NCERT 11.9.5 26Q

EE23BTECH11015 - DHANUSH V NAYAK*

Question: Show that

$$\frac{1\times 2^2+2\times 3^2+\cdots+n\times(n+1)^2}{1^2\times 2+2^2\times 3+\cdots+n^2\times(n+1)}=\frac{3n+5}{3n+1}$$

Solution:

a) Consider the Numerator of the LHS part:

$$1\times2^2+2\times3^2+\cdots+n\times(n+1)^2=\sum_{k=1}^{n}k(k+1)^2 \quad (1)$$

Now,

$$\sum_{k=1}^{n}k(k+1)^2=\sum_{k=1}^{n}k^3+2k^2+k \quad (2)$$

$$=\sum_{k=1}^{n}k^3+\sum_{k=1}^{n}2k^2+\sum_{k=1}^{n}k \quad (3)$$

$$=\sum_{k=1}^{n}k^3+2\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k \quad (4)$$

$$=\left(\frac{n(n+1)}{2}\right)^2+2\cdot\frac{n(n+1)(2n+1)}{6}+ \quad (5)$$

$$+\frac{n(n+1)}{2}$$

$$=\frac{n(n+1)}{2}\left[\frac{n(n+1)}{2}+\frac{2}{3}(2n+1)+1\right] \quad (6)$$

$$=\frac{n(n+1)}{2}\left[\frac{3n^2+11n+10}{6}\right] \quad (7)$$

$$=\frac{n(n+1)}{12}\left[3n^2+6n+5n+10\right] \quad (8)$$

$$=\frac{n(n+1)}{12}\left[3n(n+2)+5(n+2)\right] \quad (9)$$

$$=\frac{n(n+1)(n+2)(3n+5)}{12} \quad (10)$$

Therefore,

$$1\times2^2+2\times3^2+\cdots+n\times(n+1)^2=\frac{n(n+1)(n+2)(3n+5)}{12} \quad (11)$$

$$\frac{n(n+1)(n+2)(3n+5)}{12}\cdot u[n]=x[n] \quad (12)$$

Derivation of z-transform of numerator :

$$x[n]=\frac{n(n+1)(n+2)(3n+5)}{12}\cdot u[n] \quad (13)$$

$$=\frac{3n^4+14n^3+21n^2+10n}{12}\cdot u[n] \quad (14)$$

$$X(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n} \quad (15)$$

Z-Transform is calculated as:

$$X(z)=\sum_{n=-\infty}^{\infty}x[n]\cdot z^{-n} \quad (16)$$

$$12X(z)=\sum_{n=0}^{\infty}3n^4u[n]z^{-n}+\sum_{n=0}^{\infty}14n^3u[n]z^{-n} \quad (17)$$

$$+\sum_{n=0}^{\infty}21n^2u[n]z^{-n}+\sum_{n=0}^{\infty}10nu[n]\,z^{-n}$$

Let $U(z)$ be z transform of $x[n]=u[n]$.

$$U(z)=\sum_{n=0}^{\infty}u[n]\cdot z^{-n} \quad (18)$$

$$=\sum_{n=0}^{\infty}(1)\cdot z^{-n} \quad (19)$$

$$=1+z^{-1}+z^{-2}+\ldots \quad (20)$$

$$=\frac{1}{1-z^{-1}} \quad (21)$$

$$=\frac{z}{z-1} \quad (22)$$

By the differentiation property:
Here , x[n]=u[n] and $X(z)=U(z)$

$$nx[n]\xleftrightarrow{Z}-z\frac{dX(z)}{dz}, \quad (23)$$

$$nu[n]\xleftrightarrow{Z}-z\frac{dX(z)}{dz}, \quad (24)$$

$$\frac{dX(z)}{dz}=-\frac{1}{(z-1)^2}, \quad (25)$$

$$nu(n)\xleftrightarrow{Z}\frac{z}{(z-1)^2}. \quad (26)$$

Therefore, the Z-transform of n$\cdot u[n]$,

$$X_1(z) = \frac{z^{-1}}{(1-z^{-1})^2}. \tag{27}$$

The Region of Convergence (ROC) is defined as the set of points in the complex plane for which the Z-transform summation converges, i.e., doesn't blow up in magnitude to infinity: The Region of Convergence (ROC) is defined as:

$$\text{ROC} = \left\{ z : \left| \sum_{n=-\infty}^{\infty} x[n]z^{-n} \right| < \infty \right\} \tag{28}$$

Equation (28) expresses the set of points in the complex plane for which the Z-transform summation converges.

The Z-transform of the unit step signal $u[n]$ is given by:

$$U(z) = \sum_{n=-\infty}^{\infty} u[n]z^{-n} \tag{29}$$

Since $u[n] = 1$ for $n \geq 0$ and $u[n] = 0$ for $n < 0$, the summation simplifies to:

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{30}$$

$$\text{ROC} = \left\{ z : \left| \sum_{n=0}^{\infty} z^{-n} \right| < \infty \right\} \tag{31}$$

This is an infinite GP. And it converges only if $|r| < 1$ .And here , $|r| = |z^{-1}|$ Therefore, the Region of Convergence (ROC) for this Z-transform is:

$$\text{ROC: } |z| > 1 \tag{32}$$

In the differentiation property, ROC does not change:

$$x[n] \overset{Z}{\longleftrightarrow} X(z),\ ROC = R \tag{33}$$

$$nx[n] \overset{Z}{\longleftrightarrow} -z\frac{dX(z)}{dz},\ ROC = R \tag{34}$$

Therefore,

$$X_1(z) = \frac{z^{-1}}{(1-z^{-1})^2},\ ROC = |z| > 1 \tag{35}$$

$$X_2(z) = \frac{z^{-1}(z^{-1}+1)}{(1-z^{-1})^3},\ ROC = |z| > 1 \tag{36}$$

$$X_3(z) = \frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4},\ ROC = |z| > 1 \tag{37}$$

$$X_4(z) = \frac{z^{-1}(1+11z^{-1}+11z^{-2}+z^{-3})}{(1-z^{-1})^5},\ ROC = |z| > 1 \tag{38}$$

Equation(17) can be now written as:

$$X(z) = \frac{1}{12}(3X_4(z) + 14X_3(z) + 21X_2(z) + 10X_1(z)) \tag{39}$$

$$= \frac{3}{12}\left(\frac{z^{-1}(1+11z^{-1}+11z^{-2}+z^{-3})}{(1-z^{-1})^5}\right) \tag{40}$$

$$+ \frac{14}{12}\left(\frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}\right) + \frac{21}{12}\left(\frac{z^{-1}(z^{-1}+1)}{(1-z^{-1})^3}\right)$$

$$+ \frac{10}{12}\left(\frac{z^{-1}}{(1-z^{-1})^2}.\right)$$

$$= \frac{2(z^{-2}+2z^{-1})}{(1-z^{-1})^5} \tag{41}$$

Consider the linear combination of two signals in the time domain:

$$a_1x_1(n) + a_2x_2(n) = a_1X_1(z) + a_2X_2(z) \tag{42}$$

$$\text{ROC} = \text{ROC}_1 \cap \text{ROC}_2 \tag{43}$$

Therefore, ROC of equation(17) is intersection of ROC of each signal. Every signal has ROC $|z| > 1$.So,

$$\text{ROC of } X(z) \text{ is } |z| > 1 \tag{44}$$

b) Consider the Denominator of the RHS part:

$$1^2\times2+2^2\times3+\cdots+n^2\times(n+1) = \sum_{k=1}^{n} k^2(k+1) \tag{45}$$

Now,

$$\sum_{k=1}^{n} k^2(k+1) = \sum_{k=1}^{n} k^3 + k^2 \tag{46}$$

$$= \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2 \tag{47}$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + \frac{n(n+1)(2n+1)}{6} \tag{48}$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + \frac{2n+1}{3}\right] \tag{49}$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2+7n+2}{6}\right] \tag{50}$$

$$= \frac{n(n+1)}{12}\left[3n^2+6n+n+2\right] \tag{51}$$

$$= \frac{n(n+1)}{12}\left[3n(n+2)+(n+2)\right] \tag{52}$$

$$= \frac{n(n+1)(n+2)(3n+1)}{12} \tag{53}$$

$$\frac{n(n+1)(n+2)(3n+1)}{12} \cdot u[n] = y[n] \tag{54}$$

Therefore,

$$\frac{\sum_{k=1}^{n} k(k+1)^2}{\sum_{k=1}^{n} k^2(k+1)} = \frac{\frac{n(n+1)(n+2)(3n+5)}{12}}{\frac{n(n+1)(n+2)(3n+1)}{12}} \tag{55}$$

$$= \frac{3n+5}{3n+1} \tag{56}$$

Derivation of z-transform of the denominator:

$$y[n] = \frac{n(n+1)(n+2)(3n+1)}{12} \cdot u[n] \tag{57}$$

$$= \frac{3n^4 + 10n^3 + 9n^2 + 2n}{12} \cdot u[n] \tag{58}$$

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n] \cdot z^{-n} \tag{59}$$

Z-Transform is calculated as:

$$12Y(z) = \sum_{n=0}^{\infty} 3n^4 u[n] z^{-n} + \sum_{n=0}^{\infty} 10n^3 u[n] z^{-n} + \sum_{n=0}^{\infty} 9n^2 u[n] z^{-n} + \sum_{n=0}^{\infty} 2n\, u[n] z^{-n} \tag{60}$$

We can say that Y_i denote the z-transform of n^i which will be same as respective X_i.

$$X_i = Y_i \tag{61}$$

Equation(60) can be now written as:

$$Y(z) = \frac{1}{12}(3Y_4(z) + 10Y_3(z) + 9Y_2(z) + 2Y_1(z)) \tag{62}$$

$$= \frac{3}{12}\left(\frac{z^{-1}(1 + 11z^{-1} + 11z^{-2} + z^{-3})}{(1-z^{-1})^5}\right) + \frac{10}{12}\left(\frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}\right) + \frac{9}{12}\left(\frac{z^{-1}(z^{-1}+1)}{(1-z^{-1})^3}\right) + \frac{2}{12}\left(\frac{z^{-1}}{(1-z^{-1})^2}.\right) \tag{63}$$

$$= \frac{2(z^{-1} + 2z^{-2})}{(1-z^{-1})^5} \tag{64}$$

$Y(z)$ is a linear combination of $Y_i(z)$ for $i = 1, 2, 3, 4$:

$$Y(z) = a_1 Y_1(z) + a_2 Y_2(z) + a_3 Y_3(z) + a_4 Y_4(z) \tag{65}$$

$$\mathrm{ROC}_Y = \mathrm{ROC}_1 \cap \mathrm{ROC}_2 \cap \mathrm{ROC}_3 \cap \mathrm{ROC}_4 \tag{66}$$

By equation(61) we can say that Y_i also has same ROC as respective X_i. Therefore,

$$\text{ROC of } Y(z) : |z| > 1 \tag{67}$$