NCERT: 10/5/3/18

EE23BTECH11040 - Manoj Kumar Ambatipudi*

QUESTION: A spiral is made up of successive semi circles, with centres alternatively at A and B, Starting with center at A, of radii 0.5 cm, 1.0cm, 1.5cm, 2.0cm,... . What is the length of such a spiral ,made of 13 consecutive semicircles.(Use $\pi = \frac{22}{7}$)

SOLUTION:

Variable	Description	Value
x(0)	First term	0.5
d	common difference	0.5
y(n)	Sum of $n+1$ terms	-
C_n	Length of n^{th} semicircle	$\pi x(n)$

TABLE 1
VARIABLES USED

General Term can be written as

$$x(n) = x(0) + nd \tag{1}$$

Sum upto $n+1$ terms is given by

$$y(n) = x(n) * u(n) \tag{2}$$

The corresponding Z-Transform is given by (**??**). Referring to Table 1, substituting the values in (**??**),

$$Y(z) = \frac{0.5}{(1-z^{-1})^2} + \frac{0.5z^{-1}}{(1-z^{-1})^3} \quad ROC(|z| > 1) \tag{3}$$

Finding $y(n)$ by Contour Integration,

$$y(n) = \frac{1}{2\pi j}\oint_C \left(\frac{0.5z^{n-1}}{(1-z^{-1})^2} + \frac{0.5z^{n-2}}{(1-z^{-1})^3}\right) dz \tag{4}$$

Using Residue Theorem to evaluate the integral, let

$$Y(z) = S_1 + S_2 \tag{5}$$

S_1 has 2 poles,

$$S_1 = \frac{1}{(1)!}\lim_{z\to1}\frac{d}{dz}\left((z-1)^2\frac{0.5z^{n+1}}{(z-1)^2}\right) \tag{6}$$

$$S_1 = 0.5(n+1)\lim_{z\to1}(z^n) \tag{7}$$

$$S_1 = 0.5(n+1) \tag{8}$$

Similarly, S_2 has 3 poles,

$$S_2 = \frac{1}{(2)!}\lim_{z\to1}\frac{d^2}{dz^2}\left((z-1)^3\frac{0.5z^{n+1}}{(z-1)^3}\right) \tag{9}$$

$$= \frac{0.5(n+1)}{2}\lim_{z\to1}\frac{d}{dz}(z^n) \tag{10}$$

$$= \frac{0.5(n+1)(n)}{2}\lim_{z\to1}\left(z^{n-1}\right) \tag{11}$$

$$= \frac{0.5(n)(n+1)}{2} \tag{12}$$

Finally,

$$y(n) = 0.5(n+1) + \frac{0.5(n)(n+1)}{2} \tag{13}$$

The $y(n)$ gives the sum of all the radii. Let C_n be the length of $(n+1)^{th}$ curve.

$$C_n = \pi x(n) \tag{14}$$

$$\sum_0^n C_n = \pi\sum_0^n x(n) \tag{15}$$

$$= \pi\left(0.5(n+1) + \frac{0.5(n)(n+1)}{2}\right) \tag{16}$$

$$\tag{17}$$

Substituting $n = 12$,

$$\sum_0^{13} C_n = \pi(45.5) \tag{18}$$

$$= \frac{22}{7} \times 45.5 \tag{19}$$

$$= 143 \tag{20}$$

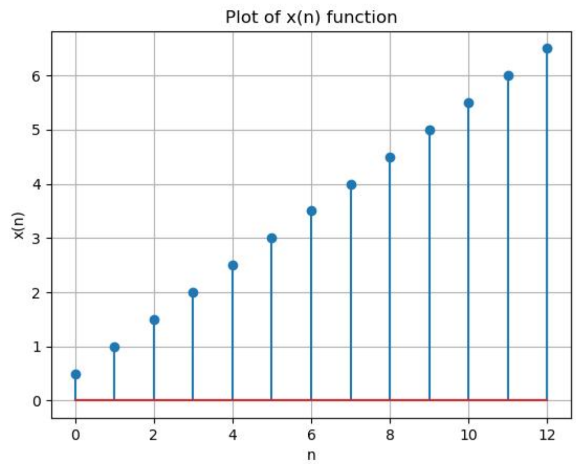

Fig. 1. Plot of general term taken from Python3

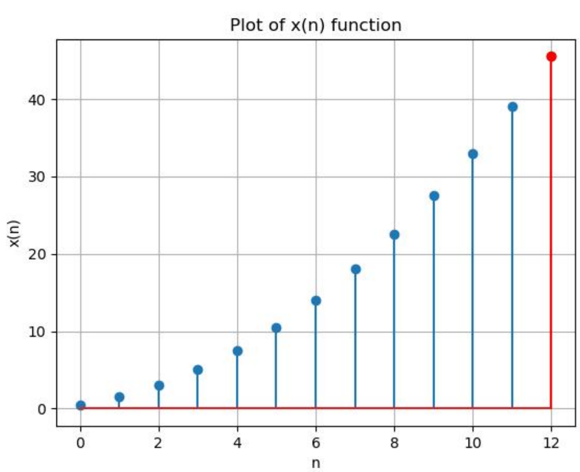

Fig. 2. Plot of Sum of n terms taken from Python3