NCERT 11.9.5 26Q

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Question:

$$\frac{1 \times 2^2 + 2 \times 3^2 + \cdots + n \times (n+1)^2}{1^2 \times 2 + 2^2 \times 3 + \cdots + n^2 \times (n+1)}$$

Find the z-transform of general term of numerator and denominator and plot them . Also find z-transform of sum of terms of numerator and denominator and plot it.

Solution:

Parameter	Description	Value
n	Integer	1, 2, 3, 4, ...
$x_1(n)$	General term of Numerator	$\left(n^3 + 5n^2 + 20n + 4\right) \cdot u(n)$
$x_2(n)$	General Term of Denominator	$\left(n^3 + 4n^2 + 5n + 2\right) \cdot u(n)$
$y_1(n)$	Sum of terms of numerator	$\frac{3n^4+26n^3+81n^2+106n+48}{12}$
$y_2(n)$	Sum of terms of denominator	$\frac{3n^4+22n^3+57n^2+62n+24}{12}$
$U(z)$	z-transform of $u(n)$	$\frac{1}{1-z^{-1}}, \{z \in \mathbb{C} : \lvert z\rvert > 1\}$
ROC	Region of convergence	$\left\{z : \left\lvert\sum_{n=-\infty}^{\infty} x(n)z^{-n}\right\rvert < \infty\right\}$

TABLE 1: Parameter Table

1. General term of Numerator of the LHS part:

$$x_1(n) = (n+1)(n+2)^2 \cdot u(n) \tag{1}$$

$$= \left(n^3 + 5n^2 + 20n + 4\right) \cdot u(n) \tag{2}$$

By the differentiation property :

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z), \; ROC = R \tag{3}$$

$$n^k x(n) \overset{\mathcal{Z}}{\longleftrightarrow} (-z)^k \frac{d^k X(z)}{dz^k}, \; ROC = R \tag{4}$$

$$\implies n^k u(n) \overset{\mathcal{Z}}{\longleftrightarrow} (-z)^k \frac{d^k U(z)}{dz^k} \{z \in \mathbb{C} : |z| > 1\} \tag{5}$$

$$n \cdot u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^2}, \{z \in \mathbb{C} : |z| > 1\} \tag{6}$$

$$n^2 \cdot u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(z^{-1}+1\right)}{(1-z^{-1})^3}, \{z \in \mathbb{C} : |z| > 1\} \tag{7}$$

$$n^3 \cdot u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(1 + 4z^{-1} + z^{-2}\right)}{(1-z^{-1})^4}, \{z \in \mathbb{C} : |z| > 1\} \tag{8}$$

$$n^4 \cdot u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(1 + 11z^{-1} + 11z^{-2} + z^{-3}\right)}{(1-z^{-1})^5} \tag{9}$$

where $\{z \in \mathbb{C} : |z| > 1\}$

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n) \cdot u(n) \cdot z^{-n} \tag{10}$$

$$= \sum_{n=-\infty}^{\infty} \left(n^3 + 5n^2 + 20n + 4\right) \cdot u(n) \cdot z^{-n} \tag{11}$$

$$\therefore X_1(z) = \frac{4 + 14z^{-1} - 24z^{-2} + 12z^{-3}}{(1-z^{-1})^4}, \{z \in \mathbb{C} : |z| > 1\} \tag{12}$$

$$y_1(n) = 1 \times 2^2 + 2 \times 3^2 + \cdots + n \times (n+1)^2 \tag{13}$$

$$y_1(n) = \sum_{k=0}^{n} (k+1)(k+2)^2 \cdot u(n) \tag{14}$$

$$y_1(n) = x_1(n) * u(n) \tag{15}$$

$$Y_1(z) = X_1(z) \cdot U(z) \tag{16}$$

$$= \frac{4 + 14z^{-1} - 24z^{-2} + 12z^{-3}}{(1-z^{-1})^5}, \{z \in \mathbb{C} : |z| > 1\} \tag{17}$$

Using partial fractions:

$$Y_1(z) = \frac{34z^{-1}}{(1-z^{-1})} + \frac{72z^{-2}}{(1-z^{-1})^2} + \frac{64z^{-3}}{(1-z^{-3})^3} \tag{18}$$
$$+ \frac{28z^{-4}}{(1-z^{-1})^4} + \frac{6z^{-5}}{(1-z^{-1})^5} + 4$$

$$Y_1(z) = \frac{z^{-1}\left(1+11z^{-1}+11z^{-2}+z^{-3}\right)}{4\cdot(1-z^{-1})^5} \tag{19}$$
$$+ \frac{26\cdot z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{12\,(1-z^{-1})^4} + \frac{27\cdot z^{-1}\left(z^{-1}+1\right)}{4\,(1-z^{-1})^3}$$
$$+ \frac{53\cdot z^{-1}}{6\,(1-z^{-1})^2} + 4\frac{1}{(1-z^{-1})}$$

Taking the inverse z-transform of each term :

$$y_1(n) = \frac{3n^4+26n^3+81n^2+106n+48}{12} \tag{20}$$

2. General term of Denominator of the RHS part:

$$x_2(n) = (n+1)^2(n+2)\cdot u(n) \tag{21}$$
$$= \left(n^3+4n^2+5n+2\right)\cdot u(n) \tag{22}$$

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2(n)\cdot u(n)\cdot z^{-n} \tag{23}$$
$$= \sum_{n=-\infty}^{\infty} \left(n^3+4n^2+5n+2\right)\cdot u(n)\cdot z^{-n} \tag{24}$$

$$\therefore X_2(z) = \frac{2+4z^{-1}}{(1-z^{-1})^4}, \{z\in\mathbb{C} : |z|>1\} \tag{25}$$

$$y_2(n) = 1^2\times 2 + 2^2\times 3 + \cdots + n^2\times(n+1) \tag{26}$$

$$y_2(n) = \sum_{k=0}^{n} (k+1)^2(k+2)\cdot u(n) \tag{27}$$

$$y_2(n) = x_2(n) * u(n) \tag{28}$$
$$Y_2(z) = X_2(z)\cdot U(z) \tag{29}$$
$$= \frac{2+4z^{-1}}{(1-z^{-1})^5}, \{z\in\mathbb{C} : |z|>1\} \tag{30}$$

Using partial fractions:

$$Y_2(z) = \frac{14z^{-1}}{(1-z^{-1})} + \frac{36z^{-2}}{(1-z^{-1})^2} + \frac{44z^{-3}}{(1-z^{-3})^3} \tag{31}$$
$$+ \frac{26z^{-4}}{(1-z^{-1})^4} + \frac{6z^{-5}}{(1-z^{-1})^5} + 2$$

$$Y_2(z) = \frac{z^{-1}\left(1+11z^{-1}+11z^{-2}+z^{-3}\right)}{4\cdot(1-z^{-1})^5} \tag{32}$$
$$+ \frac{11\cdot z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{6\,(1-z^{-1})^4} + \frac{19\cdot z^{-1}\left(z^{-1}+1\right)}{4\,(1-z^{-1})^3}$$
$$+ \frac{31\cdot z^{-1}}{6\,(1-z^{-1})^2} + 2\frac{1}{(1-z^{-1})}$$

Taking the inverse z-transform of each term:

$$y_2(n) = \frac{3n^4+22n^3+57n^2+62n+24}{12} \tag{33}$$

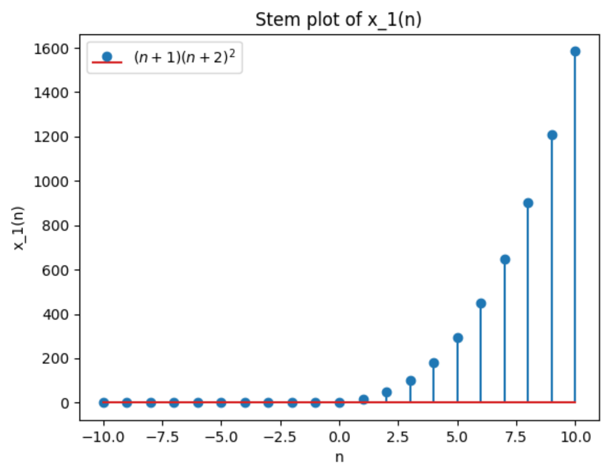

Fig. 1: Stem Plot of $x_1(n)$

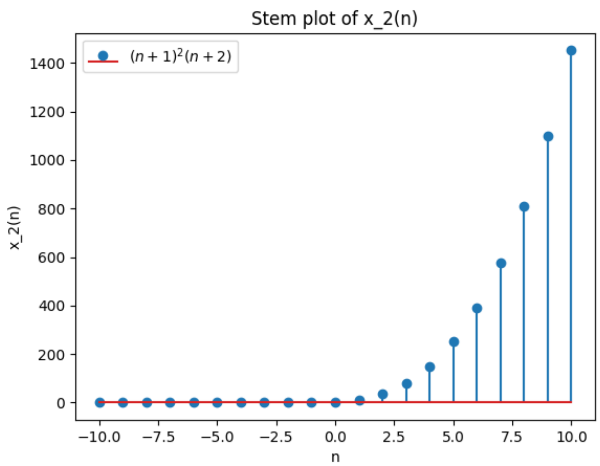

Fig. 2: Stem Plot of $x_2(n)$

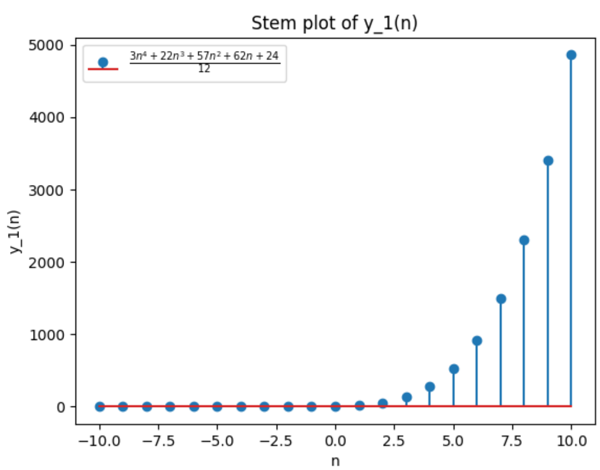

Fig. 3: Stem Plot of $y_1(n)$

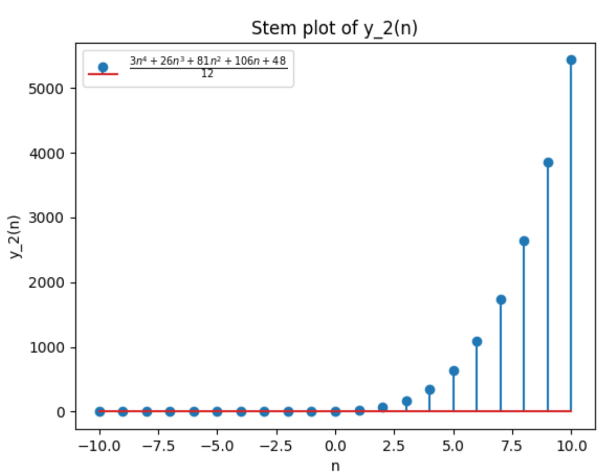

Fig. 4: Stem Plot of $y_2(n)$