \wr \mathrm{c} \int \mathbb{C}^{\mathrm{P}} \wr \int$), or $\wr \mathrm{a} \int$ or $\wr \mathrm{b} \int$ by the second rule; but $\wr \mathrm{a} \int$ and $\wr \mathrm{b} \int$ are incomparable w.r.t. $\mathbb{C}^{\mathrm{P}}$.
Define the functor $mMon\langle - \rangle : \mathrm{PO} \to \mathrm{mMon}$ by

$$\begin{aligned} &mMon\langle P \rangle = (\mathcal{M}_{\mathrm{fin}}\,|P|, \uplus, \wr \int, \mathbb{C}^{P})\,, \\ &mMon\langle \varphi : P \to Q \rangle = \lambda \wr p_1, \ldots, p_n \int \in \mathcal{M}_{\mathrm{fin}}\,|P| \,.\, \wr \varphi(p_1), \ldots, \varphi(p_n) \int\,. \end{aligned}$$

In particular, $jMon\langle P \rangle = jMon(mMon\langle P^{-1} \rangle)$ and $mMon\langle P \rangle = mMon(jMon\langle P^{-1} \rangle)$.

6. Finally, the *convex* or *Plotkin ordering* on $\mathcal{M}_{\mathrm{fin}}\,|P|$ is the intersection of $\mathbb{C}_P$ and $\mathbb{C}^P$. Then $T\ (\mathbb{C}_P \cap \mathbb{C}^P)\ U$ if, and only if, there is a bijective mapping $h : \mathcal{S}(T) \to \mathcal{S}(U)$ with $p \leq_P q$ if $h(j,p) = (k,q)$ for all $(j,p) \in \mathcal{S}(T)$; we again call such a map a *witness* for $T\ (\mathbb{C}_P \cap \mathbb{C}^P)\ U$. The relation $\mathbb{C}_P \cap \mathbb{C}^P$ is also a partial ordering on $\mathcal{M}_{\mathrm{fin}}\,|P|$, and again $\uplus$ is monotonic w.r.t. this ordering.

LEMMA. $(\mathcal{M}_{\mathrm{fin}}\,|P|, \uplus, \wr \int, \mathbb{C}_P \cap \mathbb{C}^P)$ *is a partially ordered commutative monoid.* □

Define the functor $pocMon\langle - \rangle : \mathrm{PO} \to \mathrm{pocMon}$ by

$$\begin{aligned} &pocMon\langle P \rangle = (\mathcal{M}_{\mathrm{fin}}\,|P|, \uplus, \wr \int, \mathbb{C}_P \cap \mathbb{C}^{P})\,, \\ &pocMon\langle \varphi : P \to Q \rangle = \lambda \wr p_1, \ldots, p_n \int \in \mathcal{M}_{\mathrm{fin}}\,|P| \,.\, \wr \varphi(p_1), \ldots, \varphi(p_n) \int\,. \end{aligned}$$

7. Define the functor $PO : \mathrm{pocMon} \to \mathrm{PO}$ by

$$\begin{aligned} &PO(M) = (|M|, \leq_M)\,, \\ &PO(\varphi : M \to N) = \varphi\,. \end{aligned}$$

For each $P \in |\mathrm{PO}|$ and each $x \in \{poc, j, m\}$ define the partial order homomorphisms $\eta_P^{x\mathrm{Mon}} : P \to PO(xMon\langle P \rangle)$ by $\eta_P^{x\mathrm{Mon}}(p) = \wr p \int$. Then each $\eta^{x\mathrm{Mon}} = (\eta_P^{x\mathrm{Mon}})_{P \in |\mathrm{PO}|}$ is a natural transformation from 1_{PO} to $PO \circ xMon\langle - \rangle$.
Let $x \in \{poc, j, m\}$, $P \in |\mathrm{PO}|$, $M \in |x\mathrm{Mon}|$, and $\varphi : P \to PO(M)$. Define $\varphi^{\sharp_{x\mathrm{Mon}}} : xMon\langle P \rangle \to M$ by

$$\varphi^{\sharp_{x\mathrm{Mon}}}(\wr p_1, \ldots, p_n \int) = \varphi(p_1) \cdot_M \ldots \cdot_M \varphi(p_n)$$

for all $\wr p_1, \ldots, p_n \int \in \mathcal{M}_{\mathrm{fin}}\,|P|$, where, if $n = 0$, the right hand side is to be understood as ε_M.
Then for all $x \in \{poc, j, m\}$, $PO(\varphi^{\sharp_{x\mathrm{Mon}}})(\eta_P^{x\mathrm{Mon}}(p)) = \varphi(p)$ and all $\varphi^{\sharp_{x\mathrm{Mon}}}$ are unique with this property.

LEMMA. *$pocMon\langle P \rangle$, $jMon\langle P \rangle$, and $mMon\langle P \rangle$ are the free partially ordered commutative, join, and meet monoids over the partial order P, respectively.* □

8. Each upper semi-lattice can also be viewed as a join monoid. Indeed, define the functor $jMon : \mathrm{uSL} \to \mathrm{jMon}$ by

$$\begin{aligned} &jMon(U) = (|U|, \sqcup_U, \bot_U, \leq_U)\,, \\ &jMon(\varphi : U \to V) = \varphi\,. \end{aligned}$$

In particular, by §2 and §4, for each partial order P, $jMon(uSL\langle P \rangle) = (\mathcal{I}_{\mathrm{fin}}(P), \cup_P, \emptyset, \subseteq_P)$ is a join monoid.

3.1. *Constructing Meet Monoids from Dags*

1. Let G be a dag. Consider the *single-predecessor lifting* $\rightsquigarrow_G^{\mathrm{SPD}} \subseteq (\mathcal{M}_{\mathrm{fin}}\,|G|) \times (\mathcal{M}_{\mathrm{fin}}\,|G|)$ of G to the finite multisets $\mathcal{M}_{\mathrm{fin}}\,|G|$ over the elements of G, given by

$$T \rightsquigarrow_G^{\mathrm{SPD}} T \uplus \wr g \int ,$$
$$g \to_G h \text{ implies } T \uplus \wr g \int \rightsquigarrow_G^{\mathrm{SPD}} T \uplus \wr h \int .$$

Then $G^{\mathrm{SPD}} = (\mathcal{M}_{\mathrm{fin}}\,|G|, \rightsquigarrow_G^{\mathrm{SPD}})$ is a dag, where $T \rightsquigarrow_G^{\mathrm{SPD}} U$ expresses that U is *worse* than T. We write $\leq_G^{\mathrm{SPD}}$ for $\geq_{PO\langle G^{\mathrm{SPD}}\rangle} = ((\rightsquigarrow_G^{\mathrm{SPD}})^*)^{-1}$. The defining clauses for $\rightsquigarrow_G^{\mathrm{SPD}}$ correspond to the defining clauses for the upper ordering $\Subset^{PO\langle G\rangle^{-1}}$. Thus, $(\mathcal{M}_{\mathrm{fin}}\,|G|, \uplus, \wr\int, \leq_G^{\mathrm{SPD}})$ is a meet monoid, and

LEMMA. *For each dag G, $mMon\langle PO\langle G\rangle^{-1}\rangle \cong (\mathcal{M}_{\mathrm{fin}}\,|G|, \uplus, \wr\int, \leq_G^{\mathrm{SPD}})$.* ☐

2. Let G be a dag. Consider the *transitive-predecessors lifting* $\rightsquigarrow_G^{\mathrm{TPD}} \subseteq (\mathcal{M}_{\mathrm{fin}}\,|G|) \times (\mathcal{M}_{\mathrm{fin}}\,|G|)$ of G to the finite multisets $\mathcal{M}_{\mathrm{fin}}\,|G|$ over the elements of G, given by

$$T \rightsquigarrow_G^{\mathrm{TPD}} T \uplus \wr g \int ,$$
$$g_1, \dots, g_n \to_G^+ h \text{ implies } T \uplus \wr g_1, \dots, g_n \int \rightsquigarrow_G^{\mathrm{TPD}} T \uplus \wr h \int .$$

Then $G^{\mathrm{TPD}} = (\mathcal{M}_{\mathrm{fin}}\,|G|, \rightsquigarrow_G^{\mathrm{TPD}})$ is a dag; we write $\leq_G^{\mathrm{TPD}}$ for $\geq_{PO\langle G^{\mathrm{TPD}}\rangle} = ((\rightsquigarrow_G^{\mathrm{TPD}})^*)^{-1}$. Furthermore, $\leq_G^{\mathrm{TPD}}$ is monotonic w.r.t. multiset union and thus $(\mathcal{M}_{\mathrm{fin}}\,|G|, \uplus, \wr\int, \leq_G^{\mathrm{TPD}})$ is a meet monoid. We have

LEMMA. *Let G be a dag, let $S_1, S_2 \in \mathcal{P}_{\mathrm{fin}}\,|G|$, and let $\bar{S}_1, \bar{S}_2 \in \mathcal{M}_{\mathrm{fin}}\,|G|$ be the finite multisets corresponding to S_1 and S_2, respectively. Then*

$$\bar{S}_1 \leq_G^{\mathrm{TPD}} \bar{S}_2 \textit{ implies } \mathrm{Max}^{PO\langle G\rangle}(S_2) \Subset_{PO\langle G\rangle} \mathrm{Max}^{PO\langle G\rangle}(S_1) .$$

Proof. Let $\bar{S}_1 \leq_G^{\mathrm{TPD}} \bar{S}_2$ hold. It suffices to prove the claim for the two defining clauses for $\rightsquigarrow_G^{\mathrm{TPD}}$. For the case that there is a $g \in S_1$ such that $S_2 = S_1 \setminus \{g\}$, we have

$$\mathrm{Max}^{PO\langle G\rangle}(S_2) = \mathrm{Max}^{PO\langle G\rangle}(S_1 \setminus \{g\}) \Subset_{PO\langle G\rangle} \mathrm{Max}^{PO\langle G\rangle}(S_1) \setminus \{g\} \Subset_{PO\langle G\rangle} \mathrm{Max}^{PO\langle G\rangle}(S_1) .$$

For the case that there is an $h \in S_1$ and $g_1, \dots, g_n \in S_2$ with $g_1, \dots, g_n \to_G^+ h$ such that $S_2 = (S_1 \setminus \{h\}) \cup \{g_1, \dots, g_n\}$, we have

$$\begin{aligned}&\mathrm{Max}^{PO\langle G\rangle}(S_2) = \mathrm{Max}^{PO\langle G\rangle}(S_1 \cup \{g_1, \dots, g_n\}) \setminus \{h\} \Subset_{PO\langle G\rangle} \\ &\qquad \mathrm{Max}^{PO\langle G\rangle}(S_1) \setminus \{h\} \Subset_{PO\langle G\rangle} \mathrm{Max}^{PO\langle G\rangle}(S_1) ,\end{aligned}$$

since $h \in S_1$. ☐

The converse is in general wrong: Let $G = (\{g, h\}, \{(g, h)\})$, i.e., $g \to_G h$; then indeed we have $\mathrm{Max}^{PO\langle G\rangle}(\{g, h\}) \Subset_{PO\langle G\rangle} \mathrm{Max}^{PO\langle G\rangle}(\{h\})$, but $\wr h \int \not\leq_G^{\mathrm{TPD}} \wr g, h \int$. However, if $S_2 \in \mathcal{I}_{\mathrm{fin}}(PO\langle G\rangle)$, then the converse follows from the map $f : \mathrm{Max}^{PO\langle G\rangle}(S_2) \to \mathrm{Max}^{PO\langle G\rangle}(S_1)$ witnessing $S_2 = \mathrm{Max}^{PO\langle G\rangle}(S_2) \Subset_{PO\langle G\rangle} \mathrm{Max}^{PO\langle G\rangle}(S_1)$.

3. Let P be a partial order. Define $\sqsubseteq_P \subseteq (\mathcal{M}_{\mathrm{fin}}\,|P|) \times (\mathcal{M}_{\mathrm{fin}}\,|P|)$ by

$$T \uplus V \sqsubseteq_P U \uplus V \iff \forall p \in \mathrm{Max}^{\leq_P}(U) \,.\, \exists q \in \mathrm{Max}^{\leq_P}(T) \,.\, p <_P q$$

where T and U have no common elements. Then $\sqsubseteq_P$ is obviously reflexive and antisymmetric. In order to show transitivity, let $W_1 \sqsubseteq_P W_2$ and $W_2 \sqsubseteq_P W_3$ hold. Then there are unique T_i, U_i, V_i for $1 \leq i \leq 3$ such that T_i and U_i have no common elements and

$$W_1 = T_1 \uplus V_1 = T_3 \uplus V_3$$

$$W_2 = U_1 \uplus V_1 = T_2 \uplus V_2$$
$$W_3 = U_3 \uplus V_3 = U_2 \uplus V_2$$

Without loss of generality, we assume that there is no common element in all three of W_1, W_2, and W_3. We have to prove that for every $p \in \mathrm{Max}^{\leq_P}(U_3)$ there is a $q \in \mathrm{Max}^{\leq_P}(T_3)$ such that $p <_P q$. The plan of the proof is shown is the following figure, where $\hat{T}_i$ and $\hat{U}_i$ stand for $\mathrm{Max}^{\leq_P}(T_i)$ and $\mathrm{Max}^{\leq_P}(U_i)$ for $1 \leq i \leq 3$ respectively, and we follow a single element from the left to the right according to the arrows, where the dashed arrows stand for $\leq_P$, the solid arrows for $<_P$, and the double arrows for equality:

$$\begin{tikzcd}
\hat{U}_3 \arrow[r, dashed] \arrow[dr, Rightarrow] & \hat{U}_2 \arrow[r] & \hat{T}_2 \arrow[r, dashed] \arrow[dr, Rightarrow] & \hat{U}_1 \arrow[r] & \hat{T}_1 \arrow[r, dashed] \arrow[dr, Rightarrow] & \hat{T}_3 \\
V_3 \arrow[r, Rightarrow] \arrow[ur, dashed] & V_2 \arrow[r, Rightarrow] & V_2 \arrow[r, Rightarrow] \arrow[ur, dashed] & V_1 \arrow[r, Rightarrow] & V_1 \arrow[r, Rightarrow] \arrow[ur, dashed] & V_3 \arrow[lllll, Rightarrow, bend left]
\end{tikzcd}$$

Let thus $p_3 \in \mathrm{Max}^{\leq_P}(U_3)$. We make a case distinction on where p_3 resides w.r.t. the partition $W_3 = U_2 \uplus V_2$:

1. $p_3 \in U_2$: Then there is a $p_2 \in \hat{U}_2$ with $p_3 \leq_P p_2$ and a $q_2 \in \hat{T}_2$ with $p_2 <_P q_2$.
1.1. $q_2 \in U_1$: Then there is a $p_1 \in \hat{U}_1$ with $q_2 \leq_P p_1$ and a $q_1 \in \hat{T}_1$ with $p_1 <_P q_1$. If $q_1 \in T_3$, then there is a $q_3 \in \hat{T}_3$ with $q_1 \leq_P q_3$, and we are done; or $q_1 \in V_3$.
1.2. $q_2 \in V_1$: If $q_2 \in T_3$, then there is a $q_3 \in \hat{T}_3$ with $q_2 \leq_P q_3$, and we are done; or $q_2 \in V_3$.

2. $p_3 \in V_2$: Then $p_3 \in U_1$ or $p_3 \in V_1$.
2.1. $p_3 \in U_1$: Then there is a $p_1 \in \hat{U}_1$ with $p_3 \leq_P p_1$ and a $q_1 \in \hat{T}_1$ with $p_1 <_P q_1$. If $q_1 \in T_3$, then there is a $q_3 \in \hat{T}_3$ with $q_1 \leq_P q_3$, and we are done; or $q_1 \in V_3$.
2.2. $p_3 \in V_1$: If $p_3 \in T_3$, then there is a $q_3 \in \hat{T}_3$ with $p_3 \leq_P q_3$; or $p_3 \in V_3$. But both $p_3 = q_3$ and $p_3 \in V_3$ are impossible, since otherwise there is a common element of all W_1, W_2, and W_3; thus it must be $p_3 <_P q_3$ with $q_3 \in \hat{T}_3$, and we are done.

In the remaining unsettled cases of (1.1), (1.2), and (2.1) we get a $q \in V_3$ with $p_3 <_P q$. But for each $q \in V_3$ there is a q' with $q \leq_P q'$ and $q' \in \hat{T}_3$ or $q' \in V_3$, by the same reasoning as for p_3 above. Again, $q' = q$ is impossible since then q is a common element of all W_1, W_2, and W_3. If $q' \in \hat{T}_3$ with $q <_P q'$, then $p_3 <_P q <_P q'$, and we are done. Finally, if otherwise $q' \in V_3$, then, by repeating the argument, we could build an infinite strictly ascending chain in V_3 w.r.t. $\leq_P$ which is impossible since V_3 is finite.

3.2. *Initial, Terminal, and Direct Product Meet Monoids*

1. Consider the singleton meet monoid $\mathrm{S} = (\{*\}, \cdot, *, \{(*, *)\})$ where $* \cdot * = *$.

LEMMA. S *is initial and terminal in* mMon. □

2. Let $M = (|M|, \cdot_M, \varepsilon_M, \leq_M)$ and $N = (|N|, \cdot_N, \varepsilon_N, \leq_N)$ be meet monoids. Define $\cdot_{M \times N} : (|M| \times |N|) \times (|M| \times |N|) \to |M| \times |N|$ and $\leq_{M \times N} \subseteq (|M| \times |N|) \times (|M| \times |N|)$ by

$$(m_1, n_1) \cdot_{M \times N} (m_2, n_2) = (m_1 \cdot_M m_2, n_1 \cdot_N n_2)$$
$$(m_1, n_1) \leq_{M \times N} (m_2, n_2) \iff m_1 \leq_M m_2 \wedge n_1 \leq_N n_2$$

Then $M \times N = (|M| \times |N|, \cdot_{M \times N}, (\varepsilon_M, \varepsilon_N), \leq_{M \times N})$ is a meet monoid.
Define $\pi_1 : M \times N \to M$ by $\pi_1(m, n) = m$ and $\pi_2 : M \times N \to N$ by $\pi_2(m, n) = n$. Then π_1 and π_2 are meet monoid homomorphisms. Furthermore, for any meet monoid $P = (|P|, \cdot_P, \varepsilon_P, \leq_P)$ and two meet monoid homomorphisms $\varphi_1 : P \to M$ and $\varphi_2 : P \to N$, the meet monoid homomorphism

$\langle \varphi_1, \varphi_2 \rangle : P \to M \times N$ defined by $\langle \varphi_1, \varphi_2 \rangle(p) = (\varphi_1(p), \varphi_2(p))$ is unique for the property $\varphi_1 = \langle \varphi_1, \varphi_2 \rangle \mathbin{;} \pi_1$ and $\varphi_2 = \langle \varphi_1, \varphi_2 \rangle \mathbin{;} \pi_2$.

LEMMA. mMon *has finite products.* $\square$

3.3. *Lexicographic Products of Meet Monoids*

1. For a meet monoid M, define its set of *collapsing elements* [**Gadducci2013**] by

$$\mathcal{C}(M) = \{m \in |M| \mid \exists m_1, m_2 \in |M| \,.\, m_1 <_M m_2 \wedge m_1 \cdot_M m = m_2 \cdot_M m\} \,.$$

EXAMPLE. Let P be a partial order.

(1) The set of collapsing elements of the meet monoid $mMon\langle P\rangle = (\mathcal{M}_{\mathrm{fin}}\,|P|, \uplus, \wr\int, \Subset^P)$ is empty: If $T \Subset^P U$, then $T \uplus \wr p \int = U \uplus \wr p \int$ for some $p \in |P|$ would imply that $T = U$.

(2) The set of collapsing elements of the meet monoid $mMon(jMon(uSL\langle P\rangle)) = (\mathcal{I}_{\mathrm{fin}}(P), \cup_P, \emptyset, \supseteq_P)$ is $\mathcal{I}_{\mathrm{fin}}(P) \setminus \{\emptyset\}$: If $\emptyset \neq I \in \mathcal{I}_{\mathrm{fin}}(P)$, then $I \supset_P \emptyset$, but $\emptyset \cup_P I = I = I \cup_P I$. $\square$

Generalising the first example, if M is a *strict* meet monoid, i.e., $m <_M n$ implies $m \cdot_M o <_M n \cdot_M o$ for all $m, n, o \in |M|$, then $\mathcal{C}(M) = \emptyset$. Generalising the second example, all idempotent elements of a meet monoid M which are different from ε_M are collapsing: If $m \in |M|$ such that $m \neq \varepsilon_M$ and $m \cdot_M m = m$, then $m <_M \varepsilon_M$ but $m \cdot_M m = m = \varepsilon_M \cdot_M m$.

LEMMA. $|M| \setminus \mathcal{C}(M)$ *is closed under* $\cdot_M$.

Proof. We show that $m \cdot_M n \in \mathcal{C}(M)$ if, and only if, $m \in \mathcal{C}(M)$ or $n \in \mathcal{C}(M)$: If $m \in \mathcal{C}(M)$ or $n \in \mathcal{C}(M)$, then obviously $m \cdot_M n \in \mathcal{C}(M)$. Conversely, if $m \cdot_M n \in \mathcal{C}(M)$, then there are $m_1, m_2 \in |M|$ with $m_1 <_M m_2$, and $m_1 \cdot_M (m \cdot_M n) = m_2 \cdot_M (m \cdot_M n)$; but if $m \notin \mathcal{C}(M)$, then $m_1 \cdot_M m <_M m_2 \cdot_M m$, and if also $n \notin \mathcal{C}(M)$, then $(m_1 \cdot_M m) \cdot_M n <_M (m_2 \cdot_M m) \cdot_M n$. $\square$

Note that in particular $\varepsilon_M \notin \mathcal{C}(M)$ since $m_1 <_M m_2$ implies $m_1 \cdot_M \varepsilon_M <_M m_2 \cdot_M \varepsilon_M$. Thus, $(|M| \setminus \mathcal{C}(M), \cdot_M, \varepsilon_M, \leq_M)$ forms a meet monoid.

2. A meet monoid M is *bounded* if $|M|$ has a smallest element w.r.t. $\leq_M$; we denote this element by $\bot_M$ if it exists.
In a bounded meet monoid M it holds that $m \cdot_M \bot_M = \bot_M$ for all $m \in |M|$, i.e., $\bot_M$ is an absorbing element, since $m \cdot_M \bot_M \leq_M \varepsilon_M \cdot_M \bot_M = \bot_M$. Furthermore, $\bot_M \in \mathcal{C}(M)$ if $\bot_M \neq \varepsilon_M$. We call a bounded meet monoid *weakly strict* if $m <_M n$ implies $m \cdot_M o <_M n \cdot_M o$ for all $m, n \in |M|$ and $\bot_M \neq o \in |M|$. Each meet monoid M can be *lifted* into a bounded meet monoid $M_\bot = (|M| \uplus \{\bot\}, \cdot_{M_\bot}, \varepsilon_{M_\bot}, \leq_{M_\bot})$ setting $m \cdot_{M_\bot} \bot = \bot$, $\varepsilon_{M_\bot} = \varepsilon_M$, and $\bot \leq_{M_\bot} m$ for all $m \in |M| \uplus \{\bot\}$.

3. Let M be a meet monoid and let N be a bounded meet monoid. Let

$$L = ((|M| \setminus \mathcal{C}(M)) \times |N|) \cup (\mathcal{C}(M) \times \{\bot_N\}) \,,$$

i.e., the subset of pairs $(m, n) \in |M| \times |N|$ such that if $m \in \mathcal{C}(M)$, then $n = \bot_N$. Define the binary operation $\cdot_L : L \times L \to L$ by

$$(m_1, n_1) \cdot_L (m_2, n_2) = (m_1 \cdot_M m_2, n_1 \cdot_N n_2) \,.$$

This is well-defined, i.e., for all $(m_1, n_1), (m_2, n_2) \in L$, $(m_1 \cdot_M m_2, n_1 \cdot_N n_2) \in L$: Either $m_1 \cdot_M m_2 \in |M| \setminus \mathcal{C}(M)$ or $m_1 \cdot_M m_2 \in \mathcal{C}(M)$ and then we have to ensure that $n_1 \cdot_N n_2 = \bot_N$. But if $m_1 \cdot_M m_2 \in \mathcal{C}(M)$, then $m_1 \in \mathcal{C}(M)$ or $m_2 \in \mathcal{C}(M)$ by §1. Let first $m_1 \in \mathcal{C}(M)$. Then $n_1 = \bot_N$, and $n_1 \cdot_N n_2 = \bot_N \cdot_N n_2 = \bot_N$. The case of $m_2 \in \mathcal{C}(M)$ is symmetric.

The binary operation $\cdot_L$ inherits associativity and commutativity from M and N. Further define the element $\varepsilon_L \in L$ by

$$\varepsilon_L = (\varepsilon_M, \varepsilon_N) ,$$

which is also well-defined, since $\varepsilon_M \notin \mathcal{C}(M)$. Also $(m, n) \cdot_L \varepsilon_L = (m, n)$.
Finally, define the *lexicographic ordering* $\leq_L \subseteq L \times L$ on L by

$$(m_1, n_1) \leq_L (m_2, n_2) \iff (m_1 \neq m_2 \text{ and } m_1 \leq_M m_2) \text{ or } (m_1 = m_2 \text{ and } n_1 \leq_N n_2) .$$

In order to show monotonicity of $\cdot_L$ w.r.t. $\leq_L$, i.e., that $(m_1, n_1) \cdot_L (m, n) \leq_L (m_2, n_2) \cdot_L (m, n)$ follows from $(m_1, n_1) \leq_L (m_2, n_2)$ for all $(m, n) \in L$, the crucial insight is to note that m might be collapsing and there is nothing that then forces order-preservation in the second component. More specifically, it might be the case that $m_1 <_M m_2$ but $n_1 >_N n_2$ which still yields $(m_1, n_1) \leq_L (m_2, n_2)$ as we omit the second component. Were $m_1 \cdot_M m$ equal to $m_2 \cdot_M m$, that is m collapsing, we would have that $(m_1, n_1) \cdot_L (m, n) >_L (m_2, n_2) \cdot_L (m, n)$, clearly violating monotonicity.

LEMMA. *$(L, \cdot_L, \varepsilon_L, \leq_L)$ is a meet monoid.*

Proof. Since $(L, \cdot_L, \varepsilon_L)$ is a commutative monoid, it only remains to show the monotonicity of $\cdot_L$ w.r.t. $\leq_L$. Let $(m_1, n_1) \leq_L (m_2, n_2)$ and an $(m, n) \in L$ be given.
Case $m_1 <_M m_2$: If not $m_1 \cdot_M m <_M m_2 \cdot_M m$, i.e., $m_1 \cdot_M m = m_2 \cdot_M m$, then $m \in \mathcal{C}(M)$ and thus $n = \bot_N$. Hence

$$\begin{aligned}(m_1, n_1) \cdot_L (m, n) = (m_1 \cdot_M m, n_1 \cdot_N n) = (m_1 \cdot_M m, \bot_N) = \\ (m_2 \cdot_M m, \bot_N) = (m_2 \cdot_M m, n_2 \cdot_N n) = (m_2, n_2) \cdot_L (m, n) .\end{aligned}$$

Case $m_1 = m_2$ and $n_1 \leq_N n_2$: Then $m_1 \cdot_M m = m_2 \cdot_M m$ and $n_1 \cdot_N n \leq_N n_2 \cdot_N n$. □

Let us write $M \ltimes N$ for $(L, \cdot_L, \varepsilon_L, \leq_L)$. If both M and N are bounded, then $M \ltimes N$ is bounded with $\bot_{M \ltimes N} = (\bot_M, \bot_N)$.

4. Let M and N be meet monoids such that N is bounded. Then the set of collapsing elements for the lexicographic combination of M and N is

$$\mathcal{C}(M \ltimes N) = (\mathcal{C}(M) \times \{\bot_N\}) \cup ((|M| \setminus \mathcal{C}(M)) \times \mathcal{C}(N)) .$$

Indeed, let $(m, n) \in \mathcal{C}(M \ltimes N)$. Then there are $(m_1, n_1), (m_2, n_2) \in |M \ltimes N|$ with $(m_1, n_1) <_{M \ltimes N} (m_2, n_2)$, i.e., $m_1 <_M m_2$ or $m_1 = m_2$ and $n_1 <_N n_2$, and $(m_1, n_1) \cdot_{M \ltimes N} (m, n) = (m_2, n_2) \cdot_{M \ltimes N} (m, n)$, i.e., $m_1 \cdot_M m = m_2 \cdot_M m$ and $n_1 \cdot_N n = n_2 \cdot_N n$. If $m_1 <_M m_2$, then $m \in \mathcal{C}(M)$ and therefore $n = \bot_N$; if $m_1 = m_2$, then $n \in \mathcal{C}(N)$. — Conversely, let first $(m, n) \in \mathcal{C}(M) \times \{\bot_N\}$; then there are $m_1, m_2 \in |M|$ with $m_1 <_M m_2$ and $m_1 \cdot_M m = m_2 \cdot_M m$, hence $(m_1, \bot_N) <_{M \ltimes N} (m_2, \bot_N)$ with $(m_1, \bot_N) \cdot_{M \ltimes N} (m, n) = (m_2, \bot_N) \cdot_{M \ltimes N} (m, n)$. Now, for the second case, let $(m, n) \in (|M| \setminus \mathcal{C}(M)) \times \mathcal{C}(N)$; then there are $n_1, n_2 \in |N|$ with $n_1 <_N n_2$ and $n_1 \cdot_N n = n_2 \cdot_N n$, hence $(m, n_1) <_{M \ltimes N} (m, n_2)$ with $(m, n_1) \cdot_{M \ltimes N} (m, n) = (m, n_2) \cdot_{M \ltimes N} (m, n)$.
Abbreviate $|M| \setminus \mathcal{C}(M)$ by $R(M)$. Then

$$\begin{aligned}|M \ltimes N| \setminus \mathcal{C}(M \ltimes N) = R(M \ltimes N) = \\ ((R(M) \times |N|) \cup (\mathcal{C}(M) \times \{\bot_N\})) \setminus ((R(M) \times \mathcal{C}(N)) \cup (\mathcal{C}(M) \times \{\bot_N\})) = \\ R(M) \times R(N) = (|M| \setminus \mathcal{C}(M)) \times (|N| \setminus \mathcal{C}(N)) .\end{aligned}$$

Thus, for meet monoids M, N, and O, where N and O are bounded

$$\begin{aligned}|(M \ltimes N) \ltimes O| = \\ (R(M \ltimes N) \times |O|) \cup (\mathcal{C}(M \ltimes N) \times \{\bot_O\}) =\end{aligned}$$

$$
\begin{aligned}
&(R(M) \times R(N) \times |O|) \cup (R(M) \times \mathcal{C}(N) \times \{\bot_O\}) \cup (\mathcal{C}(M) \times \{(\bot_N, \bot_O)\}) = \\
&(R(M) \times ((R(N) \times |O|) \cup (\mathcal{C}(N) \times \{\bot_O\}))) \cup (\mathcal{C}(M) \times \{\bot_{N \ltimes O}\}) = \\
&(R(M) \times |N \ltimes O|) \cup (\mathcal{C}(M) \times \{\bot_{N \ltimes O}\}) = \\
&|M \ltimes (N \ltimes O)| \ ,
\end{aligned}
$$

from which it follows that $\ltimes$ is associative.

3.4. *Cancellative Partially Ordered Commutative Monoids*

1. A partially ordered commutative monoid M is *cancellative* if $m_1 \cdot_M n = m_2 \cdot_M n$ implies $m_1 = m_2$ for all $m_1, m_2, n \in |M|$.
The full sub-category of partially ordered commutative monoids with the cancellative partially ordered commutative monoids as objects is denoted by $\mathrm{cpocMon}$.

2. Let M be a partially ordered commutative monoid. Consider the binary relation $\lesssim^{\cdot}_M \subseteq |M| \times |M|$ with $m \lesssim^{\cdot}_M n$ iff there is an $o \in |M|$ with $m \cdot_M o \leq_M n \cdot_M o$. Then $\lesssim^{\cdot}_M$ is an admissible preordering for $\leq_M$, i.e., $\leq_M \ \subseteq \ \lesssim^{\cdot}_M$ holds and $\lesssim^{\cdot}_M$ is reflexive and transitive. In particular, for transitivity, let $m_1 \lesssim^{\cdot}_M m_2$ and $m_2 \lesssim^{\cdot}_M m_3$, and let $o_1, o_2 \in |M|$ be such that $m_1 \cdot_M o_1 \leq_M m_2 \cdot_M o_1$ and $m_2 \cdot_M o_2 \leq_M m_3 \cdot_M o_2$. Then $m_1 \cdot_M (o_1 \cdot_M o_2) = (m_1 \cdot_M o_1) \cdot_M o_2 \leq_M (m_2 \cdot_M o_1) \cdot_M o_2 = (m_2 \cdot_M o_2) \cdot_M o_1 \leq_M (m_3 \cdot_M o_2) \cdot_M o_1 = m_3 \cdot_M (o_1 \cdot_M o_2)$. The relation $\lesssim^{\cdot}_M$ also is a precongruence w.r.t. $\cdot_M$, i.e., if $m_1 \lesssim^{\cdot}_M m_2$ and $n_1 \lesssim^{\cdot}_M n_2$, then $m_1 \cdot_M n_1 \lesssim^{\cdot}_M m_2 \cdot_M n_2$. In fact, let $o, p \in |M|$ be such that $m_1 \cdot_M o \leq_M m_2 \cdot_M o$ and $n_1 \cdot_M p \leq_M n_2 \cdot_M p$. Then $m_1 \cdot_M n_1 \cdot_M o \cdot_M p \leq_M m_2 \cdot_M n_2 \cdot_M o \cdot_M p$. Let $\sim^{\cdot}_M = \lesssim^{\cdot}_M \cap \lesssim^{\cdot\,-1}_M$. Then $m \sim^{\cdot}_M n$ iff there is an $o \in |M|$ with $m \cdot_M o = n \cdot_M o$. Let $[m]_{\sim^{\cdot}_M}$ denote the equivalence class of $m \in |M|$ w.r.t. $\sim^{\cdot}_M$. Then

$$
M/{\sim^{\cdot}} = (\{[m]_{\sim^{\cdot}_M} \mid m \in |M|\}, \cdot_{M/\sim^{\cdot}}, [\varepsilon_M]_{\sim^{\cdot}_M}, \leq_{M/\sim^{\cdot}})
$$

with $[m]_{\sim^{\cdot}_M} \cdot_{M/\sim^{\cdot}} [n]_{\sim^{\cdot}_M} = [m \cdot_M n]_{\sim^{\cdot}_M}$ and $[m]_{\sim^{\cdot}_M} \leq_{M/\sim^{\cdot}} [n]_{\sim^{\cdot}_M}$ iff $m \lesssim^{\cdot}_M n$ forms a cancellative partially ordered commutative monoid: the multiplication and the partial ordering on $|M/{\sim^{\cdot}}|$ are well-formed [**wechler:1992**] and if $[m]_{\sim^{\cdot}_M} \cdot_{M/\sim^{\cdot}} [o]_{\sim^{\cdot}_M} = [n]_{\sim^{\cdot}_M} \cdot_{M/\sim^{\cdot}} [o]_{\sim^{\cdot}_M}$, then $m \cdot_M o \sim^{\cdot}_M n \cdot_M o$, i.e., $m \cdot_M o \cdot_M p = n \cdot_M o \cdot_M p$ for some $p \in |M|$, hence $m \sim^{\cdot}_M n$, and thus $[m]_{\sim^{\cdot}_M} = [n]_{\sim^{\cdot}_M}$.
Define the functor $\mathit{cpocMon} : \mathrm{pocMon} \to \mathrm{cpocMon}$ by

$$
\begin{aligned}
&\mathit{cpocMon}(M) = M/{\sim^{\cdot}} \ , \\
&\mathit{cpocMon}(\varphi : M \to N) = \lambda [m]_{\sim^{\cdot}_M} \,.\, [\varphi(m)]_{\sim^{\cdot}_N} \ ,
\end{aligned}
$$

which is well defined, since if $m \sim^{\cdot}_M m'$, then there is an $o \in |M|$ with $m \cdot_M o = m' \cdot_M o$ and $\varphi(m) \cdot_N \varphi(o) = \varphi(m \cdot_M o) = \varphi(m' \cdot_M o) = \varphi(m') \cdot_N \varphi(o)$, i.e., $\varphi(m) \sim^{\cdot}_N \varphi(m')$.

3. Let M be a partially ordered commutative monoid. Let N be a cancellative partially ordered commutative monoid and let $\varphi : M \to N$ be a partially ordered commutative monoid morphism. Define $\varphi^{\cdot} : M/{\sim^{\cdot}} \to N$ by $\varphi^{\cdot}([m]_{\sim^{\cdot}_M}) = \varphi(m)$; this is well defined, since if $m \sim^{\cdot}_M m'$, then there is an $o \in |M|$ with $m \cdot_M o = m' \cdot_M o$ and thus $\varphi(m) \cdot_N \varphi(o) = \varphi(m \cdot_M o) = \varphi(m' \cdot_M o) = \varphi(m') \cdot_N \varphi(o)$, which implies $\varphi(m) = \varphi(m')$ since N is cancellative. Further, define the partially ordered commutative monoid morphism $\delta_M : M \to M/{\sim^{\cdot}}$ by $\delta_M(m) = [m]_{\sim^{\cdot}_M}$. Then $\varphi^{\cdot}$ is the unique meet monoid morphism such that $\varphi^{\cdot} \circ \delta_M = \varphi$.
The family $\delta = (\delta_M : M \to M/{\sim^{\cdot}})_{M \in |\mathrm{pocMon}|}$ forms a natural transformation from 1_{pocMon} to $\mathit{cpocMon}$. Furthermore, each δ_M is surjective.

LEMMA. $(\delta, -^{\cdot})$ *forms an epireflection of* $\mathrm{cpocMon}$ *in* pocMon. $\square$

3.5. *Idempotent Meet Monoids*

1. A meet monoid M is *idempotent* if $m \cdot_M m = m$ for all $m \in |M|$. For each meet monoid morphism $\varphi : M \to N$ with M and N idempotent, it holds that $\varphi(m) = \varphi(m \cdot_M m) = \varphi(m) \cdot_N \varphi(m) = \varphi(m)$.
The full sub-category of meet monoids with the idempotent meet monoids as objects is denoted by imMon.

2. Let M be a meet monoid. An element $m \in |M|$ is *archimedean smaller* than an element $n \in |M|$, written as $m \lesssim^\alpha_M n$, if there is a $k \in \mathbb{N}$ such that $m^{(k)_M} \leq_M n$ (where we write $m^{(k)_M}$ for the k-fold product of m w.r.t. $\cdot_M$). Two elements $m, n \in |M|$ are *archimedean equivalent*, written as $\sim^\alpha_M$, if $m \lesssim^\alpha_M n$ and $n \lesssim^\alpha_M m$.
Then $\lesssim^\alpha_M$ is an admissible preordering for $\leq_M$, i.e., $\leq_M \subseteq \lesssim^\alpha_M$ holds and $\lesssim^\alpha_M$ is reflexive and transitive. In particular, for transitivity, let $m_1 \lesssim^\alpha_M m_2$ and $m_2 \lesssim^\alpha_M m_3$, and let $k_1, k_2 \in \mathbb{N}$ be such that $m_1^{(k_1)_M} \leq_M m_2$ and $m_2^{(k_2)_M} \leq_M m_3$. Then $m_1^{(k_1 \cdot k_2)_M} = (m_1^{(k_1)_M})^{(k_2)_M} \leq_M m_2^{(k_2)_M} \leq_M m_3$.
The relation $\lesssim^\alpha_M$ also is a precongruence w.r.t. $\cdot_M$, i.e., if $m_1 \lesssim^\alpha_M m_2$ and $n_1 \lesssim^\alpha_M n_2$, then $m_1 \cdot_M n_1 \lesssim^\alpha_M m_2 \cdot_M n_2$. In fact, let $k, l \in \mathbb{N}$ be such that $m_1^{(k)_M} \leq_M m_2$ and $n_1^{(l)_M} \leq_M n_2$. Then $(m_1 \cdot_M n_1)^{(k+l)_M} = m_1^{(k)_M} \cdot_M m_1^{(l)_M} \cdot_M n_1^{(k)_M} \cdot_M n_1^{(l)_M} \leq_M m_1^{(k)_M} \cdot_M n_1^{(l)_M} \leq_M m_2 \cdot_M n_2$ as M is a meet monoid.
Let $[m]_{\sim^\alpha_M}$ denote the equivalence class of $m \in |M|$ w.r.t. archimedean equivalence $\sim^\alpha_M$. Then

$$M/{\sim^\alpha} = (\{[m]_{\sim^\alpha_M} \mid m \in |M|\}, \cdot_{M/\sim^\alpha}, [\varepsilon_M]_{\sim^\alpha_M}, \leq_{M/\sim^\alpha})$$

with $[m]_{\sim^\alpha_M} \cdot_{M/\sim^\alpha} [n]_{\sim^\alpha_M} = [m \cdot_M n]_{\sim^\alpha_M}$ and $[m]_{\sim^\alpha_M} \leq_{M/\sim^\alpha} [n]_{\sim^\alpha_M}$ iff $m \lesssim^\alpha_M n$ forms an idempotent meet monoid: the multiplication and the partial ordering on $|M/{\sim^\alpha}|$ are well-formed [**wechler:1992**] and $m \cdot_M m \sim^\alpha_M m$, i.e., $[m]_{\sim^\alpha_M} \cdot_{M/\sim^\alpha} [m]_{\sim^\alpha_M} = [m \cdot_M m]_{\sim^\alpha_M} = [m]_{\sim^\alpha_M}$.
Define the functor $\mathit{imMon} : \mathrm{mMon} \to \mathrm{imMon}$ by

$$\mathit{imMon}(M) = M/{\sim^\alpha} ,$$
$$\mathit{imMon}(\varphi : M \to N) = \lambda [m]_{\sim^\alpha_M} \,.\, [\varphi(m)]_{\sim^\alpha_N} ,$$

which is well defined, since if $m \sim^\alpha_M m'$, then there are $k, k' \in \mathbb{N}$ with $m^{(k)_M} \leq_M m'$ and $m'^{(k')_M} \leq_M m$ and thus $\varphi(m^{(k)_M}) \leq_N \varphi(m')$ and $\varphi(m'^{(k')_M}) \leq_N \varphi(m)$, i.e., $(\varphi(m))^{(k)_N} \leq_N \varphi(m')$ and $(\varphi(m'))^{(k')_N} \leq_N \varphi(m)$, that is, $\varphi(m) \sim^\alpha_N \varphi(m')$.

3. Let M be a meet monoid. Let N be an idempotent meet monoid and let $\varphi : M \to N$ be a meet monoid morphism. Define $\varphi^\alpha : M/{\sim^\alpha} \to N$ by $\varphi^\alpha([m]_{\sim^\alpha_M}) = \varphi(m)$; this is well-defined, since if $m \sim^\alpha_M m'$, then there are $k, k' \in \mathbb{N}$ with $m^{(k)_M} \leq_M m'$ and $m'^{(k')_M} \leq_M m$ and thus $\varphi(m^{(k)_M}) \leq_N \varphi(m')$ and $\varphi(m'^{(k')_M}) \leq_N \varphi(m)$, i.e., $(\varphi(m))^{(k)_N} \leq_N \varphi(m')$ and $(\varphi(m'))^{(k')_N} \leq_N \varphi(m)$ which implies $\varphi(m) \leq_N \varphi(m')$ and $\varphi(m') \leq_N \varphi(m)$ as N is idempotent, that is, $\varphi(m) = \varphi(m')$. Further, define the meet monoid morphism $\alpha_M : M \to M/{\sim^\alpha}$ by $\alpha_M(m) = [m]_{\sim^\alpha_M}$. Then φ^α is the unique meet monoid morphism such that $\varphi^\alpha \circ \alpha_M = \varphi$.
The family $\alpha = (\alpha_M : M \to M/{\sim^\alpha})_{M \in |\mathrm{mMon}|}$ forms a natural transformation from 1_{mMon} to imMon. Furthermore, each α_M is surjective.

LEMMA. $(\alpha, -^\alpha)$ *forms an epireflection of* imMon *in* mMon. □

3.6. *Complete Meet Monoids*

1. A meet monoid M is *complete* if every subset X of $|M|$ has a supremum w.r.t. $\leq_M$, written as $\bigvee_M X$, and $m \cdot_M \bigvee_M X = \bigvee_M m \cdot_M X$, where $m \cdot_M X = \{m \cdot_M n \mid n \in X\}$. In particular, a

complete meet monoid M is bounded with $\bigvee_M \emptyset$ the bottom element. A meet monoid morphism $\varphi : M \to N$ with M and N complete meet monoids is *continuous* if $\varphi(\bigvee_M X) = \bigvee_N \varphi(X)$ for all $X \subseteq M$, where $\varphi(X) = \{\varphi(m) \mid m \in X\}$ for all $X \subseteq M$.
The sub-category of meet monoids with the complete meet monoids as objects and the continuous meet monoid morphisms as morphisms is denoted by cmMon.

2. Let M be a meet monoid. An $I \subseteq |M|$ is a *downset* of M if $m' \in I$ implies $m \in I$ for all $m \leq_M m'$. For an $X \subseteq |M|$ we write $X{\downarrow_M}$ for the smallest downset containing X, i.e., $X{\downarrow_M} = \{m \in |M| \mid \exists m' \in X \,.\, m \leq_M m'\}$. For a set $\mathcal{I}$ of downsets of M, $\bigcup \mathcal{I}$ is a downset of M, and it is the supremum of $\mathcal{I}$ w.r.t. $\subseteq$.
Define $I \mathbin{\tilde{\cdot}_M} J = \{m \cdot_M n \mid m \in I,\ n \in J\}{\downarrow_M}$ for two downsets I, J of M, and $I \mathbin{\tilde{\cdot}_M} \mathcal{J} = \{I \mathbin{\tilde{\cdot}_M} J \mid J \in \mathcal{J}\}$ for a downset I of M and $\mathcal{J}$ a set of downsets of M.
Let I be a downset of M and $\mathcal{J}$ a set of downsets of M. Then $I \mathbin{\tilde{\cdot}_M} \bigcup \mathcal{J} = \bigcup I \mathbin{\tilde{\cdot}_M} \mathcal{J}$. Indeed, let first $m \in I \mathbin{\tilde{\cdot}_M} \bigcup \mathcal{J}$. Then there is an $m_I \in I$ and an $m_{\mathcal{J}} \in \bigcup \mathcal{J}$ such that $m \leq_M m_I \cdot_M m_{\mathcal{J}}$. Furthermore, there is a $J \in \mathcal{J}$ such that $m_{\mathcal{J}} \in J$. Thus $m_I \cdot_M m_{\mathcal{J}} \in I \mathbin{\tilde{\cdot}_M} \mathcal{J}$, which yields $m \in I \mathbin{\tilde{\cdot}_M} \mathcal{J}$, and hence $m \in \bigcup I \mathbin{\tilde{\cdot}_M} \mathcal{J}$. Conversely, let $m \in \bigcup I \mathbin{\tilde{\cdot}_M} \mathcal{J}$. Then there is a $J \in \mathcal{J}$ such that $m \in I \mathbin{\tilde{\cdot}_M} J$ and $m_I \in I$ and $m_J \in J$ with $m \leq_M m_I \cdot_M m_J$. Thus $m_J \in \bigcup \mathcal{J}$ and hence $m \in I \mathbin{\tilde{\cdot}_M} \bigcup \mathcal{J}$.

LEMMA. $M{\downarrow} = (\{X{\downarrow_M} \mid X \subseteq |M|\}, \tilde{\cdot}_M, \{\varepsilon_M\}{\downarrow_M}, \subseteq)$ *is a complete meet monoid.* $\square$

Define the functor $\mathit{cmMon} : \mathrm{mMon} \to \mathrm{cmMon}$ by

$$\begin{aligned} &\mathit{cmMon}(M) = M{\downarrow}\ , \\ &\mathit{cmMon}(\varphi : M \to N) = \lambda X{\downarrow_M} \,.\, \varphi(X){\downarrow_N}\ . \end{aligned}$$

Therein $\mathit{cmMon}(\varphi : M \to N)$ is (1) well-defined, i.e., if $X{\downarrow_M} = X'{\downarrow_M}$, then $\varphi(X){\downarrow_N} = \varphi(X'){\downarrow_N}$, and (2) continuous: (1) If $n \in \varphi(X){\downarrow_N}$, there is an $m \in X$ such that $n \leq_N \varphi(m)$, and hence an $m' \in X'$ such that $m \leq_M m'$, i.e., $n \leq_N \varphi(m) \leq_N \varphi(m')$, from which $n \in \varphi(X'){\downarrow_N}$ follows; and the converse case is symmetric. (2) For each set of subsets $\mathcal{X}$ of M, $n \in \varphi(\bigcup \mathcal{X}{\downarrow_M})$ (where $\mathcal{X}{\downarrow_M} = \{X{\downarrow_M} \mid X \in \mathcal{X}\}$) if, and only if, there is an $X \in \mathcal{X}$ and an $m \in X$ such that $n \leq_N \varphi(m)$ if, and only if, $n \in \bigcup \varphi(\mathcal{X}){\downarrow_N}$ (where $\varphi(\mathcal{X}) = \{\varphi(X) \mid X \in \mathcal{X}\}$).

3. Let M be a meet monoid. Let N be a complete meet monoid and $\varphi : M \to N$ a meet monoid morphism. Define $\varphi^{\downarrow} : M{\downarrow} \to N$ by $\varphi^{\downarrow}(X{\downarrow_M}) = \bigvee_N \varphi(X)$. This is well-defined, since $X{\downarrow_M} = X'{\downarrow_M}$ implies $\bigvee_N \varphi(X) = \bigvee_N \varphi(X')$: If $\bigvee_N \varphi(X) \leq_N n$, i.e., $\varphi(m) \leq_N n$ for all $m \in X$, then also $\bigvee_N \varphi(X') \leq_N n$ as for each $m' \in X'$ there is an $m \in X$ with $m' \leq_M m$, from which $\varphi(m') \leq_N \varphi(m) \leq_N n$ follows; and the converse is symmetric. Further, define the meet monoid morphism $\gamma_M : M \to M{\downarrow}$ by $\gamma_M(m) = \{m\}{\downarrow_M}$. Then $\varphi^{\downarrow}$ is the unique continuous meet monoid morphism such that $\varphi^{\downarrow} \circ \gamma_M = \varphi$. Indeed, let $\psi : M{\downarrow} \to N$ be a continuous meet monoid morphism with $\psi \circ \gamma_M = \varphi$. Then

$$\begin{aligned} &\varphi^{\downarrow}(X{\downarrow_M}) = \textstyle\bigvee_N\{\varphi(m) \mid m \in X\}) = \bigvee_N\{\varphi^{\downarrow}(\{m\}{\downarrow_M}) \mid m \in X\} = \\ &\qquad \textstyle\bigvee_N\{\psi(\{m\}{\downarrow_M}) \mid m \in X\} = \psi(\bigcup\{\{m\}{\downarrow_M} \mid m \in X\}) = \psi(X{\downarrow_M})\ . \end{aligned}$$

LEMMA. $(\gamma, -^{\downarrow})$ *forms a reflection of* cmMon *in* mMon. $\square$

4. C-SEMIRINGS

1. A *c-semiring* [**BistarelliMRSVF99**] $(X, \oplus, \otimes, \mathbf{0}, \mathbf{1})$ is given by a set X, two binary operations $\otimes, \oplus : X \times X \to X$, and two constants $\mathbf{0}, \mathbf{1} \in X$ such that the following axioms are satisfied for all $x, y, z \in X$:

(1) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$
(2) $x \oplus y = y \oplus x$
(3) $x \oplus \mathbf{1} = \mathbf{1}$
(4) $x \oplus \mathbf{0} = x$
(5) $(x \otimes y) \otimes z = x \otimes (y \otimes z)$
(6) $x \otimes y = y \otimes x$
(7) $x \otimes \mathbf{0} = \mathbf{0}$
(8) $x \otimes \mathbf{1} = x$
(9) $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$

In words, $\oplus$ is associative and commutative, has $\mathbf{1}$ as annihilator and $\mathbf{0}$ as neutral element; $\otimes$ is associative and commutative, has $\mathbf{0}$ as annihilator and $\mathbf{1}$ as neutral element; and $\otimes$ distributes over $\oplus$.
A *c-semiring homomorphism* $\varphi : A \to B$ from a c-semiring $A = (|A|, \oplus_A, \otimes_A, \mathbf{0}_A, \mathbf{1}_A)$ to a c-semiring $B = (|B|, \oplus_B, \otimes_B, \mathbf{0}_B, \mathbf{1}_B)$ is given by a map $\varphi : |A| \to |B|$ such that for all $a_1, a_2 \in |A|$:
(1) $\varphi(a_1 \oplus_A a_2) = \varphi(a_1) \oplus_B \varphi(a_2)$
(2) $\varphi(a_1 \otimes_A a_2) = \varphi(a_1) \otimes_B \varphi(a_2)$
(3) $\varphi(\mathbf{0}_A) = \mathbf{0}_B$
(4) $\varphi(\mathbf{1}_A) = \mathbf{1}_B$

The category cSRng of c-semirings has the c-semirings as objects and the c-semiring homomorphisms as morphisms.

2. In a c-semiring $(X, \oplus, \otimes, \mathbf{0}, \mathbf{1})$ the operation $\oplus$ is idempotent:

$$x \oplus x = (x \otimes \mathbf{1}) \oplus (x \otimes \mathbf{1}) = x \otimes (\mathbf{1} \oplus \mathbf{1}) = x \otimes \mathbf{1} = x \ .$$

Thus, there is a functor $uSL : \mathrm{cSRng} \to \mathrm{uSL}$, defined by

$$uSL(A) = (|A|, \oplus_A, \mathbf{0}) \ ,$$
$$uSL(\varphi : A \to B) = \varphi \ .$$

For a c-semiring A, the thereby induced ordering $\leq_{uSL(A)}$, explicitly given by $a \leq_{uSL(A)} b$ if, and only if, $a \oplus_A b = b$, will be written as $\preceq_A$.
With this definition, for all $a, b, c \in |A|$ it holds that
(1) $\mathbf{0} \preceq_A a \preceq_A \mathbf{1}$;
(2) $a \preceq_A a \oplus_A b$ and $b \preceq_A a \oplus_A b$;
(3) if $a \preceq_A c$ and $b \preceq_A c$, then $a \oplus_A b \preceq_A c$.

Also $\oplus_A$ is monotone w.r.t. $\preceq_A$ in both arguments, i.e.,

$$a \preceq_A a' \text{ and } b \preceq_A b' \text{ implies } a \oplus_A b \preceq_A a' \oplus_A b' \ .$$

3. In a c-semiring $(X, \oplus, \otimes, \mathbf{0}, \mathbf{1})$ the operation $\otimes$ is monotone w.r.t. the induced ordering $\preceq$, since if $x \preceq x'$, i.e., $x \oplus x' = x'$, then

$$(x \otimes y) \oplus (x' \otimes y) = (x \oplus x') \otimes y = x' \otimes y \ ,$$

i.e., $x \otimes y \preceq x' \otimes y$, from which it follows that

$$x \preceq x' \text{ and } y \preceq y' \text{ implies } x \otimes y \preceq x' \otimes y' \ .$$

Furthermore, for all $x, y \in X$

$$x \otimes y \preceq x \text{ and } x \otimes y \preceq y \ ,$$

since

$$(x \otimes y) \oplus x = (x \otimes y) \oplus (x \otimes \mathbf{1}) = x \otimes (y \oplus \mathbf{1}) = x \otimes \mathbf{1} = x \ .$$

Thus, there is a functor $mMon : \mathrm{cSRng} \to \mathrm{mMon}$, given by

$$mMon(A) = (|A|, \otimes_A, \mathbf{1}_A, \preceq_A) \ ,$$
$$mMon(\varphi : A \to B) = \varphi \ .$$

Note that $mMon(A)$ is a bounded meet monoid with $\bot_{mMon(A)} = \mathbf{0}_A$.

4.1. *Constructing C-Semirings from Meet Monoids*

1. Let M be meet monoid. We write $\mathcal{I}_{\text{fin}}(M)$ for $\mathcal{I}_{\text{fin}}(PO(M))$. Define the operations $\tilde{\cup}_M, \tilde{\cdot}_M : \mathcal{I}_{\text{fin}}(M) \times \mathcal{I}_{\text{fin}}(M) \to \mathcal{I}_{\text{fin}}(M)$ by

$$I \mathbin{\tilde{\cup}_M} J = \mathrm{Max}^{\leq_M}(I \cup J) \ ,$$
$$I \mathbin{\tilde{\cdot}_M} J = \mathrm{Max}^{\leq_M}\{m \cdot_M n \mid m \in I,\ n \in J\} \ .$$

LEMMA. $(\mathcal{I}_{\text{fin}}(M), \tilde{\cup}_M, \tilde{\cdot}_M, \emptyset, \{\varepsilon_M\})$ *is a c-semiring.*

Proof. Let $I, J, K \in \mathcal{I}_{\text{fin}}(M)$.
The operation $\tilde{\cup}_M$ is associative and commutative and has $\emptyset$ as neutral element by §2. Furthermore, $I \mathbin{\tilde{\cup}_M} \{\varepsilon_M\} = \{\varepsilon_M\}$, since ε_M is the greatest element of $|M|$ w.r.t. $\leq_M$.
For the associativity of $\tilde{\cdot}_M$ we have

$$\begin{aligned}
&I \mathbin{\tilde{\cdot}_M} (J \mathbin{\tilde{\cdot}_M} K) = \\
&\quad \mathrm{Max}^{\leq_M}\{m_I \cdot_M m_{JK} \mid m_I \in I,\ m_{JK} \in \mathrm{Max}^{\leq_M}\{m_J \cdot_M m_K \mid m_J \in J,\ m_K \in K\}\} = \\
&\quad \mathrm{Max}^{\leq_M}\{m_I \cdot_M m_J \cdot_M m_K \mid m_I \in I,\ m_J \in J,\ m_K \in K\}\} = \\
&\quad \mathrm{Max}^{\leq_M}\{m_{IJ} \cdot_M m_K \mid m_{IJ} \in \mathrm{Max}^{\leq_M}\{m_I \cdot_M m_J \mid m_I \in I,\ m_J \in J\},\ m_K \in K\} = \\
&\quad (I \mathbin{\tilde{\cdot}_M} J) \mathbin{\tilde{\cdot}_M} K \ ,
\end{aligned}$$

since

$$\mathrm{Max}^{\leq_M}\{m \cdot_M n \mid m \in I,\ n \in \mathrm{Max}^{\leq_M}(X)\} = \mathrm{Max}^{\leq_M}\{m \cdot_M n \mid m \in I,\ n \in X\}$$

for all $X \in \mathcal{P}_{\text{fin}}\,|M|$. Assume $n \in X$ but $n \notin \mathrm{Max}^{\leq_M}(X)$; then there exists some (maximal) $n' \in X$ such that $n <_M n'$; Since $n \leq n'$ implies $m \cdot_M n \leq m \cdot_M n'$ by the monotonicity of $\cdot_M$, $m \cdot_M n'$
Also $\tilde{\cdot}_M$ inherits commutativity from $\cdot_M$; $I \mathbin{\tilde{\cdot}_M} \emptyset = \emptyset$ by definition; and $I \mathbin{\tilde{\cdot}_M} \{\varepsilon_M\} = I$, since ε_M is neutral in M.
Finally, $\tilde{\cdot}_M$ distributes over $\tilde{\cup}_M$:

$$\begin{aligned}
&I \mathbin{\tilde{\cdot}_M} (J \mathbin{\tilde{\cup}_M} K) = \\
&\quad \mathrm{Max}^{\leq_M}\{m_I \cdot_M m_{JK} \mid m_I \in I,\ m_{JK} \in \mathrm{Max}^{\leq_M}(J \cup K)\} = \\
&\quad \mathrm{Max}^{\leq_M}\{m_I \cdot_M m_{JK} \mid m_I \in I,\ m_{JK} \in J \cup K\} = \\
&\quad \mathrm{Max}^{\leq_M}(\{m_I \cdot_M m_J \mid m_I \in I,\ m_J \in J\} \cup \{m_I \cdot_M m_K \mid m_I \in I,\ m_K \in K\}) = \\
&\quad \mathrm{Max}^{\leq_M}(\mathrm{Max}^{\leq_M}\{m_I \cdot_M m_J \mid m_I \in I,\ m_J \in J\} \cup \\
&\qquad\qquad \mathrm{Max}^{\leq_M}\{m_I \cdot_M m_K \mid m_I \in I,\ m_K \in K\}) = \\
&\quad (I \mathbin{\tilde{\cdot}_M} J) \mathbin{\tilde{\cup}_M} (I \mathbin{\tilde{\cdot}_M} K) \ ,
\end{aligned}$$

since

$$\mathrm{Max}^{\leq_M}(I \cup \mathrm{Max}^{\leq_M}(X)) = \mathrm{Max}^{\leq_M}(I \cup X)$$

for all $X \in \mathcal{P}_{\text{fin}}\,|M|$. ∎

Let $\varphi : M \to N$ be a meet monoid homomorphism. For $X \in \mathcal{P}_{\text{fin}}\,|M|$, we have

$$\mathrm{Max}^{\leq_N}(\varphi(\mathrm{Max}^{\leq_M}(X))) = \mathrm{Max}^{\leq_N}(\varphi(X)) .$$

Indeed, on the one hand, $\mathrm{Max}^{\leq_N}(\varphi(\mathrm{Max}^{\leq_M}(X))) \subseteq \mathrm{Max}^{\leq_N}(\varphi(X))$, since $\mathrm{Max}^{\leq_M}(X) \subseteq X$. For $\mathrm{Max}^{\leq_N}(\varphi(X)) \subseteq \mathrm{Max}^{\leq_N}(\varphi(\mathrm{Max}^{\leq_M}(X)))$ it suffices to show that for each $n \in \varphi(X)$ there is an $n' \in \varphi(\mathrm{Max}^{\leq_M}(X))$ such that $n \leq_M n'$. Thus, let $n \in \varphi(X)$, i.e., $n = \varphi(m)$ for some $m \in X$. Then there is an $m' \in \mathrm{Max}^{\leq_M}(X)$ with $m \leq_M m'$, hence $n = \varphi(m) \leq_N \varphi(m')$, and $\varphi(m') \in \varphi(\mathrm{Max}^{\leq_M}(X))$.
Define the functor $cSRng\langle - \rangle : \mathrm{mMon} \to \mathrm{cSRng}$ by

$$cSRng\langle M \rangle = (\mathcal{I}_{\text{fin}}(M), \tilde{\cup}_M, \tilde{\cdot}_M, \emptyset, \{\varepsilon_M\}) ,$$
$$cSRng\langle \varphi : M \to N \rangle = \lambda\{m_1, \dots, m_k\} \in \mathcal{I}_{\text{fin}}(M) \,.\, \mathrm{Max}^{\leq_N}\{\varphi(m_1), \dots, \varphi(m_k)\} .$$

Indeed, $cSRng\langle \varphi : M \to N \rangle$ is a c-semiring homomorphism from $cSRng\langle M \rangle$ to $cSRng\langle N \rangle$:

$$\begin{aligned}
&cSRng\langle\varphi\rangle(\emptyset) = \emptyset ,\\
&cSRng\langle\varphi\rangle(\{\varepsilon_M\}) = \{\varphi(\varepsilon_M)\} = \{\varepsilon_N\} ,\\
&cSRng\langle\varphi\rangle(I_1 \mathbin{\tilde{\cup}_M} I_2) = cSRng\langle\varphi\rangle(\mathrm{Max}^{\leq_M}(I_1 \cup I_2)) =\\
&\qquad \mathrm{Max}^{\leq_N}(\varphi(\mathrm{Max}^{\leq_M}(I_1 \cup I_2))) = \mathrm{Max}^{\leq_N}(\varphi(I_1 \cup I_2)) = \mathrm{Max}^{\leq_N}(\varphi(I_1) \cup \varphi(I_2)) =\\
&\qquad cSRng\langle\varphi\rangle(I_1) \mathbin{\tilde{\cup}_N} cSRng\langle\varphi\rangle(I_2) ,\\
&cSRng\langle\varphi\rangle(I_1 \mathbin{\tilde{\cdot}_M} I_2) = cSRng\langle\varphi\rangle(\mathrm{Max}^{\leq_M}\{m_1 \cdot_M m_2 \mid m_1 \in I_1,\ m_2 \in I_2\}) =\\
&\qquad \mathrm{Max}^{\leq_N}(\varphi(\mathrm{Max}^{\leq_M}\{m_1 \cdot_M m_2 \mid m_1 \in I_1,\ m_2 \in I_2\})) =\\
&\qquad \mathrm{Max}^{\leq_N}\{\varphi(m_1 \cdot_M m_2) \mid m_1 \in I_1,\ m_2 \in I_2\} =\\
&\qquad \mathrm{Max}^{\leq_N}\{\varphi(m_1) \cdot_N \varphi(m_2) \mid m_1 \in I_1,\ m_2 \in I_2\} =\\
&\qquad \mathrm{Max}^{\leq_N}\{n_1 \cdot_N n_2 \mid n_1 \in \varphi(I_1),\ n_2 \in \varphi(I_2)\} =\\
&\qquad cSRng\langle\varphi\rangle(I_1) \mathbin{\tilde{\cdot}_N} cSRng\langle\varphi\rangle(I_2) .
\end{aligned}$$

2. For each $M \in |\mathrm{mMon}|$, define $\eta^{\mathrm{cSRng}}_M : M \to mMon(cSRng\langle M \rangle)$ by $\eta^{\mathrm{cSRng}}_M(m) = \{m\}$. Then $\eta^{\mathrm{cSRng}} = (\eta^{\mathrm{cSRng}}_M)_{M \in |\mathrm{mMon}|}$ is a natural transformation from 1_{mMon} to $mMon \circ cSRng\langle - \rangle$.
Let $M \in |\mathrm{mMon}|$, $A \in |\mathrm{cSRng}|$, and $\varphi : M \to mMon(A)$. Define $\varphi^{\sharp_{\mathrm{cSRng}}} : cSRng\langle M \rangle \to A$ by

$$\varphi^{\sharp_{\mathrm{cSRng}}}(\{m_1, \dots, m_n\}) = \varphi(m_1) \oplus_A \cdots \oplus_A \varphi(m_n)$$

for all $\{m_1, \dots, m_n\} \in \mathcal{I}_{\text{fin}}(M)$, where, if $n = 0$, the right hand side is to be understood as $\mathbf{0}_A$; $\varphi^{\sharp_{\mathrm{cSRng}}}$ is indeed a c-semiring homomorphism, since for each $\{m'_1, \dots, m'_n\} \in \mathcal{P}_{\text{fin}}\,|M|$ we have $\varphi^{\sharp_{\mathrm{cSRng}}}(\mathrm{Max}^{\leq_M}\{m'_1, \dots, m'_n\}) = \varphi(m'_1) \oplus_A \cdots \oplus_A \varphi(m'_n)$: if $m'_i \leq_M m'_j$ then $\varphi(m'_i) \leq_{mMon(A)} \varphi(m'_j)$, i.e., $\varphi(m'_i) \preceq_A \varphi(m'_j)$, and thus $\varphi(m'_i) \oplus_A \varphi(m'_j) = \varphi(m'_j)$.
Then $mMon(\varphi^{\sharp_{\mathrm{cSRng}}})(\eta^{\mathrm{cSRng}}_M(m)) = \varphi(m)$ and $\varphi^{\sharp_{\mathrm{cSRng}}}$ is unique with this property.

LEMMA. *$cSRng\langle M \rangle$ is the free c-semiring over the meet monoid M.*

3. A similar construction of c-semiring addition and multiplication operations has been introduced by E. Rollón [**rollon:phd:2008**], though starting from a given c-semiring. She proves that when A is a c-semiring, its so-called *frontier algebra* $(\mathcal{I}(A) \setminus \{\emptyset\}, \tilde{\oplus}_A, \tilde{\otimes}_A, \{\mathbf{0}_A\}, \{\mathbf{1}_A\})$ again is a c-semiring, where $\mathcal{I}(A)$ are the non-empty subsets of $|A|$ containing only pairwise incomparable elements w.r.t. $\preceq_A$, and

$$I \mathbin{\tilde{\oplus}_A} J = \mathrm{Max}^{\preceq_A}(I \cup J) ,$$

$$I \mathbin{\tilde{\otimes}_A} J = \mathrm{Max}^{\preceq_A}\{a \otimes_A b \mid a \in I,\ b \in J\}$$

for all $I, J \in \mathcal{I}(A) \setminus \{\emptyset\}$.
The underlying set of the frontier algebra contains sets of arbitrary cardinality, not only finite sets as in our approach, since no free construction is intended. The condition that only non-empty sets have to be considered is missing in [**rollon:phd:2008**]; the empty set has to be excluded, however, since otherwise $\emptyset \mathbin{\tilde{\oplus}_A} \{\mathbf{0}_A\} = \{\mathbf{0}_A\}$ and $\emptyset \mathbin{\tilde{\otimes}_A} \{\mathbf{0}_A\} = \emptyset$. In fact, we have to consider also $\emptyset$ in order to obtain a bottom element of the free c-semiring over an arbitrary meet monoid. If we only applied the construction of a free c-semiring in §2 to the sub-category of bounded meet monoids, we also could exclude $\emptyset$ and would obtain $\{\bot_M\}$ as bottom element of the free c-semiring over the bounded meet monoid M.
The finite case discussed here has been covered by H. Fargier, E. Rollón and N. Wilson for *preference degree structures*, which are our partially ordered commutative monoids [**fargier-rollon-wilson:cj:2010**]; however, it is not characterised as a free construction.

4. A meet monoid M is *total* if for all $m_1, m_2 \in |M|$:

$$m_1 \leq_M m_2 \text{ or } m_1 = m_2 \text{ or } m_2 \leq_M m_1\ .$$

For a total bounded meet monoid M, $(|M|, \max^{\leq_M}, \cdot_M, \bot_M, \varepsilon_M)$ is a c-semiring, and it is isomorphic to the free c-semiring $cSRng\langle M\rangle$ over the bounded meet monoid M in the sense of §3.

5. Let M be a meet monoid with idempotent multiplication, i.e., $m \cdot_M m = m$ for all $m \in |M|$. Then $\tilde{\cdot}_M$ is also idempotent: Let $I \in \mathcal{I}_{\text{fin}}(M)$. For all $m, n \in I$, we have $m \cdot_M n \leq_M n$. Thus

$$\begin{aligned} I \mathbin{\tilde{\cdot}_M} I &= \mathrm{Max}^{\leq_M}\{m \cdot_M n \mid m \in I,\ n \in I\} = \mathrm{Max}^{\leq_M} \bigcup\{\{m \cdot_M n \mid m \in I\} \mid n \in I\} = \\ &\quad \mathrm{Max}^{\leq_M} \bigcup\{\mathrm{Max}^{\leq_M}\{m \cdot_M n \mid m \in I\} \mid n \in I\} = \mathrm{Max}^{\leq_M} \bigcup\{\{n\} \mid n \in I\} = \\ &\quad \mathrm{Max}^{\leq_M}\{n \mid n \in I\} = I\ . \end{aligned}$$

4.2. *Initial, Terminal, and Direct Product C-Semirings*

1. Consider the c-semiring of *boolean values* $\mathrm{B} = (\{\bot, \top\}, \vee, \wedge, \bot, \top)$ where $\vee$ and $\wedge$ have their usual meaning.

LEMMA. B *is initial in* cSRng. □

2. Consider the c-semiring $\mathrm{T} = (\{*\}, \cdot, \cdot, *, *)$ with $* \cdot * = *$.

LEMMA. T *is terminal in* cSRng. □

3. Let $A = (|A|, \oplus_A, \otimes_A, \mathbf{0}_A, \mathbf{1}_A)$ and $B = (|B|, \oplus_B, \otimes_B, \mathbf{0}_B, \mathbf{1}_B)$ be c-semirings. Define $\oplus_{A\times B}$, $\otimes_{A\times B} : (|A| \times |B|) \times (|A| \times |B|) \to |A| \times |B|$ by

$$(a_1, b_1) \oplus_{A\times B} (a_2, b_2) = (a_1 \oplus_A a_2, b_1 \oplus_B b_2)$$
$$(a_1, b_1) \otimes_{A\times B} (a_2, b_2) = (a_1 \otimes_A a_2, b_1 \otimes_B b_2)$$

Then $A \times B = (|A| \times |B|, \oplus_{A\times B}, \otimes_{A\times B}, (\mathbf{0}_A, \mathbf{0}_B), (\mathbf{1}_A, \mathbf{1}_B))$ is a c-semiring.
Define $\pi_1 : A \times B \to A$ by $\pi_1(a, b) = a$ and $\pi_2 : A \times B \to B$ by $\pi_2(a, b) = b$. Then π_1 and π_2 are c-semiring homomorphisms. Furthermore, for any c-semiring $C = (|C|, \oplus_C, \otimes_C, \mathbf{0}_C, \mathbf{1}_C)$ and two c-semiring homomorphisms $\varphi_1 : C \to A$ and $\varphi_2 : C \to B$, the c-semiring homomorphism $\langle \varphi_1, \varphi_2\rangle : C \to A \times B$ defined by $\langle \varphi_1, \varphi_2\rangle(c) = (\varphi_1(c), \varphi_2(c))$ is unique for the property $\varphi_1 = \langle \varphi_1, \varphi_2\rangle \mathbin{;} \pi_1$ and $\varphi_2 = \langle \varphi_1, \varphi_2\rangle \mathbin{;} \pi_2$.

LEMMA. cSRng *has finite products.* □

4.3. *Lexicographic Products of C-Semirings*

1. The *collapsing elements* $\mathcal{C}(A)$ of a c-semiring A are the collapsing elements of $mMon(A)$.
A c-semiring A is *total* if for all $a, a_1, a_2 \in |A|$:

$$a_1 \prec_A a_2 \text{ or } a_1 = a_2 \text{ or } a_2 \prec_A a_1 \ .$$

Let A and B be c-semirings where A is total. Let $L = ((|A| \setminus \mathcal{C}(A)) \times |B|) \cup (\mathcal{C}(A) \times \{\mathbf{0}_B\})$. Define $\oplus_{A \ltimes B}, \otimes_{A \ltimes B} : L \times L \to L$ by

$$(a_1, b_1) \oplus_{A \ltimes B} (a_2, b_2) = \begin{cases} (a_1, b_1) & \text{if } a_2 \prec_A a_1 \\ (a_2, b_2) & \text{if } a_1 \prec_A a_2 \\ (a_1, b_1 \oplus_B b_2) & \text{if } a_1 = a_2 \end{cases} ,$$

$$(a_1, b_1) \otimes_{A \ltimes B} (a_2, b_2) = (a_1 \otimes_A a_2, b_1 \otimes_B b_2) \ .$$

Then $(a, b) \otimes_{A \ltimes B} ((a_1, b_1) \oplus_{A \ltimes B} (a_2, b_2)) = ((a, b) \otimes_{A \ltimes B} (a_1, b_1)) \oplus_{A \ltimes B} ((a, b) \otimes_{A \ltimes B} (a_2, b_2))$ for all $(a, b), (a_1, b_1), (a_2, b_2) \in L$, where the right hand side is $(a \otimes_A a_1, b \otimes_B b_1) \oplus_{A \ltimes B} (a \otimes a_2, b \otimes_B b_2)$: If $a_1 = a_2$, then $(a_1, b_1) \oplus_{A \ltimes B} (a_2, b_2) = (a_1, b_1 \oplus_B b_2)$ and $a \otimes_A a_1 = a \otimes_A a_2$, from which the claim follows by the distributivity of $\otimes_B$ over $\oplus_B$. If $a_2 \prec_A a_1$ (the case $a_1 \prec_A a_2$ is symmetric), then $(a_1, b_1) \oplus_{A \ltimes B} (a_2, b_2) = (a_1, b_1)$. If additionally $a \notin \mathcal{C}(A)$, then $a \otimes_A a_2 \prec_A a \otimes_A a_1$; if $a \in \mathcal{C}(A)$, then $b = \mathbf{0}_B$, and in both cases the claim follows.
Thus $A \ltimes B = (L, \oplus_{A \ltimes B}, \otimes_{A \ltimes B}, (\mathbf{0}_A, \mathbf{0}_B), (\mathbf{1}_A, \mathbf{1}_B))$ is a c-semiring, the *lexicographic product* of A and B.

2. Let A be a c-semiring. Let Z_A be a subset of $|A|$ with: (1) if $a \otimes_A b \in Z_A$, then $a \in Z_A$ or $b \in Z_A$; and (2) if $a \oplus_A b \in Z_A$, then $a \in Z_A$ and $b \in Z_A$.
Let B be another c-semiring. Let $L_{Z_A} = ((|A| \setminus Z_A) \times |B|) \cup (Z_A \times \{\mathbf{0}_B\})$. Define $\oplus^{Z_A}_{A \ltimes B}, \otimes^{Z_A}_{A \ltimes B} : L_{Z_A} \times L_{Z_A} \to L_{Z_A}$ by

$$(a_1, b_1) \oplus^{Z_A}_{A \ltimes B} (a_2, b_2) = (a_1 \oplus_A a_2, b_1 \oplus_B b_2) \ ,$$

$$(a_1, b_1) \otimes^{Z_A}_{A \ltimes B} (a_2, b_2) = (a_1 \otimes_A a_2, b_1 \otimes_B b_2) \ .$$

Then $\oplus^{Z_A}_{A \ltimes B}$ and $\otimes^{Z_A}_{A \ltimes B}$ are well defined.

5. Soft Constraints

1. We recapitulate essential notions introduced in [**Rossi2006Handbook**]. A *constraint domain* (X, D) is given by a set X of *variables* and a family $D = (D_x)_{x \in X}$ of *variable domains* where each D_x is a set. A constraint domain (X, D) is *finite* if X and $\bigcup_{x \in X} D_x$ are finite.
A *valuation* for a constraint domain (X, D) is a dependent map $v \in \Pi x \in X \,.\, D_x$, i.e., $v(x) \in D_x$; we abbreviate $\Pi x \in X \,.\, D_x$ by $[X \to D]$.
A *constraint* c over a constraint domain (X, D), or (X, D)-*constraint*, is given by a map $c : [X \to D] \to \mathbb{B}$. We also write $v \models c$ for $c(v) = tt$.

2. Given a constraint domain (X, D) and a c-semiring G, a G-*soft constraint* γ over (X, D), or (X, D)-G-*soft constraint*, is given by a map $\gamma : [X \to D] \to |G|$. In particular, a constraint over (X, D) can be considered a $\mathbb{B}$-soft constraint over (X, D).
Let Γ be a finite set of (X, D)-G-soft constraints. For a $v \in [X \to D]$ let the *solution degree* for Γ of v be

$$\Gamma(v) = \bigotimes\nolimits_G \{\gamma(v) \mid \gamma \in \Gamma\} \ .$$

Define a binary relation $\preccurlyeq_\Gamma \subseteq [X \to D] \times [X \to D]$ by

$$w \preccurlyeq_\Gamma v \iff \Gamma(w) \preceq_G \Gamma(v) ,$$

meaning that valuation v is a *better solution* for Γ than the valuation w.
The *maximum solution degrees* of Γ are given by

$$\Gamma^* = \mathrm{Max}^{\preceq_G}\{\Gamma(v) \mid v \in [X \to D]\} ,$$

and the *maximum solutions* by

$$\mathrm{Max}^{\preccurlyeq_\Gamma}[X \to D] = \{v \in [X \to D] \mid \Gamma(v) \in \Gamma^*\} .$$

3. A set of (X, D)-G-soft constraints is *admissible* if Γ is finite and for each $v \in [X \to D]$ there is a $g \in \Gamma^*$ such that $\Gamma(v) \preceq_G g$.

EXAMPLE. Let Γ be a finite set of (X, D)-G-soft constraints.

(1) If (X, D) is finite, then Γ is admissible.

(2) If $\prec_G$ has no infinite ascending chains, then Γ is admissible.

(3) Let $X = \{\mathrm{x}\}$, $D_{\mathrm{x}} = [0, 1]$, $G = ([0, 1], \max, +^1, 1, 0)$ with $r +^1 s = \min\{1, r + s\}$. Let $\gamma : [X \to D] \to |G|$ be defined by $\gamma(\{\mathrm{x} \mapsto r\}) = r$ if $r > 0$, and $\gamma(\{\mathrm{x} \mapsto 0\}) = 1$. Let $\Gamma = \{\gamma\}$. Then $\Gamma^* = \emptyset$, since the set of solution degrees is the open interval $]0, 1]$ with 0 the optimum, which, however, cannot be obtained. Thus Γ is not admissible. □

4. For a constraint domain (X, D) we fix an *extended* constraint domain $(X, D^?)$ setting $D^? = (D^?_x)_{x\in X}$ with $D^?_x = D_x \uplus \{?\}$, where ? is fresh.
A valuation $p \in \Pi x \in X \,.\, D^?_x = [X \to D^?]$ is called a *partial valuation* for (X, D).
The *domain of definition* $\mathrm{def}(p)$ of a partial valuation p for (X, D) is the set $\{x \in X \mid p(x) \neq ?\}$. For $p, q \in [X \to D^?]$, we write $p \sqsubseteq q$ if $x \in \mathrm{def}(p)$ implies $x \in \mathrm{def}(q)$ and $q(x) = p(x)$ for each $x \in X$; by $p\!\uparrow$ we denote the set $\{v \in [X \to D] \mid p \sqsubseteq v\}$ of (X, D)-valuations.

5. An $(X, D^?)$-G-soft constraint $\omega : [X \to D^?] \to |G|$ is *bounding* if $\omega(v) \preceq_G \omega(p)$ for all $p \in [X \to D^?]$ and $v \in p\!\uparrow$. A bounding $(X, D^?)$-G-soft constraint ω is *tight* for a finite set of (X, D)-G-soft constraints Γ if $\omega(v) = \Gamma(v)$ for all $v \in [X \to D]$.
A pair (π, ω) of $(X, D^?)$-G-soft constraints forms a *bounding pair* if ω is bounding and for each $p\{x \mapsto d\} \in [X \to D^?]$ there is a $v \in p\!\uparrow$ with $\pi(p\{x \mapsto d\}) \preceq_G \omega(v)$; a bounding pair (π, ω) is *tight* for a finite set of (X, D)-G-soft constraints Γ if ω is tight for Γ.
For a bounding pair (π, ω) of $(X, D^?)$-G-soft constraints the following "branch & bound" algorithm $\mathsf{maxSolDegs}_{(\pi,\omega)}$ computes, given a partial valuation $p \in [X \to D^?]$ and a finite set of lower bounds $L \subseteq |G|$ (which we assume to contain only elements which are pairwise incomparable w.r.t. $\preceq_G$), the maximum degrees in $L \cup \{\omega(v) \mid v \in p\!\uparrow\}$ w.r.t. $\preceq_G$, i.e., in particular, if $p = (\lambda x \in X \,.\, ?)$ and $L = \emptyset$, the maximum degrees in $\{\omega(v) \mid v \in [X \to D]\}$:

Assume: – (X, D) finite constraint domain
– G c-semiring
– (π, ω) bounding pair of $(X, D^?)$-G-soft constraints
In: – $p \in [X \to D^?]$ partial valuation for (X, D)
– $L \subseteq |G|$ finite and pairwise incomparable w.r.t. $\preceq_G$
Return: $\mathrm{Max}^{\preceq_G}(L \cup \omega(p\!\uparrow))$

$\mathsf{maxSolDegs}_{(\pi,\omega)}(p, L) \equiv$
if $\forall x \in X \,.\, p(x) \neq ?$
then return $\mathrm{Max}^{\preceq_G}(L \cup \{\omega(p)\})$

$x \leftarrow$ **choose** $\{x \in X \mid p(x) = ?\}$
$L \leftarrow \mathrm{Max}^{\preceq_G}(L \cup \{\pi(p\{x \mapsto d\}) \mid d \in D_x\})$
for $d \in D_x$
do if $\neg\exists l \in L \,.\, \omega(p\{x \mapsto d\}) \preceq_G l$
  then $L \leftarrow \mathsf{maxSolDegs}_{(\pi,\omega)}(p\{x \mapsto d\}, L)$ **fi od**
return L

We prove the claim that

$$\mathsf{maxSolDegs}_{(\pi,\omega)}(p, L) = \mathrm{Max}^{\preceq_G}(L \cup \omega(p\uparrow))$$

by a first induction on the cardinality n of $\{x \in X \mid p(x) = ?\}$. If $n = 0$, i.e., $p \in [X \to D]$, then $\mathsf{maxSolDegs}_{(\pi,\omega)}(p, L) = \mathrm{Max}^{\preceq_G}(L \cup \{\omega(p)\})$ and $\{\omega(p)\} = \omega(p\uparrow)$. If $n > 0$, then let $x \in X$ with $p(x) = ?$, and let $d_1, \ldots, d_r$ be an enumeration of D_x. Let $P = \{\pi(p\{x \mapsto d_i\}) \mid 1 \leq i \leq r\}$ and define

$$L_0 = \mathrm{Max}^{\preceq_G}(L \cup P) \ ,$$

and inductively

$$L_k = \begin{cases} L_{k-1} & \text{if } \exists l \in L_{k-1} \,.\, \omega(p\{x \mapsto d_k\}) \preceq_G l \\ \mathsf{maxSolDegs}_{\omega}(p\{x \mapsto d_k\}, L_{k-1}) & \text{otherwise} \end{cases}$$

for $1 \leq k \leq r$. We prove the sub-claim that

$$L_k = \mathrm{Max}^{\preceq_G}(L \cup P \cup \textstyle\bigcup_{1 \leq j \leq k} \omega(p\{x \mapsto d_j\}\uparrow))$$

for all $0 \leq k \leq r$ by a second induction on k: For $k = 0$, $L_0 = \mathrm{Max}^{\preceq_G}(L \cup P)$ by definition. For $k > 0$, let there first be an $l \in L_{k-1}$ with $\omega(p\{x \mapsto d_k\}) \preceq_G l$. Since $\omega(v) \preceq_G \omega(p\{x \mapsto d_k\}) \preceq_G l$ for all $v \in p\{x \mapsto d_k\}\uparrow$ and $l \in L_{k-1} = \mathrm{Max}^{\preceq_G}(L \cup P \cup \bigcup_{1 \leq i \leq k-1} \omega(p\{x \mapsto d_i\}\uparrow))$ by the second induction hypothesis, the sub-claim follows. Otherwise, if no such $l \in L_{k-1}$ exists, $\mathsf{maxSolDegs}_{\omega}(p\{x \mapsto d_k\}, L_{k-1}) = \mathrm{Max}^{\preceq_G}(L_{k-1} \cup \omega(p\{x \mapsto d_k\}\uparrow))$ by the first induction hypothesis, which is applicable since, by the second induction hypothesis, $L_{k-1} = \mathrm{Max}^{\preceq_G}(L \cup P \cup \bigcup_{1 \leq i \leq k-1} \omega(p\{x \mapsto d_i\}\uparrow))$, and hence L_{k-1} is pairwise incomparable w.r.t. $\preceq_G$; therefore,

$$\begin{aligned} L_k = {} & \mathsf{maxSolDegs}_{\omega}(p\{x \mapsto d_k\}, L_{k-1}) = \\ & \mathrm{Max}^{\preceq_G}(L_{k-1} \cup \omega(p\{x \mapsto d_k\}\uparrow)) = \\ & \mathrm{Max}^{\preceq_G}((\mathrm{Max}^{\preceq_G}(L \cup P \cup \textstyle\bigcup_{1 \leq i \leq k-1} \omega(p\{x \mapsto d_i\}\uparrow))) \cup \omega(p\{x \mapsto d_k\}\uparrow)) = \\ & \mathrm{Max}^{\preceq_G}(L \cup P \cup \textstyle\bigcup_{1 \leq i \leq k} \omega(p\{x \mapsto d_i\}\uparrow)) \ , \end{aligned}$$

which establishes the sub-claim. Thus, $L_r = \mathrm{Max}^{\preceq_G}(L \cup P \cup \bigcup_{1 \leq i \leq r} \omega(p\{x \mapsto d_i\}\uparrow)) = \mathrm{Max}^{\preceq_G}(L \cup \omega(p\uparrow)) = \mathsf{maxSolDegs}_{\omega}(p, L)$, since $d_1, \ldots, d_r$ is an enumeration of D_x and for each $1 \leq i \leq r$ there is a $v \in p\uparrow$ with $\pi(p\{x \mapsto d_i\}) \preceq_G \omega(v)$, which yields the claim.
In particular, if $(\Gamma_?, \Gamma^?)$ is a tight bounding pair of $(X, D^?)$-G-soft constraints for a finite set of (X, D)-G-soft constraints Γ, then

$$\begin{aligned} & \mathsf{maxSolDegs}_{(\Gamma_?,\Gamma^?)}(\lambda x \in X \,.\, ?, \emptyset) = \mathrm{Max}^{\preceq_G}(\emptyset \cup \Gamma^?((\lambda x \in X \,.\, ?)\uparrow)) = \\ & \quad \mathrm{Max}^{\preceq_G}\{\Gamma^?(v) \mid v \in [X \to D]\} = \mathrm{Max}^{\preceq_G}\{\Gamma(v) \mid v \in [X \to D]\} = \Gamma^* \ . \end{aligned}$$

6. Given a meet monoid M and a constraint domain (X, D), an *M-soft constraint* μ over (X, D), or (X, D)-*M-soft constraint*, is given by a map $\mu : [X \to D] \to |M|$. Each (X, D)-M-soft constraint induces an (X, D)-$\mathit{cSRng}\langle M\rangle$-soft constraint $\eta_M^{\mathrm{cSRng}} \circ \mu$ (viz., $(\eta_M^{\mathrm{cSRng}} \circ \mu)(v) = \{\mu(v)\}$).
Let M be a finite set of (X, D)-M-soft constraints for a meet monoid M. Then

$$\bigotimes_{\mathit{cSRng}\langle M\rangle}\{\eta_M^{\mathrm{cSRng}}(\mu(v)) \mid \mu \in \mathsf{M}\} = \{\textstyle\prod_M\{\mu(v) \mid \mu \in \mathsf{M}\}\}\ ,$$

and thus

$$w \preccurlyeq_{\eta_M^{\mathrm{cSRng}} \circ \mathsf{M}} v \iff \{\textstyle\prod_M\{\mu(w) \mid \mu \in \mathsf{M}\}\} \preceq_{\mathit{cSRng}\langle M\rangle} \{\textstyle\prod_M\{\mu(v) \mid \mu \in \mathsf{M}\}\} \iff$$
$$\textstyle\prod_M\{\mu(w) \mid \mu \in \mathsf{M}\} \leq_M \prod_M\{\mu(v) \mid \mu \in \mathsf{M}\}\ .$$

We write $\mathit{cSRng}\langle\mathsf{M}\rangle$ for the set of (X, D)-$\mathit{cSRng}\langle M\rangle$-soft constraints $\{\eta_M^{\mathrm{cSRng}} \circ \mu \mid \mu \in \mathsf{M}\}$.
In analogy to the notions for soft constraints, we define the *solution degree* for a $v \in [X \to D]$ by

$$\mathsf{M}(v) = \textstyle\prod_M\{\mu(v) \mid \mu \in \mathsf{M}\}\ ;$$

the *better solution* relation $\lesssim_{\mathsf{M}} \subseteq [X \to D] \times [X \to D]$ by

$$w \lesssim_{\mathsf{M}} v \iff \mathsf{M}(w) \leq_M \mathsf{M}(v)\ ;$$

the *maximum solution degrees* by

$$\mathsf{M}^* = \mathrm{Max}^{\leq_M}\{\mathsf{M}(v) \mid v \in [X \to D]\}\ ;$$

and the *maximum solutions* by

$$\mathrm{Max}^{\lesssim_{\mathsf{M}}}[X \to D] = \{v \in [X \to D] \mid \mathsf{M}(v) \in \mathsf{M}^*\}\ .$$

A set M of (X, D)-M-soft constraints for a meet monoid M is *admissible* if it is finite and for all $v \in [X \to D]$ there is an $m \in \mathsf{M}^*$ such that $\mathsf{M}(v) \leq_M m$.

7. The algorithm $\mathsf{maxSolDegs}_{(\pi,\omega)}$ from §5 for a bounding pair (π, ω) of $(X, D^?)$-G-soft constraints for a c-semiring G in fact also works under the assumptions that (X, D) is a constraint domain, M is a meet monoid, and α and ζ are $(X, D^?)$-M-soft constraints, such that $(\eta_M^{\mathrm{cSRng}} \circ \alpha, \eta_M^{\mathrm{cSRng}} \circ \zeta)$ is a bounding pair. We call (α, ζ) itself a *bounding pair* if $\zeta(v) \leq_M \zeta(p)$ for all $p \in [X \to D^?]$ and $v \in p{\uparrow}$, and if for each $p\{x \mapsto d\} \in [X \to D^?]$ there is a $v \in p{\uparrow}$ such that $\alpha(p\{x \mapsto d\}) \leq_M \zeta(v)$.

Assume: – (X, D) finite constraint domain
– M meet monoid
– (α, ζ) bounding pair of $(X, D^?)$-M-soft constraints
In: – $p \in [X \to D^?]$ partial valuation for (X, D)
– $L \subseteq |M|$ finite and pairwise incomparable w.r.t. $\leq_M$
Return: $\mathrm{Max}^{\leq_M}(L \cup \zeta(p{\uparrow}))$

$\mathsf{maxSolDegs}_{(\alpha,\zeta)}(p, L) \equiv$
if $\forall x \in X\,.\,p(x) \neq {?}$
then return $\mathrm{Max}^{\leq_M}(L \cup \{\zeta(p)\})$ **fi**
$x \leftarrow$ **choose** $\{x \in X \mid p(x) = {?}\}$
$L \leftarrow \mathrm{Max}^{\leq_M}(L \cup \{\alpha(p\{x \mapsto d\}) \mid d \in D_x\})$
for $d \in D_x$
do if $\neg\exists l \in L\,.\,\zeta(p\{x \mapsto d\}) \leq_M l$
then $L \leftarrow \mathsf{maxSolDegs}_{(\alpha,\zeta)}(p\{x \mapsto d\}, L)$ **fi od**
return L

Note that $\mathsf{maxSolDegs}_{(\alpha,\zeta)}(p, L) = L \oplus_{cSRng\langle M\rangle} \mathrm{Max}^{\leq_M} \zeta(p\uparrow)$.
A bounding pair (α, ζ) of $(X, D^?)$-M-soft constraints is *tight* for a finite set of (X, D)-M-soft constraints M if $\zeta(v) = \mathsf{M}(v)$ for all $v \in [X \to D^?]$. For a tight bounding pair $(\mathsf{M}_?, \mathsf{M}^?)$ for a finite set of (X, D)-M-soft constraints M we again obtain

$$\mathsf{maxSolDegs}_{(\mathsf{M}_?,\mathsf{M}^?)}(\lambda x \in X \,.\, ?, \emptyset) = \mathrm{Max}^{\leq_M}\{\mathsf{M}(v) \mid v \in [X \to D]\} = \mathsf{M}^* \,.$$

6. Constraint Hierarchies

1. A *constraint hierarchy* [**borning1992hierarchies**] $H = (C_k)_{1\leq k\leq n}$ over a constraint domain (X, D), or (X, D)-*constraint hierarchy*, is given by a family of sets of (X, D)-constraints. The constraints in *level* $1 \leq k \leq n$ are considered as *strictly more important* than the constraints in level $k + 1$. An (X, D)-constraint hierarchy is *finite* if $\bigcup_{1\leq k\leq n} C_k$ is finite.
Let $H = (C_k)_{1\leq k\leq n}$ be a finite (X, D)-constraint hierarchy, let $L = (M_i)_{1\leq k\leq n}$ be a corresponding family of meet monoids, and let for each $1 \leq k \leq n$ and each $c \in C_k$, $\mu(c)$ be an (X, D)-M_k-soft constraint. We call $\mathsf{H} = (\mathsf{M}_k)_{1\leq k\leq n}$ with $\mathsf{M}_k = \{\mu(c) \mid c \in C_k\}$ for $1 \leq k \leq n$ an (X, D)-*L-soft constraint hierarchy*. For a $v \in [X \to D]$ the *solution degree* for $(\mathsf{M}_k)_{1\leq k\leq n}$ of v is defined to be $(\mathsf{M}_k(v))_{1\leq k\leq n}$. Define a binary relation $<_{\mathsf{H}} \subseteq [X \to D] \times [X \to D]$ by

$$w <_{\mathsf{H}} v \iff \exists 1 \leq k \leq n \,.\, (\forall 1 \leq i \leq k-1 \,.\, \mathsf{M}_i(w) = \mathsf{M}_i(v)) \land \mathsf{M}_k(w) <_{M_k} \mathsf{M}_k(v) \,,$$

saying that the valuation v is *strictly better* than the valuation w, and denote its reflexive closure on $[X \to D]$ by $\leq_{\mathsf{H}}$, which is the lexicographic order on the set $\{(\mathsf{M}_k(v))_{1\leq k\leq n} \mid v \in [X \to D]\}$. In particular,

$$w <_{\mathsf{H}} v \iff (\mathsf{M}_k(w))_{1\leq k\leq n} <_{M_1\ltimes\ldots\ltimes M_n} (\mathsf{M}_k(v))_{1\leq k\leq n}$$

if, on the one hand, every M_k is a bounded meet monoid for all $2 \leq k \leq n$, and, on the other hand, $\mathsf{M}_k(v), \mathsf{M}_k(w) \notin \mathcal{C}(M_k)$ for all $1 \leq k \leq n$, or, equivalently, if $\mu(c)(v), \mu(c)(w) \notin \mathcal{C}(M_k)$ for each $c \in C_k$, $1 \leq k \leq n$. The first requirement, that each M_k is bounded, can be achieved by moving from M_k to its lifted variant $(M_k)_\perp$.

2. Let (X, D) be a constraint domain, and M and N meet monoids. Let M be a finite set of (X, D)-M-soft constraints and let N be a finite set of (X, D)-N-soft constraints.
We say that M and N are *optima equivalent*, written as $\mathsf{M} \approx \mathsf{N}$, if $\mathrm{Max}^{\lesssim_{\mathsf{M}}}[X \to D] = \mathrm{Max}^{\lesssim_{\mathsf{N}}}[X \to D]$. A sufficient criterion for optima equivalence is that $\mathsf{M}(w) \leq_M \mathsf{M}(v)$ if, and only if, $\mathsf{N}(w) \leq_N \mathsf{N}(v)$ for all $v, w \in [X \to D]$.
N *optima simulates* M, written as $\mathsf{M} \preccurlyeq \mathsf{N}$, if for each $v_{\mathsf{M}} \in \mathrm{Max}^{\lesssim_{\mathsf{M}}}[X \to D]$ there is a $v_{\mathsf{N}} \in \mathrm{Max}^{\lesssim_{\mathsf{N}}}[X \to D]$ with $\mathsf{M}(v_{\mathsf{M}}) = \mathsf{M}(v_{\mathsf{N}})$, and, vice versa, if for each $v_{\mathsf{N}} \in \mathrm{Max}^{\lesssim_{\mathsf{N}}}[X \to D]$ there is a $v_{\mathsf{M}} \in \mathrm{Max}^{\lesssim_{\mathsf{M}}}[X \to D]$ with $\mathsf{M}(v_{\mathsf{M}}) = \mathsf{M}(v_{\mathsf{N}})$. Obviously, $\mathsf{M} \approx \mathsf{N}$ if, and only if, $\mathsf{M} \preccurlyeq \mathsf{N}$ and $\mathsf{N} \preccurlyeq \mathsf{M}$.

Lemma. *Let (X, D) be a constraint domain, and let M and N be meet monoids. Let M be an admissible set of (X, D)-M-soft constraints and N an admissible set of (X, D)-N-soft constraints such that*

$$\mathsf{M}(v) <_M \mathsf{M}(v') \quad \textit{implies} \quad \mathsf{N}(v) <_N \mathsf{N}(v')$$
$$\mathsf{M}(v) \parallel_M \mathsf{M}(v') \quad \textit{implies} \quad \mathsf{N}(v) \parallel_N \mathsf{N}(v')$$

for all $v, v' \in [X \to D]$. Then $\mathsf{M} \preccurlyeq \mathsf{N}$.

Proof. Let first $v_{\mathsf{M}} \in \mathrm{Max}^{\lesssim_{\mathsf{M}}}[X \to D]$. Let $v_{\mathsf{M}} \notin \mathrm{Max}^{\lesssim_{\mathsf{N}}}[X \to D]$. Then, since N is admissible, there is a $v_{\mathsf{N}} \in \mathrm{Max}^{\lesssim_{\mathsf{N}}}[X \to D]$ with $\mathsf{N}(v_{\mathsf{M}}) <_N \mathsf{N}(v_{\mathsf{N}})$. Moreover, there is a $v'_{\mathsf{M}} \in \mathrm{Max}^{\lesssim_{\mathsf{M}}}[X \to D]$ with $\mathsf{M}(v_{\mathsf{N}}) \leq_M \mathsf{M}(v'_{\mathsf{M}})$, since M is admissible. But $\mathsf{M}(v_{\mathsf{N}}) <_M \mathsf{M}(v'_{\mathsf{M}})$ is impossible, since then also

$\mathsf{N}(v_\mathsf{N}) <_N \mathsf{N}(v'_\mathsf{M})$ contradicting $\mathsf{N}(v_\mathsf{N}) \in \mathsf{N}^*$. Thus $\mathsf{M}(v_\mathsf{N}) = \mathsf{M}(v'_\mathsf{M})$. Moreover, either $\mathsf{M}(v_\mathsf{M}) \parallel_M \mathsf{M}(v'_\mathsf{M})$ or $\mathsf{M}(v_\mathsf{M}) = \mathsf{M}(v'_\mathsf{M})$ since both $\mathsf{M}(v_\mathsf{M})$ and $\mathsf{M}(v'_\mathsf{M})$ are elements of M^*. But $\mathsf{M}(v_\mathsf{M}) \parallel_M \mathsf{M}(v'_\mathsf{M})$ is impossible, since we would have $\mathsf{M}(v_\mathsf{M}) \parallel_M \mathsf{M}(v_\mathsf{N}) = \mathsf{M}(v'_\mathsf{M})$ and $\mathsf{N}(v_\mathsf{M}) <_N \mathsf{N}(v_\mathsf{N})$. Thus $\mathsf{M}(v_\mathsf{N}) = \mathsf{M}(v'_\mathsf{M}) = \mathsf{M}(v_\mathsf{M})$.
Now let $v_\mathsf{N} \in \mathrm{Max}^{\lesssim_\mathsf{N}}[X \to D]$. If $v_\mathsf{N} \notin \mathrm{Max}^{\lesssim_\mathsf{M}}[X \to D]$, there would be, since M is admissible, a $v_\mathsf{M} \in \mathrm{Max}^{\lesssim_\mathsf{M}}[X \to D]$ such that $\mathsf{M}(v_\mathsf{N}) <_M \mathsf{M}(v_\mathsf{M})$, i.e. $\mathsf{N}(v_\mathsf{N}) <_N \mathsf{N}(v_\mathsf{M})$, contradicting $\mathsf{N}(v_\mathsf{N}) \in \mathsf{N}^*$. ☐

From the conditions of the lemma it follows that $\mathsf{N}(v) \leq_N \mathsf{N}(v')$ implies $\mathsf{M}(v) \leq_M \mathsf{M}(v')$ for all $v, v' \in [X \to D]$. Indeed, let the conditions of the lemma hold and let $\mathsf{N}(v) \leq_N \mathsf{N}(v')$ but $\mathsf{M}(v) \not\leq_M \mathsf{M}(v')$. Then either $\mathsf{M}(v) >_M \mathsf{M}(v')$ or $\mathsf{M}(v) \parallel_M \mathsf{M}(v')$, implying $\mathsf{N}(v) >_N \mathsf{N}(v')$ or $\mathsf{N}(v) \parallel_N \mathsf{N}(v')$ which both contradict $\mathsf{N}(v) \leq_N \mathsf{N}(v')$. Moreover, from the requirement that $\mathsf{N}(v) \leq_N \mathsf{N}(v')$ implies $\mathsf{M}(v) \leq_M \mathsf{M}(v')$ for all $v, v' \in [X \to D]$ it follows that $\mathsf{M}(v) \parallel_M \mathsf{M}(v')$ implies $\mathsf{N}(v) \parallel_N \mathsf{N}(v')$ for all $v, v' \in [X \to D]$. Thus, the conditions of the lemma can be equivalently replaced by

$$\begin{aligned} &\mathsf{M}(v) <_M \mathsf{M}(v') \quad \text{implies} \quad \mathsf{N}(v) <_N \mathsf{N}(v') \\ &\mathsf{N}(v) \leq_N \mathsf{N}(v') \quad \text{implies} \quad \mathsf{M}(v) \leq_M \mathsf{M}(v') \end{aligned}$$

for all $v, v' \in [X \to D]$.
(S. Bistarelli, Ph. Codognet, and F. Rossi discuss abstractions of c-semiring-based soft constraint problems by means of Galois connections [**bistarelli-codognet-rossi:ai:2002**].)

3. Let C be a finite set of (X, D)-constraints. The *locally-predicate-better* (*LPB*) *level comparator* for C corresponds to requiring

$$w <_C^{\mathrm{LPB}} v \iff \{c \in C \mid v \not\models c\} \subset \{c \in C \mid w \not\models c\} \ .$$

This can be expressed by choosing the meet monoid $M = (\mathcal{P}_{\mathrm{fin}}(C), \cup, \emptyset, \supseteq)$ and the set of (X, D)-M-soft constraints $\mathsf{M} = \{\mu(c) \mid c \in C\}$ with $\mu(c)(v) = \{c\}$ if $v \not\models c$ and $\mu(c)(v) = \emptyset$ otherwise, for each $c \in C$. However, all elements of M are idempotent, and thus the collapsing elements of M are $\mathcal{P}_{\mathrm{fin}}(C) \setminus \{\emptyset\}$. Hence, M is not suitable for a lexicographic product.
Choosing instead the meet monoid $N = (\mathcal{M}_{\mathrm{fin}}(C), \uplus, \wr\!\int, \supseteq\!\!\!\!\supseteq)$ which has no collapsing elements and the set of (X, D)-N-soft constraints $\mathsf{N} = \{\nu(c) \mid c \in C\}$ with $\nu(c)(v) = \wr c \!\int$ if $v \not\models c$ and $\nu(c)(v) = \wr\!\int$ otherwise, for each $c \in C$, deviates this situation, since we have

$$\mathsf{M}(w) \leq_M \mathsf{M}(v) \iff \mathsf{N}(w) \leq_N \mathsf{N}(v)$$

for all $v, w \in [X \to D]$. Thus $\mathsf{M} \approx \mathsf{N}$, i.e., M and N are optima equivalent.

4. A *real* meet monoid R has $0 \in |R| \subseteq \mathbb{R}_{\geq 0}$ for its underlying set, has 0 as its neutral element, and the (inverted) usual ordering on the real numbers $\geq$ as its ordering.
If $|R| = \mathbb{R}_{\geq 0}$ and $(t \cdot r) \cdot_R (t \cdot s) = t \cdot (r \cdot_R s)$ holds in the real meet monoid R for all $r, s, t \in \mathbb{R}_{\geq 0}$ (where $\cdot$ is the usual multiplication), then, according to a theorem by H. F. Bohnenblust [**bohnenblust:duke:1940**], either $1 \cdot_R 1 = 1$ and $r \cdot_R s = \max\{r, s\}$ for all $r, s \in \mathbb{R}_{\geq 0}$, or $1 \cdot_R 1 > 1$ and $r \cdot_R s = (r^p + s^p)^{1/p}$ for all $r, s \in \mathbb{R}_{\geq 0}$ for some $p > 0$.
Thus, the following are examples of real meet monoids:

– *Weighted sum*: $\mathrm{R}_1 = (\mathbb{R}_{\geq 0}, \cdot_1, 0, \geq)$ with $r \cdot_1 s = r + s$;
– *Least squares*: $\mathrm{R}_2 = (\mathbb{R}_{\geq 0}, \cdot_2, 0, \geq)$ with $r \cdot_2 s = \sqrt{r^2 + s^2}$;
– *p-norm* for $p > 0$: $\mathrm{R}_p = (\mathbb{R}_{\geq 0}, \cdot_p, 0, \geq)$ with $r \cdot_p s = (r^p + s^p)^{1/p}$;
– *Worst case*: $\mathrm{R}_\infty = (\mathbb{R}_{\geq 0}, \cdot_\infty, 0, \geq)$ with $r \cdot_\infty s = \max\{r, s\}$.

The notation R_∞ is justified by the well-known fact that $\lim_{p\to\infty}(r^p + s^p)^{1/p} = \max\{r, s\}$.
For a $V \subseteq \mathbb{R}_{\geq 0}$ and $p > 0$, let $\langle V\rangle_p$ be the smallest subset of $\mathbb{R}_{\geq 0}$ such that $0 \in \langle V\rangle_p$ and $r \cdot_p s \in \langle V\rangle_p$ if $r, s \in \langle V\rangle_p$. Then we obtain a real meet monoid $(\langle V\rangle_p, \cdot_p, 0, \geq)$. For a $V \subseteq \mathbb{R}_{\geq 0}$, let V_∞ denote the meet monoid $(V \cup \{0\}, \cdot_\infty, 0, \geq)$, and let V_p denote the meet monoids $(\langle V\rangle_p, \cdot_p, 0, \geq)$ for $p > 0$. All real meet monoids R with $\cdot_R = \cdot_p$ for some $p > 0$ have no collapsing elements, since $r \cdot_p s = (r^p + s^p)^{1/p}$ is strictly monotonic in both arguments. For real meet monoids with $\cdot_R = \cdot_\infty$, however, $\mathcal{C}(R) = |R| \setminus \{0\}$, since $\cdot_\infty$ is idempotent.

5. Let $0 \in V \subseteq \mathbb{R}_{\geq 0}$, M a meet monoid, and $\tau : V_\infty \to M$ a meet monoid homomorphism. For a $\vec{r} = (r_i)_{1\leq i\leq n} \in V^n$, we write $\tau(\vec{r})$ for $(\tau(r_i))_{1\leq i\leq n}$; for an $m \in |M|$, we write $m^{(n)_M}$ for the n-fold product of m w.r.t. $\cdot_M$. For each $n \geq 1$ we have

$(*)\quad \prod_\infty \vec{r} < \prod_\infty \vec{s} \quad \text{implies} \quad \prod_M \tau(\vec{r}) >_M \prod_M \tau(\vec{s}) \quad \text{for all } \vec{r}, \vec{s} \in V^n$

if, and only if,

$(**)\quad r < s \quad \text{implies} \quad \tau(r)^{(n)_M} >_M \tau(s) \quad \text{for all } r, s \in V.$

Indeed, let first $(*)$ hold and let $r, s \in V$ with $r < s$. Choose $r_1 = \ldots = r_n = r$, $s_1 = \ldots = s_{n-1} = 0$, and $s_n = s$. Then $\prod_\infty(r_i)_{1\leq i\leq n} = r < s = \prod_\infty(s_i)_{1\leq i\leq n}$, and thus $\tau(r)^{(n)_M} = \prod_M \tau((r_i)_{1\leq i\leq n}) >_M \prod_M \tau((s_i)_{1\leq i\leq n}) = \tau(s)$ since $\tau(s_i) = \varepsilon_M$ if $1 \leq i < n$ and $\tau(s_n) = \tau(s)$. — Now, let $(**)$ hold and let $r = \prod_\infty(r_i)_{1\leq i\leq n} < \prod_\infty(s_i)_{1\leq i\leq n} = s$. Define $r'_1 = \ldots = r'_n = r$ and $s'_1 = \ldots = s'_{n-1} = 0$, $s'_n = s$. Then $\prod_M \tau((r_i)_{1\leq i\leq n}) \geq_M \prod_M \tau((r'_i)_{1\leq i\leq n}) = \tau(r)^{(n)_M}$, since $\tau(r_i) \geq_M \tau(r)$ for all $1 \leq i \leq n$, and $\tau(s) = \prod_M \tau((s'_i)_{1\leq i\leq n}) \geq_M \prod_M \tau((s_i)_{1\leq i\leq n})$, since $\tau(0) \geq_M \tau(s_i)$ for all $1 \leq i \leq n$. Then $\prod_M \tau((r_i)_{1\leq i\leq n}) \geq_M \tau(r)^{(n)_M} >_M \tau(s) \geq_M \prod_M \tau((s_i)_{1\leq i\leq n})$.
Now $(**)$ yields some restrictions on useful choices of V, M, and τ when it comes to lexicographic products. We have $\tau(r) \geq_M \tau(r)^{(n)_M}$. If $\tau(r) = \tau(r)^{(n)_M}$, then $\tau(r)$ is idempotent, since $\tau(r) \geq_M \tau(r) \cdot_M \tau(r) = \tau(r)^{(2)_M} \geq_M \ldots \geq_M \tau(r)^{(n)_M} = \tau(r)$. If $\tau(r) >_M \tau(r)^{(n)_M}$, then there must be no $r' \in V$ with $r < r'$ and $\tau(r) \geq_M \tau(r') \geq_M \tau(r)^{(n)_M}$, since otherwise, by choosing r' for s in $(**)$, $\tau(r') \geq_M \tau(r)^{(n)_M} >_M \tau(r')$ would have to hold.

6. Consider $\overline{\mathrm{R}} = (\mathcal{M}_{\text{fin}}(\mathbb{R}_{>0}), \uplus, \wr\!\int, \sqsubseteq)$ with $T \sqsubseteq U$ if, and only if, there is a $q > 0$ such that $\prod_p T \geq \prod_p U$ for all $p > q$ (where $\prod_p \wr\!\int = 0$ and $\prod_p (T \uplus \wr r\!\int) = r \cdot_p \prod_p T$).
Then $\sqsubseteq$ is a partial order, where reflexivity and transitivity are obvious, and we thus only have to demonstrate antisymmetry: Let $T, U \in \mathcal{M}_{\text{fin}}(\mathbb{R}_{>0})$ with $T \sqsubseteq U$ and $U \sqsubseteq T$. Then there is a $q_{TU} > 0$ such that $\prod_p T \geq \prod_p U$ for all $p > q_{TU}$, and a $q_{UT} > 0$ such that $\prod_p U \geq \prod_p T$ for all $p > q_{UT}$. Hence $\prod_p T = \prod_p U$ for all $p > \max\{q_{TU}, q_{UT}\}$. Since $\lim_{p\to\infty} \prod_p T = \prod_\infty T$, we either have that $T = \wr\!\int = U$ or that there is an $r \in \mathbb{R}_{>0}$ with $\max T = r = \max U$. In the latter case, with $T = T' \uplus \wr r\!\int$, $U = U' \uplus \wr r\!\int$, we have $r \cdot_p \prod_p T' = r \cdot_p \prod_p U'$ for all $p > \max\{q_{TU}, q_{UT}\}$ and thus $\prod_p T' = \prod_p U'$ for all $p > \max\{q_{TU}, q_{UT}\}$. Therefore, $T = U$ follows by induction on the size of T.
Furthermore, if $T \sqsubseteq U$, then $T \uplus V \sqsubseteq U \uplus V$: Let $\prod_p T \geq \prod_p U$ for all $p > q$ for $q > 0$. Then $\prod_p (T \uplus V) = (\prod_p T) \cdot_p (\prod_p V) \geq (\prod_p U) \cdot_p (\prod_p V) = \prod_p (U \uplus V)$ for all $p > q$, since $\cdot_p$ is (strictly) monotonic in both arguments.
Summing up, $\overline{\mathrm{R}}$ is a meet monoid. Define $\tau : |\mathrm{R}_\infty| \to |\overline{\mathrm{R}}|$ by $\tau(r) = \wr r\!\int$ for $r \neq 0$ and $\tau(0) = \wr\!\int$; then $\tau : \mathrm{R}_\infty \to \overline{\mathrm{R}}$ is a meet monoid homomorphism. Moreover, let $r, s \in \mathbb{R}_{\geq 0}$ with $r < s$, and let $n \geq 1$. If $r = 0$, then $\tau(s) = \wr s\!\int \sqsubset \wr\!\int = \tau(r)^{(n)_{\overline{\mathrm{R}}}}$. If $r \neq 0$, then there is a $q > 0$ such that $s > n^{1/p} \cdot r$ for all $p > q$, and hence $\tau(s) = \wr s\!\int \sqsubset \wr nr\!\int = \tau(r)^{(n)_{\overline{\mathrm{R}}}}$. Thus $(**)$ holds for all $n \geq 1$.
In fact, checking whether $T \sqsubseteq U$ need not involve any p-norms: Let $T, U \in \mathcal{M}_{\text{fin}}(\mathbb{R}_{>0})$ be given. If $U = \wr\!\int$, then $T \sqsubseteq U$, and if $T = \wr\!\int$, then $T \sqsupseteq U$. Thus we are left with the case that $T \neq \wr\!\int$

and $U \neq \wr\int$; in particular, $\max T > 0$ and $\max U > 0$. If $\max T > \max U$, then $T \sqsubset U$, since then $\lim_{p\to\infty} \prod_p T = \max T > \max U = \lim_{p\to\infty} \prod_p U$; conversely, if $\max T < \max U$, then $T \sqsupset U$. Hence, we are now left with the case that $\max T = \max U$. But then we proceed recursively for T' and U' with $T = \wr \max T \int \uplus T'$ and $U = \wr \max U \int \uplus U'$.

However, a soft constraint problem that is admissible when working in R_∞ need not be admissible when transferring the problem to $\overline{\mathrm{R}}$. Let $\mathrm{X} = \{\mathrm{x}\}$, $\mathrm{D} = [0, 1]$; $\mu : [\mathrm{X} \to \mathrm{D}] \to \mathbb{R}_{\geq 0}$ with $\mu(\{\mathrm{x} \mapsto r\}) = r$ if $r \neq 0$, and $\mu(\{\mathrm{x} \mapsto 0\}) = 1$; and $\nu : [\mathrm{X} \to \mathrm{D}] \to \mathbb{R}_{\geq 0}$ with $\nu(\{\mathrm{x} \mapsto r\}) = 1$. Then $\mathsf{M} = \{\mu, \nu\}$ is a finite set of (X, D)-R_∞-soft constraints, which is also admissible, since $\mathsf{M}^* = \{1\}$ and $\mathsf{M}(v) = 1$ for all $v \in [\mathrm{X} \to \mathrm{D}]$. However, for $\mathsf{N} = \{\tau \circ \mu, \tau \circ \nu\}$, which is a finite set of (X, D)-$\overline{\mathrm{R}}$-soft constraints, we get $\mathsf{N}^* = \emptyset$, since each $v_r = \{\mathrm{x} \mapsto r\}$ with $\mathsf{N}(v_r) = \wr r, 1\int$ can be improved to some $v_{r/2} = \{\mathrm{x} \mapsto r/2\}$ with $\mathsf{N}(v_{r/2}) = \wr r/2, 1\int$ and there is no $v \in [\mathrm{X} \to \mathrm{D}]$ with $\mathsf{N}(v) = \wr 1 \int$.

7. As special cases of the equivalence of $(*)$ and $(**)$ of §5 for $0 \in V \subseteq \mathbb{R}_{\geq 0}$, $p > 0$, and $\tau : V_\infty \to V_p$ with $\tau(r) = r$ we obtain for each $n \geq 1$:

$(*_p)$ $\prod_\infty \vec{r} < \prod_\infty \vec{s}$ implies $\prod_p \vec{r} < \prod_p \vec{s}$ for all $\vec{r}, \vec{s} \in V^n$

if, and only if,

$(**_p)$ $r < s$ implies $n^{1/p} \cdot r < s$ for all $r, s \in V$.

A $0 \in V \subseteq \mathbb{R}_{\geq 0}$ is δ-*separated* for some $\delta > 1$ if $s/r \geq \delta$ for all $0 \neq r < s \in V$. For each δ-separated V and $n \geq 1$, $(**_p)$ holds if $p > \ln n / \ln \delta$, i.e. $n^{1/p} < \delta$: Let $r < s$ for $r, s \in V$. Then either $r = 0$, and thus $n^{1/p} \cdot r = 0 = r < s$, or $r \neq 0$, and thus $n^{1/p} \cdot r < \delta \cdot r \leq s$. Conversely, if $0 \in V \subseteq \mathbb{R}_{\geq 0}$ for each $\delta > 1$ shows $0 \neq r < s \in V$ with $s/r < \delta$, then $(**_p)$ is violated for each $p > 0$: Let $p > 0$ be given and choose $0 \neq r < s \in V$ such that $s/r < n^{1/p}$. Then $n^{1/p} \cdot r > s$.

EXAMPLE. (1) Let $0 \in V \subseteq \mathbb{R}_{\geq 0}$ be finite. Then there is a $\varepsilon > 0$ such that $|r_1 - r_2| \geq \varepsilon$ for all $r_1 \neq r_2 \in V$. Let $0 \neq r < s \in V$. Then $s/r \geq (r + \varepsilon)/r = 1 + \varepsilon/r \geq 1 + \varepsilon/\max V$. Thus V is $(1 + \varepsilon/\max V)$-separated.

(2) Let $c \in \mathbb{R}$ with $c > 1$ and let $V^c = \{c^n \mid n \in \mathbb{N}\} \cup \{0\}$. If $0 \neq r < s \in V^c$, then there are $m < n$ with $r = c^m$ and $s = c^n$; then $c^n/c^m = c^{n-m} \geq c$. Thus V^c is c-separated and unbounded.

(3) Let $d \in \mathbb{R}$ with $d > 1$ and let $V^{1/d} = \{d^{-n} \mid n \in \mathbb{N}\} \cup \{0\}$. If $0 \neq r < s \in V^{1/d}$, then there are $m < n$ with $r = d^{-n}$ and $s = d^{-m}$. Then $d^{-m}/d^{-n} = d^{n-m} \geq d$ holds. In addition, $0 < d^{-n} \leq d$ for all $n \in \mathbb{N}$. Hence $V^{1/d}$ is d-separated and bounded. □

8. A *weighting* for a finite set C of (X, D)-constraints is given by a function $g : C \times [X \to D] \to \mathbb{R}_{\geq 0}$ with $g(c, v) = 0$ if, and only if, $v \models c$ for $v \in [X \to D]$ and $c \in C$. Given a real meet monoid R and a weighting $g : C \times [X \to D] \to |R|$, the (R, g)-*weight* of a $v \in [X \to D]$ is given by $W_R^g(v) = \prod_R \{g(c, v) \mid c \in C\}$. Several level comparators can be defined in terms of weights:

– *Weighted sum*: $W_{\mathrm{R}_1}^g(v) = \sum_{c \in C} g(c, v)$.
– *Least squares*: $W_{\mathrm{R}_2}^g(v) = \sqrt{\sum_{c \in C} g(c, v)^2}$.
– *Worst case*: $W_{\mathrm{R}_\infty}^g(v) = \max\{g(c, v) \mid c \in C\}$.

In each case,

$$w <_C^{W_R^g} v \iff W_R^g(w) > W_R^g(v)$$

for all $v, w \in [X \to D]$.

The cases of weighted sums and least squares use real meet monoids without collapsing elements, and thus these are readily usable in lexicographic products. The worst case, however, involves

R_∞ where $\mathcal{C}(\mathrm{R}_\infty) = \mathbb{R}_{\geq 0} \setminus \{0\}$. Assume for this case that C has three different constraints c_1, c_2, and c_3; that there are valuations v_1 violating only c_1, v_2 violating only c_2, v_{13} violating exactly c_1 and c_3, and v_{23} violating exactly c_2 and c_3; that $g(c, v) = 0$ if, and only if, $c \in \{c_1, c_2, c_3\}$ is satisfied by $v \in \{v_1, v_2, v_{13}, v_{23}\}$; and that the weightings are independent of the valuation, i.e., $g(c_1, v_1) = g(c_1, v_{13})$ and $g(c_2, v_2) = g(c_2, v_{23})$ and $g(c_3, v_{13}) = g(c_3, v_{23})$. Also assume that the level weightings for the valuations v_1, v_2, v_{13}, and v_{23} for the worst case are related by

$$W^g_{\mathrm{R}_\infty}(v_1) = g(c_1, v_1) > g(c_2, v_2) = W^g_{\mathrm{R}_\infty}(v_2)\ ,$$
$$W^g_{\mathrm{R}_\infty}(v_{13}) = \max\{g(c_1, v_{13}), g(c_3, v_{13})\} = \max\{g(c_2, v_{23}), g(c_3, v_{23})\} = W^g_{\mathrm{R}_\infty}(v_{23})\ .$$

Any set of (X, D)-M-soft constraints $\mathsf{M} = \{\mu(c) \mid c \in C\}$ reflecting the ordering induced by $W^g_{\mathrm{R}_\infty}$ on valuations, i.e., $\mathsf{M}(w) \leq_M \mathsf{M}(v) \iff W^g_{\mathrm{R}_\infty}(w) \geq W^g_{\mathrm{R}_\infty}(v)$, would thus have $\mu(c_3)$ as collapsing element in M.

PROPOSITION. *Let (X, D) be a constraint domain, $0 \in V \subseteq \mathbb{R}_{\geq 0}$ δ-separated, M_∞ an admissible set of (X, D)-V_∞-soft constraints, and $p > \ln|\mathsf{M}_\infty|/\ln\delta$. Define $\tau_p : |V_\infty| \to |V_p|$ by $\tau_p(r) = r$ and the finite set of (X, D)-V_p-soft constraints M_p by $\mathsf{M}_p = \{\tau_p \circ \mu \mid \mu \in \mathsf{M}_\infty\}$. If M_p is admissible, then $\mathsf{M}_\infty \preccurlyeq \mathsf{M}_p$.*

Proof. The claim that $\mathsf{M}_\infty \preccurlyeq \mathsf{M}_p$ follows from Lem. §2 by the choice of p and the totality of the order in V_∞. □

For a finite set of (X, D)-constraints C and a weighting $g : C \times [X \to D] \to \mathbb{R}_{\geq 0}$, let $V_0 = \{g(c, v) \mid c \in C,\ v \in [X \to D]\}$. Assume that $V_0 \cup \{0\}$ is δ-separated for some $\delta > 1$ and let $p > \ln|C|/\ln\delta$. For each $c \in C$, define the (X, D)-$(V_0)_\infty$-soft constraint c^g_∞ by $c^g_\infty(v) = g(c, v)$ and the (X, D)-$(V_0)_p$-soft constraint c^g_p by $c^g_p(v) = g(c, v)$. Then, since C is finite, we obtain $\{c^g_\infty \mid c \in C\} \preccurlyeq \{c^g_p \mid c \in C\}$, provided that both $\{c^g_\infty \mid c \in C\}$ and $\{c^g_p \mid c \in C\}$ are admissible.

7. CONSTRAINT RELATIONSHIPS

1. A *constraint relationship* over a constraint domain (X, D), or (X, D)-*constraint relationship*, is given by a dag C, where $|C|$ is a set of (X, D)-constraints. We think of a constraint $c' \in |C|$ as *more important* than another constraint $c \in |C|$ if $c \to_C c'$. An (X, D)-constraint relationship C is *finite* if $|C|$ is finite.

2. Let C be an (X, D)-constraint relationship and let M be a meet monoid with a partial order homomorphism $\varphi : PO\langle C\rangle \to PO(M)$. For each $c \in |C|$, define the (X, D)-M-soft constraint $c_{M,\varphi} : [X \to D] \to |M|$ by

$$c_{M,\varphi}(v) = \begin{cases} \varphi(c) & \text{if } v \not\models c \\ \varepsilon_M & \text{otherwise} \end{cases}\ .$$

We write $C_{M,\varphi}$ for the set of (X, D)-M-soft constraints $\{c_{M,\varphi} \mid c \in |C|\}$.

EXAMPLE. Let C be a finite constraint relationship over (X, D).

(1) We first consider the single-predecessor lifting introduced in §1.
Let $M_C = mMon(jMon\langle PO\langle C\rangle\rangle) = mMon\langle PO\langle C\rangle^{-1}\rangle$ and define $m_C : PO\langle C\rangle \to PO(M_C)$ by $m_C(c) = \eta^{\mathrm{mMon}}_{PO\langle C\rangle^{-1}}(c) = \wr c \int$; in particular $\varepsilon_{M_C} = \wr\int$. Then for $v, w \in [X \to D]$

$$w \lesssim_{C_{M_C,m_C}} v \iff$$
$$\textstyle\prod_{M_C}\{c_{M_C,m_C}(w) \mid c \in |C|\} \leq_{M_C} \prod_{M_C}\{c_{M_C,m_C}(v) \mid c \in |C|\} \iff$$
$$\wr c \mid w \not\models c,\ c \in |C| \int \sqsubseteq^{PO\langle C\rangle^{-1}} \wr c \mid v \not\models c,\ c \in |C| \int \iff$$

$$\wr c \mid v \not\models c,\ c \in |C| \int \sqsubseteq_{PO\langle C\rangle} \wr c \mid w \not\models c,\ c \in |C| \int .$$

Thus, v is considered a better solution than w if each constraint that is not satisfied by v can be paired off with a constraint that is not satisfied by w and which is more important.

(2) We now consider the transitive-predecessors lifting introduced in §2.

Let $U_C = \mathit{mMon}(\mathit{jMon}(\mathit{uSL}\langle PO\langle C\rangle\rangle))$ and define $u_C : PO\langle C\rangle \to PO(U_C)$ by $u_C(c) = \{c\}$; in particular, $\varepsilon_{U_C} = \emptyset$. Then for $v, w \in [X \to D]$

$$\begin{aligned}
& w \lesssim_{C_{U_C,u_C}} v \iff \\
& \quad \textstyle\prod_{U_C}\{c_{U_C,u_C}(w) \mid c \in |C|\} \leq_{U_C} \prod_{U_C}\{c_{U_C,u_C}(v) \mid c \in |C|\} \iff \\
& \quad \mathrm{Max}^{\leq_{PO\langle C\rangle}}\{c \mid w \not\models c,\ c \in |C|\} \supseteq_{PO\langle C\rangle} \mathrm{Max}^{\leq_{PO\langle C\rangle}}\{c \mid v \not\models c,\ c \in |C|\} \iff \\
& \quad \mathrm{Max}^{\leq_{PO\langle C\rangle}}\{c \mid v \not\models c,\ c \in |C|\} \subseteq_{PO\langle C\rangle} \mathrm{Max}^{\leq_{PO\langle C\rangle}}\{c \mid w \not\models c,\ c \in |C|\} \iff \\
& \quad \forall c \in \{c_v \in |C| \mid v \not\models c_v\} \,.\, \exists c' \in \{c_w \in |C| \mid w \not\models c_w\} \,.\, c \leq_{PO\langle C\rangle} c' \,.
\end{aligned}$$

Thus, v is considered a better solution than w if each constraint that is not satisfied by v can be covered by a constraint that is not satisfied by w and which is more important. □

3. The *scope* of a constraint c over a constraint domain (X, D) is given by the set of variables it depends on, i.e.,

$$\mathrm{sc}(c) = \{x \in X \mid \exists v \in [X \to D], d_1 \neq d_2 \in D_x \,.\, c(v\{x \mapsto d_1\}) \neq c(v\{x \mapsto d_2\})\} \,.$$

For a partial valuation $p \in [X \to D^?]$, we write $p \not\models c$ if $\mathrm{sc}(c) \subseteq \mathrm{def}(p)$ and $v \not\models c$ for some $v \in p\!\uparrow$ (which is well-defined, since then c only depends on variables that are in the domain of definition of p).

Let C be a finite constraint relationship over (X, D), let M be a meet monoid, and let $\varphi : PO\langle C\rangle \to PO(M)$. Define $\alpha_{M,\varphi}, \zeta_{M,\varphi} : [X \to D^?] \to |M|$ by

$$\begin{aligned}
\alpha_{M,\varphi}(p) &= \textstyle\prod_M\{\varphi(c) \mid c \in |C|,\ \mathrm{sc}(c) \subseteq \mathrm{def}(p),\ p \not\models c\} \cdot_M \prod_M\{\varphi(c) \mid \mathrm{sc}(c) \not\subseteq \mathrm{def}(p)\} \,, \\
\zeta_{M,\varphi}(p) &= \textstyle\prod_M\{\varphi(c) \mid c \in |C|,\ \mathrm{sc}(c) \subseteq \mathrm{def}(p),\ p \not\models c\} \,.
\end{aligned}$$

LEMMA. *$(\alpha_{M,\varphi}, \zeta_{M,\varphi})$ is a tight bounding pair of $(X, D^?)$-M-soft constraints for $C_{M,\varphi}$.*

Proof. For a $p \in [X \to D^?]$ let $V(p) = \{c \in |C| \mid \mathrm{sc}(c) \subseteq \mathrm{def}(p),\ p \not\models c\}$ and $W(p) = \{c \in |C| \mid \mathrm{sc}(c) \not\subseteq \mathrm{def}(p)\}$. Then $\alpha_{M,\varphi}(p) = \prod_M \varphi(V(p)) \cdot_M \prod_M \varphi(W(p))$ and $\zeta_{M,\varphi}(p) = \prod_M \varphi(V(p))$ for all $p \in [X \to D^?]$.

Let $p \in [X \to D^?]$ and $v \in p\!\uparrow$ be given. Then $V(p) \subseteq V(v)$, and thus $\zeta_{M,\varphi}(v) \leq_M \zeta_{M,\varphi}(p)$: For a $c \in V(p)$, i.e., $\mathrm{sc}(c) \subseteq \mathrm{def}(p)$ and $p \not\models c$, also $c \in V(v)$, since $v \in p\!\uparrow$ and thus $v \not\models c$.

Now let $p' = p\{x \mapsto d\} \in [X \to D^?]$ and let $v \in p\!\uparrow$ be arbitrary. Then $V(v) \subseteq V(p') \cup W(p')$, and thus $\alpha_{M,\varphi}(p') \leq_M \zeta_{M,\varphi}(v)$: Let $c \in V(v)$, i.e., $v \not\models c$. If $\mathrm{sc}(c) \subseteq \mathrm{def}(p')$, then $p' \not\models c$, and hence $c \in V(p')$; otherwise $c \in W(p')$.

Finally, $\zeta_{M,\varphi}(v) = C_{M,\varphi}(v)$ and thus $(\alpha_{M,\varphi}, \zeta_{M,\varphi})$ is tight for $C_{M,\varphi}$. □

EXAMPLE. Consider the constraint domain (X, D) given by

$$\begin{aligned}
&\mathrm{X} = \{\mathrm{x}, \mathrm{y}, \mathrm{z}\} \,, \\
&\mathrm{D_x} = \mathrm{D_y} = \mathrm{D_z} = \{1, 2, 3\}
\end{aligned}$$

as well as the constraint relationship $\mathrm{C} = (\{\mathrm{c_1}, \mathrm{c_2}, \mathrm{c_3}\}, \{(\mathrm{c_2}, \mathrm{c_1}), (\mathrm{c_3}, \mathrm{c_1})\})$, i.e., $\mathrm{c_2} \to_{\mathrm{C}} \mathrm{c_1}$ and $\mathrm{c_3} \to_{\mathrm{C}} \mathrm{c_1}$, with

$$\begin{aligned}
&\mathrm{c_1} : \mathrm{x} + 1 = \mathrm{y} \,, \\
&\mathrm{c_2} : \mathrm{z} = \mathrm{y} + 2 \,,
\end{aligned}$$

$$\mathrm{c}_3 : \mathrm{x} + \mathrm{y} \leq 3 \,.$$

Let $M_{\mathrm{C}} = mMon\langle PO\langle \mathrm{C}\rangle^{-1}\rangle$ and $m_{\mathrm{C}} = \eta^{\mathrm{mMon}}_{PO\langle \mathrm{C}\rangle^{-1}}$. Then

$$\begin{aligned}
&\alpha_{M_{\mathrm{C}},m_{\mathrm{C}}}(p) = \lbag c \in |\mathrm{C}| \mid \mathrm{sc}(c) \subseteq \mathrm{def}(p),\ p \not\models c \rbag \uplus \lbag c \in |\mathrm{C}| \mid \mathrm{sc}(c) \not\subseteq \mathrm{def}(p) \rbag \,, \\
&\zeta_{M_{\mathrm{C}},m_{\mathrm{C}}}(p) = \lbag c \in |\mathrm{C}| \mid \mathrm{sc}(c) \subseteq \mathrm{def}(p),\ p \not\models c \rbag \,,
\end{aligned}$$

such that, for example,

$$\begin{aligned}
&\alpha_{M_{\mathrm{C}},m_{\mathrm{C}}}(\{\mathrm{x} \mapsto 1, \mathrm{y} \mapsto 1, \mathrm{z} \mapsto ?\}) = \lbag \mathrm{c}_1, \mathrm{c}_2 \rbag \,, \qquad &&\zeta_{M_{\mathrm{C}},m_{\mathrm{C}}}(\{\mathrm{x} \mapsto 1, \mathrm{y} \mapsto 1, \mathrm{z} \mapsto ?\}) = \lbag \mathrm{c}_1 \rbag \,, \\
&\alpha_{M_{\mathrm{C}},m_{\mathrm{C}}}(\{\mathrm{x} \mapsto 1, \mathrm{y} \mapsto 2, \mathrm{z} \mapsto ?\}) = \lbag \mathrm{c}_2 \rbag \,, &&\zeta_{M_{\mathrm{C}},m_{\mathrm{C}}}(\{\mathrm{x} \mapsto 1, \mathrm{y} \mapsto 2, \mathrm{z} \mapsto ?\}) = \lbag \rbag \,, \\
&\alpha_{M_{\mathrm{C}},m_{\mathrm{C}}}(\{\mathrm{x} \mapsto 2, \mathrm{y} \mapsto 3, \mathrm{z} \mapsto ?\}) = \lbag \mathrm{c}_2, \mathrm{c}_3 \rbag \,, &&\zeta_{M_{\mathrm{C}},m_{\mathrm{C}}}(\{\mathrm{x} \mapsto 2, \mathrm{y} \mapsto 3, \mathrm{z} \mapsto ?\}) = \lbag \mathrm{c}_3 \rbag \,;
\end{aligned}$$

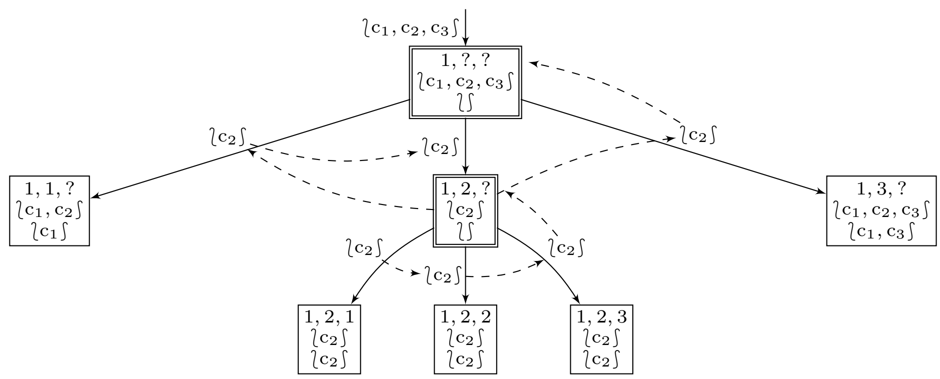
in particular

$$\zeta_{M_{\mathrm{C}},m_{\mathrm{C}}}(\{\mathrm{x} \mapsto 1, \mathrm{y} \mapsto 1, \mathrm{z} \mapsto ?\}) = \lbag \mathrm{c}_1 \rbag <_{M_{\mathrm{C}}} \lbag \mathrm{c}_2 \rbag = \alpha_{M_{\mathrm{C}},m_{\mathrm{C}}}(\{\mathrm{x} \mapsto 1, \mathrm{y} \mapsto 2, \mathrm{z} \mapsto ?\}) \,.$$

We abbreviate $\alpha_{M_{\mathrm{C}},m_{\mathrm{C}}}$ by α and $\zeta_{M_{\mathrm{C}},m_{\mathrm{C}}}$ by ζ. We follow an execution of $\mathsf{maxSolDegs}_{(\alpha,\zeta)}(\{\mathrm{x} \mapsto ?, \mathrm{y} \mapsto ?, \mathrm{z} \mapsto ?\}, \emptyset)$, choosing the variables x, y, and z in this order and running through $\{1, 2, 3\}$ in the natural order; we select x and y first, since $\mathrm{sc}(\mathrm{c}_1) = \{\mathrm{x}, \mathrm{y}\}$ and c_1 is the top element in C.
The first step in evaluating $\mathsf{maxSolDegs}_{(\alpha,\zeta)}(\lambda x \in \mathrm{X} \,.\, ?, \emptyset)$ is to evaluate $\mathsf{maxSolDegs}_{(\alpha,\zeta)}(\{\mathrm{x} \mapsto 1, \mathrm{y} \mapsto ?, \mathrm{z} \mapsto ?\}, \lbag \mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3 \rbag)$, since

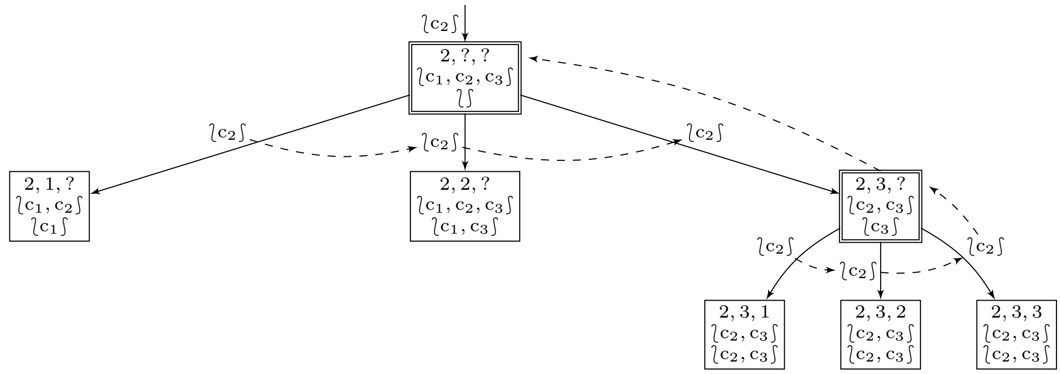
$$\alpha((\lambda x \in \mathrm{X} \,.\, ?)\{x \mapsto d\}) = \lbag \mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3 \rbag \,,$$

for all $d \in \{1, 2, 3\}$ and $\zeta((\lambda x \,.\, \mathrm{X})\{x \mapsto 1\}) = \lbag \rbag$; which leads to the following graph:

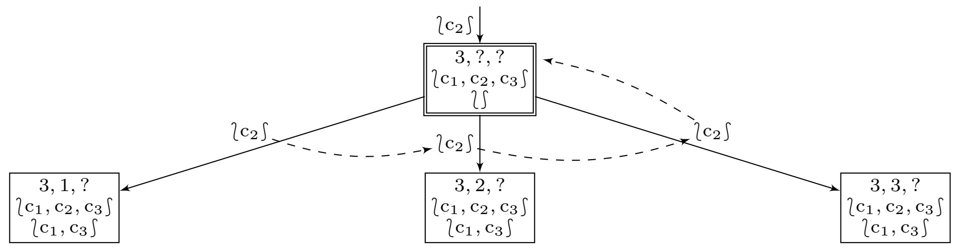


The annotations on the solid edges represent the current values of the lower bound set L, where we have omitted the set braces. Each node gives the respective partial valuation p for x, y, and z at the top, $\alpha(p)$ in the middle, and $\zeta(p)$ at the bottom. Doubly outlined nodes represent calls to $\mathsf{maxSolDegs}_{(\alpha,\zeta)}(p, L)$; singly outlined nodes represent the successful test whether $\zeta(p)$ already is dominated by a lower bound in L. Finally, the dashed edges show the flow of the lower bounds.

Thus, $\mathsf{maxSolDegs}_{(\alpha,\zeta)}(\{\mathrm{x} \mapsto 1, \mathrm{y} \mapsto ?, \mathrm{z} \mapsto ?\}, \emptyset) = \{\wr \mathrm{c}_2 \int\}$, and $\mathsf{maxSolDegs}_{(\alpha,\zeta)}(\{\mathrm{x} \mapsto 2, \mathrm{y} \mapsto ?, \mathrm{z} \mapsto ?\}, \{\wr \mathrm{c}_2 \int\})$ is executed:



Hence, $\mathsf{maxSolDegs}_{(\alpha,\zeta)}(\{\mathrm{x} \mapsto 2, \mathrm{y} \mapsto ?, \mathrm{z} \mapsto ?\}, \{\wr \mathrm{c}_2 \int\}) = \{\wr \mathrm{c}_2 \int\}$ and $\mathsf{maxSolDegs}_{(\alpha,\zeta)}(\{\mathrm{x} \mapsto 3, \mathrm{y} \mapsto ?, \mathrm{z} \mapsto ?\}, \{\wr \mathrm{c}_2 \int\})$ is executed:



Therefore, $\mathsf{maxSolDegs}_{(\alpha,\zeta)}(\{\mathrm{x} \mapsto 3, \mathrm{y} \mapsto ?, \mathrm{z} \mapsto ?\}, \{\wr \mathrm{c}_2 \int\}) = \{\wr \mathrm{c}_2 \int\}$, and we have as the final result that $\mathsf{maxSolDegs}_{(\alpha,\zeta)}(\{\mathrm{x} \mapsto ?, \mathrm{y} \mapsto ?, \mathrm{z} \mapsto ?\}, \emptyset) = \{\wr \mathrm{c}_2 \int\}$. $\square$