NCERT 11.9.5 26Q

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Question:

$$\frac{1\times 2^2 + 2\times 3^2 + \cdots + n\times(n+1)^2}{1^2\times 2 + 2^2\times 3 + \cdots + n^2\times(n+1)}$$

Find the z-transform of general term of numerator and denominator and plot them . Also find z-transform of sum of terms of numerator and denominator and plot it.

Solution:

Parameter	**Description**	**Value**
n	Integer	.... -2,-1,0,1, 2, ...
$x_1(n)$	General term of Numerator	$\left(n^3+5n^2+20n+4\right)\cdot u(n)$
$x_2(n)$	General Term of Denominator	$\left(n^3+4n^2+5n+2\right)\cdot u(n)$
$y_1(n)$	Sum of terms of numerator	?
$y_2(n)$	Sum of terms of denominator	?
$U(z)$	z-transform of $u(n)$	$\frac{1}{1-z^{-1}}, \{z\in\mathbb{C}:\lvert z\rvert>1\}$
ROC	Region of convergence	$\left\{z:\left\lvert\sum_{n=-\infty}^{\infty}x(n)z^{-n}\right\rvert<\infty\right\}$

TABLE 1: Parameter Table

1. Analysis of Numerator:

By the differentiation property :

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z), ROC = R \tag{1}$$

$$n^k x(n) \overset{\mathcal{Z}}{\longleftrightarrow} (-z)^k \frac{d^k X(z)}{dz^k}, ROC = R \tag{2}$$

$$\implies n^k u(n) \overset{\mathcal{Z}}{\longleftrightarrow} (-z)^k \frac{d^k U(z)}{dz^k} \{z\in\mathbb{C}:|z|>1\} \tag{3}$$

Therefore,

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^2}, \{z\in\mathbb{C}:|z|>1\} \tag{4}$$

$$n^2u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(z^{-1}+1\right)}{(1-z^{-1})^3}, \{z\in\mathbb{C}:|z|>1\} \tag{5}$$

$$n^3u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{(1-z^{-1})^4}, \{z\in\mathbb{C}:|z|>1\} \tag{6}$$

$$n^4u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(1+11z^{-1}+11z^{-2}+z^{-3}\right)}{(1-z^{-1})^5} \tag{7}$$

where $\{z\in\mathbb{C}:|z|>1\}$

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n)\,u(n)\,z^{-n} \tag{8}$$

$$= \sum_{n=-\infty}^{\infty}\left(n^3+5n^2+20n+4\right)u(n)\,z^{-n} \tag{9}$$

$$\therefore X_1(z) = \frac{4+14z^{-1}-24z^{-2}+12z^{-3}}{(1-z^{-1})^4}, \{z\in\mathbb{C}:|z|>1\} \tag{10}$$

$$y_1(n) = \sum_{k=0}^{n} x_1(n) \tag{11}$$

$$y_1(n) = x_1(n) * u(n) \tag{12}$$

$$Y_1(z) = X_1(z)\,U(z) \tag{13}$$

$$= \frac{4+14z^{-1}-24z^{-2}+12z^{-3}}{(1-z^{-1})^5}, \{z\in\mathbb{C}:|z|>1\} \tag{14}$$

Using partial fractions:

$$Y_1(z) = \frac{34z^{-1}}{(1-z^{-1})} + \frac{72z^{-2}}{(1-z^{-1})^2} + \frac{64z^{-3}}{(1-z^{-3})^3} \tag{15}$$

$$+ \frac{28z^{-4}}{(1-z^{-1})^4} + \frac{6z^{-5}}{(1-z^{-1})^5} + 4$$

$$\frac{z^{-1}}{(1-z^{-1})} = \frac{1}{1-z^{-1}} - 1 \tag{16}$$

Taking the inverse z-transform of each term :

$$Z^{-1}\left[\frac{z^{-1}}{(1-z^{-1})}\right] = u(n) - \delta(n) = u(n-1) \tag{17}$$

$$\frac{z^{-2}}{(1-z^{-1})^2} = \left(\frac{z^{-1}}{(1-z^{-1})}\right)\left(\frac{z^{-1}}{(1-z^{-1})}\right) \tag{18}$$

$$Z^{-1}\left[\frac{z^{-2}}{(1-z^{-1})^2}\right] = u(n-1) * u(n-1) \tag{19}$$

$$\therefore Z^{-1}\left[\frac{z^{-k}}{(1-z^{-1})^k}\right] = (u(n-1) * u(n-1) * \ldots \tag{20}$$

$$\ldots * u(n-1))_{k \text{ times}}$$

$$Z^{-1}\left[\frac{z^{-2}}{(1-z^{-1})^2}\right] = (n-1)\,u(n-1) \tag{21}$$

$$Z^{-1}\left[\frac{z^{-3}}{(1-z^{-1})^3}\right] = \frac{(n-1)(n-2)}{2} u(n-1) \tag{22}$$

$$Z^{-1}\left[\frac{z^{-4}}{(1-z^{-1})^4}\right] = \frac{(n-1)(n-2)(n-3)}{6} u(n-1) \tag{23}$$

$$Z^{-1}\left[\frac{z^{-}}{(1-z^{-1})^5}\right] = \frac{(n-1)(n-2)(n-3)(n-4)}{24} \tag{24}$$

$$u(n-1)$$

Substituting above inverse transforms in equation(15)

$$y_1(n) = \frac{3n^4 + 26n^3 + 81n^2 + 106n + 48}{12} \tag{25}$$

2. Analysis of Denominator:

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2(n)\,u(n)\,z^{-n} \tag{26}$$

$$= \sum_{n=-\infty}^{\infty} \left(n^3 + 4n^2 + 5n + 2\right) u(n)\,z^{-n} \tag{27}$$

$$\therefore X_2(z) = \frac{2+4z^{-1}}{(1-z^{-1})^4}, \{z \in \mathbb{C} : |z| > 1\} \tag{28}$$

$$\tag{29}$$

$$y_2(n) = \sum_{k=0}^{n} x_2(n) \tag{30}$$

$$y_2(n) = x_2(n) * u(n) \tag{31}$$

$$Y_2(z) = X_2(z)\,U(z) \tag{32}$$

$$= \frac{2+4z^{-1}}{(1-z^{-1})^5}, \{z \in \mathbb{C} : |z| > 1\} \tag{33}$$

Using partial fractions:

$$Y_2(z) = \frac{14z^{-1}}{(1-z^{-1})} + \frac{36z^{-2}}{(1-z^{-1})^2} + \frac{44z^{-3}}{(1-z^{-3})^3} \tag{34}$$

$$+ \frac{26z^{-4}}{(1-z^{-1})^4} + \frac{6z^{-5}}{(1-z^{-1})^5} + 2$$

Taking the inverse z-transform of each term :

$$y_2(n) = \frac{3n^4 + 22n^3 + 57n^2 + 62n + 24}{12} \tag{35}$$

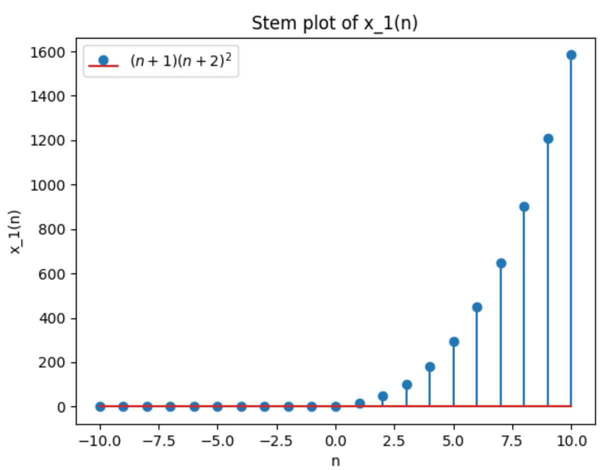

Fig. 1: Stem Plot of $x_1(n)$

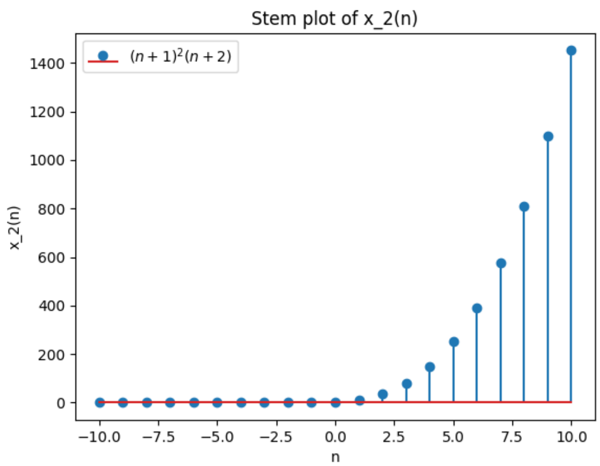

Fig. 2: Stem Plot of $x_2(n)$

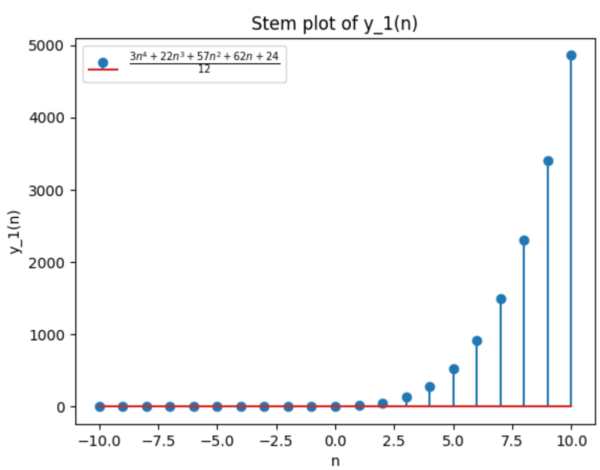

Fig. 3: Stem Plot of $y_1(n)$

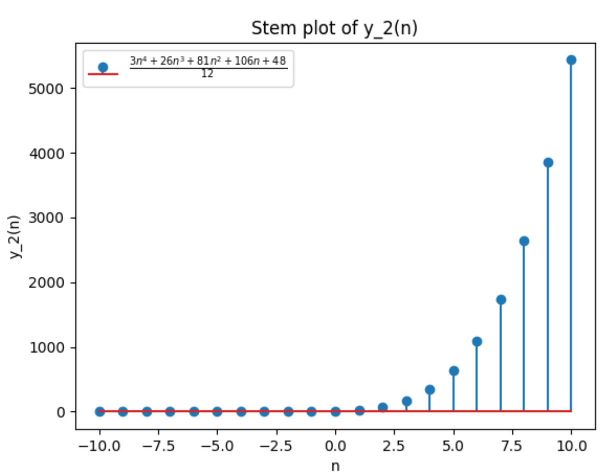

Fig. 4: Stem Plot of $y_2(n)$