NCERT 11.9.5 26Q

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Question: Show that

$$\frac{1\times 2^2+2\times 3^2+\cdots+n\times(n+1)^2}{1^2\times 2+2^2\times 3+\cdots+n^2\times(n+1)}=\frac{3n+5}{3n+1}$$

Solution:

Parameter	**Description**	**Value**
n	Integer	1, 2, 3, 4, ...
$x(n)$	Discrete-sequence	$\frac{3n^4+26n^3+81n^2+106n+48}{12}u(n)$
$y(n)$	Discrete-sequence	$\frac{3n^4+22n^3+57n^2+62n+24}{12}u(n)$
$X(z)$	z-transform of $x(n)$	$\frac{24(2+z^{-1})}{(1-z^{-1})^5}, ROC = \|z\| > 1$
$Y(z)$	z-transform of $y(n)$	$\frac{24(1+2z^{-1})}{(1-z^{-1})^5},\ ROC = \|z\| > 1$
$u(n)$	Unit step sequence	$u(n) = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$
$U(z)$	z-transform of $u(n)$	$\frac{1}{1-z^{-1}}, ROC = \|z\| > 1$
ROC	Region of convergence	$\left\{z : \left\|\sum_{n=-\infty}^{\infty} x(n)z^{-n}\right\| < \infty\right\}$
$Y_k(z), X_k(z)$	z-transform of $n^k u(n)$	$(-z)^k \frac{d^kU(z)}{dz^k},\ ROC = \|z\| > 1$
$X_1(z)$	z-transform of $n \cdot u(n)$	$\frac{z^{-1}}{(1-z^{-1})^2},\ ROC = \|z\| > 1$
$X_2(z)$	z-transform of $n^2 \cdot u(n)$	$\frac{z^{-1}(z^{-1}+1)}{(1-z^{-1})^3},\ ROC = \|z\| > 1$
$X_3(z)$	z-transform of $n^3 \cdot u(n)$	$\frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4},\ ROC = \|z\| > 1$
$X_4(z)$	z-transform of $n^4 \cdot u(n)$	$\frac{z^{-1}(1+11z^{-1}+11z^{-2}+z^{-3})}{(1-z^{-1})^5},\ ROC = \|z\| > 1$

TABLE 1: Parameter Table

1. Consider the Numerator of the LHS part:

$$1\times 2^2+2\times 3^2+\cdots+n\times(n+1)^2=\sum_{k=1}^{n}k(k+1)^2 \tag{1}$$

$$\sum_{k=1}^{n}k(k+1)^2=\left(\frac{n(n+1)}{2}\right)^2+2\cdot\frac{n(n+1)(2n+1)}{6} \tag{2}$$
$$+\frac{n(n+1)}{2}$$

$$=\frac{n(n+1)(n+2)(3n+5)}{12} \tag{3}$$

$$x(n)=\frac{(n+1)(n+2)(n+3)(3n+8)}{12}\cdot u(n) \tag{4}$$

$$=\frac{3n^4+26n^3+81n^2+106n+48}{12}\cdot u(n) \tag{5}$$

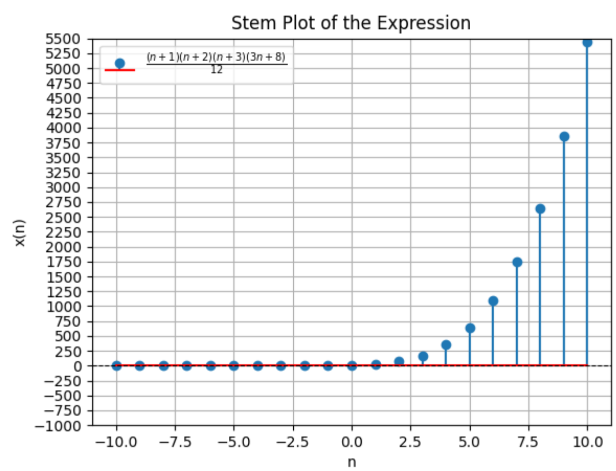

Fig. 1.: Stem Plot of $x(n)$

By the differentiation property :

$$x(n)\overset{\mathcal{Z}}{\longleftrightarrow}X(z),\ ROC=R \tag{6}$$

$$n^kx(n)\overset{\mathcal{Z}}{\longleftrightarrow}(-z)^k\frac{d^kX(z)}{dz^k},\ ROC=R \tag{7}$$

$$\implies n^ku(n)\overset{\mathcal{Z}}{\longleftrightarrow}(-z)^k\frac{d^kU(z)}{dz^k} \tag{8}$$

$$\implies X_k(z)=(-z)^k\frac{d^kU(z)}{dz^k},\ ROC=|z|>1 \tag{9}$$

$$X(z)=\frac{1}{12}\left(3X_4(z)+26X_3(z)+81X_2(z)+106X_1(z)\right)$$
$$+48U(z) \tag{10}$$

Refering to table1 for X_k and $U(z)$ values

$$X(z) = \frac{24\left(2 + z^{-1}\right)}{\left(1 - z^{-1}\right)^5},\ ROC = |z| > 1 \tag{11}$$

2. Consider the Denominator of the RHS part:

$$1^2 \times 2 + 2^2 \times 3 + \cdots + n^2 \times (n+1) = \sum_{k=1}^{n} k^2 (k+1) \tag{12}$$

$$\sum_{k=1}^{n} k^2 (k+1) = \left(\frac{n(n+1)}{2}\right)^2 + \frac{n(n+1)(2n+1)}{6} \tag{13}$$

$$= \frac{(n+1)(n+2)(n+3)(3n+4)}{12} \tag{14}$$

$$\frac{(n+1)(n+2)(n+3)(3n+4)}{12} \cdot u(n) = y(n) \tag{15}$$

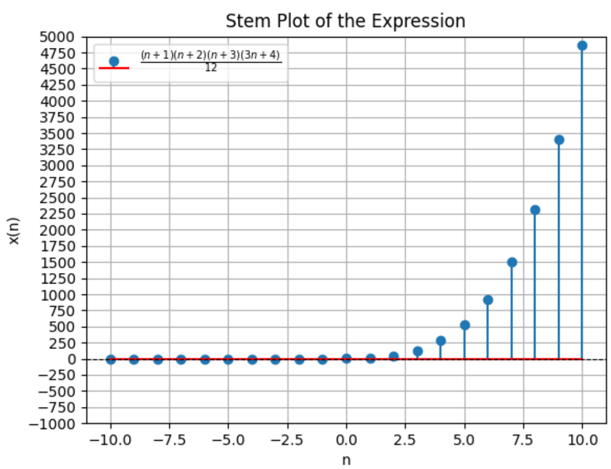

Fig. 2.: Stem Plot of $y(n)$

$$Y(z) = \sum_{n=-\infty}^{\infty} y(n) \cdot z^{-n} \tag{16}$$

$$X_k = Y_k \tag{17}$$

Refering to table1 for Y_k and $U(z)$ values

$$Y(z) = \frac{1}{12}\left(3Y_4(z) + 22Y_3(z) + 57Y_2(z) + 62Y_1(z)\right) \tag{18}$$

$$+ 24U(z)$$

$$Y(z) = \frac{24\left(1 + 2z^{-1}\right)}{\left(1 - z^{-1}\right)^5},\ ROC = |z| > 1 \tag{19}$$