Null Tetrads

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Date: 2024 年 9 月 20 日

Abstract

感谢 YouTube 博主@TensorCalculusRobertDavie的教学视频Null Vectors and Null Tetrads。鉴于笔者未能找到如此基础的中文资源，特针对其视频整理笔记如下。

1 Null Tetrads

Null Tetrads 是定义在时空上的一组基底切向量 $\{l, n, m, \overline{m}\}$，它们会随度规张量 g_{ab} 的不同而不同。

在任意时空中考虑，记其正交归一的四线元余切基矢组为 $\{v, i, j, k\}$，满足

$$v \cdot v = \eta_{\alpha\beta} v^\alpha v^\beta = v_\beta v^\beta = -1 \tag{1.1}$$

$$i \cdot i = \eta_{\alpha\beta} i^\alpha i^\beta = i_\beta i^\beta = j \cdot j = k \cdot k = 1 \tag{1.2}$$

且它们两两正交

定义

$$l = \frac{1}{\sqrt{2}}(v + i), \quad n = \frac{1}{\sqrt{2}}(v - i) \tag{1.3}$$

$$m = \frac{1}{\sqrt{2}}(j + \mathrm{i}k), \quad \overline{m} = \frac{1}{\sqrt{2}}(j - \mathrm{i}k) \tag{1.4}$$

利用基矢的关系很容易检验以下关系成立

- 均为零矢量 $l \cdot l = n \cdot n = m \cdot m = \overline{m} \cdot \overline{m} = 0$
- 正交性 $l \cdot m = l \cdot \overline{m} = n \cdot m = n \cdot \overline{m} = 0$
- 归一性 $l \cdot n = -1, \quad m \cdot \overline{m} = 1$

现在我们可以取切矢量基矢组为 $\{e_0, e_1, e_2, e_3\} = \{l, n, m, \overline{m}\}$

即有

$$e_0{}^\alpha = l^\alpha = \frac{1}{\sqrt{2}}(v^\alpha + i^\alpha), \quad e_1{}^\alpha = n^\alpha = \frac{1}{\sqrt{2}}(v^\alpha - i^\alpha) \tag{1.5}$$

$$e_2{}^\alpha = m^\alpha = \frac{1}{\sqrt{2}}(j^\alpha - \mathrm{i}k^\alpha), \quad e_3{}^\alpha = \overline{m}^\alpha = \frac{1}{\sqrt{2}}(j^\alpha + \mathrm{i}k^\alpha) \tag{1.6}$$

先考虑闵氏时空。利用 frame metric γ_{mn} 的性质，可求此基矢下的 frame metric 为

$$\gamma_{mn}=\begin{bmatrix} e_0\cdot e_0 & e_0\cdot e_1 & e_0\cdot e_2 & e_0\cdot e_3\\ e_1\cdot e_0 & e_1\cdot e_1 & e_1\cdot e_2 & e_1\cdot e_3\\ e_2\cdot e_0 & e_2\cdot e_1 & e_2\cdot e_2 & e_2\cdot e_3\\ e_3\cdot e_0 & e_3\cdot e_1 & e_3\cdot e_2 & e_3\cdot e_3 \end{bmatrix}=\begin{bmatrix} l\cdot l & l\cdot n & l\cdot m & l\cdot \overline{m}\\ n\cdot l & n\cdot n & n\cdot m & n\cdot \overline{m}\\ m\cdot l & m\cdot n & m\cdot m & m\cdot \overline{m}\\ \overline{m}\cdot l & \overline{m}\cdot n & \overline{m}\cdot m & \overline{m}\cdot \overline{m} \end{bmatrix}=\begin{bmatrix} 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{1.7}$$

容易检验 γ_{mn} 是它自己的逆，即有

$$\begin{bmatrix} 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.8}$$

故

$$\gamma_{mn}=\gamma^{mn} \tag{1.9}$$

$$\begin{bmatrix} e^0\\ e^1\\ e^2\\ e^3 \end{bmatrix}=\begin{bmatrix} 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} e_0\\ e_1\\ e_2\\ e_3 \end{bmatrix}=\begin{bmatrix} -e_1\\ -e_0\\ e_3\\ e_2 \end{bmatrix} \tag{1.10}$$

下面给出 frame metric 和 coordinate metric 的关系

$$g_\alpha=e^m{}_\alpha\gamma_m \tag{1.11}$$

$$g_{\alpha\beta}=e^m{}_\alpha\gamma_m e^n{}_\beta\gamma_n=\gamma_{mn}e^m{}_\alpha e^n{}_\beta \tag{1.12}$$

$$\gamma_m=e_m{}^\alpha\partial_\alpha=e_m{}^\alpha g_\alpha \tag{1.13}$$

特别地我们有

$$\mathrm{d}s^2=g_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta \tag{1.14}$$

$$=\gamma_{mn}e^m{}_\alpha\mathrm{d}x^\alpha e^n{}_\beta\mathrm{d}x^\beta \tag{1.15}$$

$$e^m=e^m{}_\alpha\mathrm{d}x^\alpha \tag{1.16}$$

$$e_m=e_m{}^\alpha\partial_\alpha \tag{1.17}$$

和前面类似的我们有

$$g^{\alpha\beta}=\gamma^{mn}e_m{}^\alpha e_n{}^\beta \tag{1.18}$$

现在代入 γ^{mn} 的具体形式

$$g^{\alpha\beta} = \gamma^{mn} {e_m}^{\alpha} {e_n}^{\beta} \tag{1.19}$$
$$= \gamma^{01} {e_0}^{\alpha} {e_1}^{\beta} + \gamma^{10} {e_1}^{\alpha} {e_0}^{\beta} + \gamma^{23} {e_2}^{\alpha} {e_3}^{\beta} + \gamma^{32} {e_3}^{\alpha} {e_2}^{\beta} \tag{1.20}$$
$$= -l^{\alpha} n^{\beta} - n^{\alpha} l^{\beta} + m^{\alpha} \overline{m}^{\beta} + \overline{m}^{\alpha} m^{\beta} \tag{1.21}$$

以及

$$g_{\alpha\beta} = -l_{\alpha} n_{\beta} - n_{\alpha} l_{\beta} + m_{\alpha} \overline{m}_{\beta} + \overline{m}_{\alpha} m_{\beta} \tag{1.22}$$

这就是常见的利用 Null Tetrads 表示度规张量的公式，这个公式是具有普遍性的。这个变换在研究对称性等方面具有一些明显的优势。

现在以闵氏时空为例。闵氏时空中有

$$\mathrm{d}s^2 = -c^2 \mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2 \tag{1.23}$$

故其余切基矢组为 $\{c\mathrm{d}t, \mathrm{d}r, r\mathrm{d}\theta, r\sin\theta\mathrm{d}\theta\}$。构造

$$l = \frac{1}{\sqrt{2}}(c\mathrm{d}t + \mathrm{d}r), \quad n = \frac{1}{\sqrt{2}}(c\mathrm{d}t - \mathrm{d}r) \tag{1.24}$$
$$m = \frac{1}{\sqrt{2}}(r\mathrm{d}\theta + \mathrm{i} r\sin\theta\mathrm{d}\phi), \quad \overline{m} = \frac{1}{\sqrt{2}}(r\mathrm{d}\theta - \mathrm{i} r\sin\theta\mathrm{d}\phi) \tag{1.25}$$

写成分量的形式

$$l_{\alpha} = \frac{1}{\sqrt{2}}(1,1,0,0), \quad n_{\alpha} = \frac{1}{\sqrt{2}}(1,-1,0,0) \tag{1.26}$$
$$m_{\alpha} = \frac{1}{\sqrt{2}}(0,0,r,\mathrm{i} r\sin\theta), \quad \overline{m}_{\alpha} = \frac{1}{\sqrt{2}}(0,0,r,-\mathrm{i} r\sin\theta) \tag{1.27}$$

由 $g_{\alpha\beta} = -l_{\alpha} n_{\beta} - n_{\alpha} l_{\beta} + m_{\alpha} \overline{m}_{\beta} + \overline{m}_{\alpha} m_{\beta}$

$$g_{00} = -l_0 n_0 - n_0 l_0 + m_0 \overline{m}_0 + m_0 \overline{m}_0 \tag{1.28}$$
$$= -\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + 0 + 0 \tag{1.29}$$
$$= -1 \tag{1.30}$$

类似地，可以求出 $g_{11} = 1, g_{22} = r, g_{33} = r\sin\theta$，则 $g_{\alpha\beta}$ 确为闵氏度规

$$g_{\alpha\beta} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix} \tag{1.31}$$

其逆为

$$g^{\alpha\beta} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{bmatrix} \tag{1.32}$$

利用 $g^{\alpha\beta}$ 可对 Null Tetrads 进行指标升降

$$l^\alpha = g^{\alpha\beta} l_\beta = \frac{1}{\sqrt{2}}(-1, 1, 0, 0) \tag{1.33}$$

类似地，得到

$$n^\alpha = \frac{1}{\sqrt{2}}(-1, -1, 0, 0), \quad m^\alpha = \frac{1}{\sqrt{2}}\left(0, 0, \frac{1}{r}, \frac{\mathrm{i}}{r\sin\theta}\right) \tag{1.34}$$

直接取复共轭就有

$$\overline{m}^\alpha = \frac{1}{\sqrt{2}}\left(0, 0, \frac{1}{r}, \frac{-\mathrm{i}}{r\sin\theta}\right) \tag{1.35}$$

可以检验切矢形式的 Null Tetrads 仍满足前文所提到的性质。

现在考虑一个更复杂但常用的情况。考虑 Advanced Eddington-Finkelstein 坐标 (u, r, θ, ϕ) 中的史瓦西度规

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{c^2 r}\right)\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}r + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \tag{1.36}$$

令

$$F = \left(1 - \frac{2GM}{c^2 r}\right) \tag{1.37}$$

度规的矩阵形式可以写为

$$g_{\alpha\beta} = \begin{bmatrix} -F & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix} \tag{1.38}$$

注意到

$$\begin{bmatrix} -F & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & F & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.39}$$

不妨先假设

$$l^\alpha = (l^u, l^r, l^\theta, l^\phi) = (0, 1, 0, 0) \tag{1.40}$$

则

$$l_\alpha = g_{\alpha\beta} l^\beta = (g_{ur} l^r, 0, 0, 0) = (1, 0, 0, 0) \tag{1.41}$$

m，$\overline{m}$ 与闵氏时空中的相同。现在目标是解出 n^α。

利用 $g_{\alpha\beta} = -l_\alpha n_\beta - n_\alpha l_\beta + m_\alpha \overline{m}_\beta + \overline{m}_\alpha m_\beta$

$$g_{uu} = -l_u n_u - n_u l_u + m_u \overline{m}_u + \overline{m}_u m_u \tag{1.42}$$

$$= -2 l_u n_u \tag{1.43}$$

$$= -2 n_u \tag{1.44}$$

$$= -F \tag{1.45}$$

故
$$n_u = \frac{F}{2} \tag{1.46}$$

$$\begin{aligned} g_{ur} &= -l_u n_r - n_u l_r + m_u \overline{m}_r + \overline{m}_u m_r \qquad (1.47)\\ &= -n_r \qquad (1.48)\\ &= 1 \qquad (1.49)\end{aligned}$$

故
$$n_r = -1 \tag{1.50}$$

于是得到
$$n_\alpha = (\frac{F}{2}, -1, 0, 0) \tag{1.51}$$

利用 $g^{\alpha\beta}$ 升指标
$$n^\alpha = (-1, -\frac{F}{2}, 0, 0) \tag{1.52}$$

由前文
$$m^\alpha = \frac{1}{\sqrt{2}}(0, 0, \frac{1}{r}, \frac{\mathrm{i}}{r\sin\theta}), \quad \overline{m}^\alpha = \frac{1}{\sqrt{2}}(0, 0, \frac{1}{r}, \frac{-\mathrm{i}}{r\sin\theta}) \tag{1.53}$$

综上
$$l^\alpha = (0, 1, 0, 0), \quad l_\alpha = (1, 0, 0, 0) \tag{1.54}$$
$$n^\alpha = (-1, -\frac{F}{2}, 0, 0), \quad n_\alpha = (\frac{F}{2}, -1, 0, 0) \tag{1.55}$$
$$m^\alpha = \frac{1}{\sqrt{2}}(0, 0, \frac{1}{r}, \frac{\mathrm{i}}{r\sin\theta}), \quad m_\alpha = \frac{1}{\sqrt{2}}(0, 0, r, \mathrm{i}r\sin\theta) \tag{1.56}$$
$$\overline{m}^\alpha = \frac{1}{\sqrt{2}}(0, 0, \frac{1}{r}, \frac{-\mathrm{i}}{r\sin\theta}), \quad \overline{m}_\alpha = \frac{1}{\sqrt{2}}(0, 0, r, -\mathrm{i}r\sin\theta) \tag{1.57}$$