NCERT 11.9.5 26Q

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Question: Show that

$$\frac{1 \times 2^2 + 2 \times 3^2 + \cdots + n \times (n+1)^2}{1^2 \times 2 + 2^2 \times 3 + \cdots + n^2 \times (n+1)} = \frac{3n+5}{3n+1}$$

Solution:

a) Consider the Numerator of the LHS part:

$$1 \times 2^2 + 2 \times 3^2 + \cdots + n \times (n+1)^2 = \sum_{k=1}^{n} k(k+1)^2$$

Now,

$$\sum_{k=1}^{n} k(k+1)^2 = \sum_{k=1}^{n} k^3 + 2k^2 + k$$

$$= \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} 2k^2 + \sum_{k=1}^{n} k$$

$$= \sum_{k=1}^{n} k^3 + 2\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + 2 \cdot \frac{n(n+1)(2n+1)}{6} +$$

$$+ \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + \frac{2}{3}(2n+1) + 1\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2 + 11n + 10}{6}\right]$$

$$= \frac{n(n+1)}{12}\left[3n^2 + 6n + 5n + 10\right]$$

$$= \frac{n(n+1)}{12}\left[3n(n+2) + 5(n+2)\right]$$

$$= \frac{n(n+1)(n+2)(3n+5)}{12}$$

Therefore,

$$1 \times 2^2 + 2 \times 3^2 + \cdots + n \times (n+1)^2 = \frac{n(n+1)(n+2)(3n+5)}{12}$$

$$\frac{n(n+1)(n+2)(3n+5)}{12} \cdot u[n] = x[n]$$

Derivation of z-transform of numerator :

$$x[n] = \frac{n(n+1)(n+2)(3n+5)}{12} \cdot u[n]$$

$$= \frac{3n^4 + 14n^3 + 21n^2 + 10n}{12} \cdot u[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \qquad (1)$$

Z-Transform is calculated as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$$

$$12X(z) = \sum_{n=0}^{\infty} 3n^4 u[n]z^{-n} + \sum_{n=0}^{\infty} 14n^3 u[n]z^{-n}$$

$$\qquad (2)$$

$$+ \sum_{n=0}^{\infty} 21n^2 u[n]z^{-n} + \sum_{n=0}^{\infty} 10n u[n] \, z^{-n}$$

Let $X_0(z)$ be z transform of $x[n] = u[n]$.

$$X_0(z) = \sum_{n=0}^{\infty} u[n] \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} (1) \cdot z^{-n}$$

$$= 1 + z^{-1} + z^{-2} + \ldots$$

$$= \frac{1}{1 - z^{-1}}$$

$$= \frac{z}{z - 1}$$

By the differentiation property :

$$nx[n] \xleftrightarrow{z} -z\frac{dX(z)}{dz},$$

$$nu[n] \xleftrightarrow{z} -z\frac{dX(z)}{dz},$$

$$\frac{dX(z)}{dz} = -\frac{1}{(z-1)^2},$$

$$nu(n) \xleftrightarrow{z} \frac{z}{(z-1)^2}.$$

Therefore, the Z-transform of $n \cdot u[n]$,

$$X_1(z) = \frac{z}{(z-1)^2}. \qquad (3)$$

By using the same property we can compute the z transform of other powers of n. Here,the z-transforms of $n^i \cdot u[n]$ are denoted as $X_i(z)$:

$$X_2(z) = \frac{z^{-1}(z^{-1}+1)}{(1-z^{-1})^3} \qquad (4)$$

$$X_3(z) = \frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4} \qquad (5)$$

$$X_4(z) = \frac{z^{-1}(1+11z^{-1}+11z^{-2}+z^{-3})}{(1-z^{-1})^5} \qquad (6)$$

Equation(2) can be now written as:

$$
\begin{aligned}
X(z) &= \frac{1}{12}(3X_4(z) + 14X_3(z) + 21X_2(z) + 10X_1(z)) \\
&= \frac{3}{12}\left(\frac{z^{-1}(1+11z^{-1}+11z^{-2}+z^{-3})}{(1-z^{-1})^5}\right) \\
&\quad + \frac{14}{12}\left(\frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}\right) + \frac{21}{12}\left(\frac{z^{-1}(z^{-1}+1)}{(1-z^{-1})^3}\right) \\
&\quad + \frac{10}{12}\left(\frac{z}{(z-1)^2}\right) \\
&= \frac{2(z^{-2}+2z^{-1})}{(1-z^{-1})^5}
\end{aligned}
$$

b) Consider the Denominator of the RHS part:

$$1^2 \times 2 + 2^2 \times 3 + \cdots + n^2 \times (n+1) = \sum_{k=1}^{n} k^2(k+1)$$

Now,

$$
\begin{aligned}
\sum_{k=1}^{n} k^2(k+1) &= \sum_{k=1}^{n} k^3 + k^2 \\
&= \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2 \\
&= \left(\frac{n(n+1)}{2}\right)^2 + \frac{n(n+1)(2n+1)}{6} \\
&= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + \frac{2n+1}{3}\right] \\
&= \frac{n(n+1)}{2}\left[\frac{3n^2+7n+2}{6}\right] \\
&= \frac{n(n+1)}{12}\left[3n^2+6n+n+2\right] \\
&= \frac{n(n+1)}{12}\left[3n(n+2)+(n+2)\right] \\
&= \frac{n(n+1)(n+2)(3n+1)}{12}
\end{aligned}
$$

$$\frac{n(n+1)(n+2)(3n+1)}{12} \cdot u[n] = y[n]$$

Therefore,

$$
\begin{aligned}
\frac{\sum_{k=1}^{n} k(k+1)^2}{\sum_{k=1}^{n} k^2(k+1)} &= \frac{\frac{n(n+1)(n+2)(3n+5)}{12}}{\frac{n(n+1)(n+2)(3n+1)}{12}} \\
&= \frac{3n+5}{3n+1}
\end{aligned}
$$

Derivation of z-transform of the denominator:

$$
\begin{aligned}
y[n] &= \frac{n(n+1)(n+2)(3n+1)}{12} \cdot u[n] \\
&= \frac{3n^4 + 10n^3 + 9n^2 + 2n}{12} \cdot u[n]
\end{aligned}
$$

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n] \cdot z^{-n} \qquad (7)$$

Z-Transform is calculated as:

$$12Y(z) = \sum_{n=0}^{\infty} 3n^4 u[n]z^{-n} + \sum_{n=0}^{\infty} 10n^3 u[n]z^{-n} \qquad (8)$$

$$+ \sum_{n=0}^{\infty} 9n^2 u[n]z^{-n} + \sum_{n=0}^{\infty} 2n\, u[n]z^{-n}$$

We can say that Y_i denote the z-transform of n^i which will be same as respective X_i.

$$X_i = Y_i$$

Equation(8) can be now written as:

$$
\begin{aligned}
Y(z) &= \frac{1}{12}(3Y_4(z) + 10Y_3(z) + 9Y_2(z) + 2Y_1(z)) \\
&= \frac{3}{12}\left(\frac{z^{-1}(1 + 11z^{-1} + 11z^{-2} + z^{-3})}{(1 - z^{-1})^5}\right) \\
&\quad + \frac{10}{12}\left(\frac{z^{-1}(1 + 4z^{-1} + z^{-2})}{(1 - z^{-1})^4}\right) + \frac{9}{12}\left(\frac{z^{-1}(z^{-1} + 1)}{(1 - z^{-1})^3}\right) \\
&\quad + \frac{2}{12}\left(\frac{z}{(z - 1)^2}\right) \\
&= \frac{2(z^{-1} + 2z^{-2})}{(1 - z^{-1})^5}
\end{aligned}
$$