NCERT 11.9.5 26Q

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Question: Show that

$$\frac{1\times 2^2+2\times 3^2+\cdots+n\times(n+1)^2}{1^2\times 2+2^2\times 3+\cdots+n^2\times(n+1)}=\frac{3n+5}{3n+1}$$

Solution:

a) Consider the Numerator of the LHS part:

$$1\times 2^2+2\times 3^2+\cdots+n\times(n+1)^2=\sum_{k=1}^{n}k(k+1)^2$$

Now,

$$\begin{aligned}\sum_{k=1}^{n}k(k+1)^2 &= \sum_{k=1}^{n}k^3+2k^2+k\\ &= \sum_{k=1}^{n}k^3+\sum_{k=1}^{n}2k^2+\sum_{k=1}^{n}k\\ &= \sum_{k=1}^{n}k^3+2\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k\\ &= \left(\frac{n(n+1)}{2}\right)^2+2\cdot\frac{n(n+1)(2n+1)}{6}+\\ &+\frac{n(n+1)}{2}\\ &= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2}+\frac{2}{3}(2n+1)+1\right]\\ &= \frac{n(n+1)}{2}\left[\frac{3n^2+11n+10}{6}\right]\\ &= \frac{n(n+1)}{12}\left[3n^2+6n+5n+10\right]\\ &= \frac{n(n+1)}{12}\left[3n(n+2)+5(n+2)\right]\\ &= \frac{n(n+1)(n+2)(3n+5)}{12}\end{aligned}$$

Therefore,

$$1\times 2^2+2\times 3^2+\cdots+n\times(n+1)^2=\frac{n(n+1)(n+2)(3n+5)}{12}$$

$$\frac{n(n+1)(n+2)(3n+5)}{12}\cdot u[n]=x[n]$$

b) Consider the Denominator of the RHS part:

$$1^2\times 2+2^2\times 3+\cdots+n^2\times(n+1)=\sum_{k=1}^{n}k^2(k+1)$$

Now,

$$\begin{aligned}\sum_{k=1}^{n}k^2(k+1) &= \sum_{k=1}^{n}k^3+k^2\\ &= \sum_{k=1}^{n}k^3+\sum_{k=1}^{n}k^2\\ &= \left(\frac{n(n+1)}{2}\right)^2+\frac{n(n+1)(2n+1)}{6}\\ &= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2}+\frac{2n+1}{3}\right]\\ &= \frac{n(n+1)}{2}\left[\frac{3n^2+7n+2}{6}\right]\\ &= \frac{n(n+1)}{12}\left[3n^2+6n+n+2\right]\\ &= \frac{n(n+1)}{12}\left[3n(n+2)+(n+2)\right]\\ &= \frac{n(n+1)(n+2)(3n+1)}{12}\end{aligned}$$

$$\frac{n(n+1)(n+2)(3n+1)}{12}\cdot u[n]=y[n]$$

Therefore,

$$\begin{aligned}\frac{\sum_{k=1}^{n}k(k+1)^2}{\sum_{k=1}^{n}k^2(k+1)} &= \frac{\frac{n(n+1)(n+2)(3n+5)}{12}}{\frac{n(n+1)(n+2)(3n+1)}{12}}\\ &= \frac{3n+5}{3n+1}\end{aligned}$$

c) Derivation of z-transform of numerator :

$$x[n] = \frac{n(n+1)(n+2)(3n+5)}{12} \cdot u[n]$$
$$= \frac{3n^4 + 14n^3 + 21n^2 + 10n}{12} \cdot u[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{1}$$

Z-Transform is calculated as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$$

$$12X(z) = \sum_{n=0}^{\infty} 3n^4 u[n] z^{-n} + \sum_{n=0}^{\infty} 14n^3 u[n] z^{-n} + \sum_{n=0}^{\infty} 21n^2 u[n] z^{-n} + \sum_{n=0}^{\infty} 10nu[n]\, z^{-n} \tag{2}$$

Let $X_0(z)$ be z transform of $x[n] = u[n]$.

$$\begin{aligned} X_0(z) &= \sum_{n=0}^{\infty} u[n] \cdot z^{-n} \\ &= \sum_{n=0}^{\infty} (1) \cdot z^{-n} \\ &= 1 + z^{-1} + z^{-2} + \dots \\ &= \frac{1}{1 - z^{-1}} \\ &= \frac{z}{z-1} \end{aligned}$$

By the differentiation property :

$$\begin{aligned} nx[n] &\overset{Z}{\longleftrightarrow} -z\frac{dX(z)}{dz}, \\ nu[n] &\overset{Z}{\longleftrightarrow} -z\frac{dX(z)}{dz}, \\ \frac{dX(z)}{dz} &= -\frac{1}{(z-1)^2}, \\ nu(n) &\overset{Z}{\longleftrightarrow} \frac{z}{(z-1)^2}. \end{aligned}$$

Therefore, the Z-transform of n$\cdot u[n]$,

$$X_1(z) = \frac{z}{(z-1)^2}. \tag{3}$$

By using the same property we can compute the z transform of other powers of n. Here,the z-transforms of $n^i \cdot u[n]$ are denoted as $X_i(z)$:

$$X_2(z) = \frac{z^{-1}(z^{-1} + 1)}{(1 - z^{-1})^3} \tag{4}$$

$$X_3(z) = \frac{z^{-1}(1 + 4z^{-1} + z^{-2})}{(1 - z^{-1})^4} \tag{5}$$

$$X_4(z) = \frac{z^{-1}(1 + 11z^{-1} + 11z^{-2} + z^{-3})}{(1 - z^{-1})^5} \tag{6}$$

Equation(2) can be now written as:

$$\begin{aligned} X(z) &= \frac{1}{12}(3X_4(z) + 14X_3(z) + 21X_2(z) + 10X_1(z)) \\ &= \frac{3}{12}\left(\frac{z^{-1}(1 + 11z^{-1} + 11z^{-2} + z^{-3})}{(1 - z^{-1})^5}\right) \\ &+ \frac{14}{12}\left(\frac{z^{-1}(1 + 4z^{-1} + z^{-2})}{(1 - z^{-1})^4}\right) + \frac{21}{12}\left(\frac{z^{-1}(z^{-1} + 1)}{(1 - z^{-1})^3}\right) \\ &+ \frac{10}{12}\left(\frac{z}{(z-1)^2}\right) \\ &= \frac{2(z^{-2} + 2z^{-1})}{(1 - z^{-1})^5} \end{aligned}$$

d) Derivation of z-transform of the denominator:

$$y[n] = \frac{n(n+1)(n+2)(3n+1)}{12} \cdot u[n]$$
$$= \frac{3n^4 + 10n^3 + 9n^2 + 2n}{12} \cdot u[n]$$

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n] \cdot z^{-n} \tag{7}$$

Z-Transform is calculated as:

$$12Y(z) = \sum_{n=0}^{\infty} 3n^4 u[n] z^{-n} + \sum_{n=0}^{\infty} 10n^3 u[n] z^{-n} + \sum_{n=0}^{\infty} 9n^2 u[n] z^{-n} + \sum_{n=0}^{\infty} 2n\, u[n] z^{-n} \tag{8}$$

We can say that Y_i denote the z-transform of n^i which will be same as respective X_i.

$$X_i = Y_i$$

Equation(8) can be now written as:

$$
\begin{aligned}
Y(z) &= \frac{1}{12}(3Y_4(z) + 10Y_3(z) + 9Y_2(z) + 2Y_1(z)) \\
&= \frac{3}{12}\left(\frac{z^{-1}(1 + 11z^{-1} + 11z^{-2} + z^{-3})}{(1 - z^{-1})^5}\right) \\
&+ \frac{10}{12}\left(\frac{z^{-1}(1 + 4z^{-1} + z^{-2})}{(1 - z^{-1})^4}\right) + \frac{9}{12}\left(\frac{z^{-1}(z^{-1} + 1)}{(1 - z^{-1})^3}\right) \\
&+ \frac{2}{12}\left(\frac{z}{(z - 1)^2}\right) \\
&= \frac{2(z^{-1} + 2z^{-2})}{(1 - z^{-1})^5}
\end{aligned}
$$