NCERT 11.9.5 26Q

EE23BTECH11015 - DHANUSH V NAYAK*

Question: Show that

$$\frac{1\times 2^2 + 2\times 3^2 + \cdots + n\times (n+1)^2}{1^2\times 2 + 2^2\times 3 + \cdots + n^2\times (n+1)} = \frac{3n+5}{3n+1}$$

Solution:

1. Consider the Numerator of the LHS part:

$$1\times 2^2 + 2\times 3^2 + \cdots + n\times (n+1)^2 = \sum_{k=1}^{n} k(k+1)^2 \tag{1}$$

Now,

$$\sum_{k=1}^{n} k\left(k+1\right)^2 = \sum_{k=1}^{n} k^3 + 2k^2 + k \tag{2}$$

$$= \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} 2k^2 + \sum_{k=1}^{n} k \tag{3}$$

$$= \sum_{k=1}^{n} k^3 + 2\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k \tag{4}$$

$$= \left(\frac{n\left(n+1\right)}{2}\right)^2 + 2\cdot\frac{n\left(n+1\right)\left(2n+1\right)}{6} \tag{5}$$

$$+ \frac{n\left(n+1\right)}{2}$$

$$= \frac{n\left(n+1\right)}{2}\left[\frac{n\left(n+1\right)}{2} + \frac{2}{3}\left(2n+1\right) + 1\right] \tag{6}$$

$$= \frac{n\left(n+1\right)}{2}\left[\frac{3n^2 + 11n + 10}{6}\right] \tag{7}$$

$$= \frac{n\left(n+1\right)}{12}\left[3n^2 + 6n + 5n + 10\right] \tag{8}$$

$$= \frac{n\left(n+1\right)}{12}\left[3n\left(n+2\right) + 5\left(n+2\right)\right] \tag{9}$$

$$= \frac{n\left(n+1\right)\left(n+2\right)\left(3n+5\right)}{12} \tag{10}$$

Therefore,

$$1\times2^2+2\times3^2+\cdots+n\times(n+1)^2 = \frac{n\left(n+1\right)\left(n+2\right)\left(3n+5\right)}{12} \tag{11}$$

Parameter	Description
n	Integer
x(n)	Discrete-sequence
y(n)	Discrete-sequence
X(z)	z-transform of x(n)
Y(z)	z-transform of y(n)
u(n)	Unit step sequence
U(z)	z-transform of u(n)
ROC	Region of convergence
$Y_k(z), X_k(z)$	z-transform of $n^k \cdot u(n)$

TABLE 1.: Parameter Table

$$\frac{\left(n+1\right)\left(n+2\right)\left(n+3\right)\left(3n+8\right)}{12}\cdot u\left(n\right) = x\left(n\right) \tag{12}$$

$$x\left(n\right) = \begin{cases} 0 & \text{for } n < 0 \\ \frac{(n+1)(n+2)(n+3)(3n+8)}{12} & \text{for } n \geq 0 \end{cases} \tag{13}$$

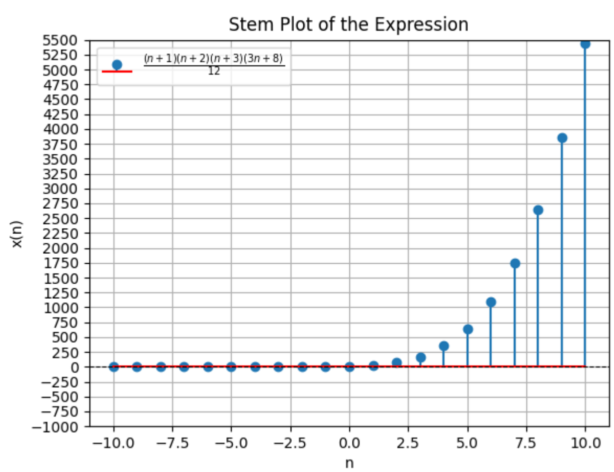

Fig. 1.: Stem Plot of $x\left(n\right)$

Derivation of z-transform of numerator :

$$x\left(n\right) = \frac{\left(n+1\right)\left(n+2\right)\left(n+3\right)\left(3n+8\right)}{12}\cdot u\left(n\right) \tag{14}$$

$$= \frac{3n^4 + 26n^3 + 81n^2 + 106n + 48}{12}\cdot u\left(n\right) \tag{15}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{16}$$

$$U(z) = \sum_{n=-\infty}^{\infty} u(n) \cdot z^{-n} \tag{17}$$

$$u(n) = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n \geq 0 \end{cases} \tag{18}$$

$$U(z) = \sum_{n=0}^{\infty} (1) \cdot z^{-n} \tag{19}$$

$$= 1 + z^{-1} + z^{-2} + \dots \tag{20}$$

$$= \frac{1}{1 - z^{-1}} \tag{21}$$

By the differentiation property:

$$x(n) \overset{Z}{\longleftrightarrow} X(z) \tag{22}$$

$$n^k x(n) \overset{Z}{\longleftrightarrow} (-z)^k \frac{d^k X(z)}{dz^k} \tag{23}$$

$$n^k u(n) \overset{Z}{\longleftrightarrow} (-z)^k \frac{d^k U(z)}{dz^k} \tag{24}$$

$$X_k(z) = (-z)^k \frac{d^k U(z)}{dz^k} \tag{25}$$

For the Z-transform of $n \cdot u(n)$, substituting k=1 in Equation (25):

$$X_1(z) = (-z) \frac{dU(z)}{dz} \tag{26}$$

$$\frac{dU(z)}{dz} = -\frac{z^{-2}}{(1 - z^{-1})^2}, \tag{27}$$

$$X_1(z) = \frac{z^{-1}}{(1 - z^{-1})^2}. \tag{28}$$

Similarly , substituting k=2,3,4 in equation (25):

$$X_2(z) = \frac{z^{-1}\left(z^{-1} + 1\right)}{(1 - z^{-1})^3} \tag{29}$$

$$X_3(z) = \frac{z^{-1}\left(1 + 4z^{-1} + z^{-2}\right)}{(1 - z^{-1})^4} \tag{30}$$

$$X_4(z) = \frac{z^{-1}\left(1 + 11z^{-1} + 11z^{-2} + z^{-3}\right)}{(1 - z^{-1})^5} \tag{31}$$

The Region of Convergence (ROC) is defined as the set of points in the complex plane for which the Z-transform summation converges, i.e., doesn't blow up in magnitude to infinity: The Region of Convergence (ROC) is defined as:

$$\text{ROC} = \left\{ z : \left| \sum_{n=-\infty}^{\infty} x(n) z^{-n} \right| < \infty \right\} \tag{32}$$

Equation (32) expresses the set of points in the complex plane for which the Z-transform summation converges.
The Z-transform of the unit step signal $u(n)$ is given by:

$$U(z) = \sum_{n=-\infty}^{\infty} u(n) z^{-n} \tag{33}$$

$$u(n) = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n \geq 0 \end{cases} \tag{34}$$

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{35}$$

$$\text{ROC} = \left\{ z : \left| \sum_{n=0}^{\infty} z^{-n} \right| < \infty \right\} \tag{36}$$

This is an infinite GP. And it converges only if $|r| < 1$ where r is the common ratio.And here , $|r| = |z^{-1}|$ Therefore, the Region of Convergence (ROC) for this Z-transform is:

$$\text{ROC: } |z| > 1 \tag{37}$$

In the differentiation property, ROC does not change:

$$x(n) \overset{Z}{\longleftrightarrow} X(z), \; ROC = R \tag{38}$$

$$n^k x(n) \overset{Z}{\longleftrightarrow} (-z)^k \frac{d^k X(z)}{dz^k}, \; ROC = R \tag{39}$$

Therefore,

$$X_1(z) = \frac{z^{-1}}{(1 - z^{-1})^2}, \; ROC = |z| > 1 \tag{40}$$

$$X_2(z) = \frac{z^{-1}\left(z^{-1} + 1\right)}{(1 - z^{-1})^3}, \; ROC = |z| > 1 \tag{41}$$

$$X_3(z) = \frac{z^{-1}\left(1 + 4z^{-1} + z^{-2}\right)}{(1 - z^{-1})^4}, \; ROC = |z| > 1 \tag{42}$$

$$X_4(z) = \frac{z^{-1}\left(1 + 11z^{-1} + 11z^{-2} + z^{-3}\right)}{(1 - z^{-1})^5}, \; ROC = |z| > 1 \tag{43}$$

Z-Transform of Equation (15) can be written as:

$$X(z) = \frac{1}{12}\left(3X_4(z) + 26X_3(z) + 81X_2(z) + 106X_1(z)\right) + 48U(z) \quad (44)$$

$$= \frac{3}{12}\left(\frac{z^{-1}\left(1 + 11z^{-1} + 11z^{-2} + z^{-3}\right)}{\left(1 - z^{-1}\right)^5}\right) + \frac{26}{12}\left(\frac{z^{-1}\left(1 + 4z^{-1} + z^{-2}\right)}{\left(1 - z^{-1}\right)^4}\right) + \frac{81}{12}\left(\frac{z^{-1}\left(z^{-1} + 1\right)}{\left(1 - z^{-1}\right)^3}\right) + \frac{106}{12}\left(\frac{z^{-1}}{\left(1 - z^{-1}\right)^2}\right) + \frac{48}{12}\left(\frac{1}{1 - z^{-1}}\right) \quad (45)$$

$$= \frac{24\left(z^4\right)(2z + 1)}{(z - 1)^5} \quad (46)$$

Consider the linear combination of two signals in the time domain:

$$a_1x_1(n) + a_2x_2(n) \xleftrightarrow{\mathcal{Z}} a_1X_1(z) + a_2X_2(z) \quad (47)$$

$$\text{ROC} = \text{ROC}_1 \cap \text{ROC}_2 \quad (48)$$

Therefore, ROC of equation(44) is the intersection of the ROC of each signal. Every signal has ROC $|z| > 1$. So,

$$\text{ROC of } X(z) \text{ is } |z| > 1 \quad (49)$$

2. Consider the Denominator of the RHS part:

$$1^2 \times 2 + 2^2 \times 3 + \cdots + n^2 \times (n + 1) = \sum_{k=1}^{n} k^2(k + 1) \quad (50)$$

Now,

$$\sum_{k=1}^{n} k^2(k + 1) = \sum_{k=1}^{n} k^3 + k^2 \quad (51)$$

$$= \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2 \quad (52)$$

$$= \left(\frac{n(n + 1)}{2}\right)^2 + \frac{n(n + 1)(2n + 1)}{6} \quad (53)$$

$$= \frac{n(n + 1)}{2}\left[\frac{n(n + 1)}{2} + \frac{2n + 1}{3}\right] \quad (54)$$

$$= \frac{n(n + 1)}{2}\left[\frac{3n^2 + 7n + 2}{6}\right] \quad (55)$$

$$= \frac{n(n + 1)}{12}\left[3n^2 + 6n + n + 2\right] \quad (56)$$

$$= \frac{n(n + 1)}{12}\left[3n(n + 2) + (n + 2)\right] \quad (57)$$

$$= \frac{n(n + 1)(n + 2)(3n + 1)}{12} \quad (58)$$

$$\frac{(n + 1)(n + 2)(n + 3)(3n + 4)}{12} \cdot u(n) = y(n) \quad (59)$$

$$y(n) = \begin{cases} 0 & \text{for } n < 0 \\ \frac{(n+1)(n+2)(n+3)(3n+4)}{12}. & \text{for } n \geq 0 \end{cases} \quad (60)$$

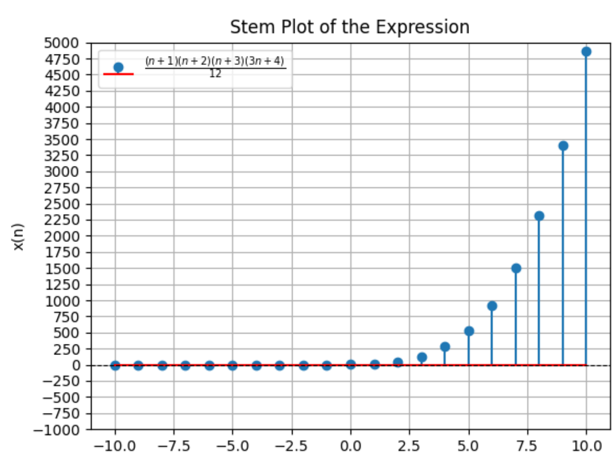

Fig. 2.: Stem Plot of $y(n)$

$$\frac{\sum_{k=1}^{n} k\left(k+1\right)^2}{\sum_{k=1}^{n} k^2\left(k+1\right)} = \frac{\frac{n(n+1)(n+2)(3n+5)}{12}}{\frac{n(n+1)(n+2)(3n+1)}{12}} \tag{61}$$

$$= \frac{3n+5}{3n+1} \tag{62}$$

Derivation of z-transform of the denominator:

$$y\left(n\right) = \frac{\left(n+1\right)\left(n+2\right)\left(n+3\right)\left(3n+4\right)}{12} \cdot u\left(n\right) \tag{63}$$

$$= \frac{3n^4 + 22n^3 + 57n^2 + 62n + 24}{12} \cdot u\left(n\right) \tag{64}$$

$$Y\left(z\right) = \sum_{n=-\infty}^{\infty} y\left(n\right) \cdot z^{-n} \tag{65}$$

We can say that $Y_k\left(z\right)$ denote the z-transform of $n^k u\left(n\right)$ which will be same as respective $X_k\left(z\right)$.

$$X_k = Y_k \tag{66}$$

$$Y_k\left(z\right) = \left(-z\right)^k \frac{d^k U\left(z\right)}{dz^k} \tag{67}$$

Now substituting k=1,2,3,4 we get :

$$Y_1\left(z\right) = \frac{z^{-1}}{\left(1-z^{-1}\right)^2},\ ROC = |z| > 1 \tag{68}$$

$$Y_2\left(z\right) = \frac{z^{-1}\left(z^{-1}+1\right)}{\left(1-z^{-1}\right)^3},\ ROC = |z| > 1 \tag{69}$$

$$Y_3\left(z\right) = \frac{z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{\left(1-z^{-1}\right)^4},\ ROC = |z| > 1 \tag{70}$$

$$Y_4\left(z\right) = \frac{z^{-1}\left(1+11z^{-1}+11z^{-2}+z^{-3}\right)}{\left(1-z^{-1}\right)^5},\ ROC = |z| > 1 \tag{71}$$

Z-Transform of equation(64)

$$Y\left(z\right) = \frac{1}{12}\left(3Y_4\left(z\right) + 22Y_3\left(z\right) + 57Y_2\left(z\right) + 62Y_1\left(z\right)\right) \tag{72}$$

$$+ 24U\left(z\right)$$

$$= \frac{3}{12}\left(\frac{z^{-1}\left(1+11z^{-1}+11z^{-2}+z^{-3}\right)}{\left(1-z^{-1}\right)^5}\right) \tag{73}$$

$$+ \frac{22}{12}\left(\frac{z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{\left(1-z^{-1}\right)^4}\right) + \frac{57}{12}\left(\frac{z^{-1}\left(z^{-1}+1\right)}{\left(1-z^{-1}\right)^3}\right)$$

$$+ \frac{62}{12}\left(\frac{z^{-1}}{\left(1-z^{-1}\right)^2}.\right) + \frac{24}{12}\left(\frac{1}{1-z^{-1}}\right)$$

$$= \frac{24\left(z^4\right)\left(2z+1\right)}{\left(z-1\right)^5} \tag{74}$$

$Y\left(z\right)$ is a linear combination of $Y_k\left(z\right)$ for $k = 1, 2, 3, 4$

$$Y\left(z\right) = a_1Y_1\left(z\right) + a_2Y_2\left(z\right) + a_3Y_3\left(z\right) + a_4Y_4\left(z\right) \tag{75}$$

$$\mathrm{ROC}_Y = \mathrm{ROC}_1 \cap \mathrm{ROC}_2 \cap \mathrm{ROC}_3 \cap \mathrm{ROC}_4 \tag{76}$$

By equation(66) we can say that Y_i also has same ROC as respective X_i. Therefore,

$$\text{ROC of } Y\left(z\right) : |z| > 1 \tag{77}$$