NCERT 11.9.5 26Q

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Question: Show that

$$\frac{1\times 2^2+2\times 3^2+\cdots+n\times(n+1)^2}{1^2\times 2+2^2\times 3+\cdots+n^2\times(n+1)}=\frac{3n+5}{3n+1}$$

Solution:
Consider the Numerator of the LHS part:

$$1\times 2^2+2\times 3^2+\cdots+n\times(n+1)^2=\sum_{k=1}^{n}k(k+1)^2$$

Now,

$$\begin{aligned}\sum_{k=1}^{n}k(k+1)^2 &= \sum_{k=1}^{n}k^3+2k^2+k\\ &= \sum_{k=1}^{n}k^3+\sum_{k=1}^{n}2k^2+\sum_{k=1}^{n}k\\ &= \sum_{k=1}^{n}k^3+2\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k\\ &= \left(\frac{n(n+1)}{2}\right)^2+2\cdot\frac{n(n+1)(2n+1)}{6}+\\ &+\frac{n(n+1)}{2}\\ &= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2}+\frac{2}{3}(2n+1)+1\right]\\ &= \frac{n(n+1)}{2}\left[\frac{3n^2+11n+10}{6}\right]\\ &= \frac{n(n+1)}{12}\left[3n^2+6n+5n+10\right]\\ &= \frac{n(n+1)}{12}\left[3n(n+2)+5(n+2)\right]\\ &= \frac{n(n+1)(n+2)(3n+5)}{12}\end{aligned}$$

Therefore,

$$1\times2^2+2\times3^2+\cdots+n\times(n+1)^2=\frac{n(n+1)(n+2)(3n+5)}{12}$$

Consider the Denominator of the RHS part:

$$1^2\times 2+2^2\times 3+\cdots+n^2\times(n+1)=\sum_{k=1}^{n}k^2(k+1)$$

Now,

$$\begin{aligned}\sum_{k=1}^{n}k^2(k+1) &= \sum_{k=1}^{n}k^3+k^2\\ &= \sum_{k=1}^{n}k^3+\sum_{k=1}^{n}k^2\\ &= \left(\frac{n(n+1)}{2}\right)^2+\frac{n(n+1)(2n+1)}{6}\\ &= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2}+\frac{2n+1}{3}\right]\\ &= \frac{n(n+1)}{2}\left[\frac{3n^2+7n+2}{6}\right]\\ &= \frac{n(n+1)}{12}\left[3n^2+6n+n+2\right]\\ &= \frac{n(n+1)}{12}\left[3n(n+2)+(n+2)\right]\\ &= \frac{n(n+1)(n+2)(3n+1)}{12}\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\sum_{k=1}^{n}k(k+1)^2}{\sum_{k=1}^{n}k^2(k+1)} &= \frac{\frac{n(n+1)(n+2)(3n+5)}{12}}{\frac{n(n+1)(n+2)(3n+1)}{12}}\\ &= \frac{3n+5}{3n+1}\end{aligned}$$

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