**Introduction to control systems**
**2020.1**

# TI0118 – Homework 1

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## Exercise 1

Consider the two tanks system arranged as shown in Figure 1.

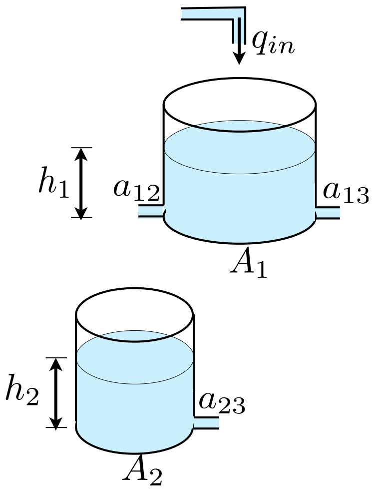



Figure 1: Two-tank system for Exercise 1.

The manipulated variable (input) is the volume flow rate $q_{in}$, while the measured variable (output) is the fluid level $h_2$ in the second tank. The areas of the two tanks are $A_1$ and $A_2$, while $a_{12}, a_{13}$ and $a_{23}$ are the areas of the orifices indicated in Figures 1. The fluid is perfect (no shear stresses, no viscosity, no heat conduction), and subject only to gravity[1]. The tanks are filled with water (incompressible fluid), and the external pressure is constant (atmospheric pressure).

1. Obtain a state-space model of the systzem, where $x_1(t) = h_1(t)$ and $x_2(t) = h_2(t)$ are state variables, $u(t) = q_{in}(t)$ is the input and $y(t) = h_2(t)$ is the output.

2. Given $a_{12} = a_{23} = a_{13} = 1$ m$^2$, $A_1 = 200$ m$^2$, $A_2 = 200$ m$^2$, find the equilibrium state $(\bar{x}_1, \bar{x}_2)$ obtained with a constant input $\bar{u} = 10$ m$^3$/s, approximating gravity acceleration as $g \approx 10$ m/s$^2$.

[1] Hint: The flow rate is given by the Torricelli law: $q_i(t) = a_{i,j}\sqrt{2gh_i(t)}$

3. Determine the linearized system $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ around the equilibrium state $(\bar{x}_1, \bar{x}_2)$.

4. In the programming language of your choice, simulate[2] the two systems response to a unit input step [3] and discuss your results.

## Solution

For the first tank, the equation that describes the variation of the level in the time is:

$$A_1 \frac{dh_1(t)}{dt} = q_{in} - q_{12} - q_{13} \tag{1}$$

For the second tank, the equation that describes the variation of the level in the time is:

$$A_2 \frac{dh_2(t)}{dt} = q_{12} - q_{23} \tag{2}$$

We can represent $q_{12}$, $q_{13}$, $q_{23}$ using Torricelli's law:

$$\begin{aligned} q_{12} &= a_{12}\sqrt{2gh_1(t)} \\ q_{13} &= a_{13}\sqrt{2gh_1(t)} \\ q_{23} &= a_{23}\sqrt{2gh_2(t)} \end{aligned}$$

Replacing in 1 and 2:

$$\frac{dh_1(t)}{dt} = \frac{1}{A_1} q_{in} - \frac{1}{A_1} a_{12}\sqrt{2gh_1} - \frac{1}{A_1} a_{13}\sqrt{2gh_1} \tag{3}$$

$$\frac{dh_2(t)}{dt} = \frac{1}{A_2} a_{12}\sqrt{2gh_1} - \frac{1}{A_2} a_{23}\sqrt{2gh_2} \tag{4}$$

The above equations describe the system behavior, but they are non-linear, as we want to represent the system using the state space model, we need to linearize these equations. To perform linearization, we can use Taylor's series to approximate the above functions to a linear model, using as a domain of the function the neighborhood of a point called a stationary point, at that point, the function is equal to zero. So, considering $x_s$ as the stationary point Taylor's series for the function going to be represent as follows:

$$f(x) = f(x_s) + (x - x_s)\frac{df(x)}{dx}\Big|_{x=x_s} + \sum_{n=2}^{\infty} \frac{1}{n!}\frac{d^n f(x)}{dx^n}\Big|_{x=x_s}(x - x_s)^n$$

$f(x_s) = 0$, because $x_s$ is the stationary point. From the neighborhood condition, we can approximate $(x - x_s)^n \ \forall n \geq 2$ to zero. In conclusion the function approximation is:

$$f(x) = (x - x_s)\frac{df(x)}{dx}\Big|_{x=x_s} \tag{5}$$

[2]Hint: In Matlab/Octave you can use the command `lsim` to simulate the linear system and `ode45` to simulate the non-linear system. Similarly, in Python you might use, respectively, `control.matlab.lsim` and `integrate.ode` specifying the integration method.

[3]Hint: In Matlab/Octave you might use the `step` command and `control.matlab.step` in Python.

Based on what was explained, the first step of the linearization process is to find a stationary point. The equations (3) and (4) have as variables: the input $q_{in}$,the level of the first tank $h_1$ and the level of the second tank $h_2$, so we need to find the values of these variables in the stationary condition, which means, find the values where functions are represented as follows:

$$0 = q_{in} - a_{12}\sqrt{2gh_1} - a_{13}\sqrt{2gh_1}$$
$$0 = a_{12}\sqrt{2gh_1} - a_{23}\sqrt{2gh_2}$$

We can use the information that $a_{12} = a_{13} = a_{23}$:

$$q_{in} = 2(a_{13}\sqrt{2gh_1})$$
$$\sqrt{2gh_1} = \sqrt{2gh_2}$$

Then:

$$\frac{q_{in}^2}{8ga_{13}^2} = h_1$$
$$\frac{q_{in}^2}{8ga_{13}^2} = h_2$$

Using the information provided by the question about the input, we found the following values for $h_1, h_2$:

$$\frac{10}{8} = 1.25 = h_1 = h_2 \tag{6}$$

In conclusion the stationary point is: $x_s = (1.25, 1.25, 10)$

Representing (3) and (4) as functions:

$$\frac{dh_1(t)}{dt} = f_1(h_1(t), h_2(t), q_{in}(t)) = \frac{1}{A_1}(q_{in} - a_{12}\sqrt{2gh_1(t)} - a_{13}\sqrt{2gh_1(t)}) \tag{7}$$
$$\frac{dh_2(t)}{dt} = f_2(h_1(t), h_2(t), q_{in}(t)) = \frac{1}{A_2}(a_{12}\sqrt{2gh_1(t)} - a_{23}\sqrt{2gh_2(t)}) \tag{8}$$

Taking the partial derivative of (7):

$$\frac{\partial f_1(h_1(t), h_2(t), q_{in}(t))}{\partial h_1} = \frac{1}{A_1}\frac{\partial(-2a_{12}\sqrt{2gh_1(t)})}{\partial h_1} = \frac{-2a_{12}g}{A_1\sqrt{2gh_1}} = 0.02$$
$$\frac{\partial f_1(h_1(t), h_2(t), q_{in}(t))}{\partial h_2} = 0$$
$$\frac{\partial f_1(h_1(t), h_2(t), q_{in}(t))}{\partial q_{in}} = \frac{1}{A_1} = 0.005$$

Taking the partial derivative of (8):

$$\frac{\partial f_2(h_1(t), h_2(t), q_{in}(t))}{\partial h_1} = \frac{1}{A_2}\frac{\partial(a_{12}\sqrt{2gh_1(t)})}{\partial h_1} = \frac{a_{12}g}{A_2\sqrt{2gh_1}} = 0.01$$
$$\frac{\partial f_2(h_1(t), h_2(t), q_{in}(t))}{\partial h_2} = \frac{1}{A_2}\frac{\partial(-a_{23}\sqrt{2gh_2(t)})}{\partial h_2} = \frac{-a_{23}g}{A_2\sqrt{2gh_2}} = -0.01$$
$$\frac{\partial f_2(h_1(t), h_2(t), q_{in}(t))}{\partial q_{in}} = 0$$

Now using the Taylor's series expansion:

$$\frac{dh_1(t)}{dt} = -0.02(h_1(t) - 1.25) + 0(h_2(t) - 1.25) + 0.005(q_{in} - 10)$$
$$\frac{dh_2(t)}{dt} = 0.01(h_1(t) - 1.25) - 0.01(h_2(t) - 1.25)$$

The last equations are linear, then we can use the state space model representation:

$$\begin{bmatrix} \frac{dh_1}{dt} \\ \frac{dh_2}{dt} \end{bmatrix} = \begin{bmatrix} -0.02 & 0 \\ 0.01 & -0.01 \end{bmatrix} \begin{bmatrix} h_1(t) - 1.25 \\ h_2(t) - 1.25 \end{bmatrix} + \begin{bmatrix} 0.005 \\ 0 \end{bmatrix} (q_{in}(t) - 10) \tag{9}$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} h_1(t) - 1.25 \\ h_2(t) - 1.25 \end{bmatrix} \tag{10}$$

Using the developed model and the nonlinear equation of $h_2(t)$ we can simulate the system output behavior due to a step with magnitude 10.

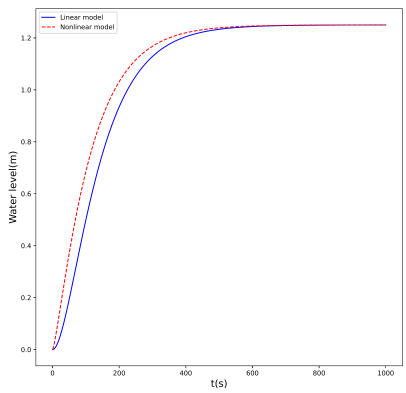



Figure 2: The linear model and nonlinear model step response

We can evaluate this model by calculating the mean square error, comparing the output of the non-linear model with the linear model.

$$MSE = \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y}_i)^2$$

The MSE of the data in the plot is 0.005118364763179614, if we modify the input magnitude and check the MSE again we get the following graph:

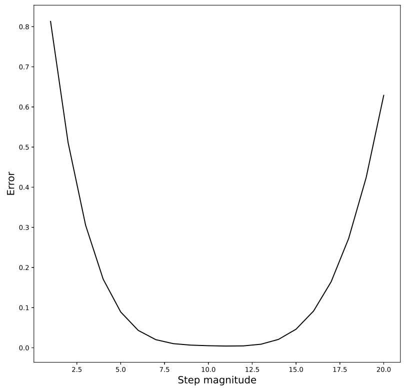



Figure 3: The Mean square error of the output comparing the linear and nonlinear model

The graph above shows that the model developed represents a good description of the system for steps with a magnitude close to 10, this is a consequence of the considerations taken in the linearization process, in it we use the input with magnitude 10, so it is expected that the model is better at his neighborhood.

## Exercise 2

Consider the input-output model in 11

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 5y(t) = \frac{du(t)}{dt} + 5u(t) \tag{11}$$

1. Define the characteristic polynomial and plot the system modes.
2. Given the initial conditions in (7), find the free evolution of the system in (5):
$$y(t)\Big|_{t=0} = 2, \quad \frac{dy(t)}{dt}\Big|_{t=0} = 1 \tag{12}$$
3. Find the forced response of the system subject to a unit step input.
4. Plot the response $y(t)$ and comment on your results.

### Solution

The equation (11) describes a linear system because it follows the superposition principle. Based on this, the system response can be divided into a forced response and a free response. The force response does not consider the initial states of the system, it only evaluates the response to the input. The free answer does not consider the input and only considers the initial conditions. We represent as follows:

$$y(t) = y_0(t) + y_u(t)$$

Where $y_0(t)$ represents the free output response and $y_u(t)$ is the forced output response. To find them we can apply the Laplace transform in the equation (11).

Before find the forced and free response due to information from the question, let's present the characteristica polynomial. To find it, it's only necessary to apply the Laplace transform in the (11). We find:

$$s^2Y(s) + 4sY(s) + 5Y(s) - (sy(0) + \dot{y}(0) + 4y(0)) = sU(s) + 5U(s)$$

We can manipulate the equation to the next:

$$Y(s)(s^2 + 4s + 5) = sU(s) + 5U(s) + (sy(0) + \dot{y}(0) + 4y(0)) \tag{13}$$

The characteristic polynomial is the term that multiply Y(s), it is:

$$p(s) = s^2 + 4s + 5 \tag{14}$$

In the equation (13) we used the forced and free response considerations:

- Considering only the step input effect:

$$\begin{aligned} Y_f(s)(s^2 + 4s + 5) &= \frac{1}{s}(s + 5) \\ Y_f(s) &= \frac{(s + 5)}{s(s^2 + 4s + 5)} \end{aligned}$$

- Considering only the initial conditions effect:

$$\begin{aligned} Y_0(s)(s^2 + 4s + 5) &= sy(0)\dot{y}(0) + 4y(0) = 2s + 9 \\ Y_0(s) &= \frac{2s + 9}{(s^2 + 4s + 5)} \end{aligned}$$

Now it's only necessary to find the inverse Laplace transform of each equation, in 3 we can find the inverse Laplace computationally, the next steps going to use partial fractions way. Let's begin with the forced response.

Before divide the equation of $Y_f(s)$ into partial fractions we need to find the roots of the characteristic polynomial, using some mathematical techniques approach the roots are:

$$\begin{aligned} s_1 &= -2 + i \\ s_2 &= -2 - i \end{aligned}$$

Then the partial fraction for the forced response is:

$$Y_f(s) = \frac{A}{s} + \frac{K}{s + 2 - i} + \frac{K^*}{s + 2 + i} \tag{15}$$

The residues are:

$$\begin{aligned} A &= \lim_{s \to 0} \frac{s + 5}{s^2 + 4s + 5} = 1 \\ K &= \lim_{s \to -2+i} = \frac{(s + 5)}{s(s + 2 + i)} = \frac{3 + i}{(-2 + i)(2i)} = \frac{3 + i}{-(2 + 4i)} = \frac{-(3 + i)(2 - 4i)}{20} = -0.5 + 0.5i \\ K^* &= -0.5 - 0.5i \end{aligned}$$

Now applying the inverse Laplace transform:

$$y_f(t) = 1 + (-0.5 + 0.5i)e^{(-2+i)t} + (-0.5 - 0.5i)e^{(-2-i)t}$$

We can represent every complex number using trigonometrical representation, for example:

$$(-0.5 + 0.5i) = -0.5(1 - i) = -0.5\sqrt{2}(\cos(0.785) - i\sin(0.785)) \tag{16}$$
$$(-0.5 - 0.5i) = -0.5(1 + i) = -0.5\sqrt{2}(\cos(0.785) + i\sin(0.785)) \tag{17}$$

We also can use the Euler's formula to find:

$$e^{(-2+i)t} = e^{-2t}(\cos(t) + i\sin(t)) \tag{18}$$
$$e^{(-2-i)t} = e^{-2t}(\cos(t) - i\sin(t)) \tag{19}$$

Then we have, for $\theta = 0.785$

$y_f(t) = 1 - 0.5\sqrt{2}e^{-2t}((\cos(t) + i\sin(t))(\cos(\theta) - i\sin(\theta)) + (\cos(\theta) + i\sin(\theta))(\cos(t) - i\sin(t)))$
$y_f(t) = 1 - 0.5\sqrt{2}e^{-2t}(2\cos(t)\cos(\theta) + 2\sin(t)\sin(\theta))$
$y_f(t) = 1 - \sqrt{2}e^{-2t}(\cos(t - \theta))$
$y_f(t) = 1 - 1.41e^{-2t}(\cos(t - 0.785))$

We can plot the last equation using the code in 4.

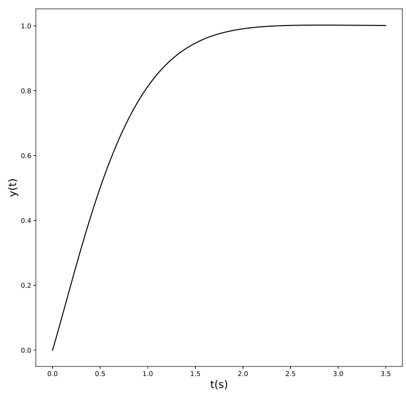



Figure 4: Forced response.

Finding the free response we need to following the same steps now for the equation:

$$Y_0(s) = \frac{2s + 9}{(s^2 + 4s + 5)}$$

The partial fractions are:

$$\frac{2s + 9}{(s^2 + 4s + 5)} = \frac{B}{(s + 2 - i)} + \frac{B^*}{(s + 2 + i)}$$

The residues are:

$$B = \lim_{s \to -2+i} \frac{(2s+9)}{s+2+i} = \frac{5+2i}{2i} = 1 - \frac{5}{2}i$$
$$B^* = 1 + \frac{5}{2}i$$

Then:

$$y_0(t) = (1 - \frac{5}{2}i)e^{-2t+it}(1 + \frac{5}{2}i)e^{-2t-it}$$

Taking the same steps that was made for the forced response we have:

$$y_0(t) = 5.38e^{-2t}\cos(t - 1.19)$$

Then the plot is:

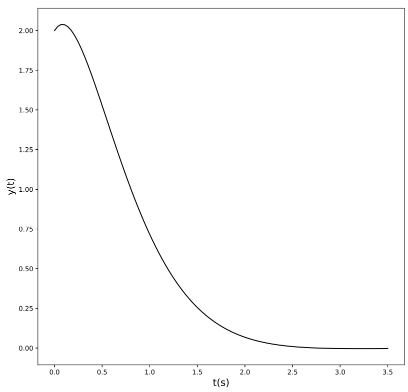



Figure 5: Free response.

In conclusion y(t) going to be:

$$y(t) = 1 + 5.38e^{-2t}\cos(t - 1.19) - 1.41e^{-2t}(\cos(t - 0.785)) \tag{20}$$

Using the code in 6 we find the system output.

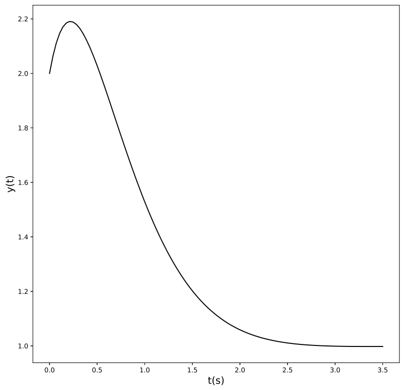



Figure 6: System response.

It's possible to see that the

# Exercise 3

For a system with the transfer function in (21):

$$G(s) = \frac{\alpha s^2 + s + 3}{s^2 + 2s + 10} \tag{21}$$

1. For $\alpha = 1$, find and plot the unit step and impulse responses.
2. For $\alpha = [-2,\ -1,\ 0,\ 1,\ 2]$, plot and compare the unit step response.
3. Discuss how the system varies its response for the different values of $\alpha$.

## Solution

For $\alpha = 1$, then we get:

$$G(s) = \frac{s^2 + s + 3}{s^2 + 2s + 10}$$

We can divide the polynomial above and get:

$$G(s) = 1 - \frac{s + 7}{s^2 + 2s + 10} \tag{22}$$

To find the response to the impulse it's only necessary applying the inverse Laplace transform of $G(s)$.

The inverse Laplace transform $G(s)$ can be represented as:

$$g(t) = \delta(t) + \mathcal{L}^{-1}\left[\frac{s + 7}{s^2 + 2s + 10}\right]$$

Where $\delta(t)$ is the impulse function. In order to calculate the other term we find the roots of the denominator, they are:

$$s_1 = -1 + 3i$$
$$s_2 = -1 - 3i$$

Then we can define the partial fractions:

$$\frac{s+7}{s^2+2s+10} = \frac{A}{s+1-3i} + \frac{A^*}{s+1+3i}$$

Where $A,\ A^*$ are complex conjugated. To find them we perform the following:

$$A = \lim_{s \to -1+3i} \frac{s+7}{s+1+3i} = \frac{6+3i}{6i} = 0.5 - i$$
$$A^* = 0.5 + i$$

Then:

$$\mathcal{L}^{-1}\left[\frac{s+7}{s^2+2s+10}\right] = (0.5-i)e^{(-1+3i)t} + (0.5+i)e^{(-1-3i)t}$$

We can represent $0.5+i$ and $0.5-i$ as:

$$0.5 + i = \sqrt{0.5^2+1}(\cos\theta + i\sin\theta)$$
$$0.5 - i = \sqrt{0.5^2+1}(\cos\theta - i\sin\theta)$$

Where: $\sqrt{0.5^2+1} = 1.118$ and $\theta = 1.107$.

$$\frac{1}{2}e^{-t}\left((1-2i)\,e^{3it} + (1+2i)\,e^{-3it}\right) = \sqrt{5}e^{-t}\cos(3t - 1.107) \tag{23}$$

Consequently the impulse response is:

$$g(t) = \delta(t) - \sqrt{5}e^{-t}\cos\left(3t - 1.107\right) \tag{24}$$

Using the code in 8 we can check the answer. We can plot the impulse response using the code in 9.

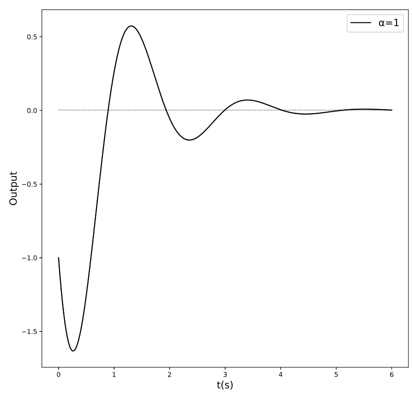



Figure 7: This is the impulse response when $\alpha$ =1.

To find the response for the step then we need to multiply G(s) by the Laplace transform of the step, $\frac{1}{s}$, then:

$$Y(s) = \frac{1}{s} - \frac{1}{s}\frac{s+7}{(s^2+2s+10)} \tag{25}$$

The inverse Laplace transform can be represented as follow:

$$y(t) = 1 - \mathcal{L}^{-1}\left[\frac{s+7}{s(s^2+2s+10)}\right]$$

Now, lets show the partial fractions for the second term:

$$\left[\frac{s+7}{s(s^2+2s+10)}\right] = \frac{B}{s} + \frac{K}{s+1-3i} + \frac{K^*}{s+1+3i}$$

Calculating the residues:

$$\begin{aligned} B &= \lim_{s\to 0}\frac{s+7}{(s^2+2s+10)} = 0.7 \\ K &= \lim_{s\to -1+3i} = \frac{6+3i}{6i(-1+3i)} = -0.35 - 0.05i \\ K^* &= -0.35 + 0.05i \end{aligned}$$

So we can represent the step response as:

$$y(t) = 0.3 + (0.35+0.05i)e^{(-1+3i)t} + (0.35-0.05i)e^{(-1-3i)t} \tag{26}$$

Set in the trigonometrical representation we have:

$$y(t) = 0.3 + 0.7e^{-t}\cos(3t+0.142) \tag{27}$$

Using the code in 9 we can check the inverse Laplace transform and using 7 we get the plot for the step response.

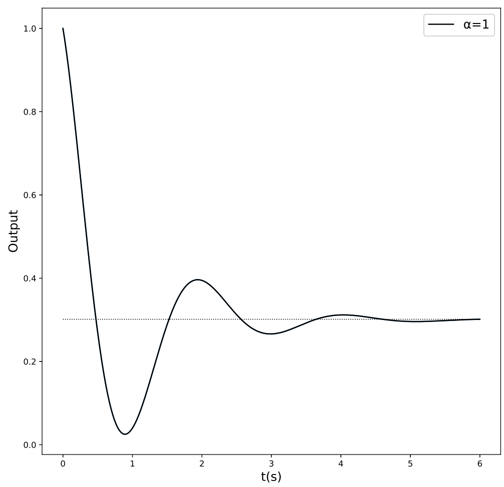



Figure 8: This is the step response when $\alpha$ =1.

Using the code in 11 we can find the step response for different values of $\alpha$:

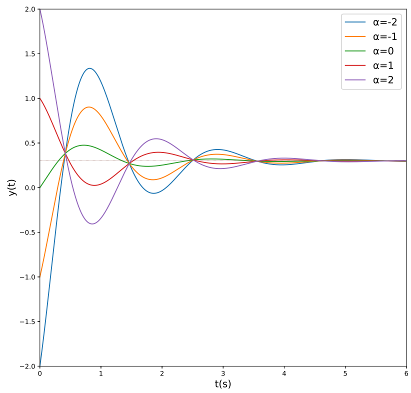



Figure 9: Step response for different values of $\alpha$

The modification of the value of alpha, modifies how the system varies it's input comparing to the steady state value, the overshoot. Also $\alpha$ modifies the initial condition value of y(t).

## Exercise 4

Given the state-space model in (12):

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ \frac{}{4} \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 13 & 9 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases} \tag{28}$$

1. Find a the corresponding transfer function.
2. Find an input-output model equivalent to the state-space model.
3. Find the state and output forced evolution as response of the input $u(t) = e^{-t}$.

### Solution

Let's find the transfer function for a state space equation, first for the state space model we have this two equations:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

The first equation is called the state equation and the second is the output equation. Considering the initial condition equal to zero and applying the Laplace transform to the state equation:

$$
\begin{aligned}
sX(s) &= AX(s) + BU(s) \\
X(s)(sI - A) &= BU(s) \\
X(s) &= (sI - A)^{-1}BU(s)
\end{aligned}
$$

Also applying the Laplace transform in the output equation and replacing the $X(s)$ value:

$$Y(s) = C(sI - A)^{-1}BU(s) + DU(s)$$

In the equation (28) D is a null matrix, then the transfer function will going to be:

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B \tag{29}$$

To calculate the transfer function we used the code in 12, and it result in:

$$\frac{Y(s)}{U(s)} = \frac{2.25s + 3.25}{s^2 + 5s + 4} \tag{30}$$

To find the input-output description, we only need to manipulate the equation (11) as:

$$Y(s)(s^2 + 5s + 4) = U(s)(2.25s + 3.25)$$

Now applying the inverse Laplace transform in both sides:

$$\ddot{y}(t) + 5\dot{y}(t) + 4y(t) = 2.25\dot{u}(t) + 3.25u(t) \tag{31}$$

To find the state forced response of the system we use the equations for each space in time:

$$
\begin{aligned}
\dot{x_1}(t) &= x_2(t) \\
\dot{x_2}(t) &= -4x_1(t) - 5x_2(t) + \frac{1}{4}u(t)
\end{aligned}
$$

For a $u(t) = e^{-t}$ and applying a Laplace transform:

$$
\begin{aligned}
sX_1(s) &= X_2(s) \\
sX_2(s) &= -4X_1(s) - 5X_2(t) + \frac{1}{4(s+1)}
\end{aligned}
$$

Finding $X_1(s)$:

$$
\begin{aligned}
X_1(s)(s^2 + 5s + 4) &= \frac{1}{4(s+1)} \\
X_1(s) &= \frac{1}{4(s+1)(s^2 + 5s + 4)}
\end{aligned}
$$

Now let's apply the inverse Laplace transform to find $x_1(t)$. First let's find the partial fraction for to represent $X_1(s)$. The roots of the numerator polynomial are:

$$
\begin{aligned}
s_1 &= -1 \\
s_2 &= -4
\end{aligned}
$$

The $X_1(s)$ is:

$$X_1(s) = \frac{1}{4(s+4)(s+1)^2} = \frac{A}{(s+4)} + \frac{B}{(s+1)} + \frac{C}{(s+1)^2}$$

The residues are:

$$A = \lim_{s \to -4} \frac{1}{4(s+1)^2} = \frac{1}{36} = 0.028$$
$$C = \lim_{s \to -1} \frac{1}{4(s+4)} = \frac{1}{12} = 0.083$$
$$B = \lim_{s \to -1} \frac{d\frac{1}{4(s+4)}}{ds} = -0.028$$

Then:

$$x_1(t) = 0.028e^{-4t} + 0.083te^{-t} - 0.028e^{-t} \tag{32}$$

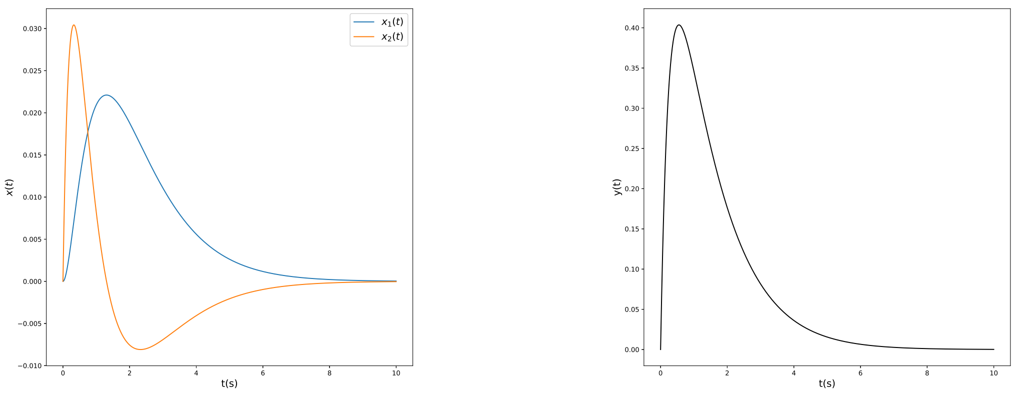
$x_2$ is derivative of $x_1(t)$, then:

$$x_2(t) = 0.028e^{-t} - 0.083te^{-t} - 0.112te^{-4t} \tag{33}$$

The $y(t)$ value is:

$$y(t) = 13x_1(t) + 9x_2(t) = (1.833t - 0.611)e^{-t} + 0.611e^{-4t} \tag{34}$$

The plot of the two states and the output are:




(a) This the state variation in time

(b) This is the output response in time

Figure 10: System output and state variation in time

## Exercise 5

Find the inverse Laplace transform by hand calculations and verify your results using the Symbolic toolbox for the following functions:

1. $F_1(s) = \dfrac{s-10}{(s+2)(s+5)}$

2. $F_2(s) = \dfrac{100}{(s+1)(s^2+4s+13)}$

3. $F_3(s) = \dfrac{s+18}{s(s+3)^2}$

Solution

First Item

The item already informed the roots of the denominator. So the partial fractions representation of $F_1(s)$ is:

$$F_1(s) = \frac{s-10}{(s+2)(s+5)} = \frac{A}{(s+2)} + \frac{B}{(s+5)} \tag{35}$$

The residues of the partial fractions are:

$$\begin{aligned} A &= \lim_{s\to -2} \frac{s-10}{s+5} = -4 \\ B &= \lim_{s\to -5} \frac{s-10}{s+2} = 5 \end{aligned}$$

Then the inverse Laplace transform is:

$$f(t) = -4e^{-2t} + 5e^{-5t} \tag{36}$$

We can check this value using the code in 16.

Second Item

The item gives just one root of the denominator, $s_1 = 1$, consequently we need to find the roots of the polynomial before to find the partial fraction representation. Then we can calculate the discriminant:

$$\Delta = 16 - 52 = -36 \tag{37}$$

Consequently the roots of the polynomial are:

$$\begin{aligned} s_2 &= \frac{-4+\sqrt{-36}}{2} = \frac{-4+6i}{2} = -2+3i \\ s_3 &= \frac{-4-\sqrt{-36}}{2} = \frac{-4-6i}{2} = -2-3i \end{aligned}$$

So the partial fractions are:

$$\frac{100}{(s+1)(s^2+4s+13)} = \frac{100}{(s+1)(s+2-3i)(s+2+3i)} = \frac{A}{s+1} + \frac{B}{s+2-3i} + \frac{B^*}{s+2+3i}$$

We can prove that $B$ and $B^*$ are complex conjugate, then we only need to calculate one of them to find the other. Based on this we calculate the residues as:

$$A = \lim_{s \to -1} \frac{100}{(s+2-3i)(s+2+3i)} = \frac{100}{(1-3i)(1+3i)} = \frac{100}{10} = 10$$

$$B = \lim_{s \to -2+3i} \frac{100}{(s+1)(s+2+3i)} = \frac{100}{(-1+3i)(6i)} = \frac{100}{-(6i+18)} = \frac{-100(18-6i)}{360} = -5+1.67i$$

$$B^* = -5 - 1.67i$$

Now the inverse transform is:

$$f_2(t) = 10e^{-t} + (-5+1.67i)e^{(-2+3i)t} + (-5-1.67i)e^{(-2-3i)t}$$
$$f_2(t) = 10e^{-t} - ((5-1.67i)e^{(-2+3i)t} + (5+1.67i)e^{(-2-3i)t}$$

We represent $(5+1.67i)$ and $5-1.67i$ as:

$$5+1.67i = \sqrt{5^2+1.67^2}(\cos\theta + i\sin\theta)$$
$$5-1.67i = \sqrt{5^2+1.67^2}(\cos\theta - i\sin\theta)$$

Where $\theta = arctg(1.67/5) = 0.322$ and $\sqrt{5^2+1.67^2} = 5.27$.

Also, we can use Euler's formula:

$$e^{(-2+3i)t} = e^{-2t}(\cos(3t) + i\sin(3t))$$
$$e^{(-2-3i)t} = e^{-2t}(\cos(3t) - i\sin(3t))$$

Then:

$$f_2(t) = 10e^{-t} - 5.27e^{-2t}((\cos\theta - i\sin\theta)(\cos(3t) + i\sin(3t)) + (\cos\theta + i\sin\theta)(\cos(3t) - i\sin(3t))$$
$$f_2(t) = 10e^{-t} - 5.27e^{-2t}(2\cos\theta\cos(3t) + 2\sin\theta\sin(3t))$$
$$f_2(t) = 10e^{-t} - 10.54e^{-2t}(\cos(3t-\theta))$$

In conclusion:

$$f_2(t) = 10e^{-t} - 10.54(e^{-2t}(\cos(3t-0.32))) \tag{38}$$

We can check the same answer using the code in 17.

### Third item

The item already informed the roots of the denominator. So the partial fractions representation of $F_3(s)$ is:

$$\frac{s+18}{s(s+3)^2} = \frac{A}{s} + \frac{B}{(s+3)} + \frac{C}{(s+3)^2}$$

The residues are:

$$A = \lim_{s \to 0} \frac{s+18}{(s+3)^2} = \frac{18}{9} = 2$$
$$C = \lim_{s \to -3} \frac{s+18}{s} = \frac{15}{-3} = -5$$
$$B = \lim_{s \to -3} \frac{d\frac{(s+18)}{s}}{ds} = \lim_{s \to -3} \frac{-18}{s^2} = -2$$

In conclusion :

$$f(t) = 2 - 2e^{-3t} - 5e^{-3t}t \tag{39}$$

We can check the answer using the code in 18

# Appendix

- Obs 1: I created a online Notebook with the code for solving each question of this homework : Notebook with the answer. The notebook is a living representation of the next section source code.

## A Source code for exercise 1

Listing 1: System Nonlinear function simulation

```
def nonlineartank(x,t,params):
  [x1,x2]=x
  [A1,A2,u,mi1,mi2,mi3,g] = params
  f1 = (1/A1)*(u-2*mi1*((2*g*x1)**(0.5)))
  f2 = (1/A2)*((mi1*((2*g*x1)**0.5))-(mi2*((2*g*x2)**0.5)))
  dx =[f1,f2]
  return dx
from scipy.integrate import odeint
T=np.arange(0,1000,0.001)
A1 =200
A2 =200
mi1 = 1
mi2 = 1
mi3 = 1
g = 10
u = 1
yode=odeint(nonlineartank,[0,0],T,args=([A1,A2,u,mi1,mi2,mi3,g],))
plt.plot(T,yode[:,1],'k')
plt.xlabel('t(s)',fontsize=15)
plt.ylabel('Water level (m)',fontsize=15)
plt.rcParams["figure.figsize"] = (10,10)
plt.savefig('Question1Nonlinear.eps', format='eps')
```

Listing 2: System linear simulation function

```
import control.matlab as ctrlmatlab
import matplotlib.pyplot as plt
A = np.matrix([[-0.02,0],[0.01,-0.01]])
B = np.matrix([[5*(10**-3)],[0]])
C = np.matrix([0,1])
D = 0
sys = ctrl.ss(A,B,C,D)
T=np.arange(0,1000,0.001)
U = 10*np.ones_like(T)-10*np.ones_like(T)
y=ctrlmatlab.lsim(sys,U,T,[-1.25,-1.25])
y = np.array(y[0])
plt.plot(T,y+1.25*np.ones_like(y),'k')
plt.xlabel('t(s)',fontsize=15)
plt.ylabel('Water level(m)',fontsize=15)
plt.rcParams["figure.figsize"] = (10,10)
plt.savefig('Question1Linear.eps', format='eps')
```

## B Source code for exercise 2

Listing 3: The inverse Laplace transform of the forced response

```
import sympy as sp
s, t,alfa = sp.symbols('s,t,alfa')
t = sp.Symbol('t', positive=True)
expression = (s+5)/(s*(s**2+4*s+5))
print(sp.inverse_laplace_transform(expression,s,t))
```

Listing 4: The inverse Laplace transform of the free response

```
import sympy as sp
s, t,alfa = sp.symbols('s,t,alfa')
t = sp.Symbol('t', positive=True)
expression = (2*s+9)/(s**2+4*s+5)
print(sp.inverse_laplace_transform(expression, s, t))
```

Listing 5: The forced response simulation

```
sysforced=ctrl.tf([1,5],[1,4,5])
Time,yforced=ctrl.step_response(sysforced)
plt.plot(Time,yforced,'k')
plt.xlabel('t(s)',fontsize=15)
plt.ylabel('y(t)',fontsize=15)
plt.rcParams["figure.figsize"] = (10,10)
plt.rc('legend', fontsize=15)
```

Listing 6: The free response simulation

```
sysfree=ctrl.tf([2,9],[1,4,5])
Time,yfree=ctrl.impulse_response(sysfree)
plt.plot(Time,yfree,'k')
plt.xlabel('t(s)',fontsize=15)
plt.ylabel('y(t)',fontsize=15)
plt.rcParams["figure.figsize"] = (10,10)
plt.rc('legend', fontsize=15)
```

## C Source code for exercise 3

Listing 7: Code to plot the step response for the first item

```
import control as ctrl
import matplotlib.pyplot as plt
import numpy as np
sys3 = ctrl.tf([1,1,3],[1,2,10])
T = np.arange(0,7,0.000001)
Time,Y = ctrl.step_response(sys3,T)
plt.plot(Time,Y)
label = str("\u03B1="+str(1))
plt.plot(Time,Y,'k',label=label)
plt.xlabel('t(s)',fontsize=15)
plt.ylabel('Output',fontsize=15)
plt.rcParams["figure.figsize"] = (10,10)
plt.rc('legend', fontsize=15)
#plt.grid(True)
plt.legend()
plt.savefig('Question3firstitem.eps', format='eps')
```

Listing 8: Code to verify the inverse Laplace transform for the impulse response

```
import sympy as sp
```

```
s, t,alfa = sp.symbols('s,t,alfa')
t = sp.Symbol('t', positive=True)
expression = (s+7)/(s**2+2*s+10)
sp.inverse_laplace_transform(expression, s, t)
```

Listing 9: Code to verify the inverse Laplace transform for the step response

```
import sympy as sp
s, t,alfa = sp.symbols('s,t,alfa')
t = sp.Symbol('t', positive=True)
expression = (s+7)/(s*(s**2+2*s+10))
sp.inverse_laplace_transform(expression, s, t)
```

Listing 10: Code to plot the impulse response

```
sys3 = ctrl.tf([1,1,3],[1,2,10])
Time,Y = ctrl.impulse_response(sys3)
plt.plot(Time,Y)
label = str("\u03B1="+str(1))
plt.plot(Time,Y,'k',label=label)
plt.xlabel('t(s)',fontsize=15)
plt.ylabel('Output',fontsize=15)
plt.rcParams["figure.figsize"] = (20,20)
plt.rc('legend', fontsize=15)
plt.legend()
plt.savefig('3impulseResponse.eps', format='eps')
```

Listing 11: Code to plot the step response for different alphas

```
vector = [-2,-1,0,1,2]
for x in vector:
  sys3 = ctrl.tf([x,1,3],[1,2,10])
  Time,Y = ctrl.step_response(sys3)
  label = str("\u03B1="+str(x))
  plt.plot(Time,Y,label=label)
  plt.plot(Time,Y[len(Y)-1]*np.ones(size),':k',linewidth=0.8)
  left, right = plt.xlim()
  plt.xlim(left=0)
  plt.xlim(right=6)
  bottom, top = plt.ylim()
  plt.ylim(top=2)
  plt.ylim(bottom=-2)
  plt.xlabel('t(s)',fontsize=15)
  plt.ylabel('y(t)',fontsize=15)
  plt.rcParams["figure.figsize"] = (10,10)
  plt.rc('legend', fontsize=15)
  plt.legend()
plt.savefig('Question3seconditem.eps', format='eps')
```

# D Source code for exercise 4

Listing 12: This is the code to find the transfer function of the system

```
A = np.matrix([[0,1],[-4,-5]])
B = np.matrix([[0],[1/4]])
C = [13,9]
D = [0]
ctrl.ss2tf(A,B,C,D)
```

Listing 13: This is the code to find the inverse Laplace transform of $x_1(t)$

```
import sympy as sp
s, t,alfa = sp.symbols('s,t,alfa')
t = sp.Symbol('t', positive=True)
expression = 1/(4*(s+4)*(s+1)**2)
print(sp.inverse_laplace_transform(expression,s,t))
```

Listing 14: This is the code to simulate the state in time

```
A = np.matrix([[0,1],[-4,-5]])
B = np.matrix([[0],[1/4]])
C = [13,9]
D = [0]
sys4 = ctrl.ss(A,B,C,D)
T = np.arange(0,10,0.001)
U = np.exp(-T)
y=ctrlmatlab.lsim(sys4,U,T)
x1 = []
x2 = []
for i in range(0,len(y[2])):
  x1.append(y[2][i][0])
  x2.append(y[2][i][1])
plt.plot(T,x1,label='$x_1(t)$')
plt.plot(T,x2,label='$x_2(t)$')
plt.xlabel('t(s)',fontsize=15)
plt.ylabel('$x(t)$',fontsize=15)
plt.rcParams["figure.figsize"] = (10,10)
plt.rc('legend', fontsize=15)
plt.legend()
plt.savefig("4stateplot.eps",format='eps')
```

Listing 15: This is the code to simulate the output in time

```
A = np.matrix([[0,1],[-4,-5]])
B = np.matrix([[0],[1/4]])
C = [13,9]
D = [0]
sys4 = ctrl.ss2tf(A,B,C,D)
T = np.arange(0,10,0.001)
U = np.exp(-T)
y=ctrlmatlab.lsim(sys4,U,T)
plt.plot(T,y[0],'k')
plt.xlabel('t(s)',fontsize=15)
plt.ylabel('y(t)',fontsize=15)
plt.rcParams["figure.figsize"] = (10,10)
plt.rc('legend', fontsize=15)
plt.savefig("4output.eps",format='eps')
```

# E Source code for exercise 5

Listing 16: This is the code to find the inverse Laplace transform of the first item

```
import sympy as sp
s, t = sp.symbols('s,t')
t = sp.Symbol('t', positive=True)
expression = (s-10)/((s+2)*(s+5))
sp.inverse_laplace_transform(expression, s, t)
```

Listing 17: This is the code to find the inverse Laplace transform of the second item

```
import sympy as sp
s, t = sp.symbols('s,t')
t = sp.Symbol('t', positive=True)
expression = 100/((s+1)*((s**2)+(4*s)+13))
sp.inverse_laplace_transform(expression, s, t)
```

Listing 18: This is the code to find the inverse Laplace transform of the third item

```
import sympy as sp
s, t = sp.symbols('s,t')
t = sp.Symbol('t', positive=True)
expression = (s+18)/(s*(s+3)**2)
sp.inverse_laplace_transform(expression, s, t)
```