17-02

- Dual cone and Examples
- Dual generalised inequalities
- minimum element $\Rightarrow$ unique minimizer $\Leftarrow$ unique but not minimum element
- minimal element $\Rightarrow$ contradiction $\Leftarrow$ convexity of s, seperating hyperplane

- Dual cone:- Given a cone k, its dual cone k^* is defined as

$$k^* = \{ \underline{y} \mid \underbrace{x^T y}_{\text{inner product}} \geq 0 \quad \forall\, \underline{x} \in k \}$$

Example:- $k = \{ (\underline{x}, t) \mid \|\underline{x}\| \leq t,\ t > 0 \}$, $\underline{x} \in \mathbb{R}^n,\ t > 0$ —— ①

$k^* = \{ (\underline{u}, v) \mid \|\underline{u}\|_* \leq v \}$, where $\|\underline{u}\|_* = \sup\{ \underline{u}^T \underline{x} \mid \|\underline{x}\| \leq 1 \}$ —— ②

$\Rightarrow \|\underline{x}/t\| < 1 \quad \because (\underline{x}, t) \in k$

$\Leftarrow \quad \underline{u}^T \underline{x} + vt \geq 0$ —— ③

Suppose $\|\underline{u}\|_* > v$, then there exists an $\underline{x}$ such that $\|\underline{x}\| < 1$

$\underline{u}^T \underline{x} > v$ —— ④

The dual condition states that $\underline{u}^T \underline{x} + vt \geq 0 \quad \forall\, t > 0$ including $t = 1$

However from ④ we see that $\underline{u}^T \underline{x} - v > 0$

Contradiction $\quad v + \underline{u}^T(-\underline{x}) \leq 0$

$v + \underline{u}^T \underline{x} \leq 0 \quad$ (from dual cone condition when $t = 1$)

- Dual generalised Inequalities:- Let k be a proper cone and let k^* be its dual. Let k induces the ordering $\preceq_k$ and k^* induces the ordering $\preceq_{k^*}$.

- **Property** :- $\underline{x} \preceq_K \underline{y}$ iff $\underline{\lambda}^T\underline{x} \leq \underline{\lambda}^T\underline{y} \quad \forall \lambda \succeq_{K^*} 0$

Let's revisit minimum and minimal elements.

<u>Minimum element</u>:- Recall: $\underline{x}$ is a minimum element of S if $x \preceq_K y \quad \forall \underline{y} \in S$.

- $\underline{x}$ is a minimum element of S if $\underline{x}$ is the unique minimizer of $\underline{\lambda}^T\underline{z}$ over all $z \in S$, $\forall \boxed{\lambda \succeq_{K^*} 0}$

<u>Minimal element</u> : Recall: $\underline{x}$ is a minimal element of S if $\underline{y} \preceq_K \underline{x}$ only if $\underline{y} = \underline{x}$

- $\underline{x}$ is a minimal element of S if $\underline{x}$ minimizes $\underline{\lambda}^T\underline{z}$, $\forall \lambda \succeq_{K^*} 0$ overall $z \in S$

- **Note**:- These relations are useful when we deal with the dual problems.

- Minimum element:- Show that if $\underline{x}$ is a minimum element of S then it is the unique minimizer of $\underline{\lambda}^T\underline{z}$ over $z \in S$ for $\lambda \succeq_{K^*} 0$.
  - from def^n of minimum element $(\underline{z} - \underline{x}) \succeq_{K^*} 0$ — ⓐ
  - from def^n of dual cone and dual generalised inequality — ⓑ

$$\underline{\lambda}^T\underline{x} \leq \underline{\lambda}^T\underline{z} \quad \forall \lambda \succeq_{K^*} 0.$$