NCERT 11.9.5 26Q

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Question: Show that

$$\frac{1 \times 2^2 + 2 \times 3^2 + \cdots + n \times (n+1)^2}{1^2 \times 2 + 2^2 \times 3 + \cdots + n^2 \times (n+1)} = \frac{3n+5}{3n+1}$$

Solution:

Parameter	Description	Value
n	Integer	1, 2, 3, 4, ...
$x_1(n)$	Numerator sequence	$\left(n^3 + 5n^2 + 20n + 4\right) \cdot u(n)$
$x_2(n)$	Denominator sequence	$\left(n^3 + 4n^2 + 5n + 2\right) \cdot u(n)$
$X_1(z)$	z-transform of $x_1(n)$	$\frac{4+14z^{-1}-24z^{-2}+12z^{-3}}{\left(1-z^{-1}\right)^4}, \{z \in \mathbb{C} : \lvert z\rvert > 1\}$
$X_2(z)$	z-transform of $x_2(n)$	$\frac{2+4z^{-1}}{\left(1-z^{-1}\right)^4}, \{z \in \mathbb{C} : \lvert z\rvert > 1\}$
$U(z)$	z-transform of $u(n)$	$\frac{1}{1-z^{-1}}, \{z \in \mathbb{C} : \lvert z\rvert > 1\}$
ROC	Region of convergence	$\left\{z : \left\lvert\sum_{n=-\infty}^{\infty} x(n)z^{-n}\right\rvert < \infty\right\}$
$X_k(z)$	z-transform of $n^k u(n)$	$(-z)^k \frac{d^k U(z)}{dz^k}, \{z \in \mathbb{C} : \lvert z\rvert > 1\}$
$X_1(z)$	z-transform of $n \cdot u(n)$	$\frac{z^{-1}}{\left(1-z^{-1}\right)^2}, \{z \in \mathbb{C} : \lvert z\rvert > 1\}$
$X_2(z)$	z-transform of $n^2 \cdot u(n)$	$\frac{z^{-1}\left(z^{-1}+1\right)}{\left(1-z^{-1}\right)^3}, \{z \in \mathbb{C} : \lvert z\rvert > 1\}$
$X_3(z)$	z-transform of $n^3 \cdot u(n)$	$\frac{z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{\left(1-z^{-1}\right)^4}, \{z \in \mathbb{C} : \lvert z\rvert > 1\}$
$X_4(z)$	z-transform of $n^4 \cdot u(n)$	$\frac{z^{-1}\left(1+11z^{-1}+11z^{-2}+z^{-3}\right)}{\left(1-z^{-1}\right)^5}, \{z \in \mathbb{C} : \lvert z\rvert > 1\}$

TABLE 1: Parameter Table

1. General term of Numerator of the LHS part:

$$x_1(n) = (n+1)(n+2)^2 \cdot u(n) \tag{1}$$

$$= \left(n^3 + 5n^2 + 20n + 4\right) \cdot u(n) \tag{2}$$

By the differentiation property :

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z), \; ROC = R \tag{3}$$

$$n^k x(n) \overset{\mathcal{Z}}{\longleftrightarrow} (-z)^k \frac{d^k X(z)}{dz^k}, \; ROC = R \tag{4}$$

$$\implies n^k u(n) \overset{\mathcal{Z}}{\longleftrightarrow} (-z)^k \frac{d^k U(z)}{dz^k} \tag{5}$$

$$\implies X_k(z) = (-z)^k \frac{d^k U(z)}{dz^k}, \; \{z \in \mathbb{C} : |z| > 1\} \tag{6}$$

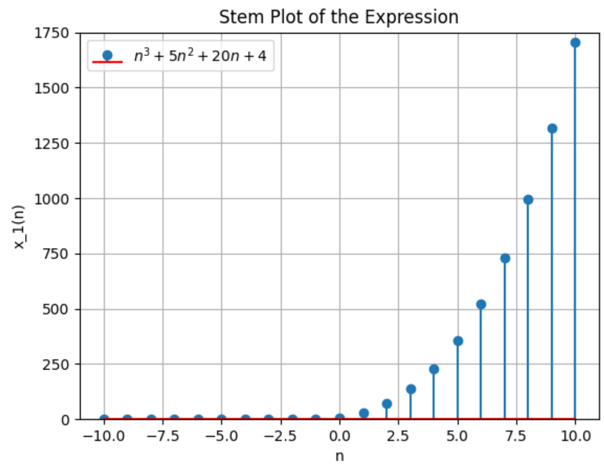

Fig. 1.: Stem Plot of $x(n)$

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n) \cdot z^{-n} \tag{7}$$

From equation (2)

$$X_1(z) = X_3(z) + 5X_2(z) + 20X_1(z) + 4U(z) \tag{8}$$

Refering to table1 for X_k and $U(z)$ values

$$X_1(z) = \frac{4 + 14z^{-1} - 24z^{-2} + 12z^{-3}}{(1 - z^{-1})^4}, \quad \{z \in \mathbb{C} : |z| > 1\} \tag{9}$$

2. General term of Denominator of the RHS part:

$$x_2(n) = (n+1)^2 (n+2) \cdot u(n) \tag{10}$$

$$= \left(n^3 + 4n^2 + 5n + 2\right) \cdot u(n) \tag{11}$$

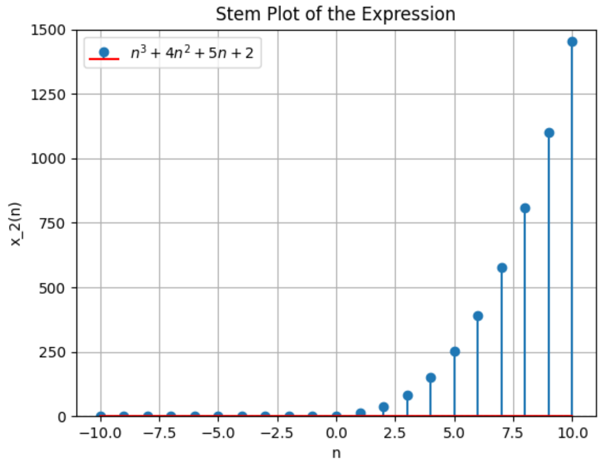

Fig. 2.: Stem Plot of $y(n)$

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2(n) \cdot z^{-n} \tag{12}$$

From equation (11) (13)

$$X_2(z) = X_3(z) + 4X_2(z) + 5X_1(z) + 2U(z) \tag{14}$$

Refering to table1 for X_k and $U(z)$ values

$$X_2(z) = \frac{2 + 4z^{-1}}{(1 - z^{-1})^4}, \{z \in \mathbb{C} : |z| > 1\} \tag{15}$$