NCERT: 10/5/3/18

EE23BTECH11040 - Manoj Kumar Ambatipudi[*]

QUESTION: A spiral is made up of successive semi circles, with centres alternatively at A and B, Starting with center at A, of radii 0.5 cm, 1.0cm, 1.5cm, 2.0cm,... . What is the length of such a spiral ,made of 13 consecutive semicircles.(Use $\pi = \frac{22}{7}$)

SOLUTION:

Variable	Description	Value
$x(0)$	First term	0.5
d	common difference	0.5
$y(n)$	Sum of $n+1$ terms	-
C_n	Length of nth semicircle	$\pi x(n)$

TABLE 1
VARIABLES USED

General Term can be written as

$$x(n) = x(0) + nd \tag{1}$$

Sum upto $n+1$ terms is given by

$$y(n) = x(n) * u(n) \tag{2}$$

The corresponding Z-Transform is given by (**??**). Referring to Table 1, substituting the values in (**??**),

$$Y(z) = \frac{0.5}{(1 - z^{-1})^2} + \frac{0.5z^{-1}}{(1 - z^{-1})^3} \quad ROC(|z| > 1) \tag{3}$$

Finding $y(n)$ by Contour Integration,

$$y(12) = \frac{1}{2\pi j} \oint_C \left(\frac{0.5z^{12-1}}{(1 - z^{-1})^2} + \frac{0.5z^{12-2}}{(1 - z^{-1})^3} \right) dz \tag{4}$$

Using Residue Theorem to evaluate the integral, let

$$Y(z) = S_1 + S_2 \tag{5}$$

S_1 has 2 poles,

$$S_1 = \frac{1}{(1)!} \lim_{z \to 1} \frac{d}{dz} \left((z - 1)^2 \frac{0.5z^{12+1}}{(z - 1)^2} \right) \tag{6}$$

$$S_1 = 0.5(12 + 1) \lim_{z \to 1} \left(z^1 2 \right) \tag{7}$$

$$S_1 = 0.5(12 + 1) \tag{8}$$

Similarly, S_2 has 3 poles,

$$S_2 = \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left((z - 1)^3 \frac{0.5z^{12+1}}{(z - 1)^3} \right) \tag{9}$$

$$= \frac{0.5(12 + 1)}{2} \lim_{z \to 1} \frac{d}{dz} \left(z^1 2 \right) \tag{10}$$

$$= \frac{0.5(12 + 1)(12)}{2} \lim_{z \to 1} \left(z^{12-1} \right) \tag{11}$$

$$= \frac{0.5(12)(12 + 1)}{2} \tag{12}$$

Finally,

$$y(12) = 0.5(12 + 1) + \frac{0.5(12)(12 + 1)}{2} \tag{13}$$

$$y(12) = 45.5 \tag{14}$$

$$\sum_{n=0}^{12} C_n = \pi y(12) \tag{15}$$
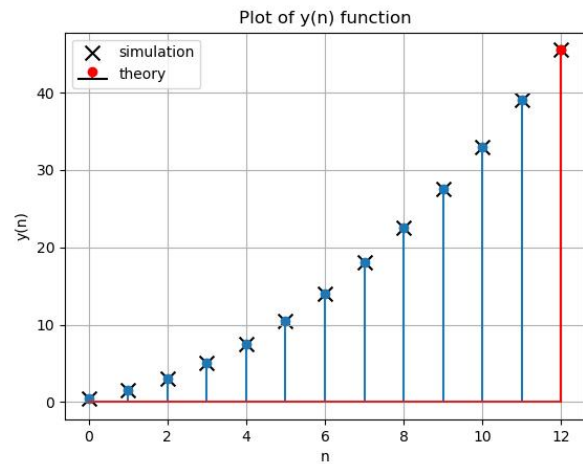
$$\sum_{n=0}^{12} C_n = \pi(45.5) \tag{16}$$

$$= 143 \tag{17}$$

Fig. 1. Plot of Sum of n terms taken from Python3