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Muon g-2 Note No. 123

Title: Multipole Expansion of the Magnetic Field in Toroidal Coordinates.

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Date: August 24, 1992

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I. Introduction

A central problem for the g-2 experiment is the calculation of the average magnetic field seen by the stored muons. Finding a multipole expansion of the field will facilitate this calculation and give a representation of the field content which may be useful during shimming. In this note, I show what this expansion would look like and propose what field measurements should be made to determine the expansion coefficients.

The vertical component of the field $B_z \approx |\vec{B}|$ satisfies Laplace's equation in the storage region and is in principle fully determined by its boundary values. If the field is represented by functions with zero laplacian (multipoles), then the errors of representation are largest at the boundary of the storage region. If these errors are acceptable, then the computed values inside the region are even more so. The best way to determine the multipole coefficients from a fixed number of measurements is to measure the field on the boundary.[1]

II. Toroidal Coordinates

Separation of Laplace's equation in toroidal coordinates[2] results in a boundary value problem appropriate for our case. The coordinates (σ, ψ, ϕ) (Fig. 1) are defined by the equations,

$$x = \frac{a \sinh\sigma \cos\phi}{\cosh\sigma - \cos\psi},\ y = \frac{a \sinh\sigma \sin\phi}{\cosh\sigma - \cos\psi},\ z = \frac{a \sin\psi}{\cosh\sigma - \cos\psi}. \tag{1}$$

As $s \equiv \cosh\sigma$ approaches infinity, the coordinate point approaches the circle of radius a in the x, y plane. We choose $a = 711.2$ cm to coincide with the beam axis. In the muon storage region, $s >> 1$, and to a good approximation

$$s = \frac{a}{r} + \frac{1}{2}\cos\theta,\ \tau = \cos\theta + \frac{r}{2a}\sin^2\theta, \tag{2}$$

where $\tau = \cos\psi$. Here r and θ are the 'polar' coordinates relative to the center of the storage region familiar from the two-dimensional problem (Fig. 2) . A solution to Laplace's equation which is finite at $r = 0$ is

$$u = (s-\tau)^{1/2}\cos n(\psi-\psi_o)\cos m(\phi-\phi_o)Q^m_{n-\frac{1}{2}}(s), \tag{3}$$

where $Q^m_{n-\frac{1}{2}}(s)$ is the associated Legendre function of the second kind. The Legendre function can be written as

$$Q^m_{n-\frac{1}{2}}(s) = K_{nm}\frac{(s^2-1)^{m/2}}{s^{n+m+1/2}}F\left(\frac{n+m}{2}+\frac{1}{4}, \frac{n+m}{2}+\frac{3}{4}; n+1; s^{-2}\right) \tag{4}$$

where K_{nm} is a normalization constant and F is the hypergeometric series

$$F = 1 + \frac{\left(\frac{n+m}{2}+\frac{1}{4}\right)\left(\frac{n+m}{2}+\frac{3}{4}\right)}{n+1}\frac{1}{s^2} + \cdots. \tag{5}$$

If we keep only the first term of the series, then, for large s, $(s-\tau)^{1/2}Q^m_{n-\frac{1}{2}}(s)$ becomes proportional to r^n, and the solution in Eq. 3 is approximately

$$u = r^n\cos n(\theta-\theta_o)\cos m(\phi-\phi_o), \tag{6}$$

and we can see the connection to the two-dimensional multipoles. This is a good approximation if s is large and if m and n are not too large.

III. Multipole Expansion

Assume that $(2N+1)$ NMR probes are mounted on the beam-tube trolley on a circle of radius $r_o \approx 4.5$ cm with the center of the circle in the x, y plane at radius R from the center of the storage ring. The trolley moves around the ring in $(2M+1)$ steps. The field will be mapped on the surface $\sigma = \sigma_o$ if we chose $R = a \coth \sigma_o = \sqrt{a^2 + r_o^2}$ (Fig. 3). Note that R differs from a by only 0.14 mm. The general solution is

$$\begin{aligned} B_z = (s-\tau)^{1/2} \Bigg\{ & \frac{b_{00}}{4} Q_{-\frac{1}{2}}(s) + \frac{1}{2} \sum_{n=1}^{N} (a_{n0} \sin n\psi + b_{n0} \cos n\psi) Q_{n-\frac{1}{2}}(s) \\ & + \frac{1}{2} \sum_{m=1}^{M} (d_{0m} \sin m\phi + b_{0m} \cos m\phi) Q^{m}_{-\frac{1}{2}}(s) \\ & + \sum_{n=1}^{N} \sum_{m=1}^{M} \left[a_{nm} \sin n\psi \cos m\phi + b_{nm} \cos n\psi \cos m\phi \right. \\ & \left. + \; c_{nm} \sin n\psi \sin m\phi + d_{nm} \cos n\psi \sin m\phi \right] Q^{m}_{n-\frac{1}{2}}(s) \Bigg\} , \end{aligned} \tag{7}$$

with the coefficients to be determined from the field measurements at $s_o = \cosh \sigma_o$,i.e.

$$\left. \begin{array}{c} a_{nm} \\ c_{nm} \end{array} \right\} = \frac{4}{Q^{m}_{n-\frac{1}{2}}(s_o)(2N+1)(2M+1)} \sum_{l=1}^{2M+1} \left\{ \begin{array}{c} \cos m\phi_l \\ \sin m\phi_l \end{array} \right\} \sum_{k=1}^{2N+1} \frac{\sin n\psi_k B_z(s_o, \psi_k, \phi_l)}{(s_o - \cos\psi_k)^{1/2}}, \tag{8}$$

and

$$\left. \begin{array}{c} b_{nm} \\ d_{nm} \end{array} \right\} = \frac{4}{Q^{m}_{n-\frac{1}{2}}(s_o)(2N+1)(2M+1)} \sum_{l=1}^{2M+1} \left\{ \begin{array}{c} \cos m\phi_l \\ \sin m\phi_l \end{array} \right\} \sum_{k=1}^{2N+1} \frac{\cos n\psi_k B_z(s_o, \psi_k, \phi_l)}{(s_o - \cos\psi_k)^{1/2}}. \tag{9}$$

The expressions in Eqs. 8 and 9 are true only if the field measurements are done at equal intervals in ϕ and ψ. From Fig. 3, however, equal intervals in ψ imply equal intervals in θ_o to within about 3 mrad.

The separation of variables in toroidal coordinates is not as complete as it is in cartesian, cylindrical, or spherical coordinates. This is reflected in the factor $(s-\tau)^{1/2}$ in Eq. 7. As a consequence of this, the representation of a constant field requires more than one nonzero multipole coefficient, e.g.[3]

$$B_o = \frac{B_o\sqrt{2}}{\pi}(s-\tau)^{1/2} \left[Q_{-\frac{1}{2}}(s) + 2 \sum_{n=1}^{N} \cos n\psi Q_{n-\frac{1}{2}}(s) \right]. \tag{10}$$

This can be seen by setting $B_z(s, \psi, \phi) = B_o$ in Eqs. 8 and 9 and using the identity[1]

$$\frac{1}{2N+1} \sum_{k=1}^{2N+1} \frac{\cos n\psi_k}{(s - \cos\psi_k)^{1/2}} = \frac{\sqrt{2}}{\pi} Q_{n-\frac{1}{2}}(s). \tag{11}$$

For the g-2 storage region, where the field is nearly uniform, the terms b_{n0} will have large contributions from the constant dipole field. The numerical accuracy required for calculating these terms will be reduced if the dipole field is subtracted. Note that on the beam axis ($s \to \infty$) only the $n = 0$ multipoles contribute to B_z, and the field on the axis which is independent of ϕ involves only the b_{00} term. From Eqs. 10 and 7, we see that a new expansion for $B_z - B_o$ will not include the b_{00} term if we choose $B_o = b_{00}\pi/(4\sqrt{2})$. From Eq. 9 and Eq. 11 for $n = 0$, B_o is the average of the field measurements weighted by $(s_o - \cos\psi_k)^{1/2}$. Since $s_o >> 1$, the weighting factor is nearly constant, and B_o is approximately the average of the field measurements.

IV. Conclusion

If we mount the NMR probes on a circle near the edge of the storage region, we can determine the multipole expansion of the field. For example, 13 NMR probes would allow us to measure normal and skew components up to $N = 6$. If the trolley moves 2 cm between measurements, then $M \approx 1100$, and the total number of measurements approaches 30,000. The most rapid azimuthal field variations will probably occur near the inflector, where the field changes significantly over distances of about $d = 20$ cm. Hence, the field multipoles with $m > 2\pi a/d \approx 200$ will be negligible.[5]

For the calculation of the average field, the muon distribution can be expanded in a double Fourier series with terms like

$$I_{nm} = \gamma_{nm}(s) \cos n\psi \cos m\phi, \tag{12}$$

and the field average will be a sum over terms with non-zero field multipoles and non-zero γ's.[6] The largest terms should be from the normal ($\cos n\psi$) terms with $m = 0, 4$.

Figures

FIG. 1. Toroidal coordinates. The coordinate surfaces are toroids (σ = const.), spherical bowls (ψ = const.), and meridian planes (ϕ = const.).

FIG. 2. Relation between toroidal coordinates and r and θ.

FIG. 3. Definition of R and r_o to give surfaces of constant σ for the NMR probes, and the relation between ψ and the angle θ_o of the probes.

References

[1]H. Wind, *Nucl. Inst. Methods* 84, 117 (1970).

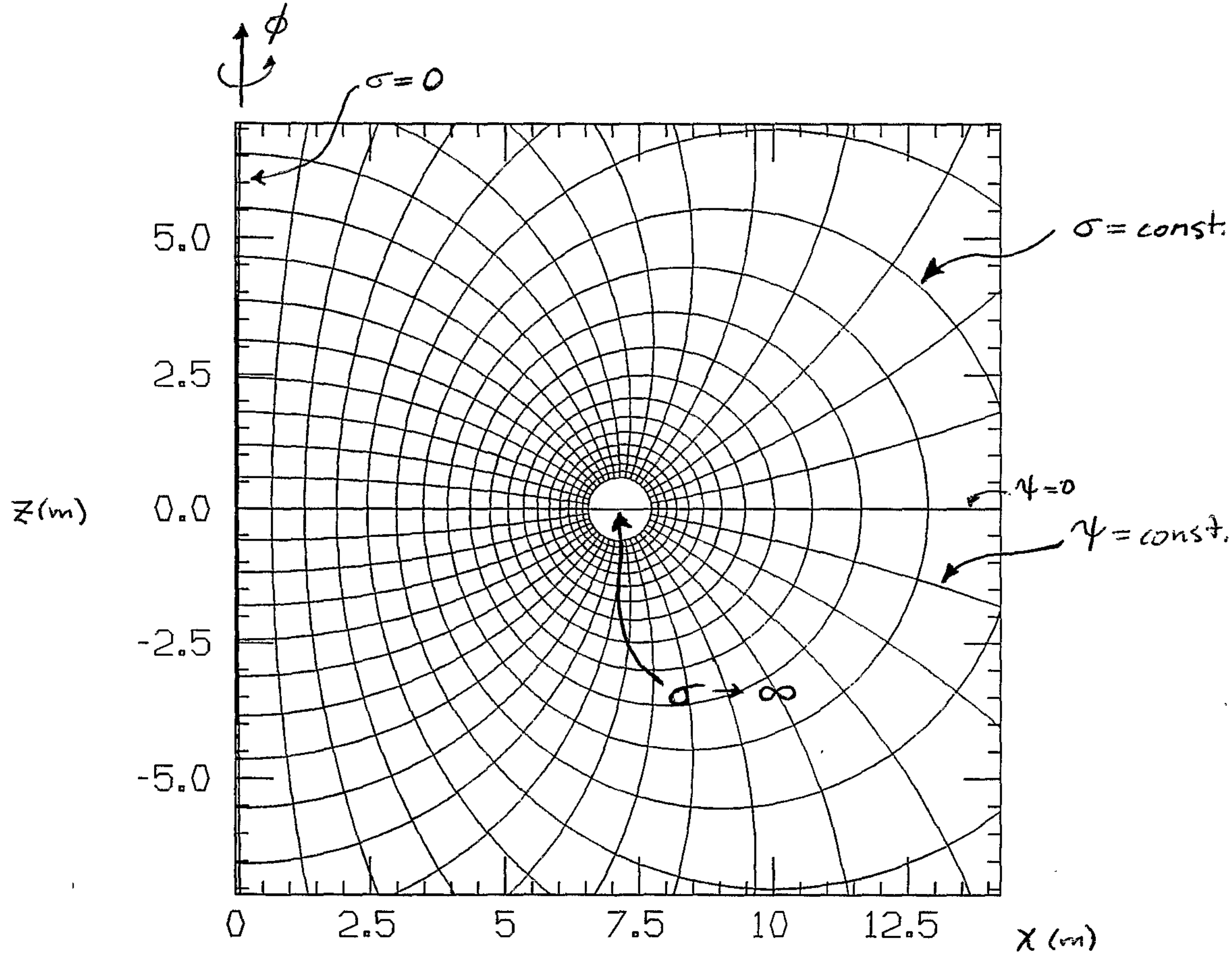
[2]H. Bateman, *Partial Differential Equations of Mathematical Physics*, (Dover, New York, 1944), pp. 461ff.

[3]Bateman, *op. cit.*, p. 465.

[4]Bateman, *op. cit.*, p. 464.

[5]William Morse and Francis Farley, g-2 note No. 116.

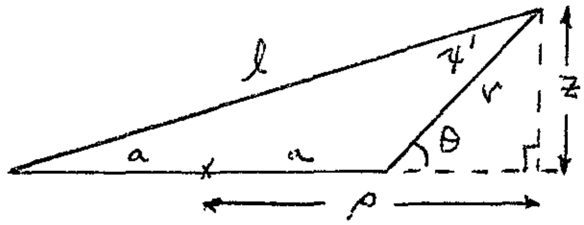
[6]Francis Farley and Walter Lysenko, g-2 note No. 23.



ϕ
$\sigma = 0$
$\sigma = const.$
$\psi = 0$
$\psi = const.$
$\sigma \to \infty$
Z(m)
5.0
2.5
0.0
-2.5
-5.0
0
2.5
5
7.5
10
12.5
X (m)

Fig. 1

Fig. 2



from Eq. 1 $\quad \rho = \dfrac{a \sinh\sigma}{\cosh\sigma - \cos\psi} \equiv \dfrac{a\alpha}{s-\tau}$, $\quad z = \dfrac{a\sin\psi}{\cosh\sigma - \cos\psi} \equiv \dfrac{ag}{s-\tau}$

with $l^2 = (\rho+a)^2 + z^2$, $\quad r^2 = (\rho-a)^2 + z^2$

and $\alpha^2 = s^2 - 1$, $\quad g^2 = 1 - \tau^2$ $\quad$ one can show

$$\frac{l}{r} = e^{\sigma} \ , \quad s = \frac{1}{2}\left(\frac{l}{r} + \frac{r}{l}\right), \text{ and } \ \frac{2a^2}{lr} = s - \tau$$

also using $4a^2 = l^2 + r^2 - 2lr\cos\psi'$, it follows $\psi' = \psi$

from $l^2 = r^2 + 4a^2 + 4ar\cos\theta$, $\quad l \approx 2a\left(1 + \frac{r}{2a}\cos\theta + \frac{r^2}{8a^2}\sin^2\theta\right)$

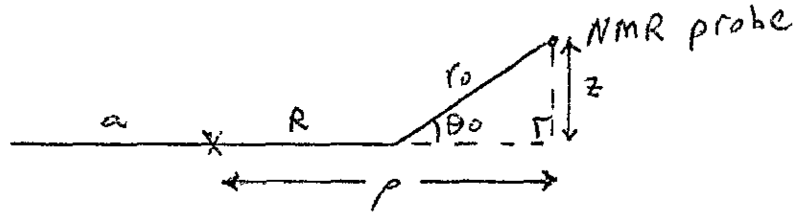
and $s = \frac{l}{2r}\left(1 + \frac{r^2}{l^2}\right)$, $\quad s \approx \frac{a}{r} + \frac{1}{2}\cos\theta + \frac{r}{4a}\left(1 + \frac{1}{2}\sin^2\theta\right)$

from $2a\cos\theta = l\cos\psi - r$

$$\cos\psi = \left(\cos\theta + \frac{r}{2a}\right)\left(1 + \frac{r}{a}\cos\theta + \frac{r^2}{4a^2}\right)^{-1/2}$$

$$\approx \cos\theta + \frac{r}{2a}\sin^2\theta - \frac{3r^2}{8a^2}\sin^2\theta\cos\theta$$

Fig. 3



$$r_0^2 = (\rho - R)^2 + z^2 = \rho^2 + z^2 + R^2 - 2\rho R$$

Using notation from Fig. 2 $\quad r_0^2 - R^2 = \dfrac{a^2(\alpha^2+q^2)}{(s_0-\tau)^2} - \dfrac{2a\alpha R}{s_0-\tau}$

but $\quad \alpha^2 + q^2 = s^2 - \tau^2$

so $\quad r_0^2 - R^2 = \dfrac{a^2(s_0+\tau) - 2a\alpha R}{s_0 - \tau}$

$$(r_0^2 - R^2 - a^2)s_0 + 2a\alpha R = (r_0^2 - R^2 + a^2)\tau$$

Requiring R, r_0 constant to give s_0 constant, independent of τ

means $\quad r_0^2 - R^2 + a^2 = 0 \quad , \quad R^2 = a^2 + r_0^2$

so $\quad -2a^2 s_0 + 2a\alpha R = 0 \qquad R = \dfrac{a}{\alpha} s_0 = a\coth\sigma$

$$r_0^2 = R^2 - a^2 = a^2\left(\frac{s_0^2}{\alpha^2} - 1\right) = \frac{a^2}{\alpha^2} \qquad r_0 = \frac{a}{\sinh\sigma}$$

finally $\quad \sin\theta_0 = \dfrac{z}{r_0} = \dfrac{aq}{s-\tau}\,\dfrac{\alpha}{a} = \dfrac{\sin\psi\sinh\sigma}{\cosh\sigma - \cos\psi}$

$$\cong \sin\psi\left(1 - \frac{r}{a}\cos\psi\right)$$