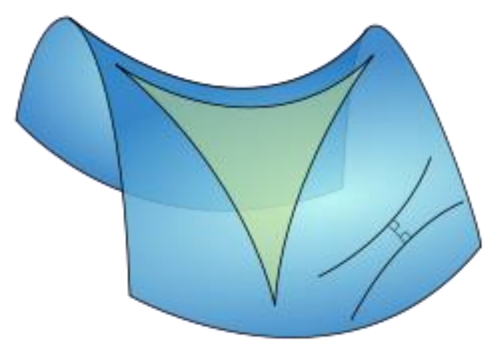
Differential Geometry of Surfaces

Some Problems

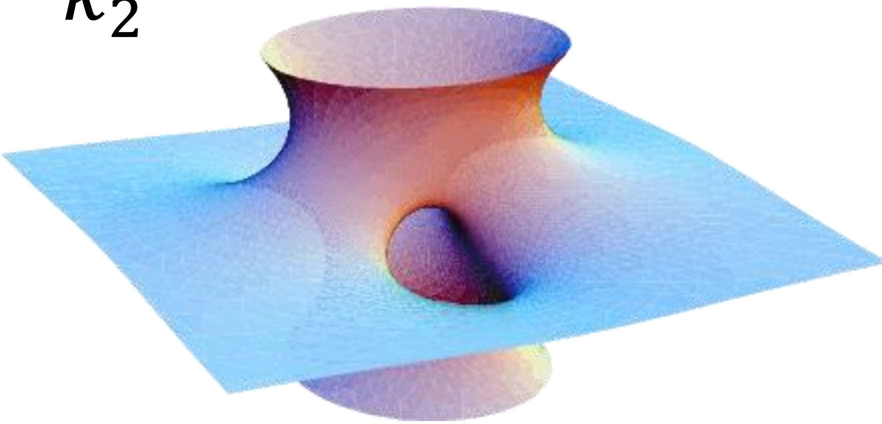


Surfaces I

- Show that minimal surfaces are either locally hyperbolic or planar.
- Answer:

$$H = 0$$

$$H = \frac{\kappa_1 + \kappa_2}{2} \;\Rightarrow\; \kappa_1 = -\kappa_2$$



Surfaces II

Reminder:

- The first fundamental form:

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} x_u^\top x_u & x_u^\top x_v \\ x_u^\top x_v & x_v^\top x_v \end{pmatrix}$$

- The second fundamental form

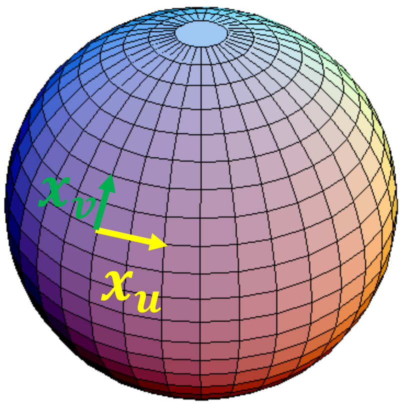
$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} x_{uu}^\top n & x_u^\top n \\ x_u^\top n & x_v^\top n \end{pmatrix}$$

Surfaces II

Determine the length of latitude lines (const v) and the surface area of the sphere, parametrized by:

$$x(u,v) = (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$$
$$(u,v) \in [0,2\pi) \times [0,\pi)$$

What do you expect to get?



Length of a curve

$$c(t) = \big(u(t), v(t)\big) \to x\big(c(t)\big)$$
$$x'\big(c(t)\big) = x_u u_t + x_v v_t$$

$$l(a,b) = \int dL = \int_a^b \left\| x'\big(u(t)\big) \right\| dt =$$
$$= \int_a^b \sqrt{\begin{pmatrix} u_t & v_t \end{pmatrix} \begin{pmatrix} x_u^\top x_u & x_u^\top x_v \\ x_u^\top x_v & x_v^\top x_v \end{pmatrix} \begin{pmatrix} u_t \\ v_t \end{pmatrix}}\, dt$$
$$= \int_a^b \sqrt{\begin{pmatrix} u_t & v_t \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} u_t \\ v_t \end{pmatrix}}\, dt$$
$$= \int_a^b \sqrt{E u_t^2 + 2FE u_t v_t + G v_t^2 dt}$$

Surfaces II

Compute the first fundamental form:

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} x_u^\top x_u & x_u^\top x_v \\ x_u^\top x_v & x_v^\top x_v \end{pmatrix}$$

The given parametrization:

$$x(u,v) = (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$$

$$(u,v) \in [0,2\pi) \times [0,\pi)$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} x_u^\top x_u & x_u^\top x_v \\ x_u^\top x_v & x_v^\top x_v \end{pmatrix}$$

Surfaces II

The first fundamental form:

$$\boldsymbol{x_u} = (-\sin(u)\sin(v), \cos(u)\sin(v), 0)$$
$$\boldsymbol{x_v} = (\cos(u)\cos(v), \sin(u)\cos(v), -\sin(v))$$

$$\boldsymbol{E} = x_u^\top x_u = \sin^2 u \cdot \sin^2 v + \cos^2 u \cdot \sin^2 v =$$
$$= \sin^2 v$$
$$\boldsymbol{F} = x_u^\top x_v = -\sin(u)\sin(v)\cos(u)\cos(v) +$$
$$+ \cos(u)\sin(v)\sin(u)\cos(v) = 0$$
$$\boldsymbol{G} = x_v^\top x_v = \cos^2 v\left(\cos^2 u + \sin^2 u\right) + \sin^2 v = 1$$

Surfaces II

The first fundamental form:

$$I = \begin{pmatrix} \sin^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

Surfaces II

Length of latitude lines (const v) of the sphere:

- Latitude lines:

$$\big(u(t), v(t)\big) = (t, v_0), t \in [0,2\pi)$$

$$u_t = 1, v_t = 0$$

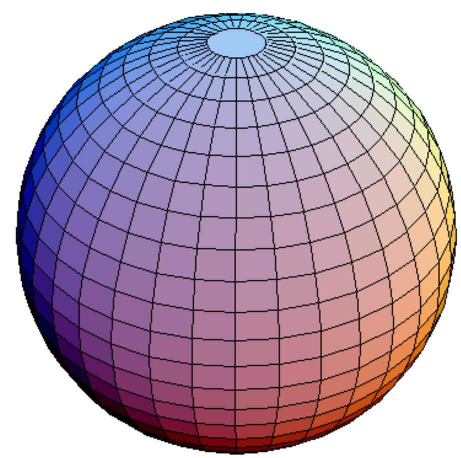
Surfaces II

Length of latitude lines (const v) of the sphere:

$$u_t = 1, v_t = 0, I = \begin{pmatrix} \sin^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

$$L = \int dL = \int_a^b \sqrt{Eu_t^2 + 2Fu_t v_t + Gv_t^2 dt} =$$

$$= \int_0^{2\pi} \sin(v) dt = 2\pi sin(v)$$



Surfaces II

- Total surface area:

$$A = \int dA = \int_0^{\pi}\int_0^{2\pi} \|x_u \times x_v\| dudv =$$

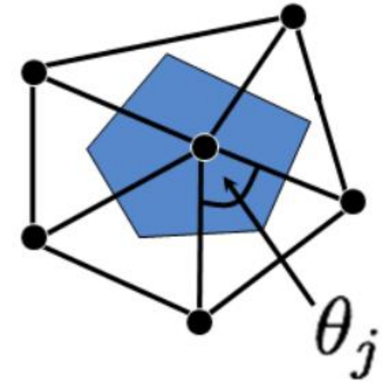
$$= \int_0^{\pi}\int_0^{2\pi} \sqrt{EG - F^2} dudv$$

$$= \int_0^{\pi}\int_0^{2\pi} sin(v) dudv = 4\pi$$

Surfaces III

Prove that the discrete Gaussian curvature (**assuming no boundary**)

$$K(v) = \left[2\pi - \sum_{f \in adj(v)} \theta_f(v)\right] / A_v$$



forms a discrete version of Gauss-Bonnet theorem:

$$2\pi\chi = \int K = \sum_v K(v)\, A_v$$

Surfaces III

Proof:

- Euler characteristic:

$$\chi = V - E + F = 2 - 2g$$

$$E = \frac{3F}{2} \Rightarrow V - E + F = \boxed{V - \frac{F}{2} = \chi}$$

Surfaces III

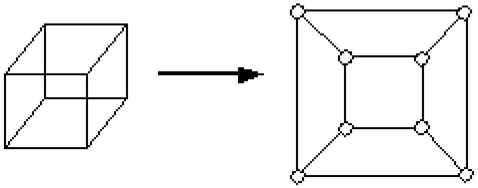
- And therefore:

$$\sum_{v} K(v)\, A_v = \sum_{v} \left(2\pi - \sum_{f \in adj(v)} \theta_f(v) \right) =$$

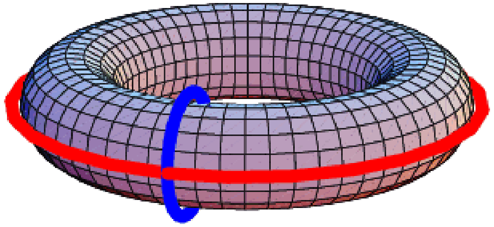
$$= 2\pi V - \sum_{f \in F} \sum_{v \in f} \theta_f(v) = 2\pi V - \sum_{f \in F} \pi =$$

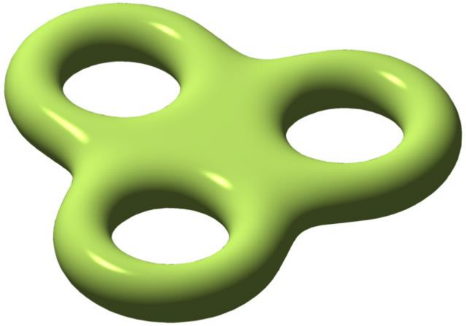
$$= 2\pi \left(V - \frac{F}{2} \right) = 2\pi\chi$$

Surfaces IV



- A planar graph – a graph that can be embedded in the plane without intersections.
- Planar graphs obey Euler's formula: V-E+F=2.
- How does one embed a closed triangular mesh of genus 0 in the plane?
  - Genus 1? (torus)
  - General Genus?





Surfaces IV

Solution: Cut the mesh

- How many face, *edges and vertices does* cutting along a boundary add?

- How many cuts do we need?

→ Cut g boundaries