# NCERT 11.9.5 26Q

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## RESULTS AND DERIVATIONS

1) By the differentiation property :

$$nx(n) \overset{\mathcal{Z}}{\longleftrightarrow} (-z)\frac{dX(z)}{dz} \tag{1}$$

$$\implies nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^2}, |z| > 1 \tag{2}$$

$$\implies n^2u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(z^{-1}+1\right)}{(1-z^{-1})^3}, |z| > 1 \tag{3}$$

$$\implies n^3u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{(1-z^{-1})^4}, |z| > 1 \tag{4}$$

$$\implies n^4u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(1+11z^{-1}+11z^{-2}+z^{-3}\right)}{(1-z^{-1})^5} \tag{5}$$

where $|z| > 1$

2) Time shifting property:

$$x(n-k) \overset{\mathcal{Z}}{\longleftrightarrow} z^{-k}X(z) \tag{6}$$

3) By (6)

$$u(n-1) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})} \tag{7}$$

By (1)

$$nu(n-1) \overset{\mathcal{Z}}{\longleftrightarrow} z\frac{2z^{-2}}{(1-z^{-1})^2} \tag{8}$$

Now ,

$$\frac{(n-1)}{2}u(n-2) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-2}}{(1-z^{-1})^2} \tag{9}$$

$$\frac{(n-1)(n-2)}{6}u(n-3) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-3}}{(1-z^{-1})^3} \tag{10}$$

$$\vdots$$

$$\frac{(n-1)(n-2)\dots(n-k+1)}{(k-1)!}u(n-k) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-k}}{(1-z^{-1})^k} \tag{11}$$

$$\implies Z^{-1}\left[\frac{z^{-2}}{(1-z^{-1})^2}\right] = (n-1)u(n-1) \tag{12}$$

$$\implies Z^{-1}\left[\frac{z^{-3}}{(1-z^{-1})^3}\right] = \frac{(n-1)(n-2)}{2}u(n-1) \tag{13}$$

$$\implies Z^{-1}\left[\frac{z^{-4}}{(1-z^{-1})^4}\right] = \frac{(n-1)(n-2)(n-3)}{6}u(n-1) \tag{14}$$

$$\implies Z^{-1}\left[\frac{z^{-5}}{(1-z^{-1})^5}\right] = \frac{(n-1)(n-2)(n-3)(n-4)}{24} \tag{15}$$

$$u(n-1)$$

4) The convolution sum is defined as

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{16}$$

from (16),

$$x(n) * u(n) = \sum_{k=0}^{n} x(k) \tag{17}$$

**Question:** Show that

$$\frac{1\times 2^2+2\times 3^2+\cdots+n\times(n+1)^2}{1^2\times 2+2^2\times 3+\cdots+n^2\times(n+1)}=\frac{3n+5}{3n+1}$$

**Solution:**

| Parameter | Description | Value |
|---|---|---|
| $n$ | Integer | .... -2,-1,0,1, 2, ... |
| $x_1(n)$ | General term of Numerator | $\left(n^3+5n^2+8n+4\right)\cdot u(n)$ |
| $x_2(n)$ | General Term of Denominator | $\left(n^3+4n^2+5n+2\right)\cdot u(n)$ |
| $y_1(n)$ | Sum of terms of numerator | ? |
| $y_2(n)$ | Sum of terms of denominator | ? |
| $U(z)$ | z-transform of $u(n)$ | $\frac{1}{1-z^{-1}}, \{z\in\mathbb{C}:\lvert z\rvert>1\}$ |
| ROC | Region of convergence | $\left\{z:\left\lvert\sum_{n=-\infty}^{\infty}x(n)z^{-n}\right\rvert<\infty\right\}$ |

TABLE 1: Parameter Table

1. Analysis of Numerator:

$$X_1(z)=\sum_{n=-\infty}^{\infty}x_1(n)z^{-n} \tag{18}$$

$$=\sum_{n=-\infty}^{\infty}\left(n^3+5n^2+8n+4\right)u(n)z^{-n} \tag{19}$$

Using results of equations (2) to (5) we get:

$$\therefore X_1(z)=\frac{4+2z^{-1}}{\left(1-z^{-1}\right)^4}, \lvert z\rvert>1 \tag{20}$$

From (17)

$$y_1(n)=x_1(n)*u(n) \tag{21}$$

$$Y_1(z)=X_1(z)U(z) \tag{22}$$

$$=\frac{4+2z^{-1}}{\left(1-z^{-1}\right)^5}, \lvert z\rvert>1 \tag{23}$$

Using partial fractions:

$$Y_1(z)=\frac{22z^{-1}}{\left(1-z^{-1}\right)}+\frac{48z^{-2}}{\left(1-z^{-1}\right)^2}+\frac{52z^{-3}}{\left(1-z^{-3}\right)^3}, \tag{24}$$

$$+\frac{28z^{-4}}{\left(1-z^{-1}\right)^4}+\frac{6z^{-5}}{\left(1-z^{-1}\right)^5}+4, \lvert z\rvert>1$$

Substituting results of equation (12) to (15) in equation (24):

$$y_1(n)=\frac{3n^4+26n^3+81n^2+106n+48}{12}u(n) \tag{25}$$

$$=\frac{(3n+8)(n+1)(n+2)(n+3)}{12}u(n) \tag{26}$$

2. Analysis of Denominator:

$$X_2(z)=\sum_{n=-\infty}^{\infty}x_2(n)z^{-n} \tag{27}$$

$$=\sum_{n=-\infty}^{\infty}\left(n^3+4n^2+5n+2\right)u(n)z^{-n} \tag{28}$$

Using results of equation (2) to (5) we get:

$$\therefore X_2(z)=\frac{2+4z^{-1}}{\left(1-z^{-1}\right)^4}, \lvert z\rvert>1 \tag{29}$$

From (17)

$$y_2(n)=x_2(n)*u(n) \tag{30}$$

$$Y_2(z)=X_2(z)U(z) \tag{31}$$

$$=\frac{2+4z^{-1}}{\left(1-z^{-1}\right)^5}, \lvert z\rvert>1 \tag{32}$$

Using partial fractions:

$$Y_2(z)=\frac{14z^{-1}}{\left(1-z^{-1}\right)}+\frac{36z^{-2}}{\left(1-z^{-1}\right)^2}+\frac{44z^{-3}}{\left(1-z^{-3}\right)^3} \tag{33}$$

$$+\frac{26z^{-4}}{\left(1-z^{-1}\right)^4}+\frac{6z^{-5}}{\left(1-z^{-1}\right)^5}+2, \lvert z\rvert>1$$

Substituting results of equation (12) to (15) in equation (33):

$$y_2(n)=\frac{3n^4+22n^3+57n^2+62n+24}{12}u(n) \tag{34}$$

$$=\frac{(3n+4)(n+1)(n+2)(n+3)}{12}u(n) \tag{35}$$

As the sequence start from n=0 , in RHS of question $n$ should be replaced by $n+1$:

$$\frac{y_1(n)}{y_2(n)}=\frac{3n+8}{3n+4} \tag{36}$$

Hence Prooved.

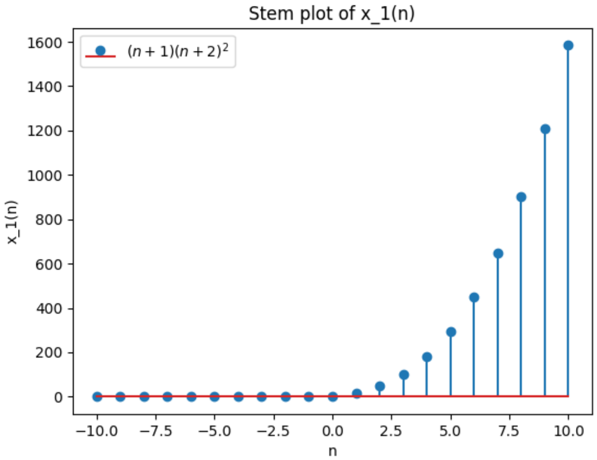


Fig. 1: Stem Plot of $x_1(n)$

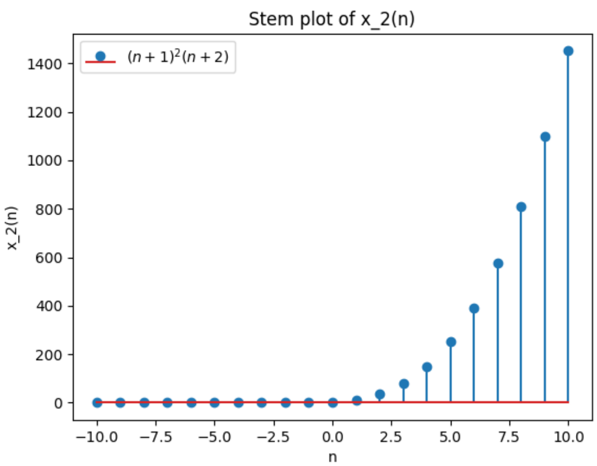


Fig. 2: Stem Plot of $x_2(n)$

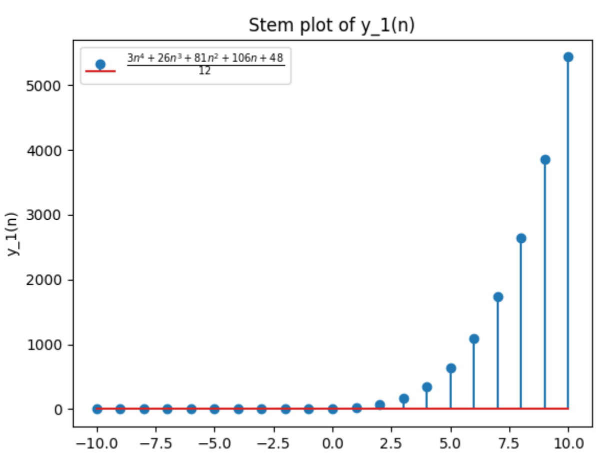


Fig. 3: Stem Plot of $y_1(n)$

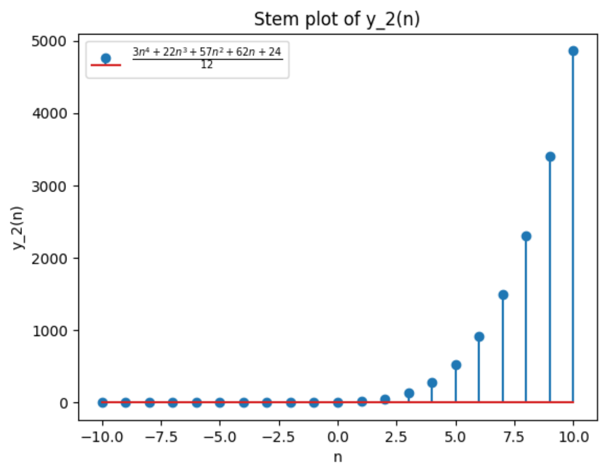


Fig. 4: Stem Plot of $y_2(n)$