NCERT 11.9.5 26Q

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Question: Show that

$$\frac{1\times 2^2+2\times 3^2+\cdots+n\times(n+1)^2}{1^2\times 2+2^2\times 3+\cdots+n^2\times(n+1)}=\frac{3n+5}{3n+1}$$

Solution:

Parameter	**Description**
n	Integer
x(n)	Discrete-sequence
y(n)	Discrete-sequence
X(z)	z-transform of x(n)
Y(z)	z-transform of y(n)
u(n)	Unit step sequence
U(z)	z-transform of u(n)
ROC	Region of convergence
$Y_k(z), X_k(z)$	z-transform of $n^k\cdot u(n)$

TABLE 1: Parameter Table

1. Consider the Numerator of the LHS part:

$$1\times 2^2+2\times 3^2+\cdots+n\times(n+1)^2=\sum_{k=1}^{n}k(k+1)^2 \tag{1}$$

Now,

$$\sum_{k=1}^{n}k(k+1)^2=\sum_{k=1}^{n}k^3+2k^2+k \tag{2}$$

$$=\left(\frac{n(n+1)}{2}\right)^2+2\cdot\frac{n(n+1)(2n+1)}{6} \tag{3}$$

$$+\frac{n(n+1)}{2}$$

$$=\frac{n(n+1)(n+2)(3n+5)}{12} \tag{4}$$

$$x(n)=\frac{(n+1)(n+2)(n+3)(3n+8)}{12}\cdot u(n) \tag{5}$$

$$=\frac{3n^4+26n^3+81n^2+106n+48}{12}\cdot u(n) \tag{6}$$

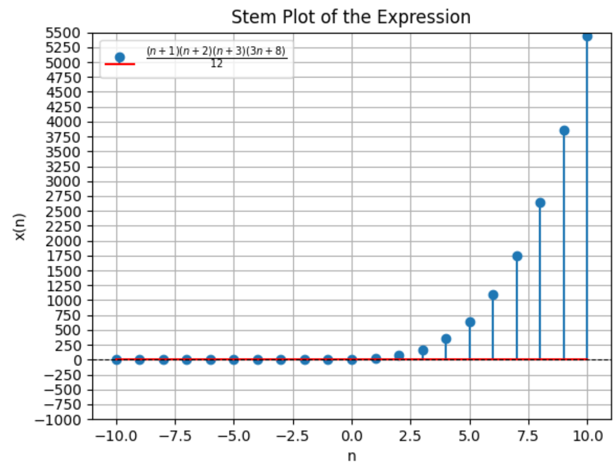

Fig. 1.: Stem Plot of $x(n)$

$$X(z)=\sum_{n=-\infty}^{\infty}x(n)z^{-n} \tag{7}$$

$$U(z)=\frac{1}{1-z^{-1}},\quad ROC=|z|>1 \tag{8}$$

The Region of Convergence (ROC) is defined as:

$$\text{ROC}=\left\{z:\left|\sum_{n=-\infty}^{\infty}x(n)z^{-n}\right|<\infty\right\} \tag{9}$$

$$\text{ROC of u}(n)=\left\{z:\left|\sum_{n=0}^{\infty}z^{-n}\right|<\infty\right\} \tag{10}$$

This is an infinite GP. And it converges only if $|r|<1$.

$$\text{ROC of u}(n)\text{: } |z|>1 \tag{11}$$

By the differentiation property :

$$x(n)\overset{\mathcal{Z}}{\longleftrightarrow}X(z),\ ROC=R \tag{12}$$

$$n^kx(n)\overset{\mathcal{Z}}{\longleftrightarrow}(-z)^k\frac{d^kX(z)}{dz^k},\ ROC=R \tag{13}$$

$$\implies n^ku(n)\overset{\mathcal{Z}}{\longleftrightarrow}(-z)^k\frac{d^kU(z)}{dz^k} \tag{14}$$

$$\implies X_k(z) = (-z)^k \frac{d^k U(z)}{dz^k}, \ ROC = |z| > 1 \tag{15}$$

Substituting k=1,2,3,4 in equation (15):

$$X_1(z) = \frac{z^{-1}}{(1-z^{-1})^2}, \ ROC = |z| > 1 \tag{16}$$

$$X_2(z) = \frac{z^{-1}\left(z^{-1}+1\right)}{(1-z^{-1})^3}, \ ROC = |z| > 1 \tag{17}$$

$$X_3(z) = \frac{z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{(1-z^{-1})^4}, \ ROC = |z| > 1 \tag{18}$$

$$X_4(z) = \frac{z^{-1}\left(1+11z^{-1}+11z^{-2}+z^{-3}\right)}{(1-z^{-1})^5}, \ ROC = |z| > 1 \tag{19}$$

$$a_1x_1(n) + a_2x_2(n) \overset{\mathcal{Z}}{\longleftrightarrow} a_1X_1(z) + a_2X_2(z) \tag{20}$$

$$\text{ROC} = \text{ROC}_1 \cap \text{ROC}_2 \tag{21}$$

$$\text{ROC of } X(z) = |z| > 1 \tag{22}$$

$$X(z) = \frac{1}{12}\left(3X_4(z) + 26X_3(z) + 81X_2(z) + 106X_1(z)\right) + 48U(z) \tag{23}$$

$$X(z) = \frac{3}{12}\left(\frac{z^{-1}\left(1+11z^{-1}+11z^{-2}+z^{-3}\right)}{(1-z^{-1})^5}\right) \tag{24}$$

$$+ \frac{26}{12}\left(\frac{z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{(1-z^{-1})^4}\right)$$

$$+ \frac{81}{12}\left(\frac{z^{-1}\left(z^{-1}+1\right)}{(1-z^{-1})^3}\right)$$

$$+ \frac{106}{12}\left(\frac{z^{-1}}{(1-z^{-1})^2}\right) + \frac{48}{12}\left(\frac{1}{1-z^{-1}}\right)$$

$$= \frac{24\left(z^4\right)(2z+1)}{(z-1)^5}, \ ROC = |z| > 1 \tag{25}$$

2. Consider the Denominator of the RHS part:

$$1^2\times 2 + 2^2\times 3 + \cdots + n^2\times(n+1) = \sum_{k=1}^{n} k^2(k+1) \tag{26}$$

Now,

$$\sum_{k=1}^{n} k^2(k+1) = \sum_{k=1}^{n} k^3 + k^2 \tag{27}$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + \frac{n(n+1)(2n+1)}{6} \tag{28}$$

$$= \frac{(n+1)(n+2)(n+3)(3n+4)}{12} \tag{29}$$

$$\frac{(n+1)(n+2)(n+3)(3n+4)}{12}\cdot u(n) = y(n) \tag{30}$$

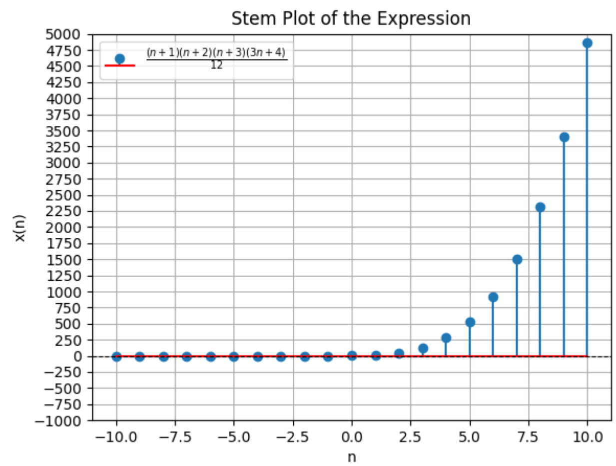

Fig. 2.: Stem Plot of $y(n)$

$$y(n) = \frac{3n^4 + 22n^3 + 57n^2 + 62n + 24}{12}\cdot u(n) \tag{31}$$

$$Y(z) = \sum_{n=-\infty}^{\infty} y(n)\cdot z^{-n} \tag{32}$$

$$X_k = Y_k \tag{33}$$

$$Y_k(z) = (-z)^k \frac{d^k U(z)}{dz^k} \tag{34}$$

Now substituting k=1,2,3,4 in equation(34):

$$Y_1(z) = \frac{z^{-1}}{(1-z^{-1})^2}, \ ROC = |z| > 1 \tag{35}$$

$$Y_2(z) = \frac{z^{-1}\left(z^{-1}+1\right)}{(1-z^{-1})^3}, \ ROC = |z| > 1 \tag{36}$$

$$Y_3(z) = \frac{z^{-1}\left(1 + 4z^{-1} + z^{-2}\right)}{(1 - z^{-1})^4},\ ROC = |z| > 1 \tag{37}$$

$$Y_4(z) = \frac{z^{-1}\left(1 + 11z^{-1} + 11z^{-2} + z^{-3}\right)}{(1 - z^{-1})^5},\ ROC = |z| > 1 \tag{38}$$

By equation(33) we can say that Y_i also has same ROC as respective X_i. Therefore,

$$\text{ROC of } Y(z) : |z| > 1 \tag{39}$$

Z-Transform of equation(31)

$$Y(z) = \frac{1}{12}\left(3Y_4(z) + 22Y_3(z) + 57Y_2(z) + 62Y_1(z)\right) \tag{40}$$

$$+ 24U(z)$$

$$= \frac{3}{12}\left(\frac{z^{-1}\left(1 + 11z^{-1} + 11z^{-2} + z^{-3}\right)}{(1 - z^{-1})^5}\right) \tag{41}$$

$$+ \frac{22}{12}\left(\frac{z^{-1}\left(1 + 4z^{-1} + z^{-2}\right)}{(1 - z^{-1})^4}\right) + \frac{57}{12}\left(\frac{z^{-1}\left(z^{-1} + 1\right)}{(1 - z^{-1})^3}\right)$$

$$+ \frac{62}{12}\left(\frac{z^{-1}}{(1 - z^{-1})^2}.\right) + \frac{24}{12}\left(\frac{1}{1 - z^{-1}}\right)$$

$$Y(z) = \frac{24\left(z^4\right)(2z + 1)}{(z - 1)^5},\ ROC = |z| > 1 \tag{42}$$