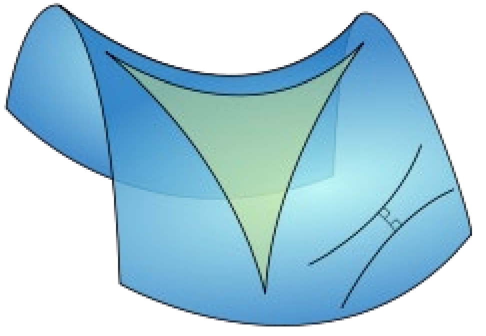
Differential Geometry of Curves

Some Problems



Space Curves I

- Reminder:
  - Parametric curve

$$C(t) = (x(t), y(t), z(t)\ )$$

  - Arc length parameterization

$$C(s) = (x(s), y(s), z(s)\ )$$

$$\left\|\dot{C}(s)\right\| = 1$$

Space Curves I

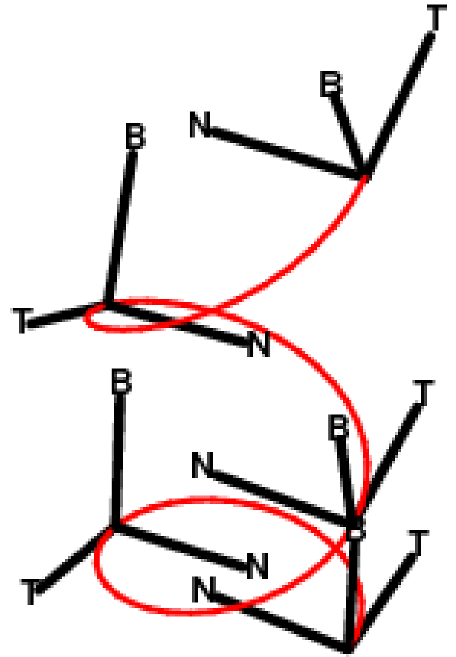
- Reminder:
  - Tangent vector:

$$T = \dot{C}(s) = \big(\dot{x}(s), \dot{y}(s), \dot{z}(s)\big)$$

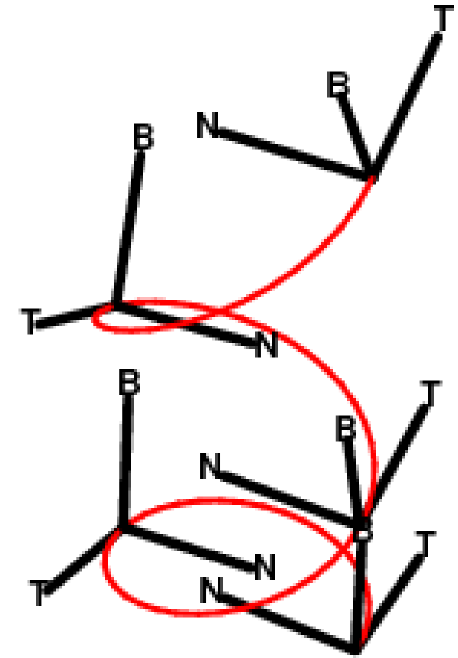
  - Curvature normal:

$$\kappa \vec{N} = \ddot{C}(s)$$

$$\kappa = \left\|\ddot{C}(s)\right\| = \frac{\left\|\dot{C}(t) \times \ddot{C}(t)\right\|}{\left\|\dot{C}(t)\right\|^3}$$



Space Curves I



- Reminder:
  - Torsion and bi-normal:

$$\vec{B} = \vec{T} \times \vec{N}$$

$$\tau = \frac{det\begin{pmatrix} | & | & | \\ \dot{C}(s) & \ddot{C}(s) & \dddot{C}(s) \\ | & | & | \end{pmatrix}}{\kappa^2} = \frac{\left(\dot{C}(s) \times \ddot{C}(s)\right) \cdot \dddot{C}(s)}{\kappa^2} =$$

$$= \frac{det\begin{pmatrix} | & | & | \\ \dot{C}(t) & \ddot{C}(t) & \dddot{C}(t) \\ | & | & | \end{pmatrix}}{\left\|\dot{C}(t) \times \ddot{C}(t)\right\|^2} = \frac{\left(\dot{C}(t) \times \ddot{C}(t)\right) \cdot \dddot{C}(t)}{\left\|\dot{C}(t) \times \ddot{C}(t)\right\|^2}$$

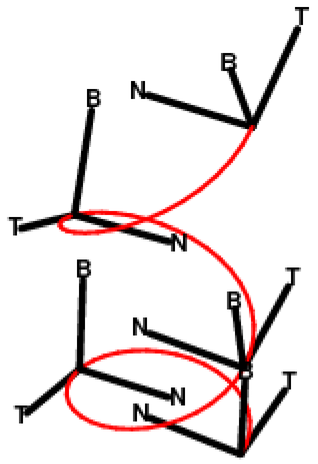
Space Curves I

- Reminder:
  - Frenet-Serret formulas:

$$\frac{dT}{ds} = \kappa N$$

$$\frac{dN}{ds} = -\kappa T + \tau B$$

$$\frac{dB}{ds} = -\tau N$$



Space Curves I

- Exercise: Find the curvature and torsion along the following curve:

$$C(t) = (3t - t^3, 3t^2, 3t + t^3)$$

- Answer:

$$\kappa = \frac{\|\dot{C}(t) \times \ddot{C}(t)\|}{\|\dot{C}(t)\|^3} =$$

$$= \frac{\|(3 - 3t^2, 6t, 3 + 3t^2) \times (-6t, 6, 6t)\|}{\left((3 - 3t^2)^2 + (6t)^2 + (3 + 3t^2)^2\right)^{\frac{3}{2}}} = \frac{1}{3(1 + t^2)^2}$$

Space Curves I

- Exercise: Find the curvature and torsion along the following curve:

$$C(t) = (3t - t^3, 3t^2, 3t + t^3)$$

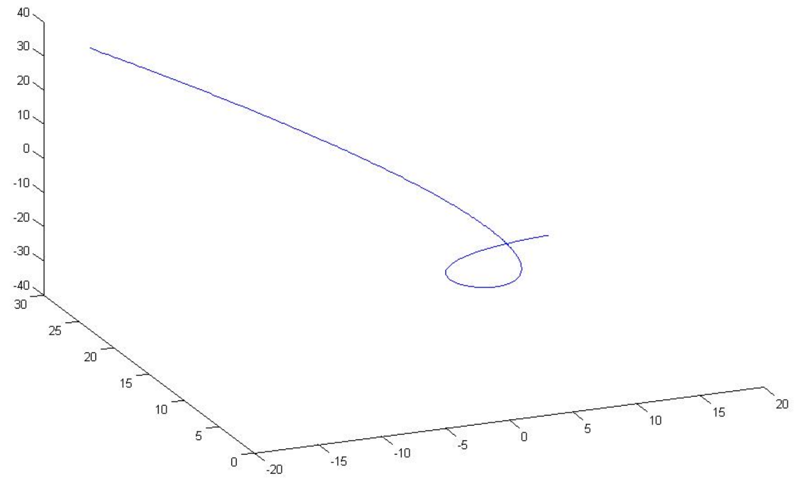
- Answer:

$$\tau = \frac{(\dot{C} \times \ddot{C}) \cdot \dddot{C}}{\|\dot{C} \times \ddot{C}\|^2} =$$

$$= \frac{\big((3 - 3t^2, 6t, 3 + 3t^2) \times (-6t, 6{,}6t)\big) \cdot (-6{,}0{,}6)}{\|(3 - 3t^2, 6t, 3 + 3t^2) \times (-6t, 6{,}6t)\|^2} = \frac{1}{3(1 + t^2)^2}$$

Space Curves I

- How does such a curve look?



- Becomes a straight line eventually on both ends, with a helical middle.

Space Curves II

- Show that two planar curves $C_1(s)$ and $C_2(s)$ are the same up to a rigid transformation iff $\kappa_1(s) = \kappa_2(s)$.

Space Curves II

- Direction 1: if two planar curves $C_1(s)$ and $C_2(s)$ are the same up to a rigid transformation **then** $\kappa_1(s) = \kappa_2(s)$
- $C_2(s) = RC_1(s) + b \Rightarrow T_2(s) = RT_1(s)$

$$\frac{dT_2(s)}{ds} = R\frac{dT_1(s)}{ds}$$

$$\kappa_2(s) = \left\|\frac{dT_2(s)}{ds}\right\| = \left\|R\frac{dT_1(s)}{ds}\right\| = \left\|\frac{dT_1(s)}{ds}\right\| = \kappa_1(s)$$

Space Curves II

- Direction 2: if two planar curves $C_1(s)$ and $C_2(s)$ has $\kappa_1(s)= \kappa_2(s)$ **then** they are the same up to a rigid transformation

- Choose a corresponding point s_0 on both curves
- Transform one curve rigidly until both tangent vectors coincide on s_0
  - Why do they perfectly coincide in length as well?

Space Curves II

$$\frac{d}{ds}(T_1 \cdot T_2) = \kappa(T_1 N_2 + T_2 N_1)$$

$$\frac{d}{ds}(N_1 \cdot N_2) = -\kappa(T_1 N_2 + T_2 N_1)$$

$$\Rightarrow$$

$$\frac{d}{ds}(T_1 \cdot T_2 + N_1 \cdot N_2) = 0$$

$$\Rightarrow T_1(s) \cdot T_2(s) + N_1(s) \cdot N_2(s) = C$$

Space Curves II

- By looking at s_0 we obtain (for all s):

$$T_1(s)\cdot T_2(s)+N_1(s)\cdot N_2(s)=2$$

- Since a dot product of two (unit-length) vectors is at most *1*, we get that

$$T_1(s)\cdot T_2(s)=N_1(s)\cdot N_2(s)=1$$

- $\Rightarrow T_1(s)=T_2(s)\Rightarrow \frac{dC_1}{ds}-\frac{dC_2}{ds}=0\Rightarrow$

$$C_1(s)=C_2(s)+\alpha$$

$$C_1(s_0)=C_2(s_0)\Rightarrow \alpha=0$$

Space Curves II

- Question: is curvature enough for general space curves?
- Answer: No. Curvature and torsion fully determine a space curve (up to rigid trans.)
- Example: Helix. The (constant) torsion defines the "stretch" of the spring.