NCERT 11.9.5 26Q

EE23BTECH11015 - DHANUSH V NAYAK*

Question: Show that

$$\frac{1\times 2^2+2\times 3^2+\cdots+n\times(n+1)^2}{1^2\times 2+2^2\times 3+\cdots+n^2\times(n+1)}=\frac{3n+5}{3n+1}$$

Solution:

Parameter	Description	Value
n	Integer	.... -2,-1,0,1, 2, ...
$x_1(n)$	General term of Numerator	$\left(n^3+5n^2+8n+4\right)\cdot u(n)$
$x_2(n)$	General Term of Denominator	$\left(n^3+4n^2+5n+2\right)\cdot u(n)$
$y_1(n)$	Sum of terms of numerator	?
$y_2(n)$	Sum of terms of denominator	?
$U(z)$	z-transform of $u(n)$	$\frac{1}{1-z^{-1}},\{z\in\mathbb{C}:\lvert z\rvert>1\}$
ROC	Region of convergence	$\left\{z:\left\lvert\sum_{n=-\infty}^{\infty}x(n)z^{-n}\right\rvert<\infty\right\}$

TABLE 1: Parameter Table

1. Analysis of Numerator:

By the differentiation property :

$$nx(n)\overset{\mathcal{Z}}{\longleftrightarrow}(-z)\frac{dX(z)}{dz}\tag{1}$$

$$\implies nu(n)\overset{\mathcal{Z}}{\longleftrightarrow}\frac{z^{-1}}{(1-z^{-1})^2},\{z\in\mathbb{C}:|z|>1\}\tag{2}$$

$$\implies n^2u(n)\overset{\mathcal{Z}}{\longleftrightarrow}\frac{z^{-1}\left(z^{-1}+1\right)}{(1-z^{-1})^3},\{z\in\mathbb{C}:|z|>1\}\tag{3}$$

$$\implies n^3u(n)\overset{\mathcal{Z}}{\longleftrightarrow}\frac{z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{(1-z^{-1})^4},\{z\in\mathbb{C}:|z|>1\}\tag{4}$$

$$\implies n^4u(n)\overset{\mathcal{Z}}{\longleftrightarrow}\frac{z^{-1}\left(1+11z^{-1}+11z^{-2}+z^{-3}\right)}{(1-z^{-1})^5}\tag{5}$$

where $\{z\in\mathbb{C}:|z|>1\}$

$$X_1(z)=\sum_{n=-\infty}^{\infty}x_1(n)z^{-n}\tag{6}$$

$$=\sum_{n=-\infty}^{\infty}\left(n^3+5n^2+8n+4\right)u(n)z^{-n}\tag{7}$$

Using results of equations (2) to (5) we get:

$$\therefore X_1(z)=\frac{4+2z^{-1}}{(1-z^{-1})^4},\{z\in\mathbb{C}:|z|>1\}\tag{8}$$

$$y_1(n)=\sum_{k=0}^{n}x_1(n)\tag{9}$$

$$y_1(n)=x_1(n)*u(n)\tag{10}$$

$$Y_1(z)=X_1(z)U(z)\tag{11}$$

$$=\frac{4+2z^{-1}}{(1-z^{-1})^5},\{z\in\mathbb{C}:|z|>1\}\tag{12}$$

Using partial fractions:

$$Y_1(z)=\frac{22z^{-1}}{(1-z^{-1})}+\frac{48z^{-2}}{(1-z^{-1})^2}+\frac{52z^{-3}}{(1-z^{-3})^3}\tag{13}$$

$$+\frac{28z^{-4}}{(1-z^{-1})^4}+\frac{6z^{-5}}{(1-z^{-1})^5}+4$$

$$\frac{z^{-1}}{(1-z^{-1})}=\frac{1}{1-z^{-1}}-1\tag{14}$$

Taking the inverse z-transform of each term:

$$u(n-1)\overset{\mathcal{Z}}{\longleftrightarrow}\frac{z^{-1}}{(1-z^{-1})}\tag{15}$$

Time shifting property:

$$x(n-k)\overset{\mathcal{Z}}{\longleftrightarrow}z^{-k}X(z)\tag{16}$$

From equation (1):

$$nu(n-1)\overset{\mathcal{Z}}{\longleftrightarrow}z\frac{2z^{-2}}{(1-z^{-1})^2}\tag{17}$$

By equation (16):

$$\frac{(n-1)}{2}u(n-2) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-2}}{(1-z^{-1})^2} \tag{18}$$

$$\frac{(n-1)(n-2)}{6}u(n-3) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-3}}{(1-z^{-1})^3} \tag{19}$$

$$\vdots$$

$$\frac{(n-1)(n-2)\dots(n-k+1)}{(k-1)!}u(n-k) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-k}}{(1-z^{-1})^k} \tag{20}$$

In equation (20) u$(n-k)$ can be replaced by u$(n-1)$.
Substituting k=2,3,4,5 in equation (20):

$$Z^{-1}\left[\frac{z^{-2}}{(1-z^{-1})^2}\right] = (n-1)\,u(n-1) \tag{21}$$

$$Z^{-1}\left[\frac{z^{-3}}{(1-z^{-1})^3}\right] = \frac{(n-1)(n-2)}{2}u(n-1) \tag{22}$$

$$Z^{-1}\left[\frac{z^{-4}}{(1-z^{-1})^4}\right] = \frac{(n-1)(n-2)(n-3)}{6}u(n-1) \tag{23}$$

$$Z^{-1}\left[\frac{z^{-5}}{(1-z^{-1})^5}\right] = \frac{(n-1)(n-2)(n-3)(n-4)}{24} \tag{24}$$

$$u(n-1)$$

Substituting results of equation (21) to (24) in equation (13):

$$y_1(n) = \frac{3n^4+26n^3+81n^2+106n+48}{12}u(n) \tag{25}$$

2. <u>Analysis of Denominator:</u>

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2(n)\,z^{-n} \tag{26}$$

$$= \sum_{n=-\infty}^{\infty} \left(n^3+4n^2+5n+2\right)u(n)\,z^{-n} \tag{27}$$

Using results of equation (2) to (5) we get:

$$\therefore X_2(z) = \frac{2+4z^{-1}}{(1-z^{-1})^4}, \{z \in \mathbb{C} : |z| > 1\} \tag{28}$$

$$y_2(n) = \sum_{k=0}^{n} x_2(n) \tag{29}$$

$$y_2(n) = x_2(n) * u(n) \tag{30}$$

$$Y_2(z) = X_2(z)\,U(z) \tag{31}$$

$$= \frac{2+4z^{-1}}{(1-z^{-1})^5}, \{z \in \mathbb{C} : |z| > 1\} \tag{32}$$

Using partial fractions:

$$Y_2(z) = \frac{14z^{-1}}{(1-z^{-1})} + \frac{36z^{-2}}{(1-z^{-1})^2} + \frac{44z^{-3}}{(1-z^{-3})^3} \tag{33}$$

$$+ \frac{26z^{-4}}{(1-z^{-1})^4} + \frac{6z^{-5}}{(1-z^{-1})^5} + 2$$

Substituting results of equation (21) to (24) in equation (33):

$$y_2(n) = \frac{3n^4+22n^3+57n^2+62n+24}{12} \tag{34}$$

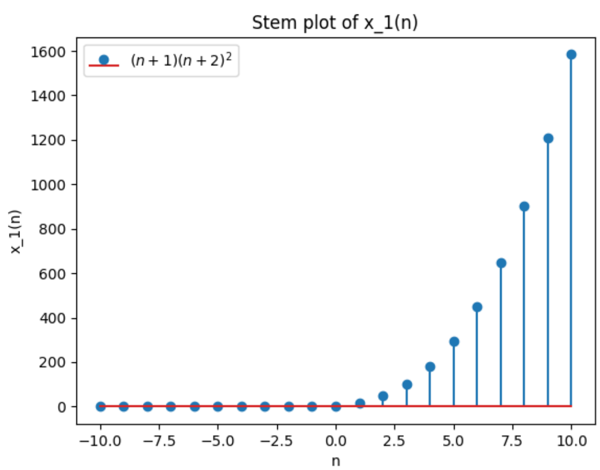

Fig. 1: Stem Plot of $x_1(n)$

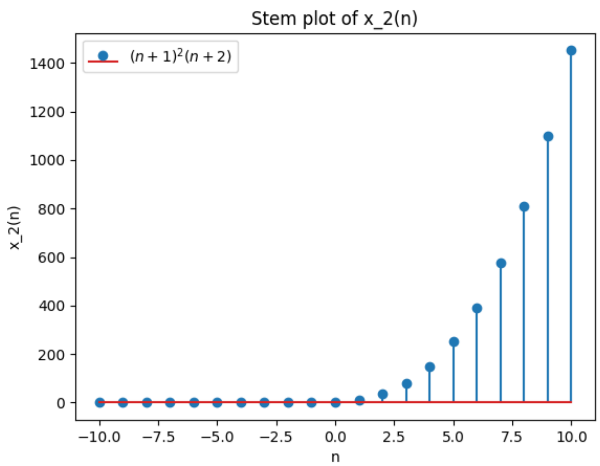

Fig. 2: Stem Plot of $x_2(n)$

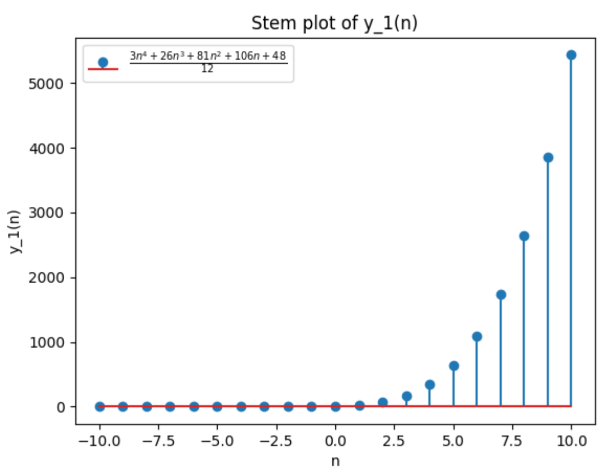

Fig. 3: Stem Plot of $y_1(n)$

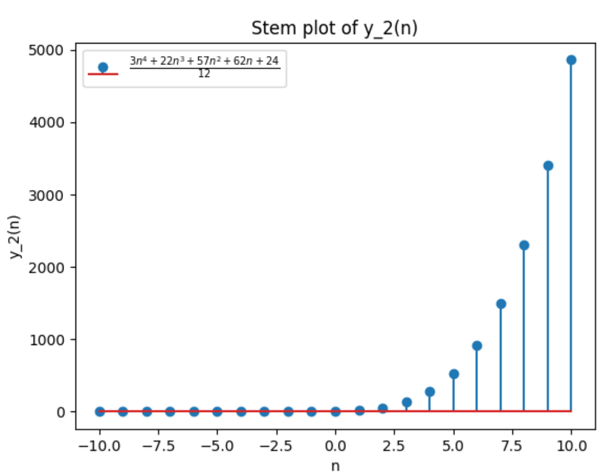

Fig. 4: Stem Plot of $y_2(n)$