NCERT 11.9.5 26Q

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Question: Show that

$$\frac{1\times 2^2+2\times 3^2+\cdots+n\times(n+1)^2}{1^2\times 2+2^2\times 3+\cdots+n^2\times(n+1)}=\frac{3n+5}{3n+1}$$

Solution:

1) Consider the Numerator of the LHS part:

$$1\times 2^2+2\times 3^2+\cdots+n\times(n+1)^2=\sum_{k=1}^{n}k\left(k+1\right)^2 \tag{1}$$

Now,

$$\sum_{k=1}^{n}k\left(k+1\right)^2=\sum_{k=1}^{n}k^3+2k^2+k \tag{2}$$
$$=\sum_{k=1}^{n}k^3+\sum_{k=1}^{n}2k^2+\sum_{k=1}^{n}k \tag{3}$$
$$=\sum_{k=1}^{n}k^3+2\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k \tag{4}$$
$$=\left(\frac{n\left(n+1\right)}{2}\right)^2+2\cdot\frac{n\left(n+1\right)\left(2n+1\right)}{6} \tag{5}$$
$$+\frac{n\left(n+1\right)}{2}$$
$$=\frac{n\left(n+1\right)}{2}\left[\frac{n\left(n+1\right)}{2}+\frac{2}{3}\left(2n+1\right)+1\right] \tag{6}$$
$$=\frac{n\left(n+1\right)}{2}\left[\frac{3n^2+11n+10}{6}\right] \tag{7}$$
$$=\frac{n\left(n+1\right)}{12}\left[3n^2+6n+5n+10\right] \tag{8}$$
$$=\frac{n\left(n+1\right)}{12}\left[3n\left(n+2\right)+5\left(n+2\right)\right] \tag{9}$$
$$=\frac{n\left(n+1\right)\left(n+2\right)\left(3n+5\right)}{12} \tag{10}$$

Therefore,

$$1\times 2^2+2\times 3^2+\cdots+n\times(n+1)^2=\frac{n\left(n+1\right)\left(n+2\right)\left(3n+5\right)}{12} \tag{11}$$

$$\frac{n\left(n+1\right)\left(n+2\right)\left(3n+5\right)}{12}\cdot u\left(n\right)=x\left(n\right) \tag{12}$$

$$x\left(n\right)=\begin{cases}0 & \text{for } n<0\\ \frac{n(n+1)(n+2)(3n+5)}{12}\cdot & \text{for } n\geq 0\end{cases} \tag{13}$$

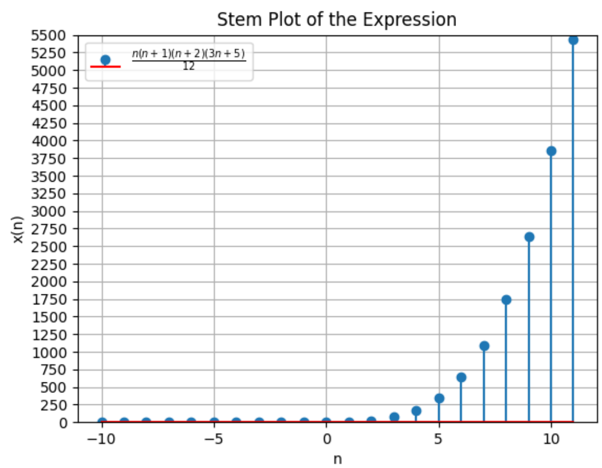

Fig. 1: Stem Plot of $x\left(n\right)$

Derivation of z-transform of numerator :

$$x\left(n\right)=\frac{n\left(n+1\right)\left(n+2\right)\left(3n+5\right)}{12}\cdot u\left(n\right) \tag{14}$$
$$=\frac{3n^4+14n^3+21n^2+10n}{12}\cdot u\left(n\right) \tag{15}$$

$$X\left(z\right)=\sum_{n=-\infty}^{\infty}x\left(n\right)z^{-n} \tag{16}$$

$$U(z) = \sum_{n=-\infty}^{\infty} u(n) \cdot z^{-n} \tag{17}$$
$$= \sum_{n=0}^{\infty} (1) \cdot z^{-n} \tag{18}$$
$$= 1 + z^{-1} + z^{-2} + \ldots \tag{19}$$
$$= \frac{1}{1 - z^{-1}} \tag{20}$$

By the differentiation property:

$$x(n) \overset{Z}{\longleftrightarrow} X(z) \tag{21}$$
$$n^k x(n) \overset{Z}{\longleftrightarrow} (-z)^k \frac{d^k X(z)}{dz^k} \tag{22}$$
$$n^k u(n) \overset{Z}{\longleftrightarrow} (-z)^k \frac{d^k U(z)}{dz^k} \tag{23}$$

$$X_k(z) = (-z)^k \frac{d^k U(z)}{dz^k} \tag{24}$$

For the Z-transform of $n \cdot u(n)$, substituting k=1 in Equation (24):

$$X_1(z) = (-z) \frac{dU(z)}{dz} \tag{25}$$
$$\frac{dU(z)}{dz} = -\frac{z^{-2}}{(1 - z^{-1})^2}, \tag{26}$$
$$X_1(z) = \frac{z^{-1}}{(1 - z^{-1})^2}. \tag{27}$$

Similarly , substituting k=2,3,4 in equation (24):

$$X_2(z) = \frac{z^{-1}\left(z^{-1} + 1\right)}{(1 - z^{-1})^3} \tag{28}$$
$$X_3(z) = \frac{z^{-1}\left(1 + 4z^{-1} + z^{-2}\right)}{(1 - z^{-1})^4} \tag{29}$$
$$X_4(z) = \frac{z^{-1}\left(1 + 11z^{-1} + 11z^{-2} + z^{-3}\right)}{(1 - z^{-1})^5} \tag{30}$$

The Region of Convergence (ROC) is defined as the set of points in the complex plane for which the Z-transform summation converges, i.e., doesn't blow up in magnitude to infinity: The Region of Convergence (ROC) is defined as:

$$\text{ROC} = \left\{ z : \left| \sum_{n=-\infty}^{\infty} x(n) z^{-n} \right| < \infty \right\} \tag{31}$$

Equation (31) expresses the set of points in the complex plane for which the Z-transform summation converges.
The Z-transform of the unit step signal $u(n)$ is given by:

$$U(z) = \sum_{n=-\infty}^{\infty} u(n) z^{-n} \tag{32}$$

$$u(n) = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n \geq 0 \end{cases} \tag{33}$$

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{34}$$

$$\text{ROC} = \left\{ z : \left| \sum_{n=0}^{\infty} z^{-n} \right| < \infty \right\} \tag{35}$$

This is an infinite GP. And it converges only if $|r| < 1$ where r is the common ratio.And here , $|r| = |z^{-1}|$ Therefore, the Region of Convergence (ROC) for this Z-transform is:

$$\text{ROC: } |z| > 1 \tag{36}$$

In the differentiation property, ROC does not change:

$$x(n) \overset{Z}{\longleftrightarrow} X(z),\ ROC = R \tag{37}$$
$$n^k x(n) \overset{Z}{\longleftrightarrow} (-z)^k \frac{d^k X(z)}{dz^k},\ ROC = R \tag{38}$$

Therefore,

$$X_1(z) = \frac{z^{-1}}{(1 - z^{-1})^2},\ ROC = |z| > 1 \tag{39}$$
$$X_2(z) = \frac{z^{-1}\left(z^{-1} + 1\right)}{(1 - z^{-1})^3},\ ROC = |z| > 1 \tag{40}$$
$$X_3(z) = \frac{z^{-1}\left(1 + 4z^{-1} + z^{-2}\right)}{(1 - z^{-1})^4},\ ROC = |z| > 1 \tag{41}$$
$$X_4(z) = \frac{z^{-1}\left(1 + 11z^{-1} + 11z^{-2} + z^{-3}\right)}{(1 - z^{-1})^5},\ ROC = |z| > 1 \tag{42}$$

$$X(z) = \frac{1}{12}\left(3X_4(z) + 14X_3(z) + 21X_2(z) + 10X_1(z)\right) \tag{43}$$

$$= \frac{3}{12}\left(\frac{z^{-1}\left(1 + 11z^{-1} + 11z^{-2} + z^{-3}\right)}{(1 - z^{-1})^5}\right) \tag{44}$$

$$+ \frac{14}{12}\left(\frac{z^{-1}\left(1 + 4z^{-1} + z^{-2}\right)}{(1 - z^{-1})^4}\right) + \frac{21}{12}\left(\frac{z^{-1}\left(z^{-1} + 1\right)}{(1 - z^{-1})^3}\right)$$

$$+ \frac{10}{12}\left(\frac{z^{-1}}{(1 - z^{-1})^2}.\right)$$

$$= \frac{2\left(z^{-2} + 2z^{-1}\right)}{(1 - z^{-1})^5} \tag{45}$$

Consider the linear combination of two signals in the time domain:

$$a_1x_1(n) + a_2x_2(n) \overset{\mathcal{Z}}{\longleftrightarrow} a_1X_1(z) + a_2X_2(z) \tag{46}$$

$$\text{ROC} = \text{ROC}_1 \cap \text{ROC}_2 \tag{47}$$

Therefore, ROC of equation(43) is the intersection of the ROC of each signal. Every signal has ROC $|z| > 1$. So,

$$\text{ROC of } X(z) \text{ is } |z| > 1 \tag{48}$$

2) Consider the Denominator of the RHS part:

$$1^2 \times 2 + 2^2 \times 3 + \cdots + n^2 \times (n + 1) = \sum_{k=1}^{n} k^2 (k + 1) \tag{49}$$

Now,

$$\sum_{k=1}^{n} k^2 (k + 1) = \sum_{k=1}^{n} k^3 + k^2 \tag{50}$$

$$= \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2 \tag{51}$$

$$= \left(\frac{n(n + 1)}{2}\right)^2 + \frac{n(n + 1)(2n + 1)}{6} \tag{52}$$

$$= \frac{n(n + 1)}{2}\left[\frac{n(n + 1)}{2} + \frac{2n + 1}{3}\right] \tag{53}$$

$$= \frac{n(n + 1)}{2}\left[\frac{3n^2 + 7n + 2}{6}\right] \tag{54}$$

$$= \frac{n(n + 1)}{12}\left[3n^2 + 6n + n + 2\right] \tag{55}$$

$$= \frac{n(n + 1)}{12}\left[3n(n + 2) + (n + 2)\right] \tag{56}$$

$$= \frac{n(n + 1)(n + 2)(3n + 1)}{12} \tag{57}$$

$$\frac{n(n + 1)(n + 2)(3n + 1)}{12} \cdot u(n) = y(n) \tag{58}$$

$$y(n) = \begin{cases} 0 & \text{for } n < 0 \\ \frac{n(n+1)(n+2)(3n+1)}{12}. & \text{for } n \geq 0 \end{cases} \tag{59}$$

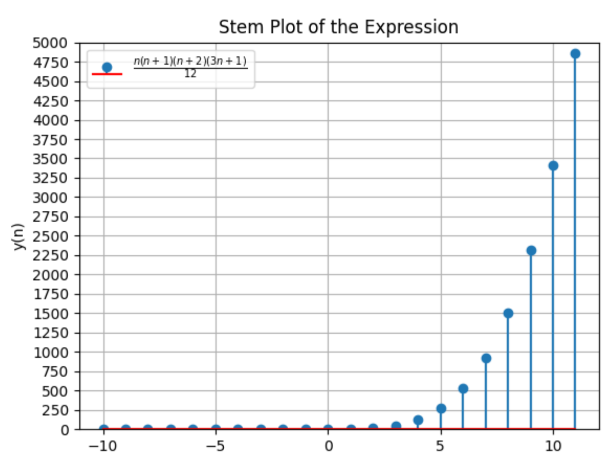

Fig. 2: Stem Plot of $y(n)$

$$\frac{\sum_{k=1}^{n} k(k+1)^2}{\sum_{k=1}^{n} k^2(k+1)} = \frac{\frac{n(n+1)(n+2)(3n+5)}{12}}{\frac{n(n+1)(n+2)(3n+1)}{12}} \quad (60)$$

$$= \frac{3n+5}{3n+1} \quad (61)$$

Derivation of z-transform of the denominator:

$$y(n) = \frac{n(n+1)(n+2)(3n+1)}{12} \cdot u(n) \quad (62)$$

$$= \frac{3n^4 + 10n^3 + 9n^2 + 2n}{12} \cdot u(n) \quad (63)$$

$$Y(z) = \sum_{n=-\infty}^{\infty} y(n) \cdot z^{-n} \quad (64)$$

We can say that $Y_i(z)$ denote the z-transform of $n^i u(n)$ which will be same as respective $X_i(z)$.

$$X_i = Y_i \quad (65)$$

$$Y_k(z) = (-z)^k \frac{d^k U(z)}{dz^k} \quad (66)$$

Now substituting k=1,2,3,4 we get :

$$Y_1(z) = \frac{z^{-1}}{(1-z^{-1})^2},\ ROC = |z| > 1 \quad (67)$$

$$Y_2(z) = \frac{z^{-1}\left(z^{-1}+1\right)}{(1-z^{-1})^3},\ ROC = |z| > 1 \quad (68)$$

$$Y_3(z) = \frac{z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{(1-z^{-1})^4},\ ROC = |z| > 1 \quad (69)$$

$$Y_4(z) = \frac{z^{-1}\left(1+11z^{-1}+11z^{-2}+z^{-3}\right)}{(1-z^{-1})^5},\ ROC = |z| > 1 \quad (70)$$

$$Y(z) = \frac{1}{12}\left(3Y_4(z) + 10Y_3(z) + 9Y_2(z) + 2Y_1(z)\right) \quad (71)$$

$$= \frac{3}{12}\left(\frac{z^{-1}\left(1+11z^{-1}+11z^{-2}+z^{-3}\right)}{(1-z^{-1})^5}\right) \quad (72)$$

$$+ \frac{10}{12}\left(\frac{z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{(1-z^{-1})^4}\right) + \frac{9}{12}\left(\frac{z^{-1}\left(z^{-1}+1\right)}{(1-z^{-1})^3}\right)$$

$$+ \frac{2}{12}\left(\frac{z^{-1}}{(1-z^{-1})^2}.\right)$$

$$= \frac{2\left(z^{-1}+2z^{-2}\right)}{(1-z^{-1})^5} \quad (73)$$

$Y(z)$ is a linear combination of $Y_i(z)$ for $i = 1, 2, 3, 4$

$$Y(z) = a_1 Y_1(z) + a_2 Y_2(z) + a_3 Y_3(z) + a_4 Y_4(z) \quad (74)$$

$$\mathrm{ROC}_Y = \mathrm{ROC}_1 \cap \mathrm{ROC}_2 \cap \mathrm{ROC}_3 \cap \mathrm{ROC}_4 \quad (75)$$

By equation(65) we can say that Y_i also has same ROC as respective X_i. Therefore,

$$\text{ROC of } Y(z) : |z| > 1 \quad (76)$$