NCERT 11.9.5 26Q

EE23BTECH11015 - DHANUSH V NAYAK*

Question:

$$\frac{1\times 2^2 + 2\times 3^2 + \cdots + n\times (n+1)^2}{1^2\times 2 + 2^2\times 3 + \cdots + n^2\times (n+1)}$$

Find the z-transform of general term of numerator and denominator and plot them . Also find z-transform of sum of terms of numerator and denominator and plot it.

Solution:

1. Analysis of Numerator:

By the differentiation property :

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z)\,, ROC = R \tag{1}$$

$$n^k x(n) \overset{\mathcal{Z}}{\longleftrightarrow} (-z)^k \frac{d^k X(z)}{dz^k}, ROC = R \tag{2}$$

$$\implies n^k u(n) \overset{\mathcal{Z}}{\longleftrightarrow} (-z)^k \frac{d^k U(z)}{dz^k} \{z \in \mathbb{C} : |z| > 1\} \tag{3}$$

Therefore,

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^2}, \{z \in \mathbb{C} : |z| > 1\} \tag{4}$$

$$n^2 u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(z^{-1}+1\right)}{(1-z^{-1})^3}, \{z \in \mathbb{C} : |z| > 1\} \tag{5}$$

$$n^3 u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{(1-z^{-1})^4}, \{z \in \mathbb{C} : |z| > 1\} \tag{6}$$

$$n^4 u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(1+11z^{-1}+11z^{-2}+z^{-3}\right)}{(1-z^{-1})^5} \tag{7}$$

where $\{z \in \mathbb{C} : |z| > 1\}$

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n)\, u(n)\, z^{-n} \tag{8}$$

$$= \sum_{n=-\infty}^{\infty} \left(n^3 + 5n^2 + 20n + 4\right) u(n)\, z^{-n} \tag{9}$$

$$\therefore X_1(z) = \frac{4 + 14z^{-1} - 24z^{-2} + 12z^{-3}}{(1-z^{-1})^4}, \{z \in \mathbb{C} : |z| > 1\} \tag{10}$$

$$y_1(n) = \sum_{k=0}^{n} x_1(n) \tag{11}$$

$$y_1(n) = x_1(n) * u(n) \tag{12}$$

$$Y_1(z) = X_1(z)\, U(z) \tag{13}$$

$$= \frac{4 + 14z^{-1} - 24z^{-2} + 12z^{-3}}{(1-z^{-1})^5}, \{z \in \mathbb{C} : |z| > 1\} \tag{14}$$

Using partial fractions:

$$Y_1(z) = \frac{34z^{-1}}{(1-z^{-1})} + \frac{72z^{-2}}{(1-z^{-1})^2} + \frac{64z^{-3}}{(1-z^{-3})^3} \tag{15}$$

$$+ \frac{28z^{-4}}{(1-z^{-1})^4} + \frac{6z^{-5}}{(1-z^{-1})^5} + 4$$

$$\frac{z^{-1}}{(1-z^{-1})} = \frac{1}{1-z^{-1}} - 1 \tag{16}$$

Taking the inverse z-transform of each term :

$$Z^{-1}\left[\frac{z^{-1}}{(1-z^{-1})}\right] = u(n) - \delta(n) = u(n-1) \tag{17}$$

$$\frac{z^{-2}}{(1-z^{-1})^2} = \left(\frac{z^{-1}}{(1-z^{-1})}\right)\left(\frac{z^{-1}}{(1-z^{-1})}\right) \tag{18}$$

$$Z^{-1}\left[\frac{z^{-2}}{(1-z^{-1})^2}\right] = u(n-1) * u(n-1) \tag{19}$$

$$\therefore Z^{-1}\left[\frac{z^{-k}}{(1-z^{-1})^k}\right] = (u(n-1) * u(n-1) * \ldots \tag{20}$$

$$\ldots * u(n-1))_{k\text{ times}}$$

$$Z^{-1}\left[\frac{z^{-2}}{(1-z^{-1})^2}\right] = (n-1)\,u\,(n-1) \tag{21}$$

$$Z^{-1}\left[\frac{z^{-3}}{(1-z^{-1})^3}\right] = \frac{(n-1)\,(n-2)}{2}u\,(n-1) \tag{22}$$

$$Z^{-1}\left[\frac{z^{-4}}{(1-z^{-1})^4}\right] = \frac{(n-1)\,(n-2)\,(n-3)}{6}u\,(n-1) \tag{23}$$

$$Z^{-1}\left[\frac{z^{-}}{(1-z^{-1})^5}\right] = \frac{(n-1)\,(n-2)\,(n-3)\,(n-4)}{24} \tag{24}$$

$$u\,(n-1)$$

Substituting above inverse transforms in equation(15)

$$y_1\,(n) = \frac{3n^4+26n^3+81n^2+106n+48}{12} \tag{25}$$

2. Analysis of Denominator:

$$X_2\,(z) = \sum_{n=-\infty}^{\infty} x_2\,(n)\,u\,(n)\,z^{-n} \tag{26}$$

$$= \sum_{n=-\infty}^{\infty} \left(n^3+4n^2+5n+2\right)u\,(n)\,z^{-n} \tag{27}$$

$$\therefore X_2\,(z) = \frac{2+4z^{-1}}{(1-z^{-1})^4}, \{z \in \mathbb{C} : |z| > 1\} \tag{28}$$

$$\tag{29}$$

$$y_2\,(n) = \sum_{k=0}^{n} x_2\,(n) \tag{30}$$

$$y_2\,(n) = x_2\,(n) * u\,(n) \tag{31}$$

$$Y_2\,(z) = X_2\,(z)\,U\,(z) \tag{32}$$

$$= \frac{2+4z^{-1}}{(1-z^{-1})^5}, \{z \in \mathbb{C} : |z| > 1\} \tag{33}$$

Using partial fractions:

$$Y_2\,(z) = \frac{14z^{-1}}{(1-z^{-1})} + \frac{36z^{-2}}{(1-z^{-1})^2} + \frac{44z^{-3}}{(1-z^{-3})^3} \tag{34}$$

$$+ \frac{26z^{-4}}{(1-z^{-1})^4} + \frac{6z^{-5}}{(1-z^{-1})^5} + 2$$

Taking the inverse z-transform of each term :

$$y_2\,(n) = \frac{3n^4+22n^3+57n^2+62n+24}{12} \tag{35}$$

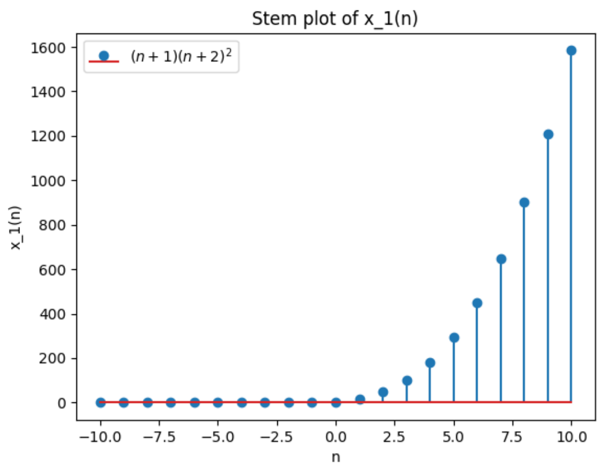

Fig. 1: Stem Plot of $x_1\,(n)$

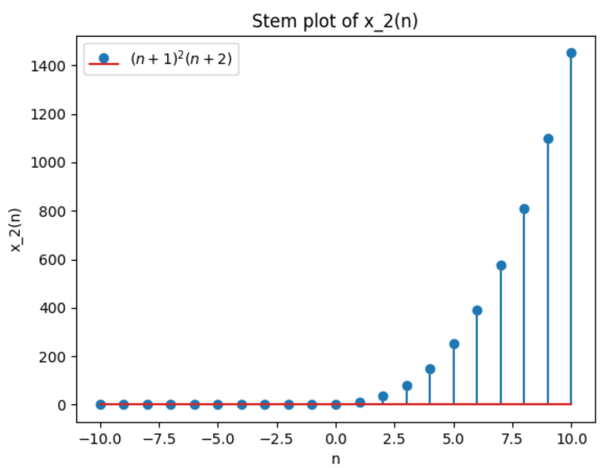

Fig. 2: Stem Plot of $x_2\,(n)$

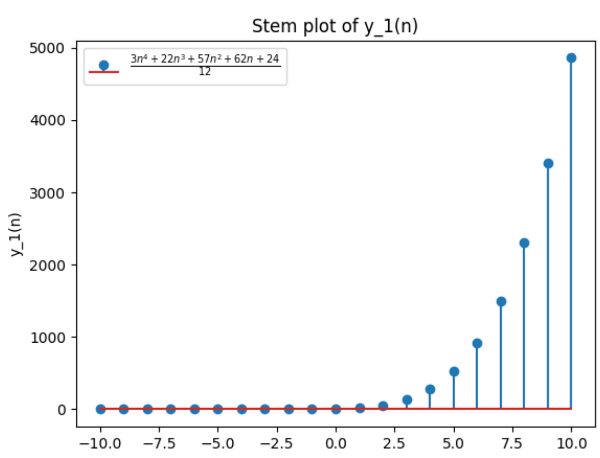

Fig. 3: Stem Plot of $y_1\,(n)$

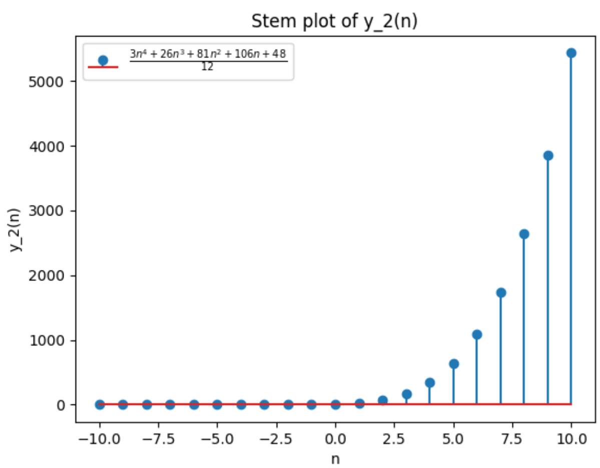

Fig. 4: Stem Plot of $y_2(n)$