

NCERT 11.9.5 26Q

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Question: Show that

$$\frac{1 \times 2^2 + 2 \times 3^2 + \dots + n \times (n+1)^2}{1^2 \times 2 + 2^2 \times 3 + \dots + n^2 \times (n+1)} = \frac{3n+5}{3n+1}$$

Solution:

Parameter	Description	Value
n	Integer -2,-1,0,1, 2, ...
$x_1(n)$	General term of Numerator	$(n^3 + 5n^2 + 8n + 4) \cdot u(n)$
$x_2(n)$	General Term of Denominator	$(n^3 + 4n^2 + 5n + 2) \cdot u(n)$
$y_1(n)$	Sum of terms of numerator	?
$y_2(n)$	Sum of terms of denominator	?
$U(z)$	z-transform of $u(n)$	$\frac{1}{1-z^{-1}}, \{z \in \mathbb{C} : z > 1\}$
ROC	Region of convergence	$\{z : \sum_{n=-\infty}^{\infty} x(n)z^{-n} < \infty\}$

TABLE 1: Parameter Table

1. Analysis of Numerator:

By the differentiation property :

$$\Rightarrow nu(n) \xleftrightarrow{z} (-z) \frac{dU(z)}{dz} \{z \in \mathbb{C} : |z| > 1\} \quad (1)$$

Therefore,

$$nu(n) \xleftrightarrow{z} \frac{z^{-1}}{(1-z^{-1})^2}, \{z \in \mathbb{C} : |z| > 1\} \quad (2)$$

$$n^2u(n) \xleftrightarrow{z} \frac{z^{-1}(z^{-1}+1)}{(1-z^{-1})^3}, \{z \in \mathbb{C} : |z| > 1\} \quad (3)$$

$$n^3u(n) \xleftrightarrow{z} \frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}, \{z \in \mathbb{C} : |z| > 1\} \quad (4)$$

$$n^4u(n) \xleftrightarrow{z} \frac{z^{-1}(1+11z^{-1}+11z^{-2}+z^{-3})}{(1-z^{-1})^5} \quad (5)$$

where $\{z \in \mathbb{C} : |z| > 1\}$

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n) u(n) z^{-n} \quad (6)$$

$$= \sum_{n=-\infty}^{\infty} (n^3 + 5n^2 + 8n + 4) u(n) z^{-n} \quad (7)$$

Using results of equation(2) to (5) we get:

$$\therefore X_1(z) = \frac{4 + 2z^{-1}}{(1-z^{-1})^4}, \{z \in \mathbb{C} : |z| > 1\} \quad (8)$$

$$y_1(n) = \sum_{k=0}^n x_1(k) \quad (9)$$

$$y_1(n) = x_1(n) * u(n) \quad (10)$$

$$Y_1(z) = X_1(z) U(z) \quad (11)$$

$$= \frac{4 + 2z^{-1}}{(1-z^{-1})^5}, \{z \in \mathbb{C} : |z| > 1\} \quad (12)$$

Using partial fractions:

$$Y_1(z) = \frac{22z^{-1}}{(1-z^{-1})} + \frac{48z^{-2}}{(1-z^{-1})^2} + \frac{52z^{-3}}{(1-z^{-1})^3} \quad (13)$$

$$+ \frac{28z^{-4}}{(1-z^{-1})^4} + \frac{6z^{-5}}{(1-z^{-1})^5} + 4$$

$$\frac{z^{-1}}{(1-z^{-1})} = \frac{1}{1-z^{-1}} - 1 \quad (14)$$

Taking the inverse z-transform of each term :

$$u(n-1) \xleftrightarrow{z} \frac{z^{-1}}{(1-z^{-1})} \quad (15)$$

Time shifting property:

$$x(n-k) \xleftrightarrow{z} z^{-k} X(z) \quad (16)$$

From equation(1):

$$nu(n-1) \xleftrightarrow{z} z \frac{2z^{-2}}{(1-z^{-1})^2} \quad (17)$$

By equation(16)

$$\frac{(n-1)}{2}u(n-2) \xleftrightarrow{Z} \frac{z^{-2}}{(1-z^{-1})^2} \quad (18)$$

$$\frac{(n-1)(n-2)}{6}u(n-3) \xleftrightarrow{Z} \frac{z^{-3}}{(1-z^{-1})^3} \quad (19)$$

⋮

$$\frac{(n-1)(n-2)\dots(n-k+1)}{(k-1)!}u(n-k) \xleftrightarrow{Z} \frac{z^{-k}}{(1-z^{-1})^k} \quad (20)$$

In equation(20) $u(n-k)$ can be replaced by $u(n-1)$.

Substituting $k=2,3,4,5$ in equation(20):

$$Z^{-1} \left[\frac{z^{-2}}{(1-z^{-1})^2} \right] = (n-1)u(n-1) \quad (21)$$

$$Z^{-1} \left[\frac{z^{-3}}{(1-z^{-1})^3} \right] = \frac{(n-1)(n-2)}{2}u(n-1) \quad (22)$$

$$Z^{-1} \left[\frac{z^{-4}}{(1-z^{-1})^4} \right] = \frac{(n-1)(n-2)(n-3)}{6}u(n-1) \quad (23)$$

$$Z^{-1} \left[\frac{z^{-5}}{(1-z^{-1})^5} \right] = \frac{(n-1)(n-2)(n-3)(n-4)}{24}u(n-1) \quad (24)$$

$$u(n-1)$$

Substituting results of equation(21) to (24) in equation(13):

$$y_1(n) = \frac{3n^4 + 26n^3 + 81n^2 + 106n + 48}{12}u(n) \quad (25)$$

2. Analysis of Denominator:

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2(n)u(n)z^{-n} \quad (26)$$

$$= \sum_{n=-\infty}^{\infty} (n^3 + 4n^2 + 5n + 2)u(n)z^{-n} \quad (27)$$

$$\therefore X_2(z) = \frac{2 + 4z^{-1}}{(1-z^{-1})^4}, \{z \in \mathbb{C} : |z| > 1\} \quad (28)$$

$$y_2(n) = \sum_{k=0}^n x_2(n) \quad (29)$$

$$y_2(n) = x_2(n) * u(n) \quad (30)$$

$$Y_2(z) = X_2(z)U(z) \quad (31)$$

$$= \frac{2 + 4z^{-1}}{(1-z^{-1})^5}, \{z \in \mathbb{C} : |z| > 1\} \quad (32)$$

Using partial fractions:

$$Y_2(z) = \frac{14z^{-1}}{(1-z^{-1})} + \frac{36z^{-2}}{(1-z^{-1})^2} + \frac{44z^{-3}}{(1-z^{-1})^3} + \frac{26z^{-4}}{(1-z^{-1})^4} + \frac{6z^{-5}}{(1-z^{-1})^5} + 2 \quad (33)$$

Substituting results of equation(21) to (24) in equation(33):

$$y_2(n) = \frac{3n^4 + 22n^3 + 57n^2 + 62n + 24}{12} \quad (34)$$

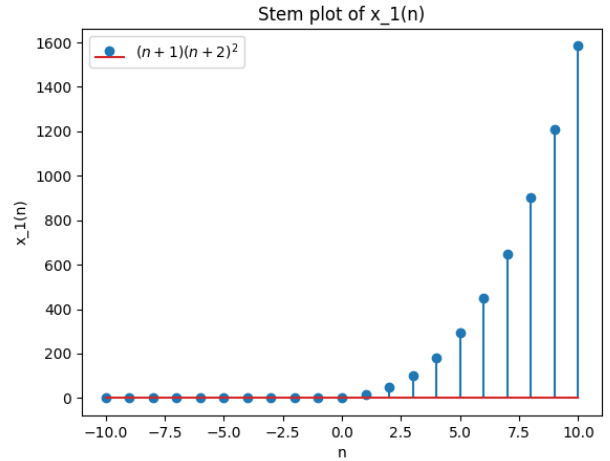
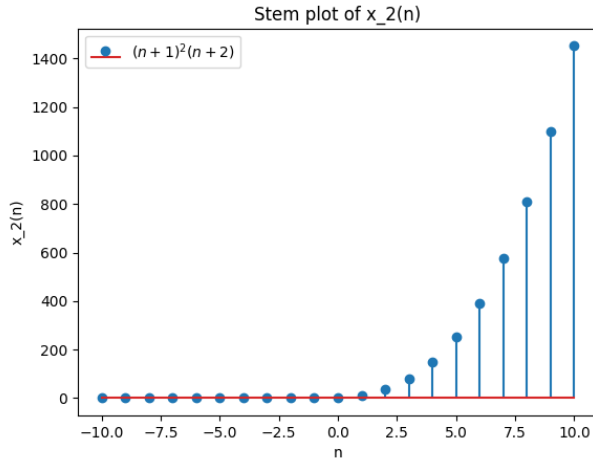
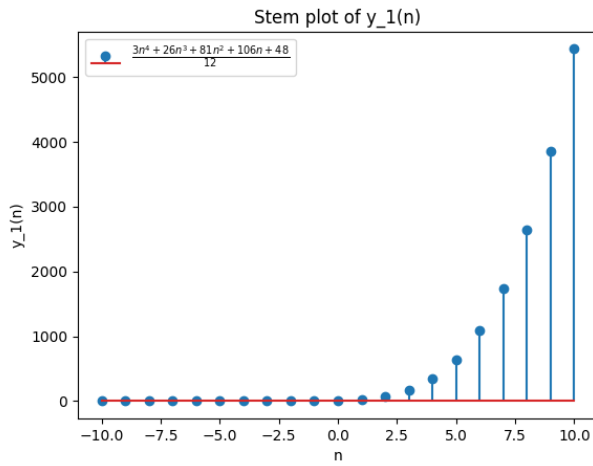
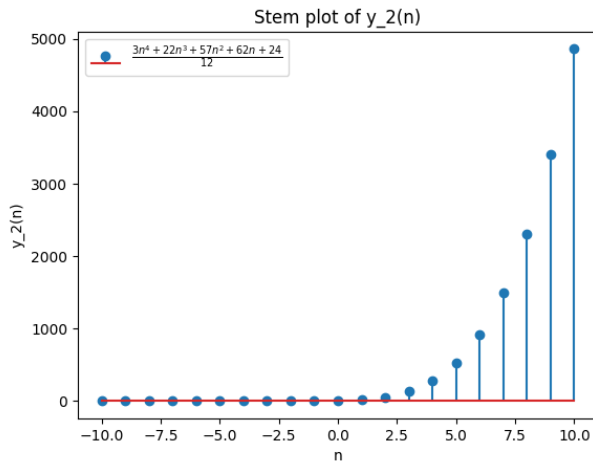


Fig. 1: Stem Plot of $x_1(n)$

Fig. 2: Stem Plot of $x_2(n)$ Fig. 3: Stem Plot of $y_1(n)$ Fig. 4: Stem Plot of $y_2(n)$