NUCLEAR INSTRUMENTS AND METHODS 127 (1975) 471–474; © NORTH-HOLLAND PUBLISHING CO.

LETTER TO THE EDITOR

MULTIPOLES IN CYLINDRICAL COORDINATES*

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Received 22 May 1975

Solutions of Laplace's equation for azimuthally symmetrical potentials in cylindrical coordinates are found which can be correlated with two-dimensional multipoles in planes $\theta = \text{const}$. Formulas are presented by which the coefficients for linear combinations of these solutions can be calculated to describe fields whose values are known along given axial or radial lines.

An azimuthally symmetrical electrostatic or magnetostatic field is described in a cylindrical coordinate system ρ, ζ, θ by a scalar potential ϕ which is a function of ρ and ζ only. Set up a rectangular coordinate system in a ρ, ζ plane with the origin at $\rho = \rho_0$, $\zeta = 0$, and having the dimensionless coordinates r, x, y, where $r = \rho/\rho_0$, $x = r-1$, $y = \zeta/\rho_0$. It is the purpose of this note to find a set of functions of these variables which are solutions of Laplace's equation in cylindrical coordinates but which, when expanded in power series in x and y, are equal to two-dimensional multipoles plus additional higher terms. The two-dimensional multipole potentials can be obtained from the complex expression $(x+iy)^m$, which is expanded and the real and imaginary parts separated, the real part giving a $2m$-pole with the field along the x-axis at $y = 0$, and the imaginary part giving a $2m$-pole with the field perpendicular to the x-axis at $y = 0$; for example, the quadrupole potentials $x^2 - y^2$ (real part) and $2xy$ (imaginary part).

The procedure followed is to seek solutions of Laplace's equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{\partial^2\phi}{\partial y^2} = 0, \tag{1}$$

in the form:

$$\phi = f_m(r) + f_{m-1}(r)\, y - f_{m-2}(r)\frac{y^2}{2!} - f_{m-3}(r)\frac{y^3}{3!} + \ldots \tag{2}$$

with the signs alternating in pairs.

It is found that eq. (2) separates into independent even and odd series, and that the functions $f_n(r)$ obey the recursion relation:

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}f_n}{\mathrm{d}r}\right) = f_{n-2}. \tag{3}$$

If any one of the functions $f_n(r)$ is given an assigned form, the functions to the right in the series are found by differentiation and to the left (i.e., in order of increasing n) by integration, involving two constants of integration at each step. Now an additional condition is put on the solutions; the series are made to terminate at $f_0(r)$ and $f_1(r)$. According to eq. (3), this occurs if $f_0(r) = 1$ or $\ln r$, and $f_1(r) = 1$ or $\ln r$. These four possibilities lead to a redundancy in the set of solutions obtained by considering all values of m in eq. (2), and it is sufficient to take $f_0(r) = 1$, $f_1(r) = \ln r$.

The continuation to higher orders of $f_n(r)$ by integration gives a branching sequence because of the constants of integration, but no generality is lost if a particular choice of the constants is made; the general case can be represented as a linear combination of the resulting set of solutions. The choice made is to treat the integrations as definite integrals starting at $r = 1$, i.e.,

$$f_n = \int_1^r \frac{1}{r}\left[\int_1^r r\, f_{n-2}\, \mathrm{d}r\right]\mathrm{d}r. \tag{4}$$

* Work performed under the auspices of the Energy Research and Development Administration.

This choice has the valuable consequence that the leading term in the power since expansion in x of $f_n(r)$ is $(1/n!)\, x^n$, satisfying the condition set in the beginning that the solutions should equal the two-dimensional multipoles in the lowest order.

It is convenient to introduce the notation $F_n(r) = n!\, f_n(r)$; then the set of solutions can be represented in the format used for the two-dimensional case, i.e., expand $(x+\mathrm{i}y)^m$, separate the real and imaginary parts, and replace x^n by $F_n(r)$, giving the solutions that will be called ϕ^{r}_m and ϕ^{i}_m. The functions $F_n(r)$ up to $n = 6$ are given below, with the power series expansions of $F_n(1+x)$ through x^7.

$$F_0 = 1,$$

$$F_1 = \ln r = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \tfrac{1}{6}x^6 + \tfrac{1}{7}x^7 + \dots,$$

$$F_2 = \tfrac{1}{2}(r^2-1) - \ln r = x^2 - \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 - \tfrac{1}{5}x^5 + \tfrac{1}{6}x^6 - \tfrac{1}{7}x^7 + \dots,$$

$$F_3 = \tfrac{3}{2}[-(r^2-1) + (r^2+1)\ln r] = x^3 - \tfrac{1}{2}x^4 + \tfrac{7}{20}x^5 - \tfrac{11}{40}x^6 + \tfrac{8}{35}x^7 + \dots,$$

$$F_4 = 3[\tfrac{1}{8}(r^4-1) + \tfrac{1}{2}(r^2-1) - (r^2+\tfrac{1}{2})\ln r] = x^2 - \tfrac{2}{5}x^5 + \tfrac{3}{10}x^6 - \tfrac{17}{70}x^7 + \dots,$$

$$F_5 = \tfrac{15}{2}[-\tfrac{3}{8}(r^4-1) + (\tfrac{1}{4}r^4 + r^2 + \tfrac{1}{4})\ln r] = x^5 - \tfrac{1}{2}x^6 + \tfrac{5}{14}x^7 + \dots,$$

$$F_6 = \tfrac{45}{4}[\tfrac{1}{36}(r^6-1) + \tfrac{1}{2}(r^4-1) - \tfrac{1}{4}(r^2-1) - (\tfrac{1}{2}r^4 + r^2 + \tfrac{1}{6})\ln r] = x^6 - \tfrac{3}{7}x^7 + \dots.$$

The magnetostatic field can equally well be represented by a vector potential A, which can be simply represented in terms of the functions $F_n(r)$ by the following procedure: expand $-\mathrm{i}(x+\mathrm{i}y)^m$, separate the real and imaginary parts, and replace x^n by:

$$\frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}r}F_{n+1}(r),$$

including the replacement of x^0 by:

$$\frac{\mathrm{d}}{\mathrm{d}r}F_1(r) = \frac{1}{r}.$$

The real part of this is a vector potential A^{r}_m which gives the same field as the scalar potential ϕ^{r}_m, and similarly for the imaginary parts.

Thus, for the quadrupole,

$$(x+\mathrm{i}y)^2 = x^2 - y^2 + 2\mathrm{i}xy,$$

and:

$$-\mathrm{i}(x+\mathrm{i}y)^2 = 2xy + \mathrm{i}(y^2 - x^2),$$

from which:

$$\phi^{\mathrm{r}}_2 = F_2 - F_0 y^2 = \tfrac{1}{2}(r^2-1) - \ln r - y^2,$$

$$\phi^{\mathrm{i}}_2 = 2F_1 y = 2y\ln r,$$

$$A^{\mathrm{r}}_2 = 2\,\frac{1}{2}\frac{\mathrm{d}F_2}{\mathrm{d}r}\,y = \left(r - \frac{1}{r}\right)y,$$

$$A^{\mathrm{i}}_2 = \frac{\mathrm{d}F_1}{\mathrm{d}r}\,y^2 - \frac{1}{3}\frac{\mathrm{d}F_3}{\mathrm{d}r} = \frac{1}{2}\left(r - \frac{1}{r}\right) - r\ln r + \frac{y^2}{r}.$$

The field components are, for the real part:

$$-B_x = \frac{\partial\phi}{\partial r} = \frac{\partial A}{\partial y} = r - \frac{1}{r} = 2x - x^2 + x^3 + \dots,$$

$$-B_y = \frac{\partial\phi}{\partial y} = -\frac{1}{r}\frac{\partial(rA)}{\partial r} = -2y,$$

and for the imaginary part:

$$-B_x = 2y/r = 2y - 2xy + 2x^2y + \ldots,$$
$$-B_y = 2\ln r = 2x - x^2 + \tfrac{2}{3}x^3 + \ldots.$$

Any field expandable as a power series in x and y can be represented by a linear combination of these solutions, which can be written:

$$\phi = \textstyle\sum_{m=1} [A_m\phi^{\mathrm{r}}_m + B_m\phi^{\mathrm{i}}_m], \tag{5}$$

where A_m and B_m are numerical coefficients. The field components B_x and B_y at $x=0$ are then given by:

$$\begin{aligned} -B_x &= A_1 + 2B_2y - 3A_3y^2 - 4B_4y^3 + 5A_5y^4 + 6B_6y^5 + \ldots, \\ -B_y &= B_1 - 2A_2y - 3B_3y^2 + 4A_4y^3 + 5B_5y^4 - 6A_6y^5 + \ldots, \end{aligned} \tag{6}$$

and at $y=0$ by:

$$\begin{aligned} -B_x &= \frac{\mathrm{d}}{\mathrm{d}r}\textstyle\sum_{m=1} A_m F_m(r), \\ -B_y &= \textstyle\sum_{m=1} m\, B_m F_{m-1}(r). \end{aligned} \tag{7}$$

From eq. (6) it is seen that there is a one-to-one relation between the coefficients A_m and B_m and the coefficients in the power-series expansions in y of the field components at $x=0$, showing that any field can be fitted by these solutions.

At $y=0$, letting:

$$-B_x = \textstyle\sum_{k=0} a_k x^k,$$

and:

$$-B_y = \textstyle\sum_{k=0} b_k x^k,$$

and using in eq. (7) the series expansion in x of $F_n(r)$, the following relations between the coefficients A_m, B_m and a_k, b_k are found:

$$\begin{aligned}
A_1 &= a_0, \\
A_2 &= \tfrac{1}{2}a_0 + \tfrac{1}{2}a_1, \\
A_3 &= -\tfrac{1}{6}a_0 + \tfrac{1}{6}a_1 + \tfrac{1}{3}a_2, \\
A_4 &= \tfrac{1}{24}a_0 - \tfrac{1}{24}a_1 + \tfrac{1}{6}a_2 + \tfrac{1}{4}a_3, \\
A_5 &= -\tfrac{1}{40}a_0 + \tfrac{1}{40}a_1 - \tfrac{1}{20}a_2 + \tfrac{1}{10}a_3 + \tfrac{1}{5}a_4, \\
A_6 &= \tfrac{1}{80}a_0 - \tfrac{1}{80}a_1 + \tfrac{1}{60}a_2 - \tfrac{1}{40}a_3 + \tfrac{1}{10}a_4 + \tfrac{1}{6}a_5, \\
A_7 &= -\tfrac{1}{112}a_0 + \tfrac{1}{112}a_1 - \tfrac{3}{280}a_2 + \tfrac{1}{70}a_3 - \tfrac{1}{35}a_4 + \tfrac{1}{14}a_5 + \tfrac{1}{7}a_6.
\end{aligned}$$

$$\begin{aligned}
B_1 &= b_0, \\
B_2 &= \tfrac{1}{2}b_1, \\
B_3 &= \tfrac{1}{6}b_1 + \tfrac{1}{3}b_2,
\end{aligned}$$

$$B_4 = -\tfrac{1}{24}b_1 + \tfrac{1}{12}b_2 + \tfrac{1}{4}b_3,$$
$$B_5 = \tfrac{1}{120}b_1 - \tfrac{1}{60}b_2 + \tfrac{1}{10}b_3 + \tfrac{1}{5}b_4,$$
$$B_6 = -\tfrac{1}{240}b_1 + \tfrac{1}{120}b_2 - \tfrac{1}{40}b_3 + \tfrac{1}{15}b_4 + \tfrac{1}{6}b_5,$$
$$B_7 = \tfrac{1}{560}b_1 - \tfrac{1}{280}b_2 + \tfrac{1}{140}b_3 - \tfrac{1}{70}b_4 + \tfrac{1}{14}b_5 + \tfrac{1}{7}b_6.$$

Any power of x in the fields thus introduces solutions of that order and higher. It is not possible to find any combination of solutions that equals the two-dimensional multipoles for more than one order, the lowest one occurring.

The solutions ϕ^{r}_m and ϕ^{i}_m can be related to solutions in terms of Bessel functions. As an example, consider the potential $\phi = J_0(r)\sinh y$. The field components at $x=0$ $(r=1)$ are compared with eq. (6), with the result that:

$$\phi = J_0(1)\left(\phi^{\mathrm{i}}_1 - \frac{1}{3!}\phi^{\mathrm{i}}_3 + \frac{1}{5!}\phi^{\mathrm{i}}_5 + \ldots\right) - J_1(1)\left(\frac{1}{2!}\phi^{\mathrm{i}}_2 - \frac{1}{4!}\phi^{\mathrm{i}}_4 + \frac{1}{6!}\phi^{\mathrm{i}}_6 + \ldots\right).$$

I wish to express my gratitude to Frank Krienen of CERN, whose solution in the case here called ϕ^{r}_2 led me to seek a generalization to other cases.