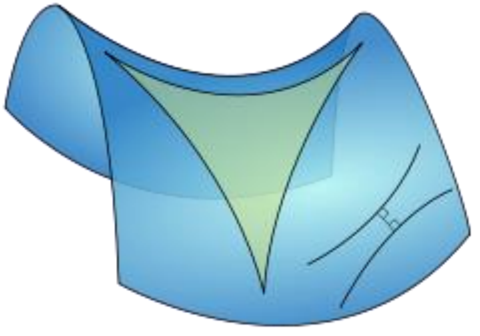
Differential Geometry of Surfaces

Some Problems



Differential Operators

Gradient ∇

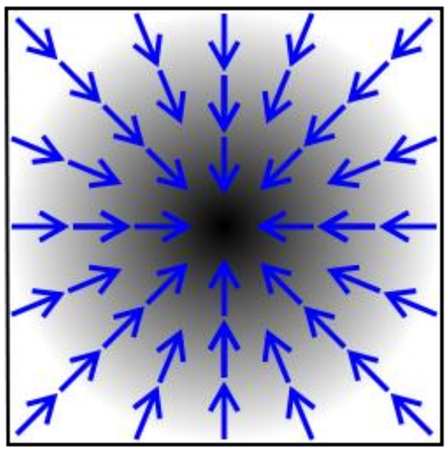
- **Input:** scalar function

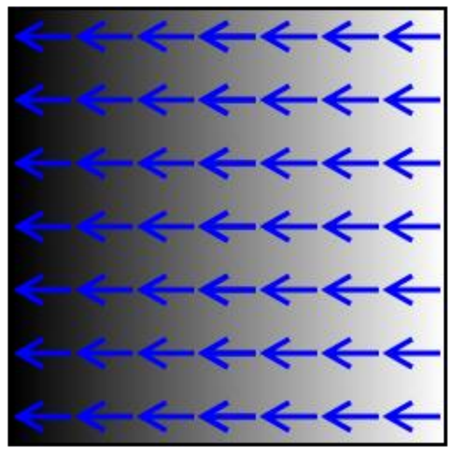
$$f: \mathbb{R}^n \to \mathbb{R}$$

- **Output:** vector field

$$\nabla f: \mathbb{R}^n \to \mathbb{R}^n$$

- **Intuition:** slope





Differential Operators

Divergence $\nabla \cdot$

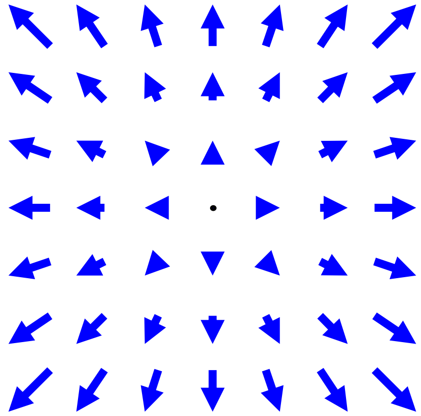
- **Input:** vector field

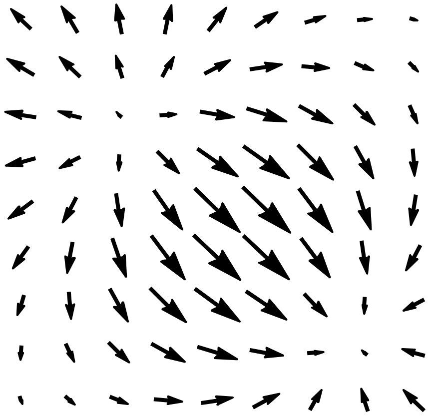
$$F\colon \mathbb{R}^n \to \mathbb{R}^n$$

- **Output:** scalar function

$$\nabla \cdot F\colon \mathbb{R}^n \to \mathbb{R}$$

- **Intuition:** sources/sinks (think Jacuzzi…)





Differential Operators

Laplacian $\mathbf{\nabla^2}, \mathbf{\Delta}$

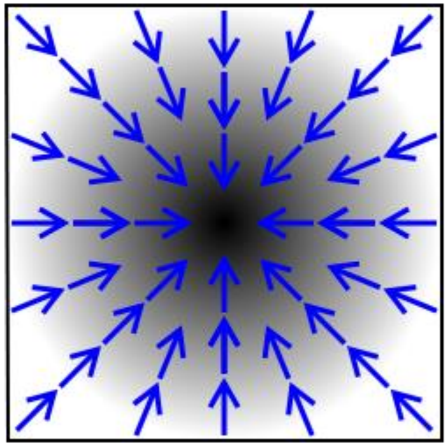
- **Input:** scalar function

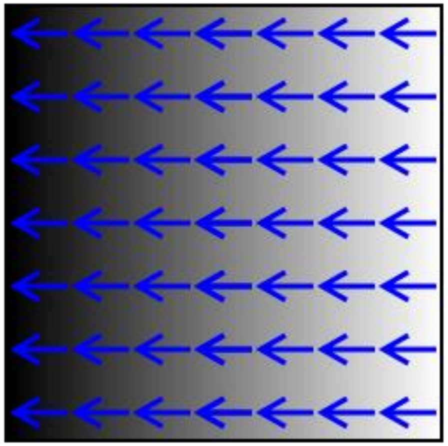
$$f: \mathbb{R}^n \to \mathbb{R}$$

- **Output:** scalar function

$$\Delta F: \mathbb{R}^n \to \mathbb{R}$$

- **Intuition:** smoothness, deviation from average

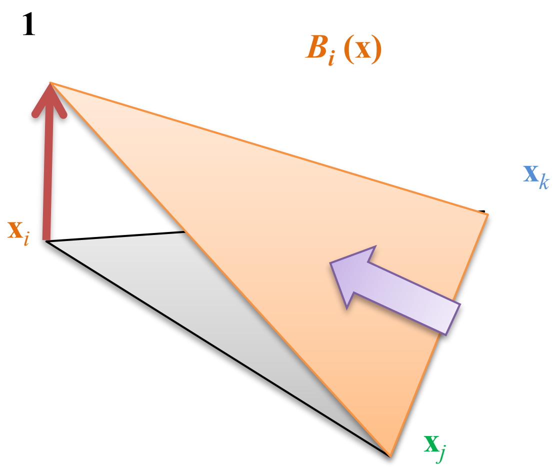




Gradient of a Function

$$f(\mathbf{x}) = f_i B_i(\mathbf{x}) + f_j B_j(\mathbf{x}) + f_k B_k(\mathbf{x})$$

$$\nabla f(\mathbf{x}) = f_i \nabla B_i(\mathbf{x}) + f_j \nabla B_j(\mathbf{x}) + f_k \nabla B_k(\mathbf{x})$$



Steepest ascent direction perpendicular to opposite edge

$$\nabla B_i(\mathbf{x}) = \nabla B_i = \frac{\left(\mathbf{x}_k - \mathbf{x}_j\right)^{\perp}}{2A_T}$$

Constant in the triangle

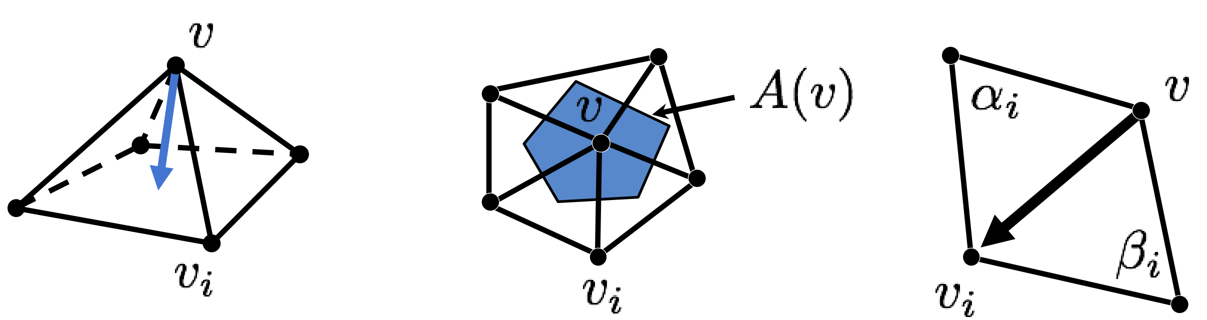
Discrete Laplace-Beltrami

Cotangent Formula

Plugging in expression for gradients gives:

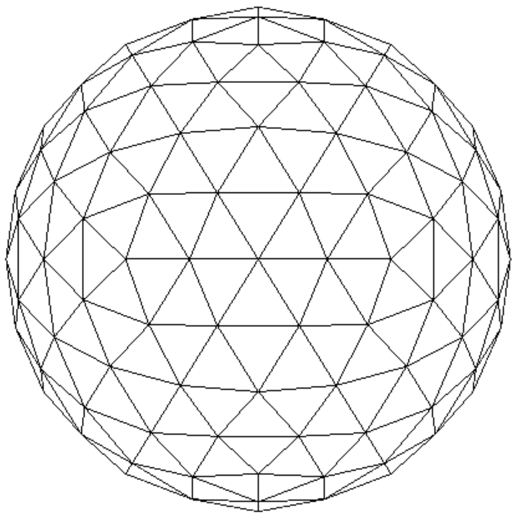
$$\Delta f(v) = \sum_{v_i \in N_1(v)} w_i (f(v_i) - f(v))$$

$$= \boxed{\frac{1}{2A(v)}} \sum_{v_i \in \mathcal{N}_1(v)} \boxed{(\cot \alpha_i + \cot \beta_i)} (f(v_i) - f(v))$$



Notations

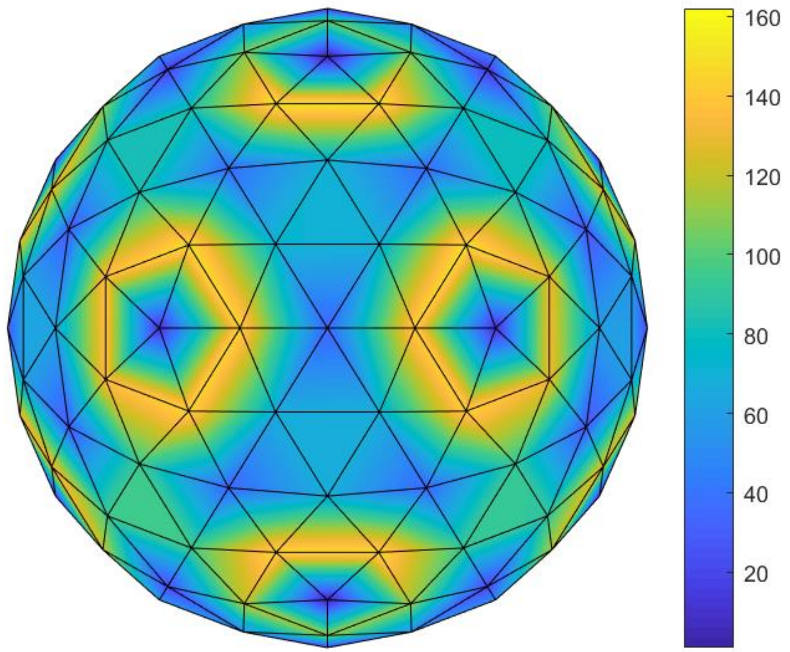
$M = (\mathcal{V}, \mathcal{F})$



Notations

$M = (\mathcal{V}, \mathcal{F})$

$f \in \mathbb{R}^{|\mathcal{V}| \times 1}$ - piecewise-linear functions



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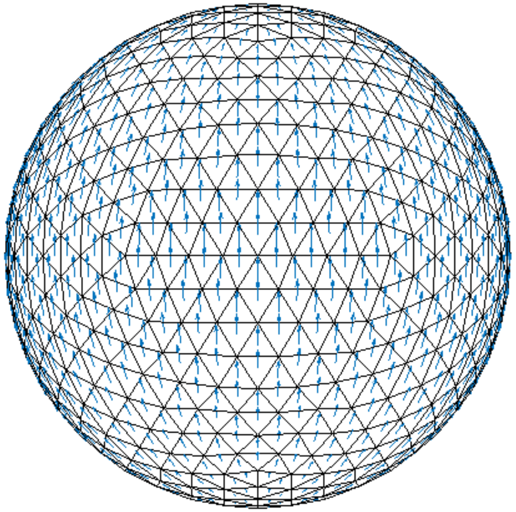
$f \in \mathbb{R}^{|\mathcal{V}| \times 1}$ - piecewise-linear functions

$v \in \mathbb{R}^{3|\mathcal{F}| \times 1}$ - piecewise-constant vector fields



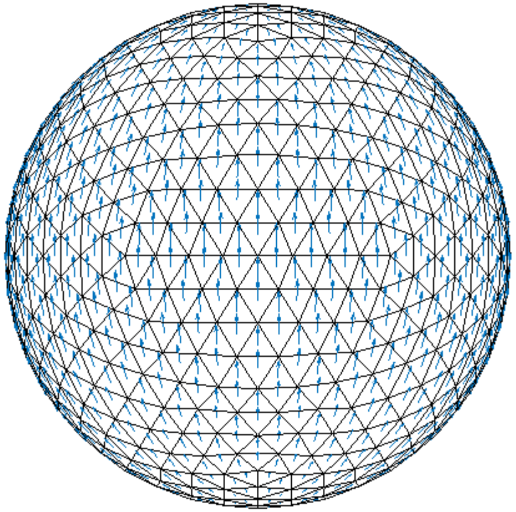
Notations

$M = (\mathcal{V}, \mathcal{F})$

$f \in \mathbb{R}^{|\mathcal{V}| \times 1}$ - piecewise-linear functions

$v \in \mathbb{R}^{3|\mathcal{F}| \times 1}$ - piecewise-constant vector fields

$$v = [\underbrace{x_1 \quad \cdots \quad x_{|\mathcal{F}|}}_{x} \quad \underbrace{y_1 \quad \cdots \quad y_{|\mathcal{F}|}}_{y} \quad \underbrace{z_1 \quad \cdots \quad z_{|\mathcal{F}|}}_{z}]^T$$



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Notations

$\text{grad} \in \mathbb{R}^{3|\mathcal{F}|\times|\mathcal{V}|}$

$\text{div} \in \mathbb{R}^{|\mathcal{V}|\times 3|\mathcal{F}|}$

$\text{L} \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$

Notations

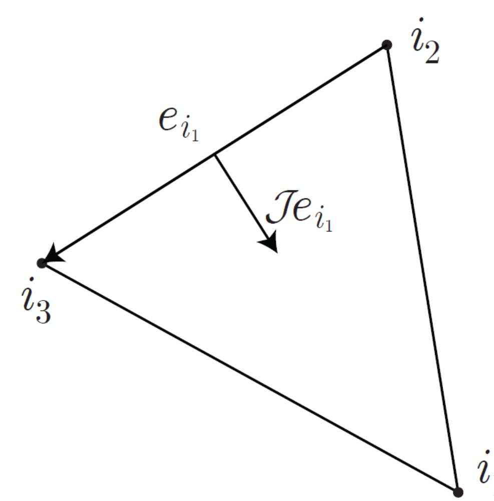
$$G_{\mathcal{V}} \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|} - \begin{bmatrix} A_{V_1} & & 0 \\ & \ddots & \\ 0 & & A_{V_{|\mathcal{V}|}} \end{bmatrix}$$

$$G_{\mathcal{F}} \in \mathbb{R}^{3|\mathcal{F}|\times 3|\mathcal{F}|} - \begin{bmatrix} A_{F_1} & & & & & & & & \\ & \ddots & & & & & & 0 & \\ & & A_{F_{|\mathcal{F}|}} & & & & & & \\ & & & A_{F_1} & & & & & \\ & & & & \ddots & & & & \\ & & & & & A_{F_{|\mathcal{F}|}} & & & \\ & & & & & & A_{F_1} & & \\ & 0 & & & & & & \ddots & \\ & & & & & & & & A_{F_{|\mathcal{F}|}} \end{bmatrix}$$

Discrete gradient

$$(\text{grad } f)(j) = \frac{1}{2G_{\mathcal{F}}(j,j)} \sum_{i=1}^{3} f_i \mathcal{J} e_{ji}$$

$\mathcal{J}$ – in-plane $\frac{\pi}{2}$ rotation



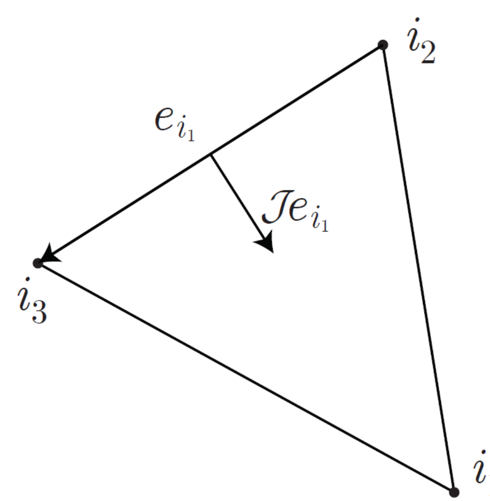
Discrete gradient

$$(\mathrm{grad}\, f)(j) = \frac{1}{2G_{\mathcal{F}}(j,j)} \sum_{i=1}^{3} f_i \mathcal{J} e_{ji}$$

$\mathcal{J}$ – in-plane $\frac{\pi}{2}$ rotation

$$\mathrm{grad} = \frac{1}{2} G_{\mathcal{F}}^{-1} E$$

$$E = \begin{array}{rc} & \begin{array}{ccc} i_1 & i_2 & \qquad\qquad i_3 \end{array} \\ \begin{array}{r} j \\[1em] j + |\mathcal{F}| \\[1em] j + 2|\mathcal{F}| \end{array} & \begin{bmatrix} (\mathcal{J}e_{ji_1})_x & (\mathcal{J}e_{ji_2})_x & \qquad\qquad (\mathcal{J}e_{ji_3})_x \\[1em] (\mathcal{J}e_{ji_1})_y & (\mathcal{J}e_{ji_2})_y & \qquad\qquad (\mathcal{J}e_{ji_3})_y \\[1em] (\mathcal{J}e_{ji_1})_z & (\mathcal{J}e_{ji_2})_z & \qquad\qquad (\mathcal{J}e_{ji_3})_z \end{bmatrix} \end{array}$$



Discrete divergence

Integration by parts:

$$\int_M v \cdot \nabla f da + \int_M f \nabla \cdot v da = \int_{\partial M} f(v \cdot n) dl = 0$$

Inner products:

- On functions: $\int_M f_1 f_2 da = f_1^T G_\mathcal{V} f_2$
- On vector fields: $\int_M v_1 \cdot v_2 da = f_1^T G_\mathcal{F} f_2$

Discrete divergence

Integration by parts:

$$\int_M v \cdot \nabla f da + \int_M f \nabla \cdot v da = \int_{\partial M} f(v \cdot n) dl = 0$$

$\rightarrow$

$$v^T G_{\mathcal{F}} \mathrm{grad} f + v^T \mathrm{div}^T G_{\mathcal{V}} f = 0 \quad \forall f, v$$

Discrete divergence

Integration by parts:

$$\int_M v \cdot \nabla f \, da + \int_M f \nabla \cdot v \, da = \int_{\partial M} f(v \cdot n) \, dl = 0$$

$\rightarrow$

$$v^T G_{\mathcal{F}} \mathrm{grad} f + v^T \mathrm{div}^T G_{\mathcal{V}} f = 0 \quad \forall f, v$$

$$G_{\mathcal{F}} \mathrm{grad} + \mathrm{div}^T G_{\mathcal{V}} = 0$$

$$\mathrm{div}^T = -G_{\mathcal{F}} \mathrm{grad} G_{\mathcal{V}}^{-1}$$

$$\boxed{\mathrm{div} = -G_{\mathcal{V}}^{-1} \mathrm{grad}^T G_{\mathcal{F}}}$$

Discrete Laplacian

$$L = -\operatorname{div}\operatorname{grad} = G_{\mathcal{V}}^{-1}\operatorname{grad}^{T} G_{\mathcal{F}}\operatorname{grad}$$
$$= G_{\mathcal{V}}^{-1}\frac{1}{2}E^{T}G_{\mathcal{F}}G_{\mathcal{F}}\frac{1}{2}G_{\mathcal{F}}^{-1}E = \frac{1}{4}G_{\mathcal{V}}^{-1}E^{T}G_{\mathcal{F}}E$$

Discrete Laplacian

$$L = -\text{div grad} = G_{\mathcal{V}}^{-1} \text{grad}^{\,T} G_{\mathcal{F}} \text{grad}$$
$$= G_{\mathcal{V}}^{-1} \frac{1}{2} E^T G_{\mathcal{F}} G_{\mathcal{F}} \frac{1}{2} G_{\mathcal{F}}^{-1} E = \frac{1}{4} G_{\mathcal{V}}^{-1} E^T G_{\mathcal{F}} E$$

$$Lf(v_i) = \frac{1}{2A_{v_i}} \sum_{v_j \in \mathcal{N}_1(v_i)} \left(cot\alpha_{ij} + cot\beta_{ij}\right)\left(f_j - f_i\right)$$