

NCERT 11.9.5 26Q

EE23BTECH11015 - DHANUSH V NAYAK*

Question: Show that

$$\frac{1 \times 2^2 + 2 \times 3^2 + \dots + n \times (n+1)^2}{1^2 \times 2 + 2^2 \times 3 + \dots + n^2 \times (n+1)} = \frac{3n+5}{3n+1}$$

Solution:

a) Consider the Numerator of the LHS part:

$$1 \times 2^2 + 2 \times 3^2 + \dots + n \times (n+1)^2 = \sum_{k=1}^n k(k+1)^2 \quad (1)$$

Now,

$$\sum_{k=1}^n k(k+1)^2 = \sum_{k=1}^n k^3 + 2k^2 + k \quad (2)$$

$$= \sum_{k=1}^n k^3 + \sum_{k=1}^n 2k^2 + \sum_{k=1}^n k \quad (3)$$

$$= \sum_{k=1}^n k^3 + 2 \sum_{k=1}^n k^2 + \sum_{k=1}^n k \quad (4)$$

$$= \left(\frac{n(n+1)}{2} \right)^2 + 2 \cdot \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \quad (5)$$

$$= \frac{n(n+1)}{2} \left[\frac{n(n+1)}{2} + \frac{2}{3}(2n+1) + 1 \right] \quad (6)$$

$$= \frac{n(n+1)}{2} \left[\frac{3n^2 + 11n + 10}{6} \right] \quad (7)$$

$$= \frac{n(n+1)}{12} [3n^2 + 6n + 5n + 10] \quad (8)$$

$$= \frac{n(n+1)}{12} [3n(n+2) + 5(n+2)] \quad (9)$$

$$= \frac{n(n+1)(n+2)(3n+5)}{12} \quad (10)$$

Therefore,

$$1 \times 2^2 + 2 \times 3^2 + \dots + n \times (n+1)^2 = \frac{n(n+1)(n+2)(3n+5)}{12} \quad (11)$$

$$\frac{n(n+1)(n+2)(3n+5)}{12} \cdot u[n] = x[n] \quad (12)$$

Derivation of z-transform of numerator :

$$x[n] = \frac{n(n+1)(n+2)(3n+5)}{12} \cdot u[n] \quad (13)$$

$$= \frac{3n^4 + 14n^3 + 21n^2 + 10n}{12} \cdot u[n] \quad (14)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \quad (15)$$

Z-Transform is calculated as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n} \quad (16)$$

$$12X(z) = \sum_{n=0}^{\infty} 3n^4 u[n]z^{-n} + \sum_{n=0}^{\infty} 14n^3 u[n]z^{-n} \quad (17)$$

$$+ \sum_{n=0}^{\infty} 21n^2 u[n]z^{-n} + \sum_{n=0}^{\infty} 10nu[n]z^{-n}$$

Let $U(z)$ be z transform of $x[n] = u[n]$.

$$U(z) = \sum_{n=0}^{\infty} u[n] \cdot z^{-n} \quad (18)$$

$$= \sum_{n=0}^{\infty} (1) \cdot z^{-n} \quad (19)$$

$$= 1 + z^{-1} + z^{-2} + \dots \quad (20)$$

$$= \frac{1}{1 - z^{-1}} \quad (21)$$

$$= \frac{z}{z - 1} \quad (22)$$

By the differentiation property:

Here , $x[n]=u[n]$ and $X(z) = U(z)$

$$nx[n] \xleftrightarrow{z} -z \frac{dX(z)}{dz}, \quad (23)$$

$$nu[n] \xleftrightarrow{z} -z \frac{dX(z)}{dz}, \quad (24)$$

$$\frac{dX(z)}{dz} = -\frac{1}{(z-1)^2}, \quad (25)$$

$$nu(n) \xleftrightarrow{z} \frac{z}{(z-1)^2}. \quad (26)$$

Therefore, the Z-transform of $n \cdot u[n]$,

$$X_1(z) = \frac{z^{-1}}{(1 - z^{-1})^2}. \quad (27)$$

The Region of Convergence (ROC) is defined as the set of points in the complex plane for which the Z-transform summation converges, i.e., doesn't blow up in magnitude to infinity: The Region of Convergence (ROC) is defined as:

$$\text{ROC} = \left\{ z : \left| \sum_{n=-\infty}^{\infty} x[n]z^{-n} \right| < \infty \right\} \quad (28)$$

Equation (??) expresses the set of points in the complex plane for which the Z-transform summation converges.

The Z-transform of the unit step signal $u[n]$ is given by:

$$U(z) = \sum_{n=-\infty}^{\infty} u[n]z^{-n} \quad (29)$$

Since $u[n] = 1$ for $n \geq 0$ and $u[n] = 0$ for $n < 0$, the summation simplifies to:

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \quad (30)$$

$$\text{ROC} = \left\{ z : \left| \sum_{n=0}^{\infty} z^{-n} \right| < \infty \right\} \quad (31)$$

This is an infinite GP. And it converges only if $|r| < 1$. And here, $|r| = |z^{-1}|$. Therefore, the Region of Convergence (ROC) for this Z-transform is:

$$\text{ROC: } |z| > 1 \quad (32)$$

In the differentiation property, ROC does not change:

$$x[n] \xrightarrow{Z} X(z), \text{ ROC} = R \quad (33)$$

$$nx[n] \xrightarrow{Z} -z \frac{dX(z)}{dz}, \text{ ROC} = R \quad (34)$$

Therefore,

$$X_1(z) = \frac{z^{-1}}{(1 - z^{-1})^2}, \text{ ROC} = |z| > 1 \quad (35)$$

$$X_2(z) = \frac{z^{-1}(z^{-1} + 1)}{(1 - z^{-1})^3}, \text{ ROC} = |z| > 1 \quad (36)$$

$$X_3(z) = \frac{z^{-1}(1 + 4z^{-1} + z^{-2})}{(1 - z^{-1})^4}, \text{ ROC} = |z| > 1 \quad (37)$$

$$X_4(z) = \frac{z^{-1}(1 + 11z^{-1} + 11z^{-2} + z^{-3})}{(1 - z^{-1})^5}, \text{ ROC} = |z| > 1 \quad (38)$$

Equation(??) can be now written as:

$$X(z) = \frac{1}{12}(3X_4(z) + 14X_3(z) + 21X_2(z) + 10X_1(z)) \quad (39)$$

$$= \frac{3}{12} \left(\frac{z^{-1}(1 + 11z^{-1} + 11z^{-2} + z^{-3})}{(1 - z^{-1})^5} \right) \quad (40)$$

$$+ \frac{14}{12} \left(\frac{z^{-1}(1 + 4z^{-1} + z^{-2})}{(1 - z^{-1})^4} \right) + \frac{21}{12} \left(\frac{z^{-1}(z^{-1} + 1)}{(1 - z^{-1})^3} \right)$$

$$+ \frac{10}{12} \left(\frac{z^{-1}}{(1 - z^{-1})^2} \right)$$

$$= \frac{2(z^{-2} + 2z^{-1})}{(1 - z^{-1})^5} \quad (41)$$

Consider the linear combination of two signals in the time domain:

$$a_1x_1(n) + a_2x_2(n) = a_1X_1(z) + a_2X_2(z) \quad (42)$$

$$\text{ROC} = \text{ROC}_1 \cap \text{ROC}_2 \quad (43)$$

Therefore, ROC of equation(??) is intersection of ROC of each signal. Every signal has ROC $|z| > 1$. So,

$$\text{ROC of } X(z) \text{ is } |z| > 1 \quad (44)$$

b) Consider the Denominator of the RHS part:

$$1^2 \times 2 + 2^2 \times 3 + \dots + n^2 \times (n+1) = \sum_{k=1}^n k^2(k+1) \quad (45)$$

Now,

$$\sum_{k=1}^n k^2(k+1) = \sum_{k=1}^n k^3 + k^2 \quad (46)$$

$$= \sum_{k=1}^n k^3 + \sum_{k=1}^n k^2 \quad (47)$$

$$= \left(\frac{n(n+1)}{2} \right)^2 + \frac{n(n+1)(2n+1)}{6} \quad (48)$$

$$= \frac{n(n+1)}{2} \left[\frac{n(n+1)}{2} + \frac{2n+1}{3} \right] \quad (49)$$

$$= \frac{n(n+1)}{2} \left[\frac{3n^2 + 7n + 2}{6} \right] \quad (50)$$

$$= \frac{n(n+1)}{12} [3n^2 + 6n + n + 2] \quad (51)$$

$$= \frac{n(n+1)}{12} [3n(n+2) + (n+2)] \quad (52)$$

$$= \frac{n(n+1)(n+2)(3n+1)}{12} \quad (53)$$

$$\frac{n(n+1)(n+2)(3n+1)}{12} \cdot u[n] = y[n] \quad (54)$$

Therefore,

$$\frac{\sum_{k=1}^n k(k+1)^2}{\sum_{k=1}^n k^2(k+1)} = \frac{\frac{n(n+1)(n+2)(3n+5)}{12}}{\frac{n(n+1)(n+2)(3n+1)}{12}} \quad (55)$$

$$= \frac{3n+5}{3n+1} \quad (56)$$

$Y(z)$ is a linear combination of $Y_i(z)$ for $i = 1, 2, 3, 4$:

$$Y(z) = a_1 Y_1(z) + a_2 Y_2(z) + a_3 Y_3(z) + a_4 Y_4(z) \quad (65)$$

$$\text{ROC}_Y = \text{ROC}_1 \cap \text{ROC}_2 \cap \text{ROC}_3 \cap \text{ROC}_4 \quad (66)$$

By equation(??) we can say that Y_i also has same ROC as respective X_i . Therefore,

$$\text{ROC of } Y(z) : |z| > 1 \quad (67)$$

Derivation of z-transform of the denominator:

$$y[n] = \frac{n(n+1)(n+2)(3n+1)}{12} \cdot u[n] \quad (57)$$

$$= \frac{3n^4 + 10n^3 + 9n^2 + 2n}{12} \cdot u[n] \quad (58)$$

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n] \cdot z^{-n} \quad (59)$$

Z-Transform is calculated as:

$$\begin{aligned} 12Y(z) &= \sum_{n=0}^{\infty} 3n^4 u[n] z^{-n} + \sum_{n=0}^{\infty} 10n^3 u[n] z^{-n} \\ &+ \sum_{n=0}^{\infty} 9n^2 u[n] z^{-n} + \sum_{n=0}^{\infty} 2n u[n] z^{-n} \end{aligned} \quad (60)$$

We can say that Y_i denote the z-transform of n^i which will be same as respective X_i .

$$X_i = Y_i \quad (61)$$

Equation(??) can be now written as:

$$Y(z) = \frac{1}{12} (3Y_4(z) + 10Y_3(z) + 9Y_2(z) + 2Y_1(z)) \quad (62)$$

$$= \frac{3}{12} \left(\frac{z^{-1}(1 + 11z^{-1} + 11z^{-2} + z^{-3})}{(1 - z^{-1})^5} \right) \quad (63)$$

$$+ \frac{10}{12} \left(\frac{z^{-1}(1 + 4z^{-1} + z^{-2})}{(1 - z^{-1})^4} \right) + \frac{9}{12} \left(\frac{z^{-1}(z^{-1} + 1)}{(1 - z^{-1})^3} \right)$$

$$+ \frac{2}{12} \left(\frac{z^{-1}}{(1 - z^{-1})^2} \right)$$

$$= \frac{2(z^{-1} + 2z^{-2})}{(1 - z^{-1})^5} \quad (64)$$