NCERT 11.9.5 26Q

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Question: Show that

$$\frac{1\times 2^2 + 2\times 3^2 + \cdots + n\times(n+1)^2}{1^2\times 2 + 2^2\times 3 + \cdots + n^2\times(n+1)} = \frac{3n+5}{3n+1}$$

Solution:

a) Consider the Numerator of the LHS part:

$$1\times2^2+2\times3^2+\cdots+n\times(n+1)^2 = \sum_{k=1}^{n} k(k+1)^2 \quad (1)$$

Now,

$$\sum_{k=1}^{n} k(k+1)^2 = \sum_{k=1}^{n} k^3 + 2k^2 + k \quad (2)$$

$$= \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} 2k^2 + \sum_{k=1}^{n} k \quad (3)$$

$$= \sum_{k=1}^{n} k^3 + 2\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k \quad (4)$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + 2\cdot\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \quad (5)$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + \frac{2}{3}(2n+1) + 1\right] \quad (6)$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2+11n+10}{6}\right] \quad (7)$$

$$= \frac{n(n+1)}{12}\left[3n^2+6n+5n+10\right] \quad (8)$$

$$= \frac{n(n+1)}{12}\left[3n(n+2)+5(n+2)\right] \quad (9)$$

$$= \frac{n(n+1)(n+2)(3n+5)}{12} \quad (10)$$

Therefore,

$$1\times2^2+2\times3^2+\cdots+n\times(n+1)^2 = \frac{n(n+1)(n+2)(3n+5)}{12} \quad (11)$$

$$\frac{n(n+1)(n+2)(3n+5)}{12}\cdot u[n] = x[n] \quad (12)$$

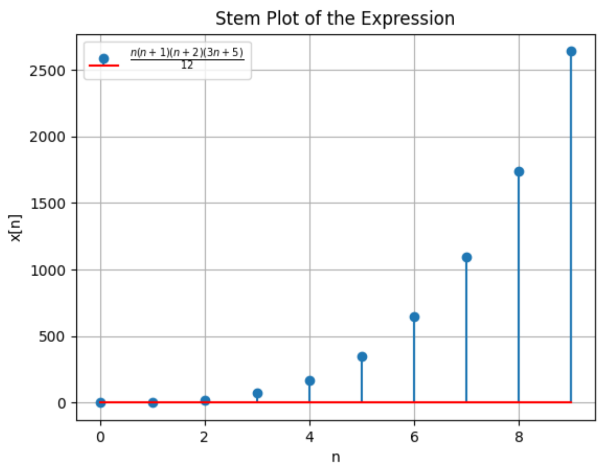

Fig. a). Stem Plot of x[n]

Derivation of z-transform of numerator :

$$x[n] = \frac{n(n+1)(n+2)(3n+5)}{12}\cdot u[n] \quad (13)$$

$$= \frac{3n^4+14n^3+21n^2+10n}{12}\cdot u[n] \quad (14)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \quad (15)$$

Z-Transform is calculated as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\cdot z^{-n} \quad (16)$$

$$X(z) = \sum_{n=-\infty}^{\infty} \frac{3n^4+14n^3+21^2+10n}{12} \quad (17)$$

$$12X(z) = \sum_{n=0}^{\infty} 3n^4z^{-n} + \sum_{n=0}^{\infty} 14n^3z^{-n} + \sum_{n=0}^{\infty} 21n^2z^{-n} + \sum_{n=0}^{\infty} 10n\, z^{-n} \quad (18)$$

Let $U(z)$ be z transform of $u[n]$.

$$U(z) = \sum_{n=0}^{\infty} u[n] \cdot z^{-n} \tag{19}$$
$$= \sum_{n=0}^{\infty} (1) \cdot z^{-n} \tag{20}$$
$$= 1 + z^{-1} + z^{-2} + \dots \tag{21}$$
$$= \frac{1}{1 - z^{-1}} \tag{22}$$

By the differentiation property:

$$x[n] \overset{Z}{\longleftrightarrow} X(z) \tag{23}$$
$$n^k x[n] \overset{Z}{\longleftrightarrow} (-z)^k \frac{d^k X(z)}{dz^k} \tag{24}$$
$$n^k u[n] \overset{Z}{\longleftrightarrow} (-z)^k \frac{d^k U(z)}{dz^k} \tag{25}$$

For the Z-transform of n $\cdot u[n]$, substituting k=1 in Equation (25):

$$n \cdot u[n] \overset{Z}{\longleftrightarrow} -z\frac{dX(z)}{dz} \tag{26}$$
$$\frac{dX(z)}{dz} = -\frac{1}{(z-1)^2}, \tag{27}$$
$$X_1(z) = \frac{z^{-1}}{(1-z^{-1})^2}. \tag{28}$$

Substituting k=2,3,4:

$$X_2(z) = \frac{z^{-1}(z^{-1}+1)}{(1-z^{-1})^3} \tag{29}$$
$$X_3(z) = \frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4} \tag{30}$$
$$X_4(z) = \frac{z^{-1}(1+11z^{-1}+11z^{-2}+z^{-3})}{(1-z^{-1})^5} \tag{31}$$

The Region of Convergence (ROC) is defined as the set of points in the complex plane for which the Z-transform summation converges, i.e., doesn't blow up in magnitude to infinity: The Region of Convergence (ROC) is defined as:

$$\text{ROC} = \left\{ z : \left| \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right| < \infty \right\} \tag{32}$$

Equation (32) expresses the set of points in the complex plane for which the Z-transform summation converges.

The Z-transform of the unit step signal $u[n]$ is given by:

$$U(z) = \sum_{n=-\infty}^{\infty} u[n] z^{-n} \tag{33}$$

Since $u[n] = 1$ for $n \geq 0$ and $u[n] = 0$ for $n < 0$, the summation simplifies to:

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{34}$$

$$\text{ROC} = \left\{ z : \left| \sum_{n=0}^{\infty} z^{-n} \right| < \infty \right\} \tag{35}$$

This is an infinite GP. And it converges only if $|r| < 1$ where r is the common ratio.And here , $|r| = |z^{-1}|$ Therefore, the Region of Convergence (ROC) for this Z-transform is:

$$\text{ROC: } |z| > 1 \tag{36}$$

In the differentiation property, ROC does not change:

$$x[n] \overset{Z}{\longleftrightarrow} X(z),\ ROC = R \tag{37}$$
$$n^k x[n] \overset{Z}{\longleftrightarrow} (-z)^k \frac{d^k X(z)}{dz^k},\ ROC = R \tag{38}$$

Therefore,

$$X_1(z) = \frac{z^{-1}}{(1-z^{-1})^2},\ ROC = |z| > 1 \tag{39}$$
$$X_2(z) = \frac{z^{-1}(z^{-1}+1)}{(1-z^{-1})^3},\ ROC = |z| > 1 \tag{40}$$
$$X_3(z) = \frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4},\ ROC = |z| > 1 \tag{41}$$
$$X_4(z) = \frac{z^{-1}(1+11z^{-1}+11z^{-2}+z^{-3})}{(1-z^{-1})^5},\ ROC = |z| > 1 \tag{42}$$

Equation(18) can be now written as:

$$X(z) = \frac{1}{12}(3X_4(z) + 14X_3(z) + 21X_2(z) + 10X_1(z)) \tag{43}$$

$$= \frac{3}{12}\left(\frac{z^{-1}(1 + 11z^{-1} + 11z^{-2} + z^{-3})}{(1 - z^{-1})^5}\right) \tag{44}$$

$$+ \frac{14}{12}\left(\frac{z^{-1}(1 + 4z^{-1} + z^{-2})}{(1 - z^{-1})^4}\right) + \frac{21}{12}\left(\frac{z^{-1}(z^{-1} + 1)}{(1 - z^{-1})^3}\right) + \frac{10}{12}\left(\frac{z^{-1}}{(1 - z^{-1})^2}.\right)$$

$$= \frac{2(z^{-2} + 2z^{-1})}{(1 - z^{-1})^5} \tag{45}$$

Consider the linear combination of two signals in the time domain:

$$a_1x_1(n) + a_2x_2(n) = a_1X_1(z) + a_2X_2(z) \tag{46}$$

$$\text{ROC} = \text{ROC}_1 \cap \text{ROC}_2 \tag{47}$$

Therefore, ROC of equation(18) is the intersection of the ROC of each signal. Every signal has ROC $|z| > 1$. So,

$$\text{ROC of } X(z) \text{ is } |z| > 1 \tag{48}$$

b) Consider the Denominator of the RHS part:

$$1^2 \times 2 + 2^2 \times 3 + \cdots + n^2 \times (n+1) = \sum_{k=1}^{n} k^2(k+1) \tag{49}$$

Now,

$$\sum_{k=1}^{n} k^2(k + 1) = \sum_{k=1}^{n} k^3 + k^2 \tag{50}$$

$$= \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2 \tag{51}$$

$$= \left(\frac{n(n + 1)}{2}\right)^2 + \frac{n(n + 1)(2n + 1)}{6} \tag{52}$$

$$= \frac{n(n + 1)}{2}\left[\frac{n(n + 1)}{2} + \frac{2n + 1}{3}\right] \tag{53}$$

$$= \frac{n(n + 1)}{2}\left[\frac{3n^2 + 7n + 2}{6}\right] \tag{54}$$

$$= \frac{n(n + 1)}{12}\left[3n^2 + 6n + n + 2\right] \tag{55}$$

$$= \frac{n(n + 1)}{12}\left[3n(n + 2) + (n + 2)\right] \tag{56}$$

$$= \frac{n(n + 1)(n + 2)(3n + 1)}{12} \tag{57}$$

$$\frac{n(n + 1)(n + 2)(3n + 1)}{12} \cdot u[n] = y[n] \tag{58}$$

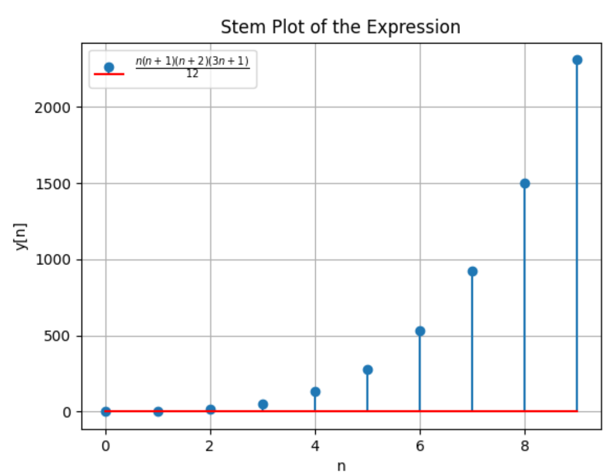

Fig. b). Stem Plot of y[n]

The Ncert question result is proved,

$$\frac{\sum_{k=1}^{n} k(k+1)^2}{\sum_{k=1}^{n} k^2(k+1)} = \frac{\frac{n(n+1)(n+2)(3n+5)}{12}}{\frac{n(n+1)(n+2)(3n+1)}{12}} \tag{59}$$

$$= \frac{3n+5}{3n+1} \tag{60}$$

Derivation of z-transform of the denominator:

$$y[n] = \frac{n(n+1)(n+2)(3n+1)}{12} \cdot u[n] \tag{61}$$

$$= \frac{3n^4 + 10n^3 + 9n^2 + 2n}{12} \cdot u[n] \tag{62}$$

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n] \cdot z^{-n} \tag{63}$$

Z-Transform is calculated as:

$$Y(z) = \sum_{n=-\infty}^{\infty} \frac{3n^4 + 10n^3 + 9n^2 + 2n}{12} \tag{64}$$

$$12Y(z) = \sum_{n=0}^{\infty} 3n^4 z^{-n} + \sum_{n=0}^{\infty} 10n^3 z^{-n} \tag{65}$$
$$+ \sum_{n=0}^{\infty} 9n^2 z^{-n} + \sum_{n=0}^{\infty} 2n\, z^{-n}$$

We can say that Y_i denote the z-transform of n^i which will be same as respective X_i.

$$X_i = Y_i \tag{66}$$

Equation(64) can be now written as:

$$Y(z) = \frac{1}{12}(3Y_4(z) + 10Y_3(z) + 9Y_2(z) + 2Y_1(z)) \tag{67}$$

$$= \frac{3}{12}\left(\frac{z^{-1}(1 + 11z^{-1} + 11z^{-2} + z^{-3})}{(1 - z^{-1})^5}\right) \tag{68}$$
$$+ \frac{10}{12}\left(\frac{z^{-1}(1 + 4z^{-1} + z^{-2})}{(1 - z^{-1})^4}\right) + \frac{9}{12}\left(\frac{z^{-1}(z^{-1} + 1)}{(1 - z^{-1})^3}\right)$$
$$+ \frac{2}{12}\left(\frac{z^{-1}}{(1 - z^{-1})^2}.\right)$$
$$= \frac{2(z^{-1} + 2z^{-2})}{(1 - z^{-1})^5} \tag{69}$$

$Y(z)$ is a linear combination of $Y_i(z)$ for $i = 1, 2, 3, 4$:

$$Y(z) = a_1 Y_1(z) + a_2 Y_2(z) + a_3 Y_3(z) + a_4 Y_4(z) \tag{70}$$

$$\mathrm{ROC}_Y = \mathrm{ROC}_1 \cap \mathrm{ROC}_2 \cap \mathrm{ROC}_3 \cap \mathrm{ROC}_4 \tag{71}$$

By equation(66) we can say that Y_i also has same ROC as respective X_i. Therefore,

$$\text{ROC of } Y(z) : |z| > 1 \tag{72}$$