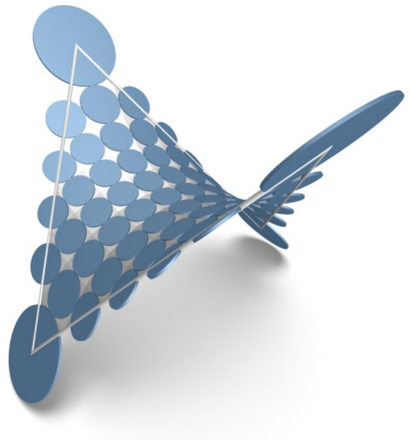
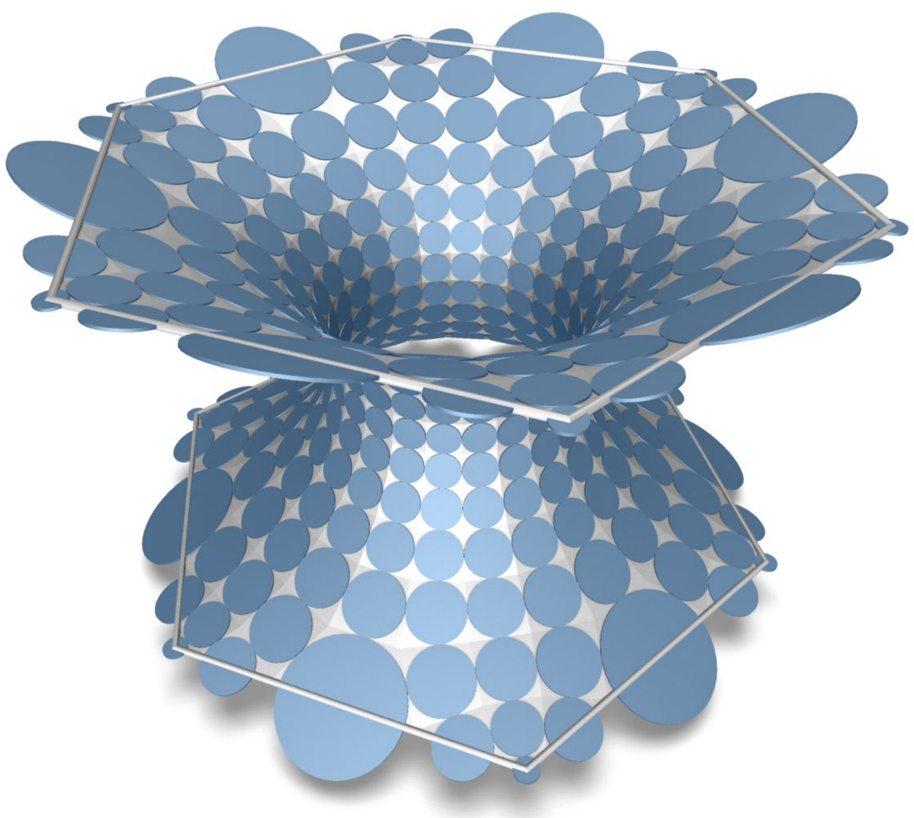
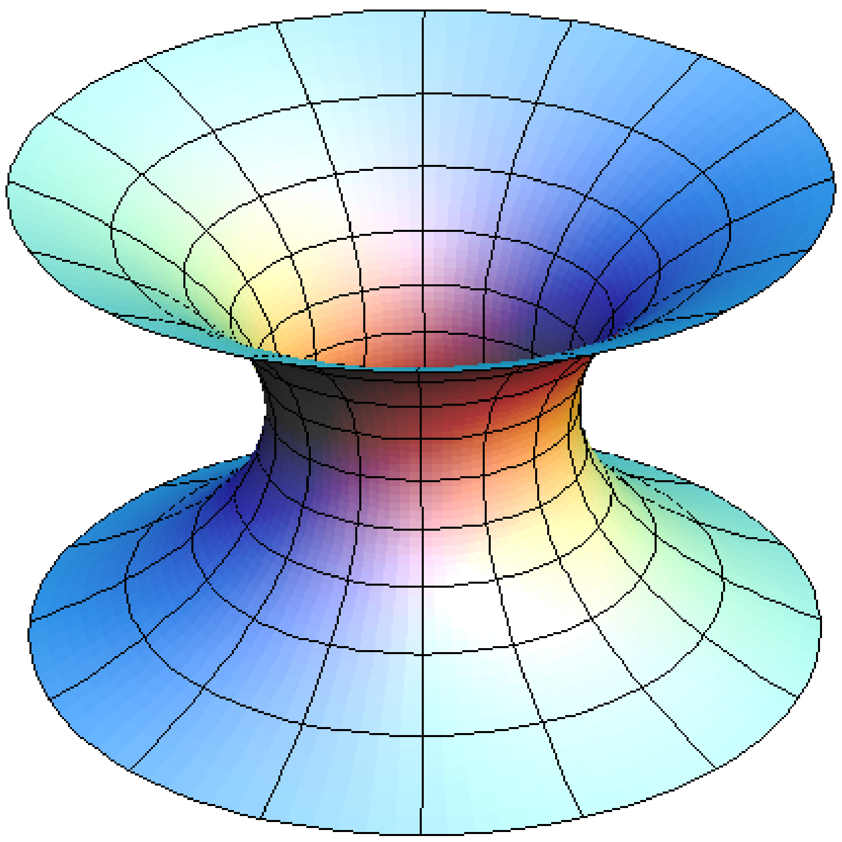


(Discrete) Differential Geometry



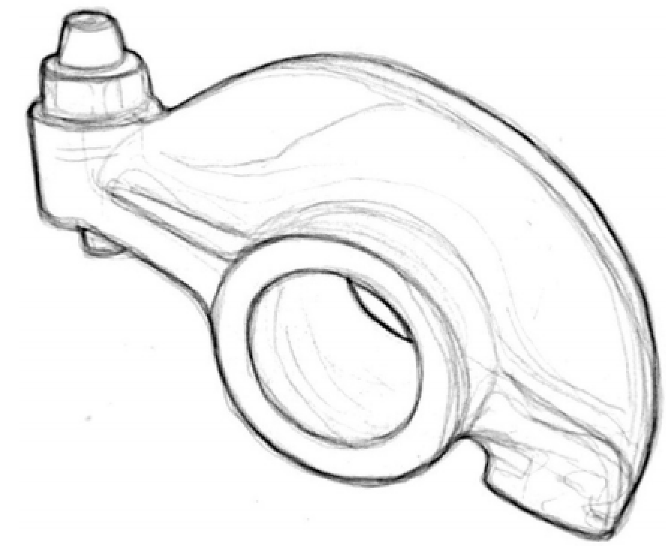


A Motivating Problem

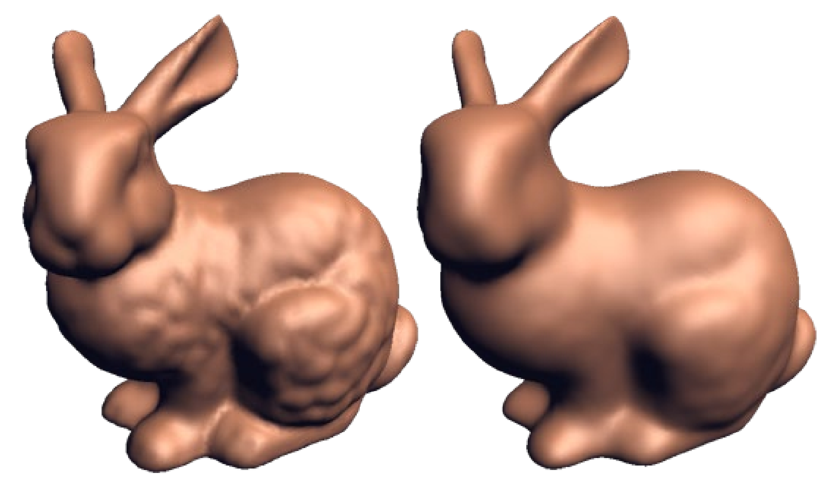
Non Photorealistic Rendering

“Where Do People Draw Lines?”, [Cole et al. 2008]

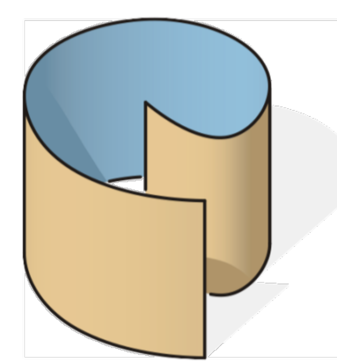
Motivation



- Understand the structure of the surface
  - Properties: smoothness, “curviness”, important directions
- How to modify the surface to change these properties



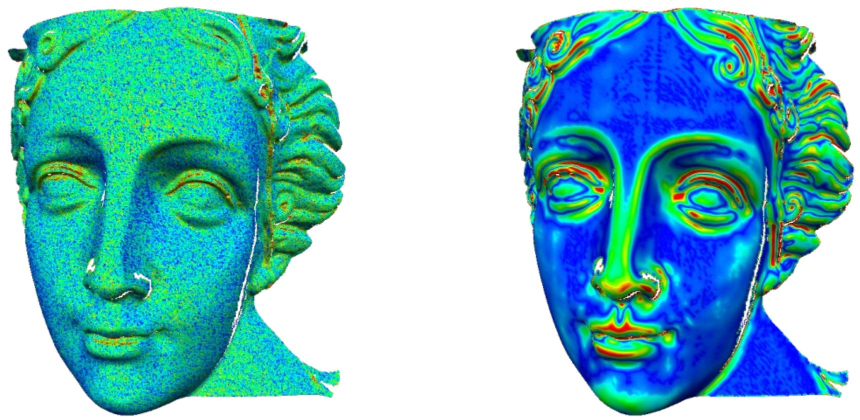
- What properties are preserved for different modifications
- The math behind the scenes for many geometry processing applications



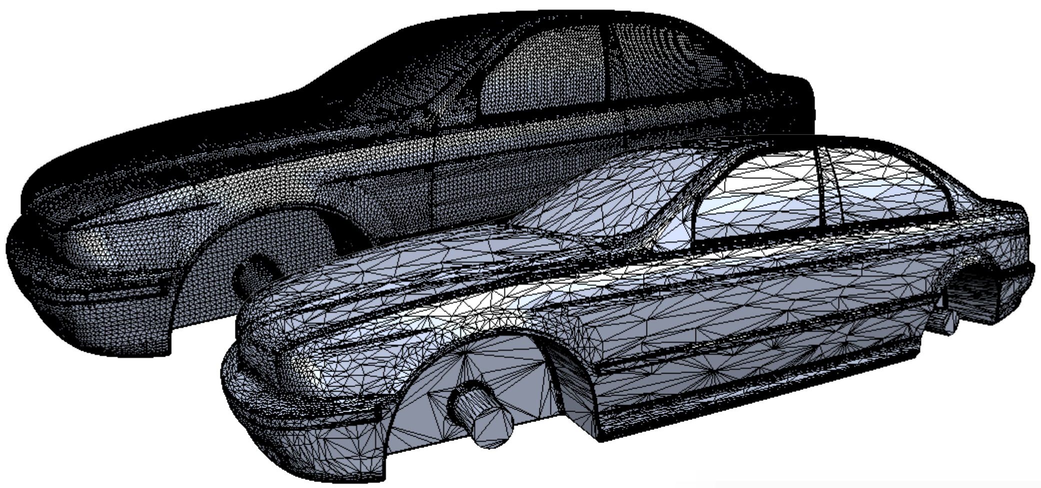
Some images from http://brickisland.net/cs177/

More Applications

- Smoothness
  - ➞ Mesh smoothing



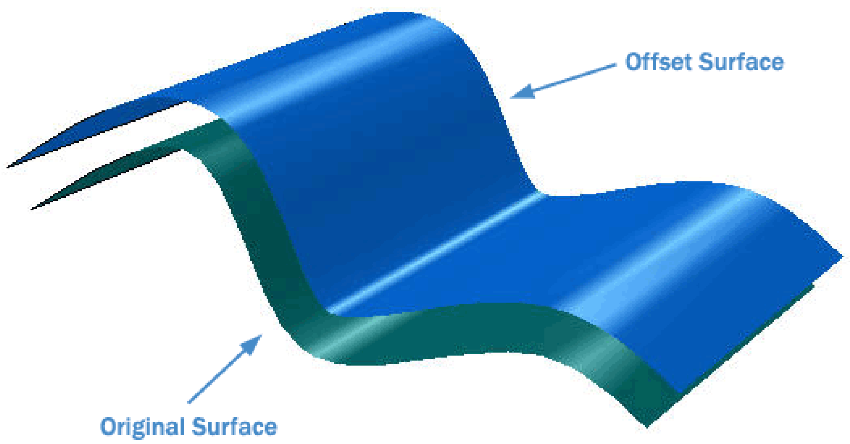
- Curvature
  - ➞ Adaptive simplification



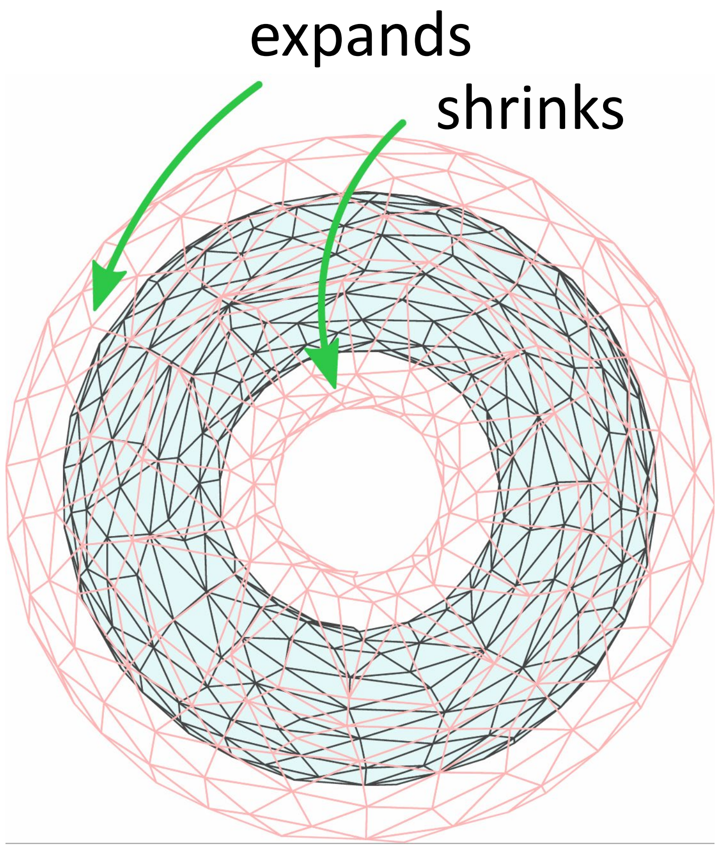
  - ➞ Physically-based simulation

Curvature?

- A plane is flat, so curvature 0
- *Offset* surface

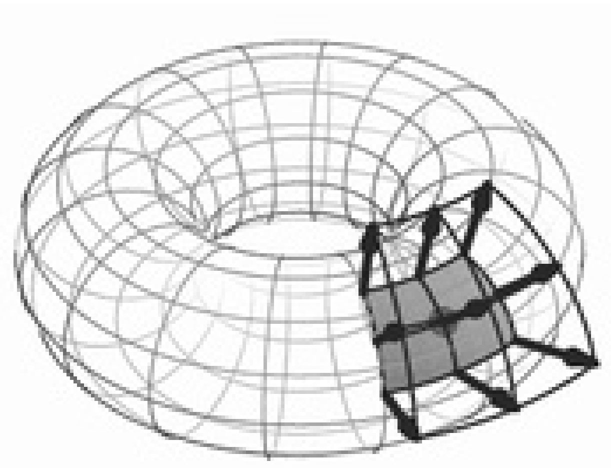


- Some areas expand, other shrink
  - Why?

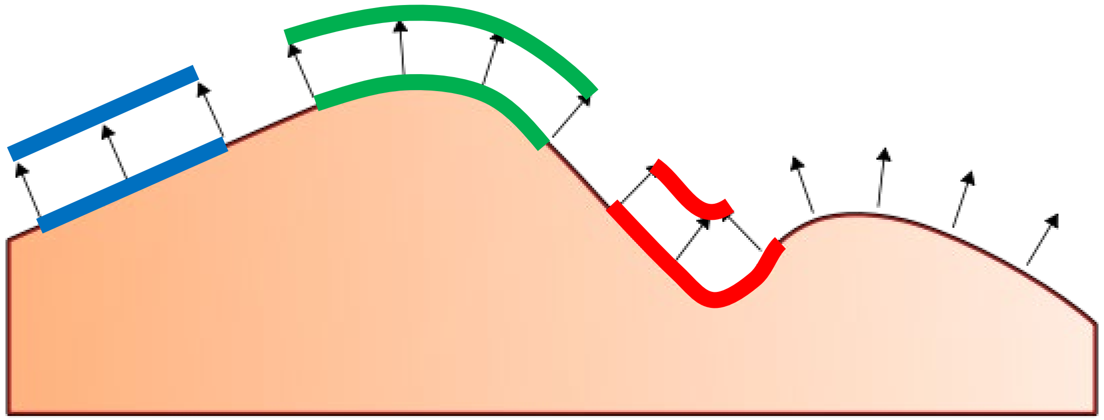


A Closer Look

- *Normal* directions define offset surface
  - The “other” direction that’s pointing off the surface



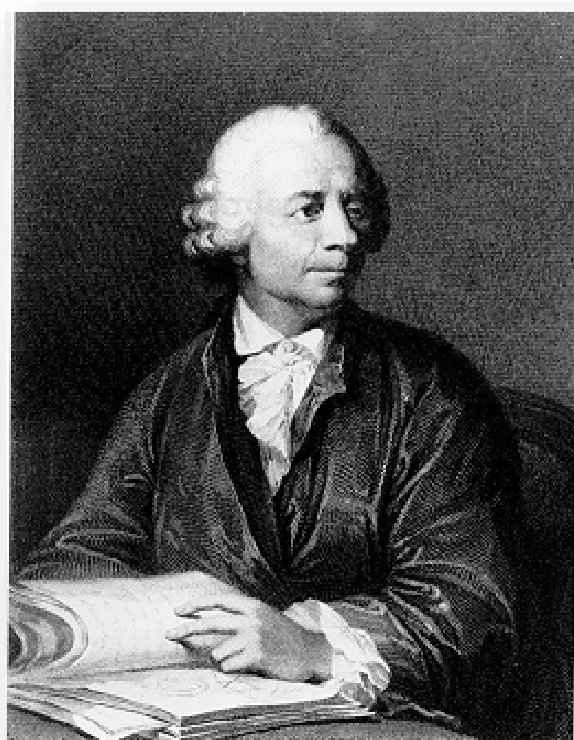
- The way they **change** determines shrinkage/expansion



Surfaces?

Differential Geometry

- M.P. do Carmo: ***Differential Geometry of Curves and Surfaces***, Prentice Hall, 1976



Leonard Euler (1707 - 1783)

Carl Friedrich Gauss (1777 - 1855)

Parametric Curves

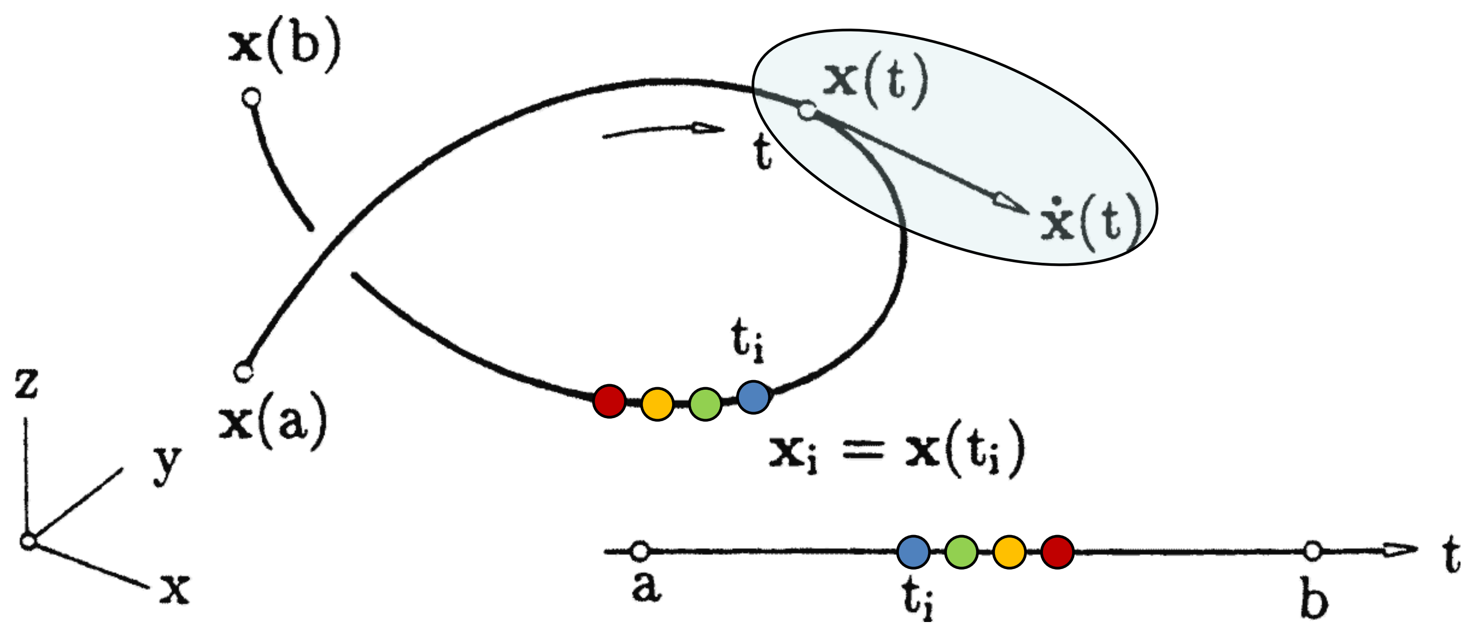
$$\dot{x} = \frac{\partial x}{\partial t}$$

$$\mathbf{x} = \mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} \neq \mathbf{0}$$

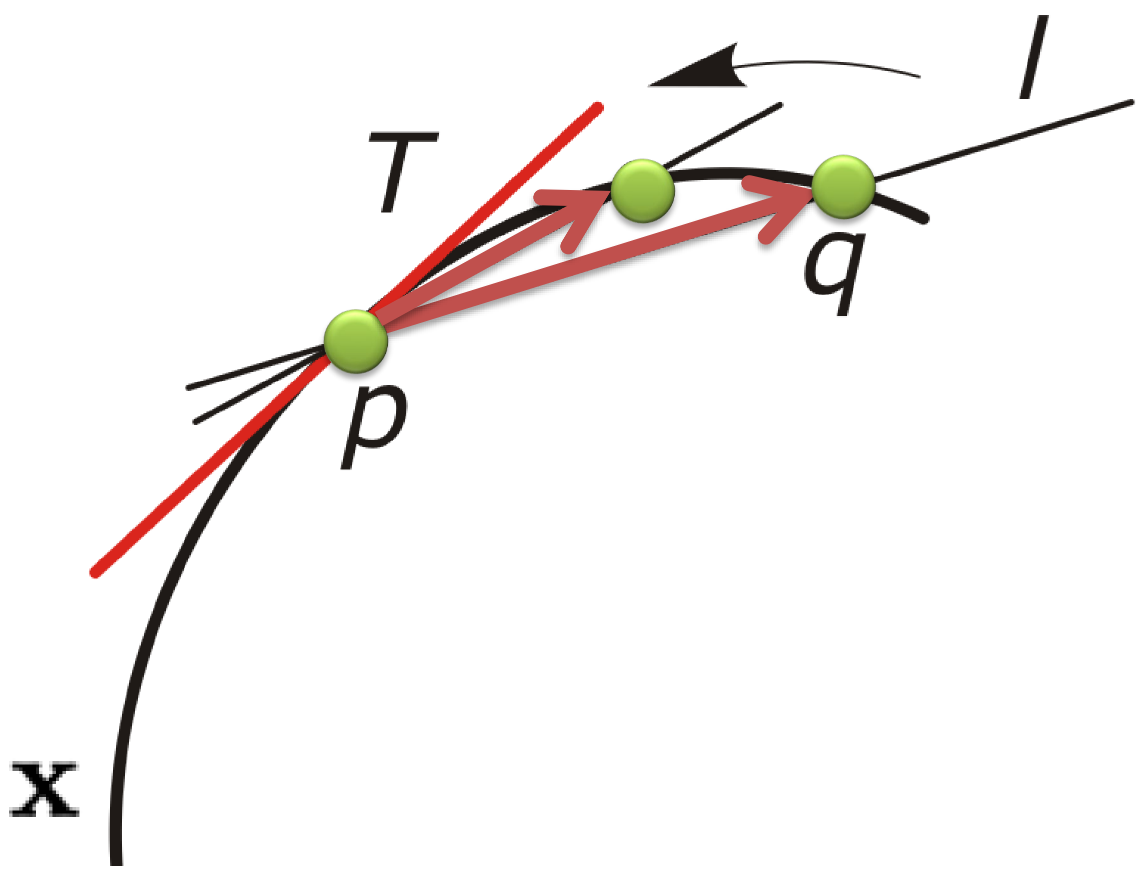
$$t \in [a, b] \subset \mathbb{R}$$

"velocity" of particle on trajectory



The Tangent Vector

As a limit



$$p = \mathbf{x}(t) = \ (x(t), y(t), z(t))$$

$$q = \mathbf{x}(t+h)$$

Connecting line l is in the direction

$$\mathbf{x}(t+h)\ - \mathbf{x}(t)$$

Limit of the direction vector

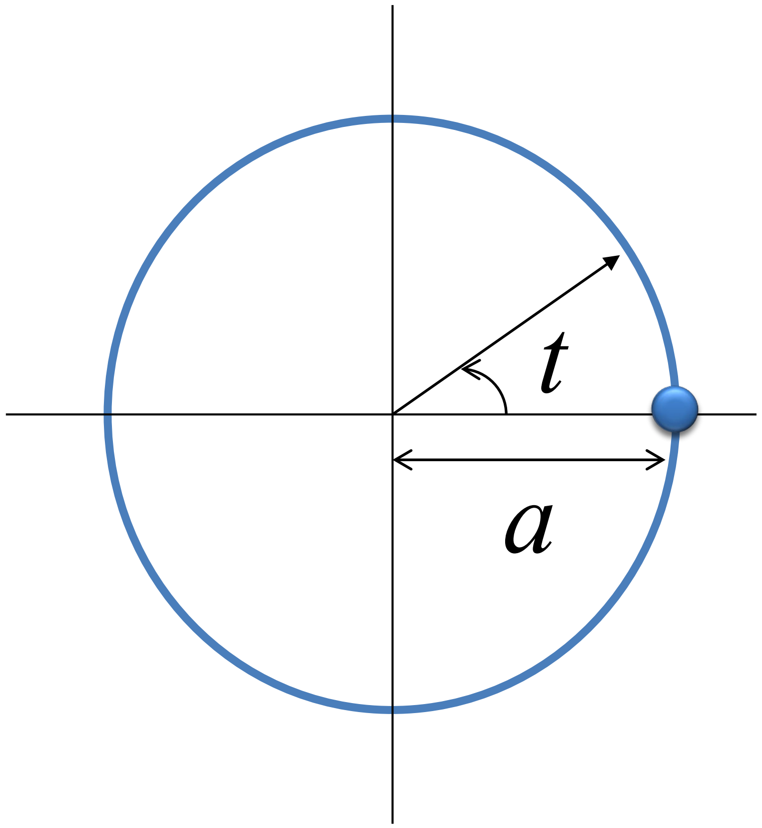
$$\lim_{h \to 0} \frac{\mathbf{x}(t+h) - \mathbf{x}(t)}{h} =$$

$$\boldsymbol{T} = \ \dot{\mathbf{x}}(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$$

Some images from "Architectural Geometry" by Pottmann, 2007

Parametric Curves

A Simple Example



$$\boldsymbol{\alpha}_1(t) = (a\cos(t),\, a\sin(t))$$

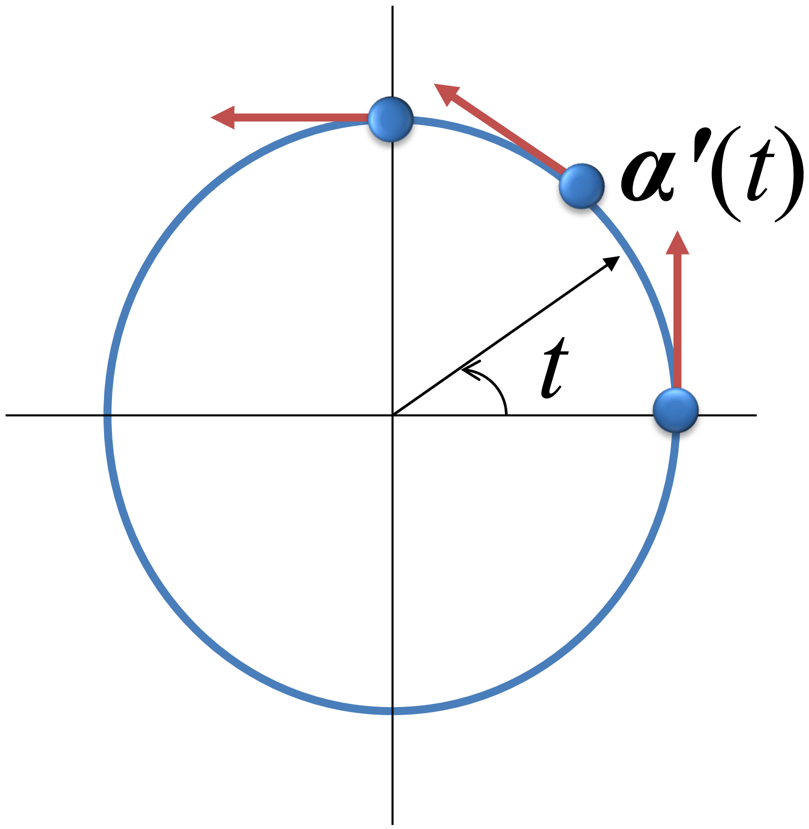
$$t \in [0, 2\pi]$$

$$\boldsymbol{\alpha}_2(t) = (a\cos(2t),\, a\sin(2t))$$

$$t \in [0, \pi]$$

Parametric Curves

A Simple Example



$$\boldsymbol{\alpha}(t) = (\cos(t), \sin(t))$$

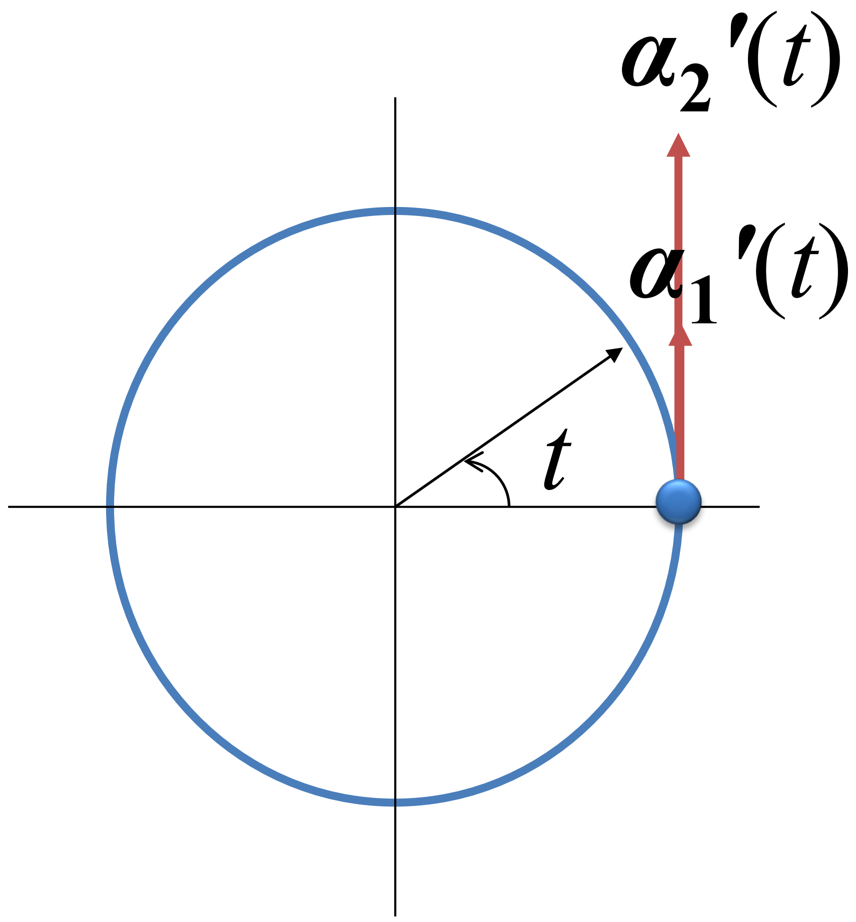
$$\boldsymbol{\alpha}'(t) = (-\sin(t), \cos(t))$$

$\boldsymbol{\alpha}'(t)$ - direction of movement

$|\boldsymbol{\alpha}'(t)|$ - speed of movement

Parametric Curves

A Simple Example



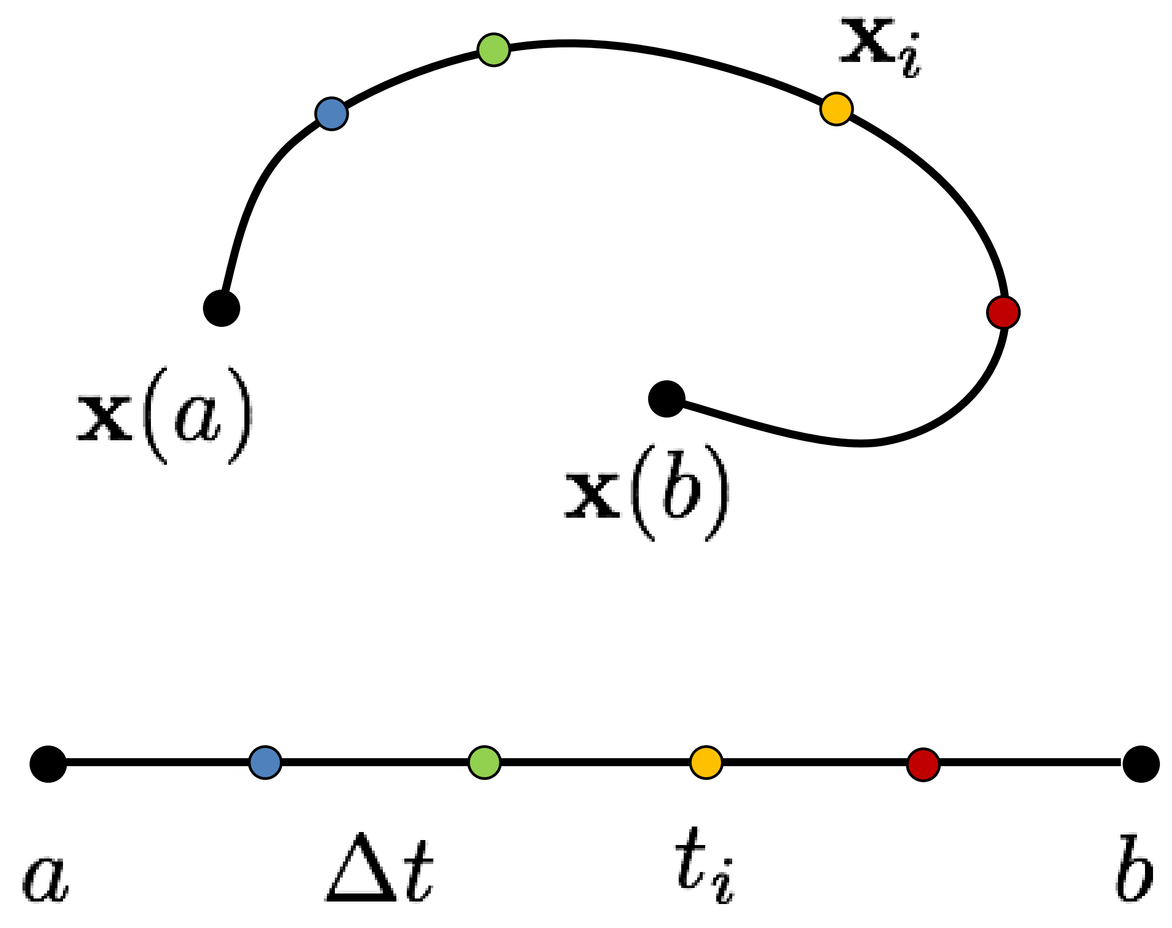
$$\alpha_1(t) = (\cos(t), \sin(t))$$

$$\alpha_2(t) = (\cos(2t), \sin(2t))$$

Same direction, different speed

Length of a Curve

- Let $t_i = a + i\Delta t$ and $\mathbf{x}_i = \mathbf{x}(t_i)$

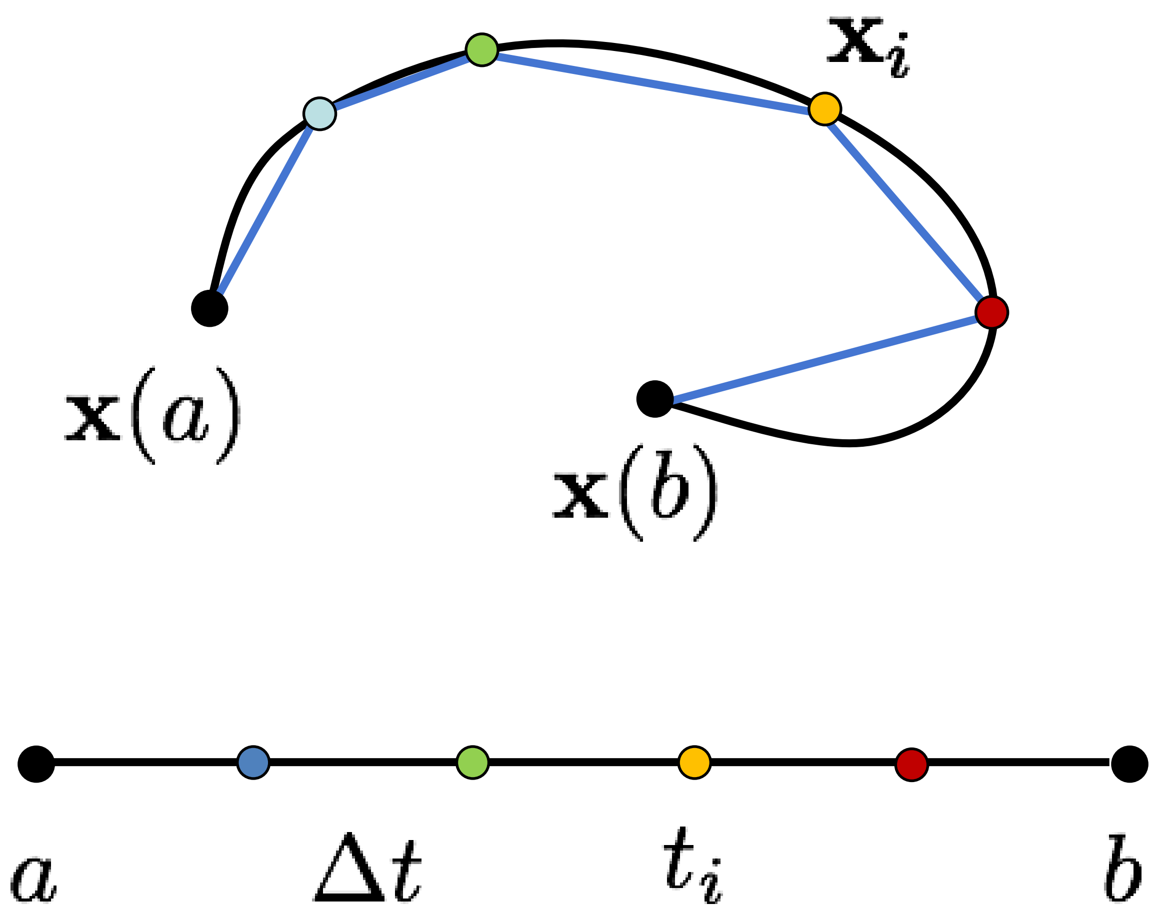


Length of a Curve

- Chord length $$S = \sum_i \|\Delta \mathbf{x}_i\| = \sum_i \|\frac{\Delta \mathbf{x}_i}{\Delta t}\|\Delta t$$

$$\Delta \mathbf{x}_i = \mathbf{x}_{i+1} - \mathbf{x}_i$$

Euclidean norm

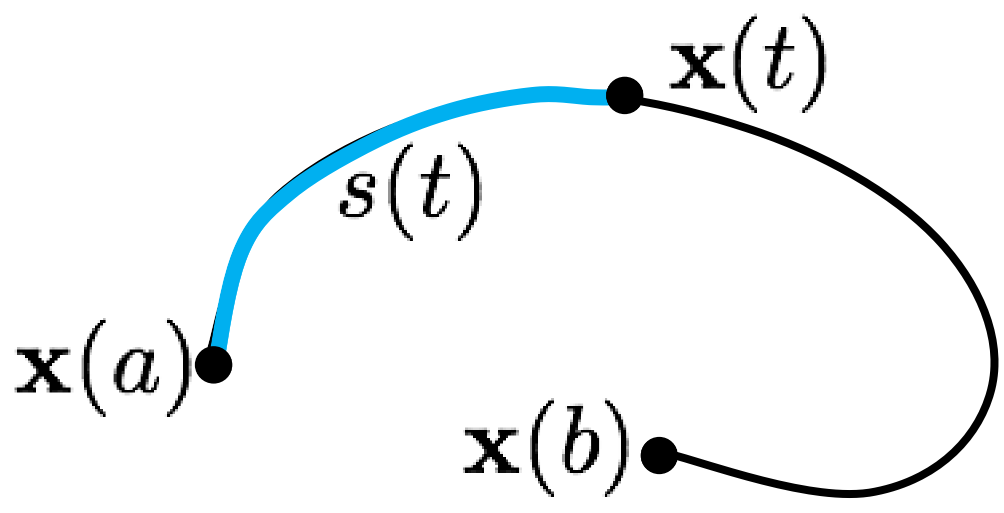


Length of a Curve

- Chord length $$S = \sum_i \|\Delta \mathbf{x}_i\| = \sum_i \|\frac{\Delta \mathbf{x}_i}{\Delta t}\|\Delta t$$

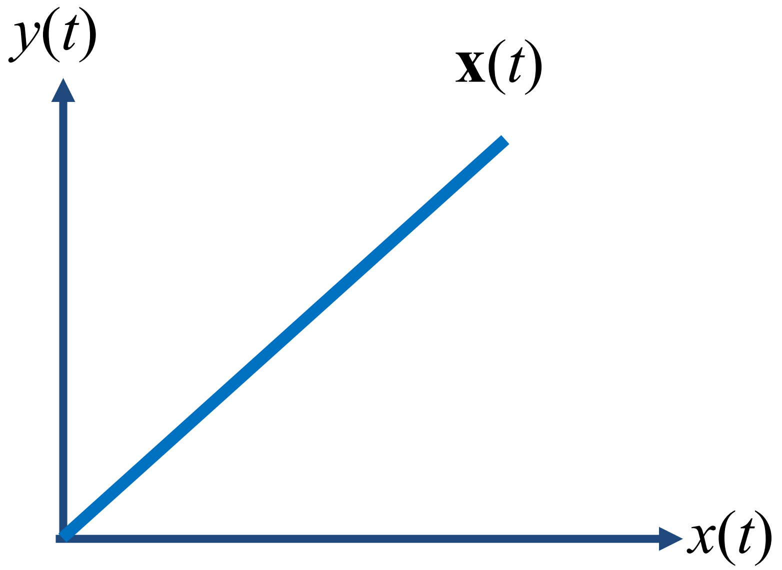
  $$\Delta \mathbf{x}_i = \mathbf{x}_{i+1} - \mathbf{x}_i$$

- Arc length $$s = s(t) = \int_a^t \|\dot{\mathbf{x}}\| dt$$

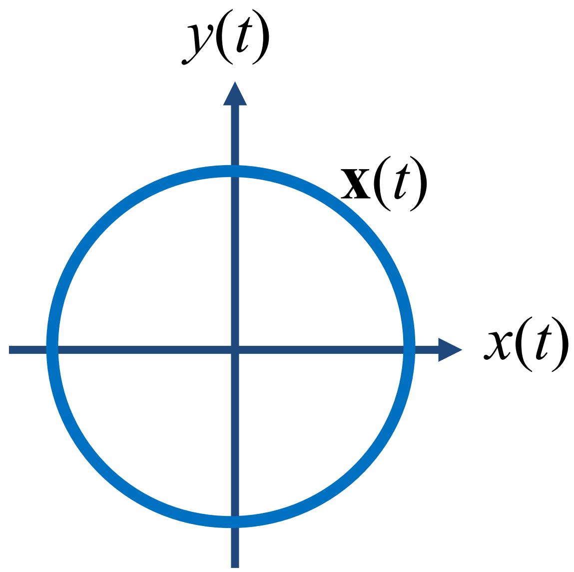


Examples

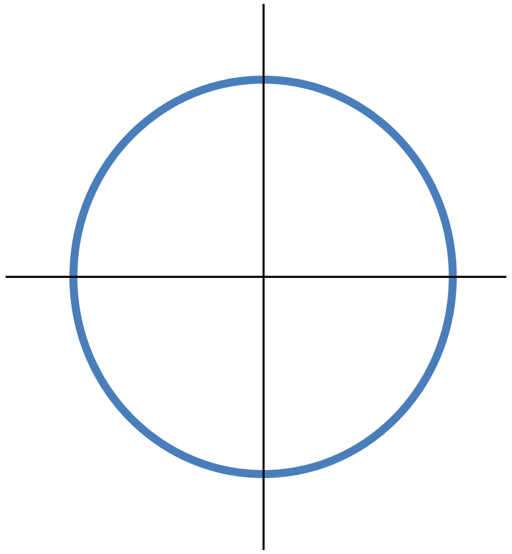
- Straight line
  - $\mathbf{x}(t) = (t,t),\ t \in [0,\infty)$
  - $\mathbf{x}(t) = (2t,2t),\ t \in [0,\infty)$
  - $\mathbf{x}(t) = (t^2,t^2),\ t \in [0,\infty)$



- Circle
  - $\mathbf{x}(t) = (\cos(t),\sin(t)),\ t \in [0,2\pi)$
  - $\mathbf{x}(t) = \left(\dfrac{t^2-1}{t^2+1}, \dfrac{2t}{t^2+1}\right) \quad t \in (-\infty,+\infty)$



Examples



$\boldsymbol{\alpha}(t) = (a\cos(t), a\sin(t)),\ t \in [0,2\pi]$

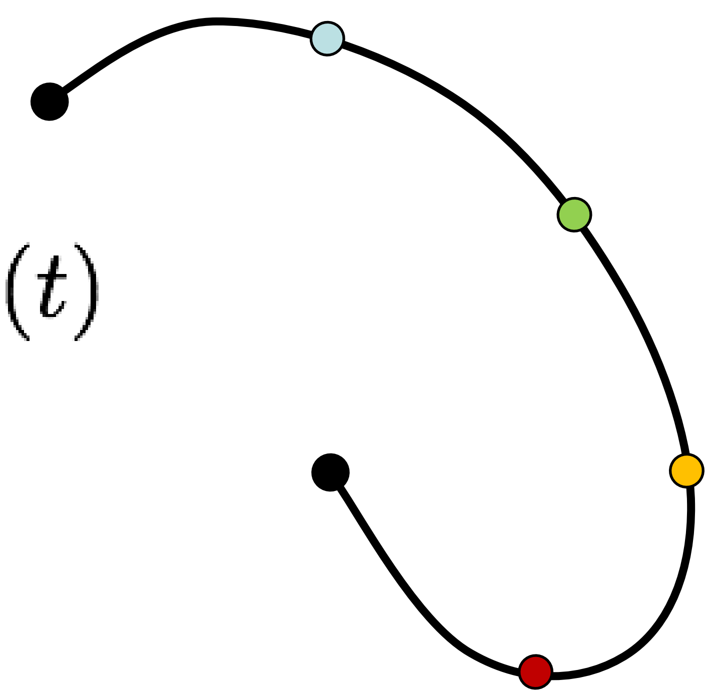
$\boldsymbol{\alpha}'(t) = (-a\sin(t), a\cos(t))$

$$
\begin{aligned}
L(\alpha) &= \int_0^{2\pi} |\alpha'(t)|\,dt \\
&= \int_0^{2\pi} \sqrt{a^2\sin^2(t) + a^2\cos^2(t)}\,dt \\
&= a\int_0^{2\pi} dt = 2\pi a
\end{aligned}
$$

Many possible parameterizations

Length of the curve does not depend on parameterization!

Arc Length Parameterization



- Re-parameterization $\mathbf{x}(u(t))$

$$\frac{d\mathbf{x}(u(t))}{dt} = \frac{d\mathbf{x}}{du}\frac{du}{dt} = \dot{\mathbf{x}}(u(t))\dot{u}(t)$$

- Arc length parameterization

$$s = s(t) = \int_a^t \|\dot{\mathbf{x}}\| dt \qquad ds = \|\dot{\mathbf{x}}\| dt$$

  – parameter value s for $\mathbf{x}(s)$ equals length of curve from $\mathbf{x}(a)$ to $\mathbf{x}(s)$

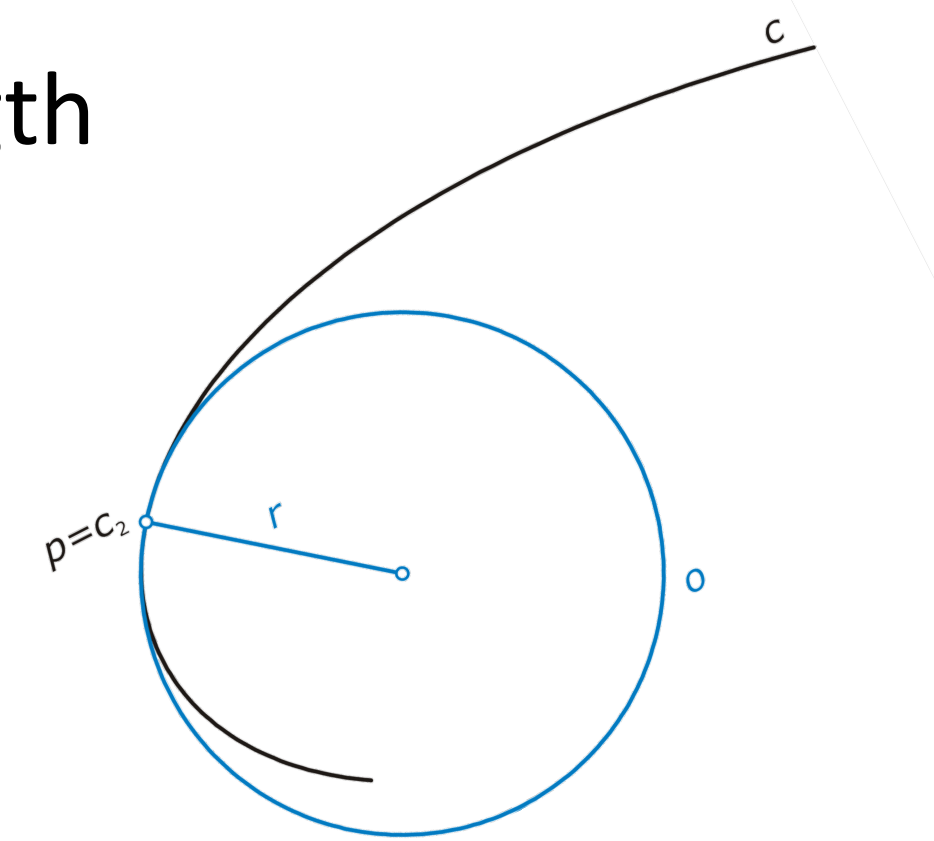
$$\|\dot{\mathbf{x}}(s)\| = 1 \rightarrow \dot{\mathbf{x}}(s) \cdot \ddot{\mathbf{x}}(s) = 0$$

19

Curvature

$\mathbf{x}(t)$ a curve parameterized by arc length

The ***curvature*** of $\mathbf{x}$ at t: $\boxed{\kappa = \left\|\ddot{\mathbf{x}}(t)\right\|}$



$\dot{\mathbf{x}}(t)$ – the tangent vector at t

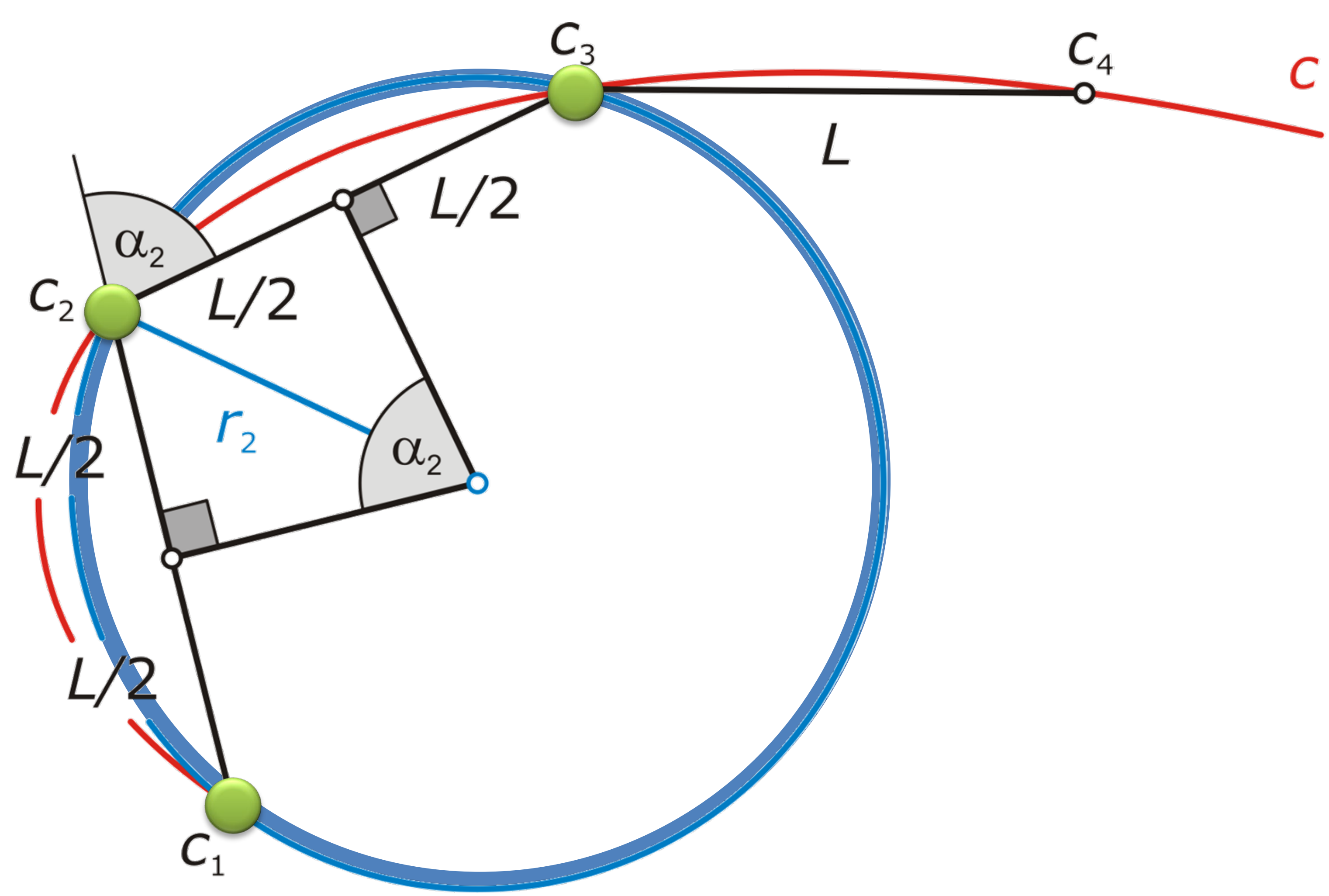
$\ddot{\mathbf{x}}(t)$ – the *change* in the tangent vector at t

$R(t) = 1/\kappa(t)$ is the ***radius of curvature*** at t

[Some driving-related intuition]

Curvature

As a Limit



3 points define a circle.

What is the *osculating circle* radius?

$$r_2 \quad = \quad L \,/\, (\, 2 \sin(\alpha_2/2)\,)$$

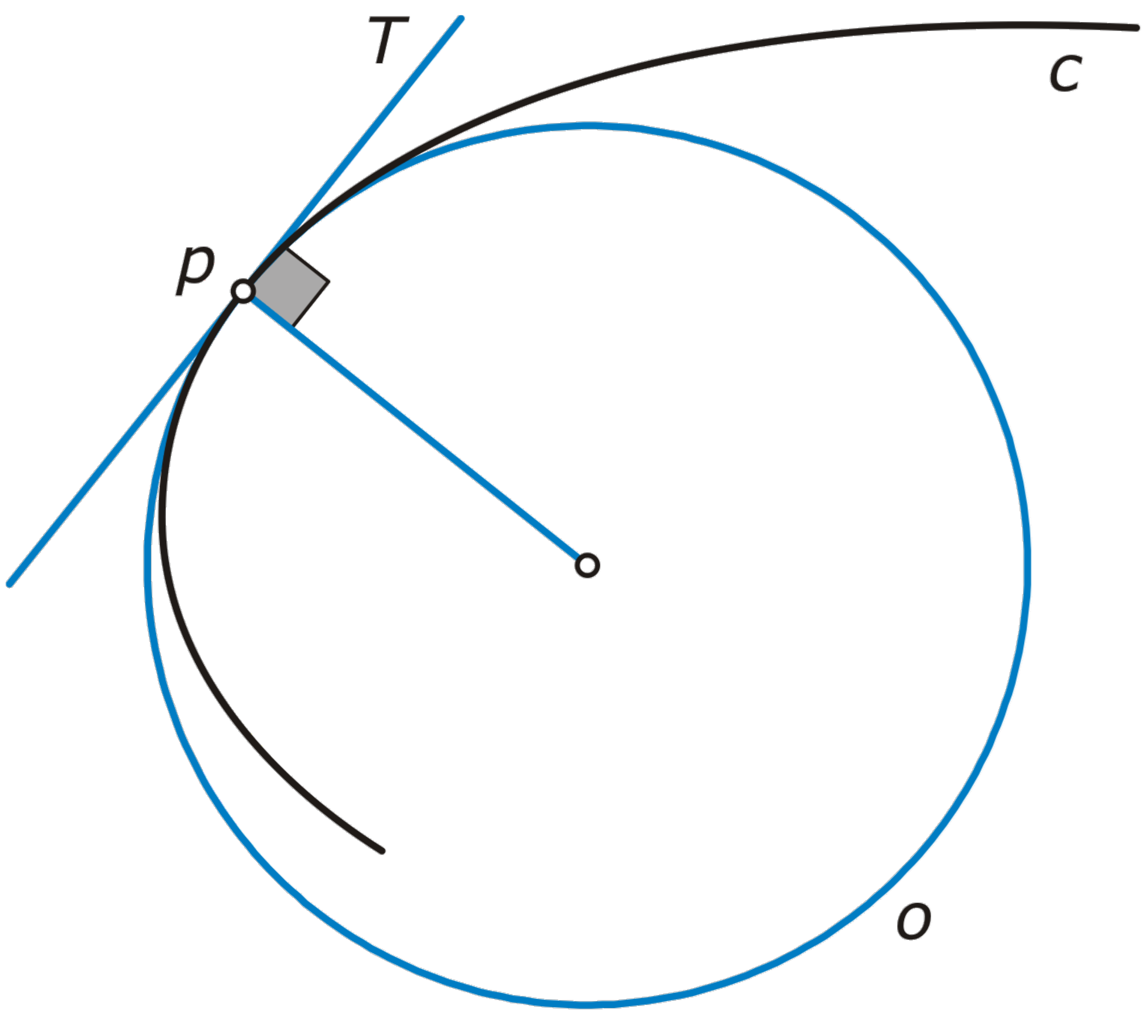
And the discrete curvature (the inverse):

$$\kappa \;=\; \lim(\,(2\sin(\alpha/2))\;/\;L\;\;)$$

$$\kappa \;=\; \lim(\alpha\;/L) \quad (for\ small\ \alpha)$$

Some images from “Architectural Geometry” by Pottmann, 2007

Signed Curvature



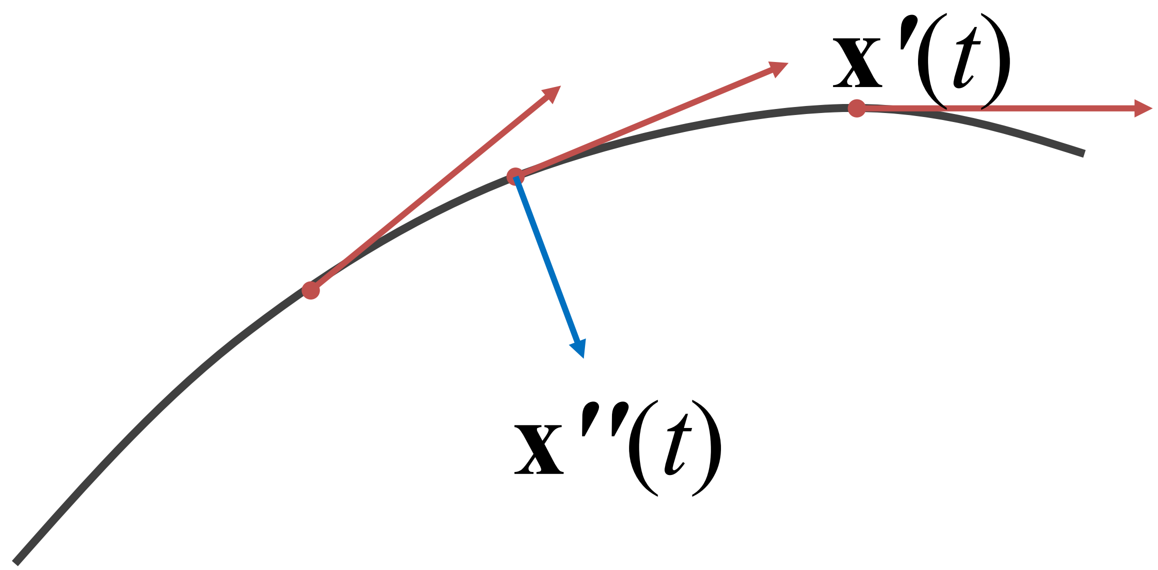
The **sign of the curvature** depends on the side of *o* w.r.t. the **tangent**.

…In CCW curves, positive = circle to the left!

The Normal Vector

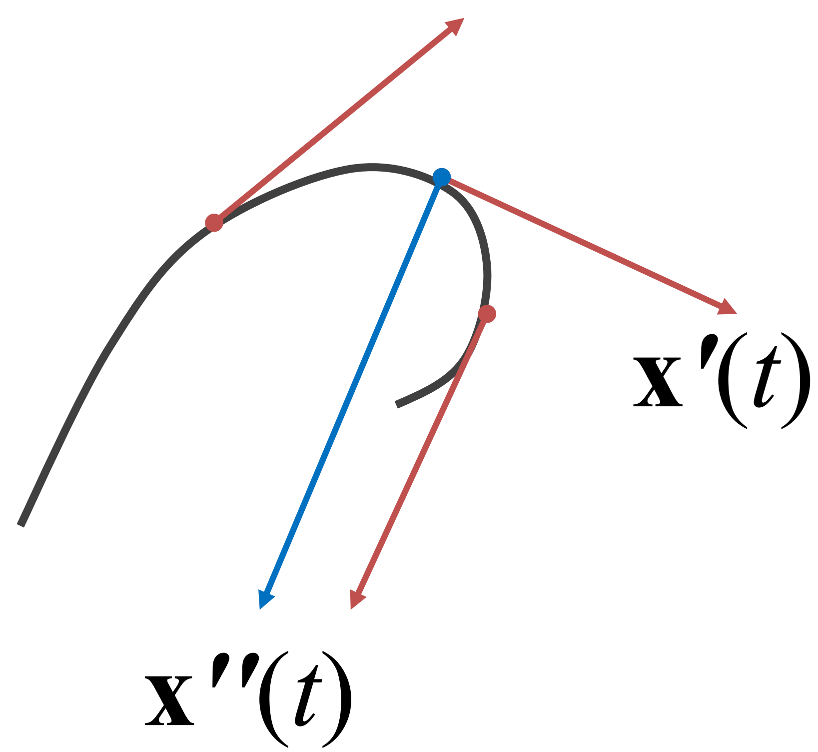
$\mathbf{x}'(t) = \boldsymbol{T}(t)$ - tangent vector

$|\mathbf{x}'(t)|$ - velocity



$\mathbf{x}''(t) = \boldsymbol{T}'(t)$ - normal direction

$|\mathbf{x}''(t)|$ - curvature

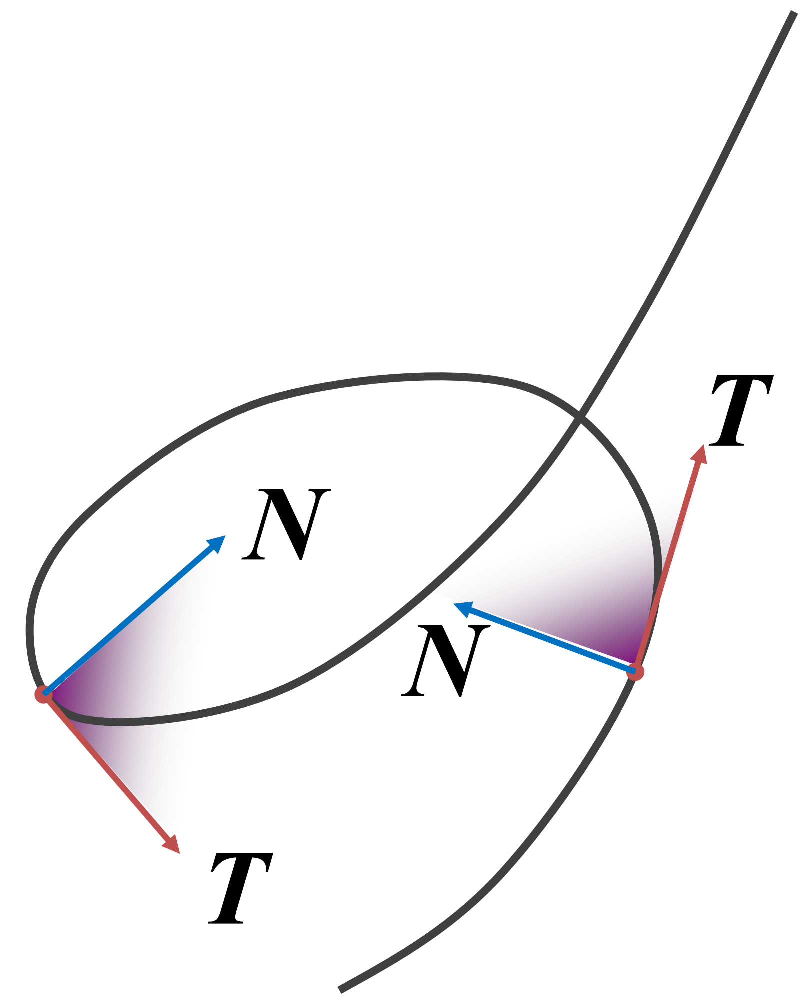


If $|\mathbf{x}''(t)| \neq \mathbf{0}$, define $N(t) = \boldsymbol{T}'(t)/|\boldsymbol{T}'(t)|$

Then $\mathbf{x}''(t) = \boldsymbol{T}'(t) = \kappa(t)\boldsymbol{N}(t)$

The Osculating Plane

The plane determined by the unit tangent and normal vectors $T(s)$ and $N(s)$ is called the *osculating plane* at s

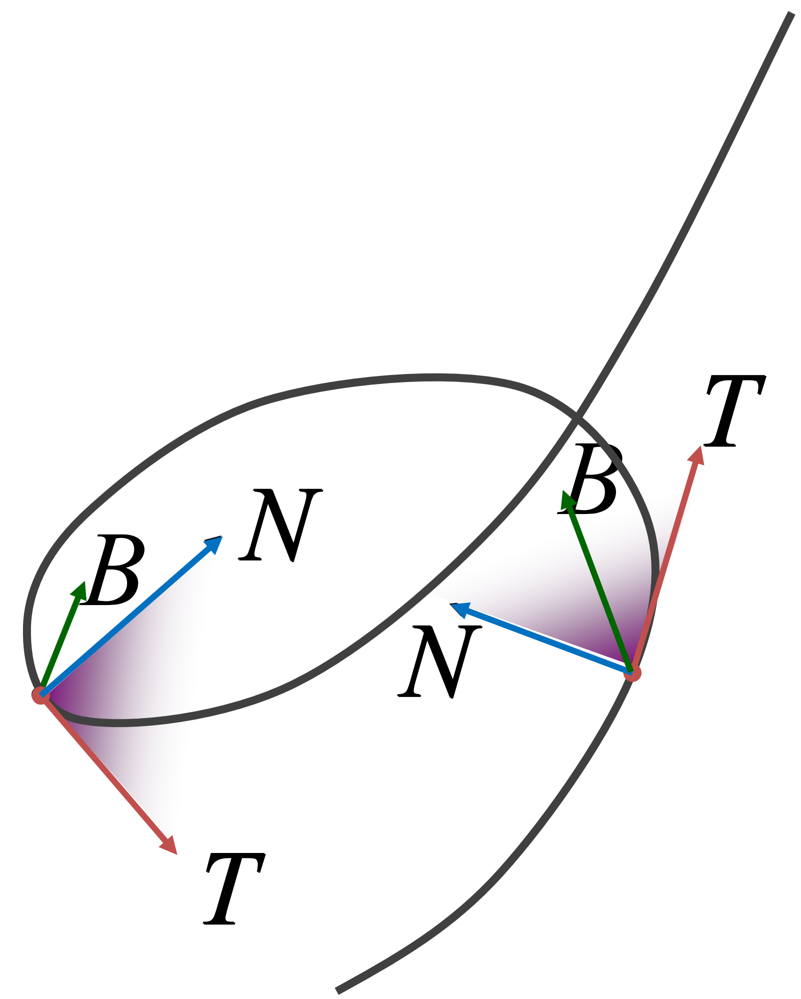


The Binormal Vector

For points s, s.t. $\kappa(s) \neq 0$, the *binormal vector* $\boldsymbol{B}(s)$ is defined as:

$$\boldsymbol{B}(s) = \boldsymbol{T}(s) \times \boldsymbol{N}(s)$$

The binormal vector defines the osculating plane



The Frenet Frame

$$T = \frac{\dot{x}}{\|\dot{x}\|}$$

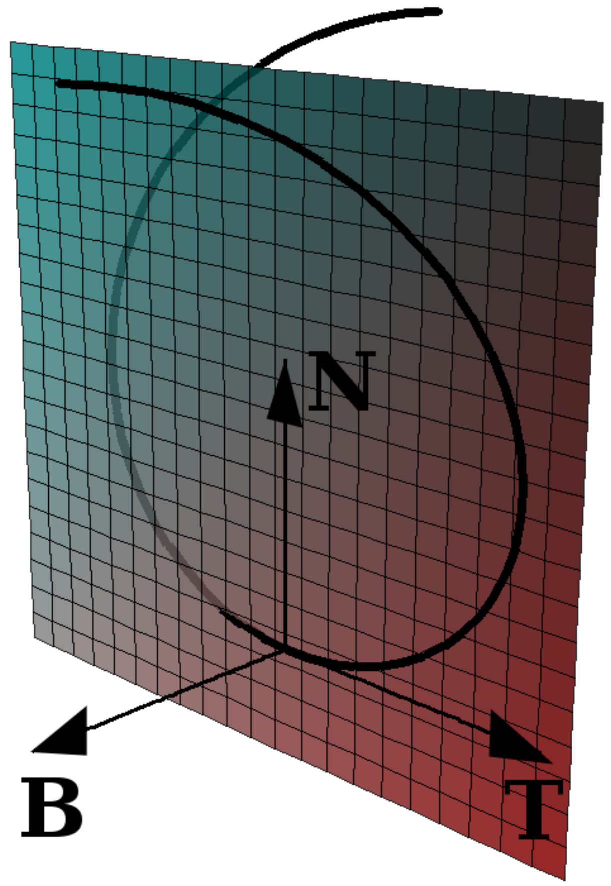
tangent

$$N = \frac{\ddot{x}}{\|\ddot{x}\|}$$

normal

$$B = T \times N$$

binormal



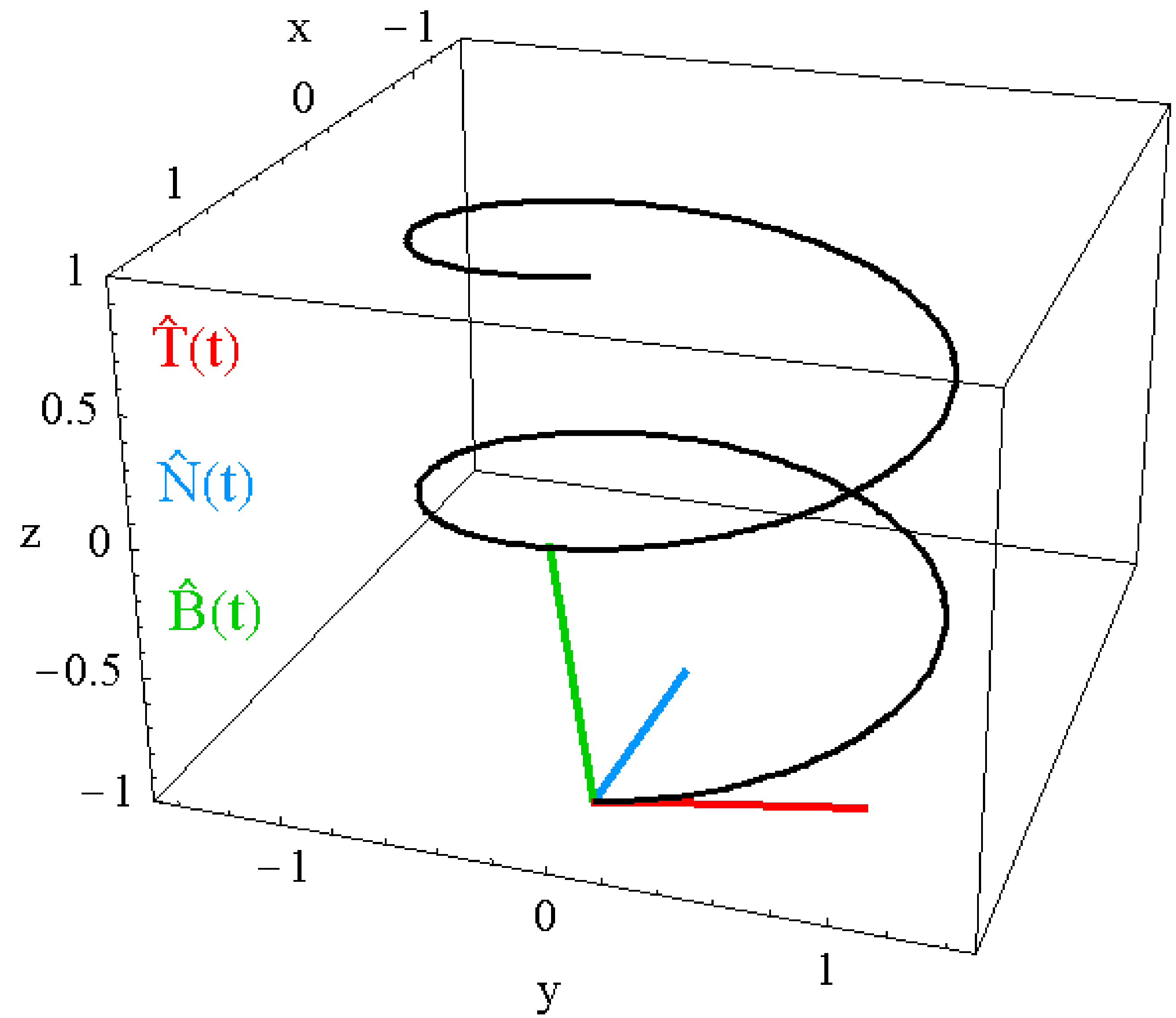
Demo

The Frenet Frame

$\{\boldsymbol{T}(s), \boldsymbol{N}(s), \boldsymbol{B}(s)\}$ form an orthonormal basis for R^3 called the *Frenet frame*

How does the frame change when the particle moves?

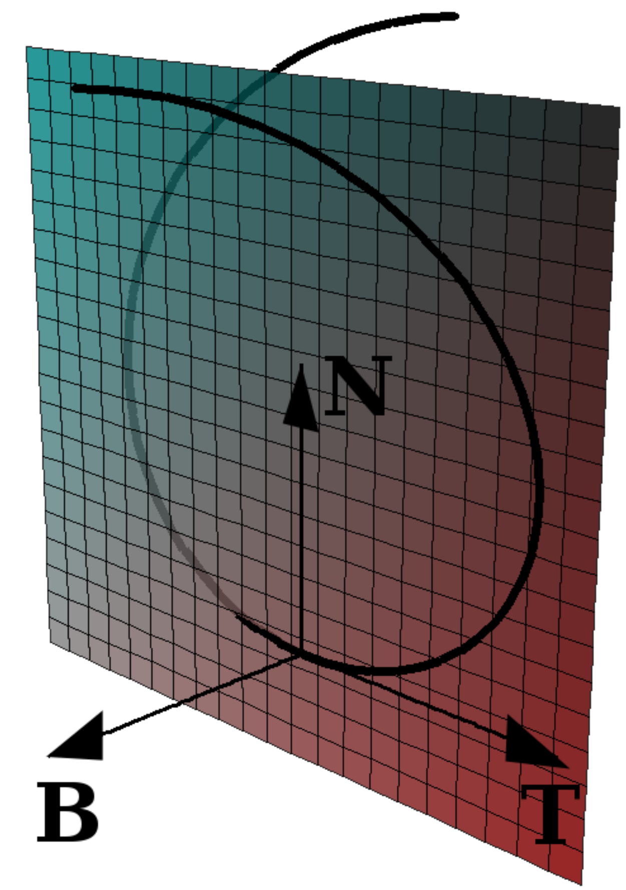
What are $\boldsymbol{T}', \boldsymbol{N}', \boldsymbol{B}'$ in terms of $\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}$?



The Frenet Frame

- Frenet-Serret formulas

$$\begin{aligned}
\dot{T} &= + \kappa N \\
\dot{N} &= -\kappa T + \tau B \\
\dot{B} &= - \tau N
\end{aligned}$$



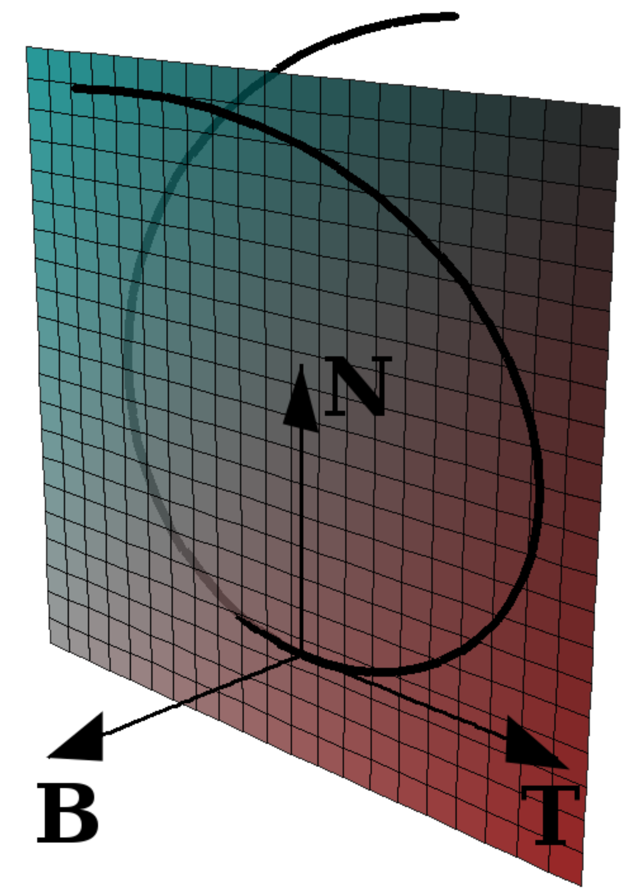
- curvature $\kappa = \|\ddot{x}\|$

- torsion $\tau = \frac{1}{\kappa^2} \det\left[\dot{x}, \ddot{x}, \dddot{x}\right]$

(arc-length parameterization)

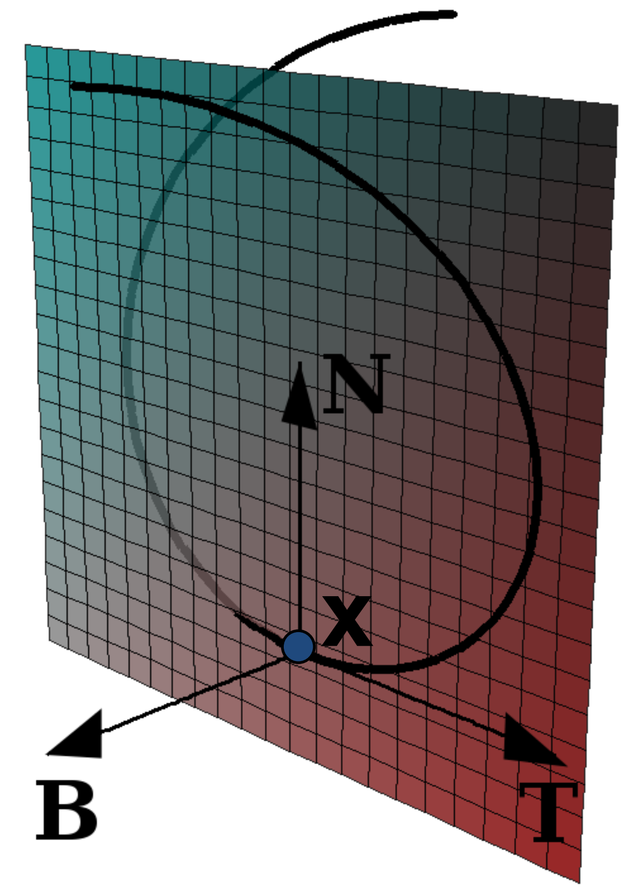
Curvature and Torsion

- Curvature: Deviation from straight line
- Torsion: Deviation from planarity
- Independent of parameterization
  - intrinsic properties of the curve
- Invariant under rigid (translation+rotation) motion
- Define curve uniquely up to rigid motion

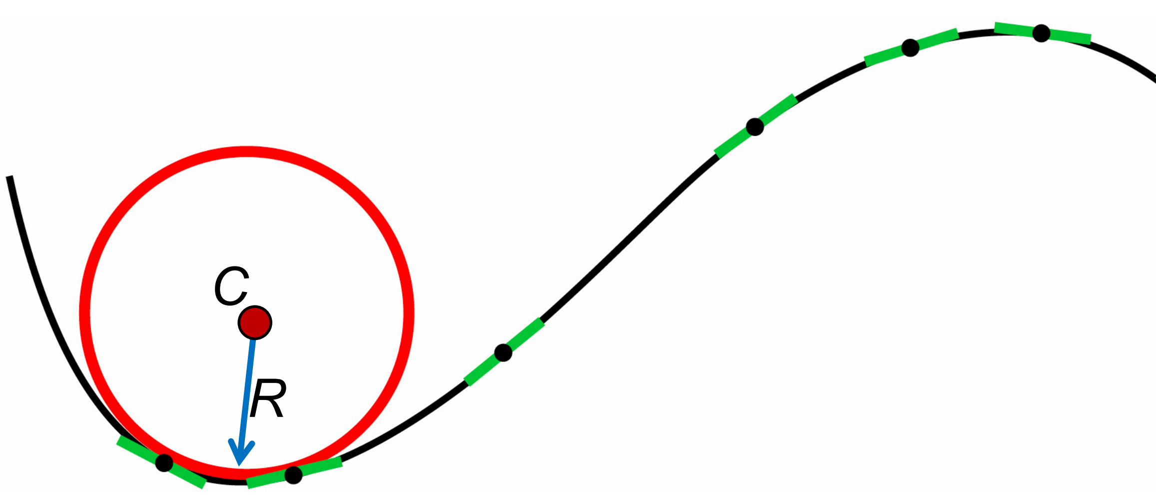


Curvature and Torsion

- Planes defined by $\mathbf{x}$ and two vectors:
  - ***osculating plane***: vectors $\mathbf{T}$ and $\mathbf{N}$
  - ***normal plane***: vectors $\mathbf{N}$ and $\mathbf{B}$
  - ***rectifying plane***: vectors $\mathbf{T}$ and $\mathbf{B}$



- Osculating circle
  - second order contact with curve
  - center $C = x + \kappa^{-1} N$
  - radius $R = \kappa^{-1}$

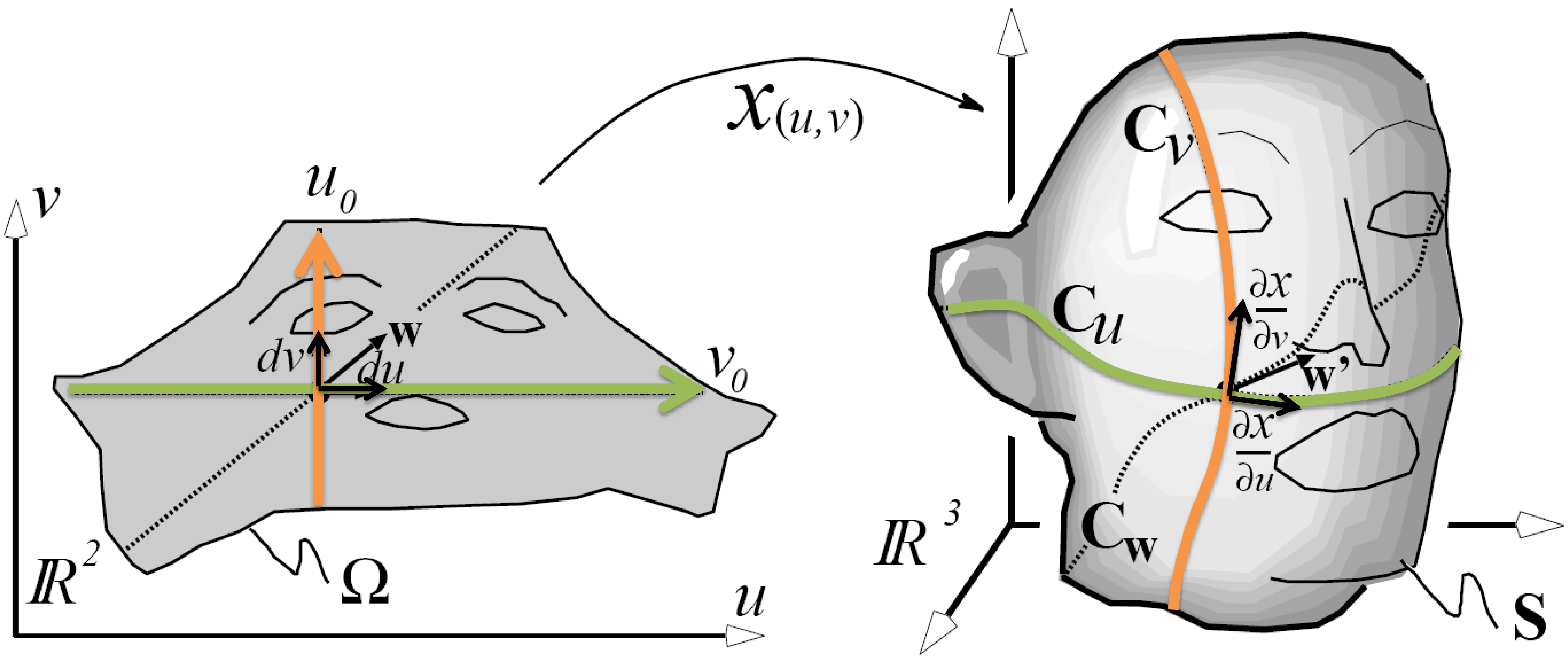


Things you can do with curves

Discrete Elastic Rods, Bergou et al., SIGGRAPH 2008

Differential Geometry: Surfaces

$$\mathbf{x}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}, \ (u,v) \in \mathbb{R}^2$$



Differential Geometry: Surfaces

- Continuous surface

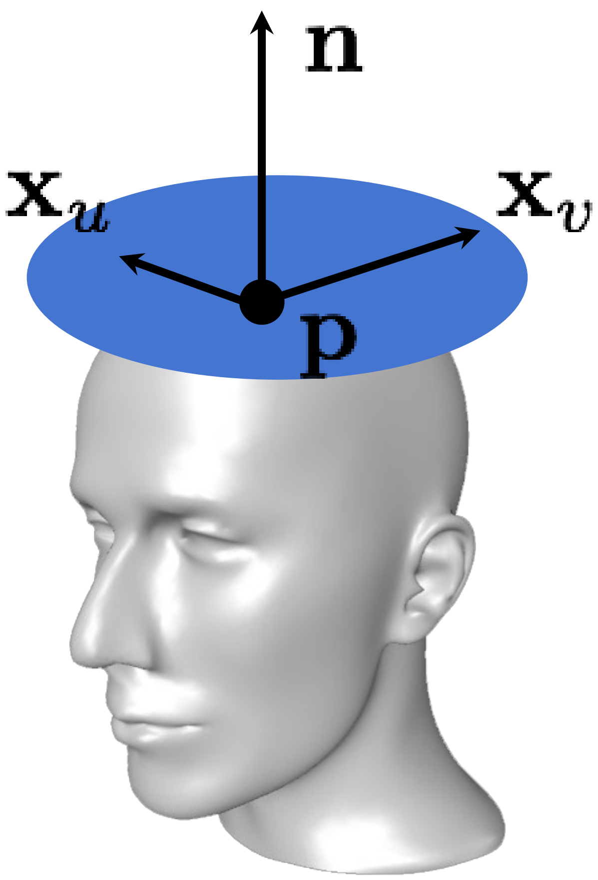
$$\mathbf{x}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}, \ (u,v) \in \mathbb{R}^2$$

- Normal vector

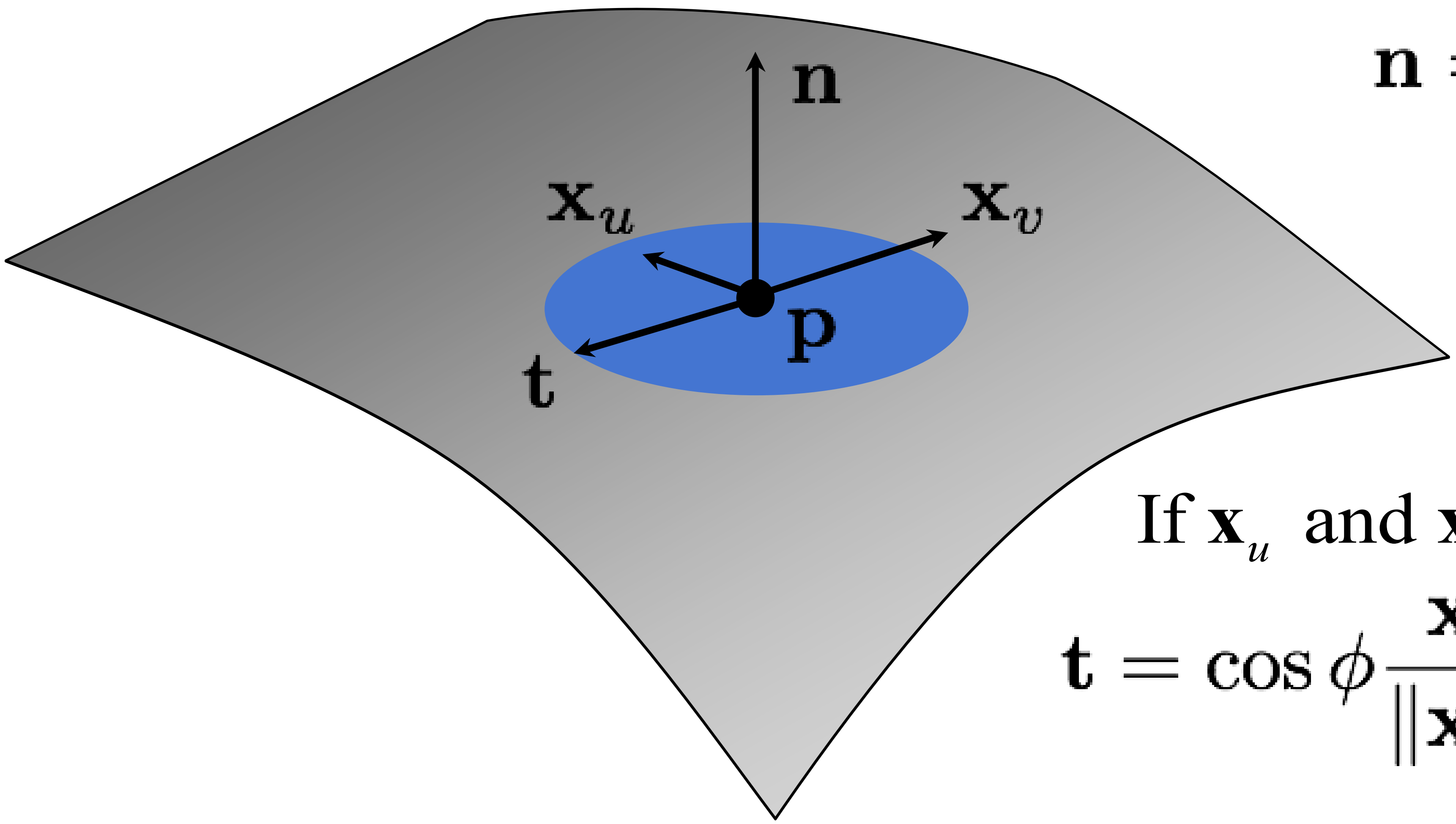
$$\mathbf{n} = (\mathbf{x}_u \times \mathbf{x}_v)/\|\mathbf{x}_u \times \mathbf{x}_v\|$$

  – assuming regular parameterization, i.e.

$$\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$$



Normal Curvature

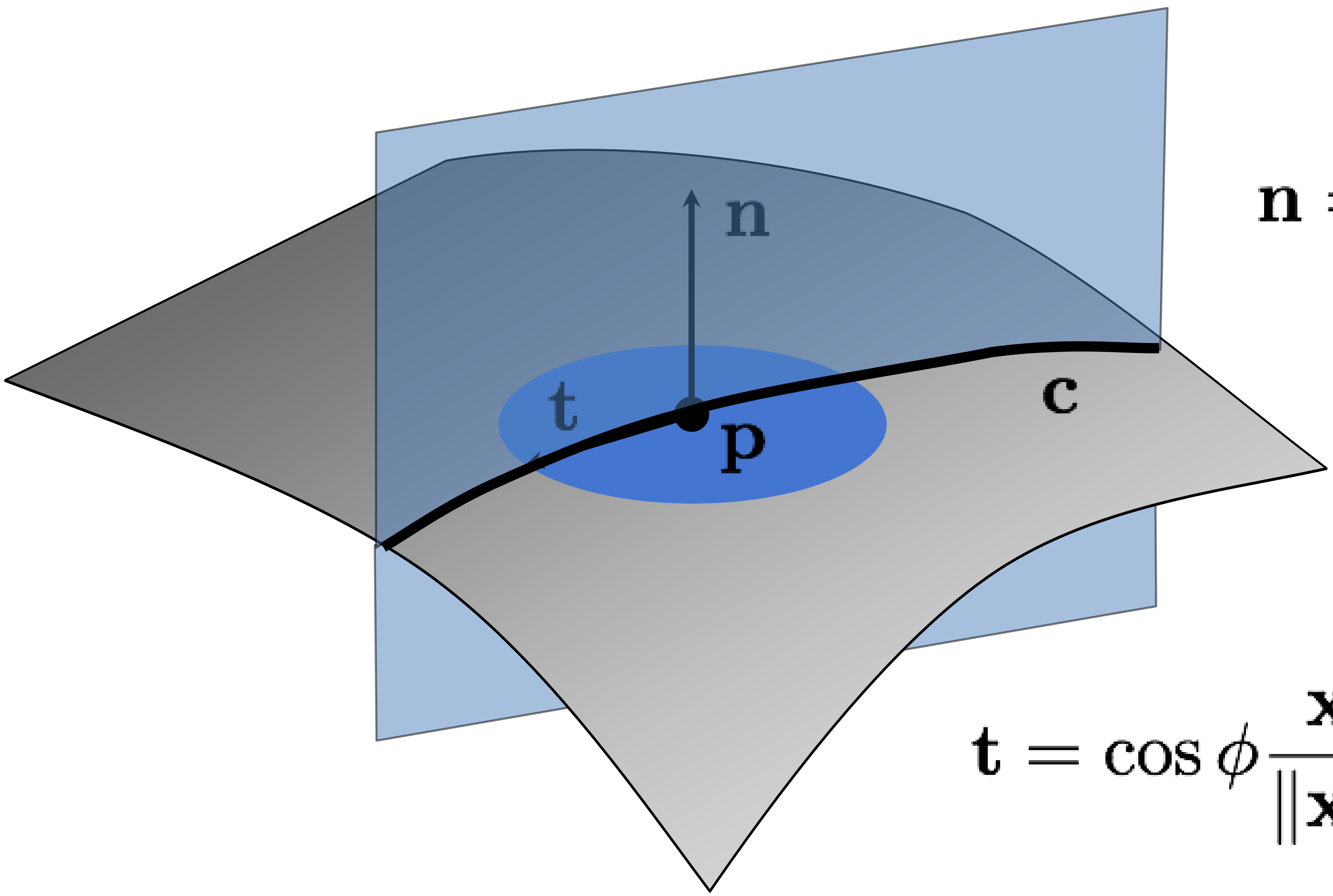


$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

If $\mathbf{x}_u$ and $\mathbf{x}_v$ are orthogonal:

$$\mathbf{t} = \cos\phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin\phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

Normal Curvature

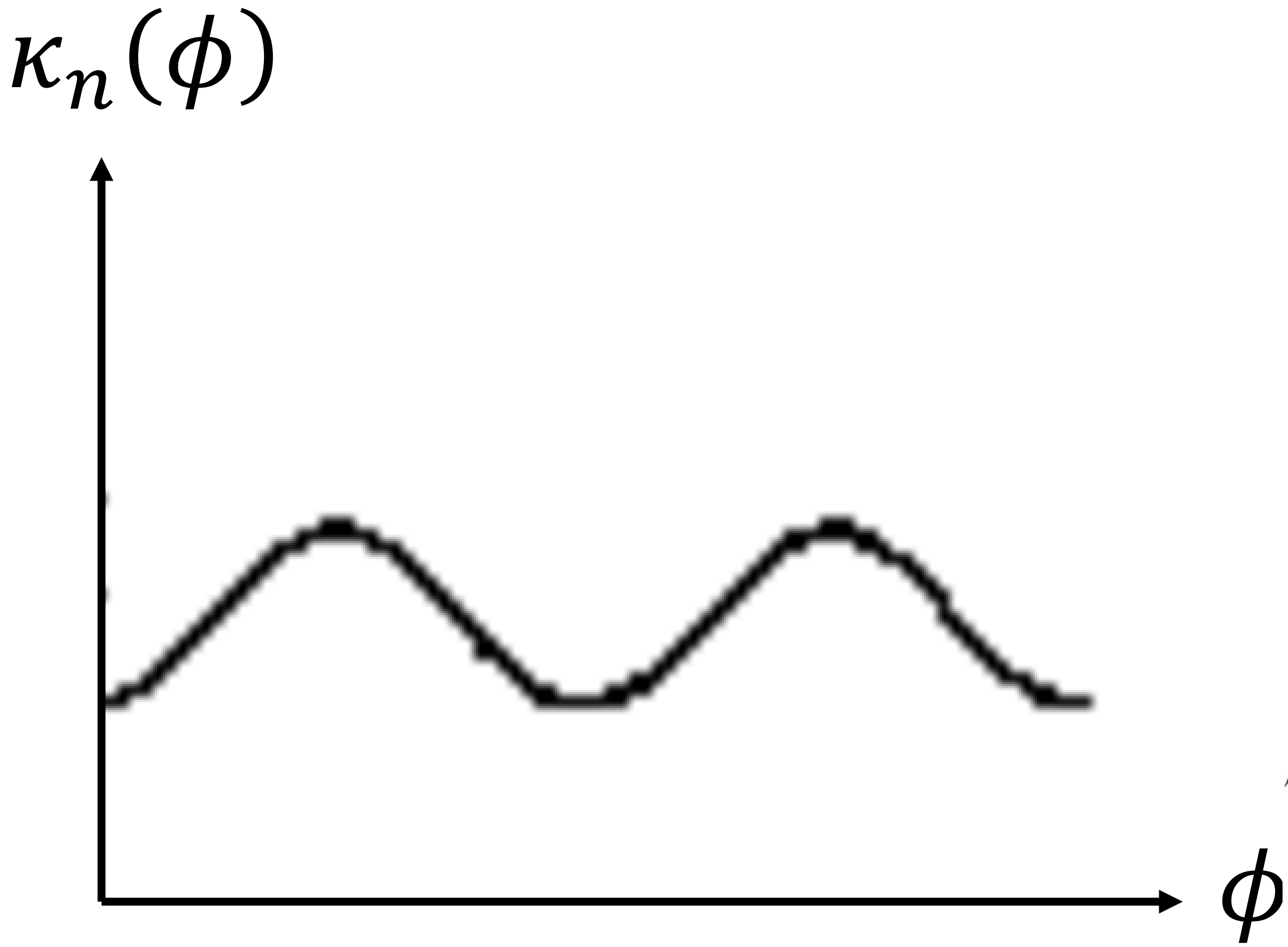


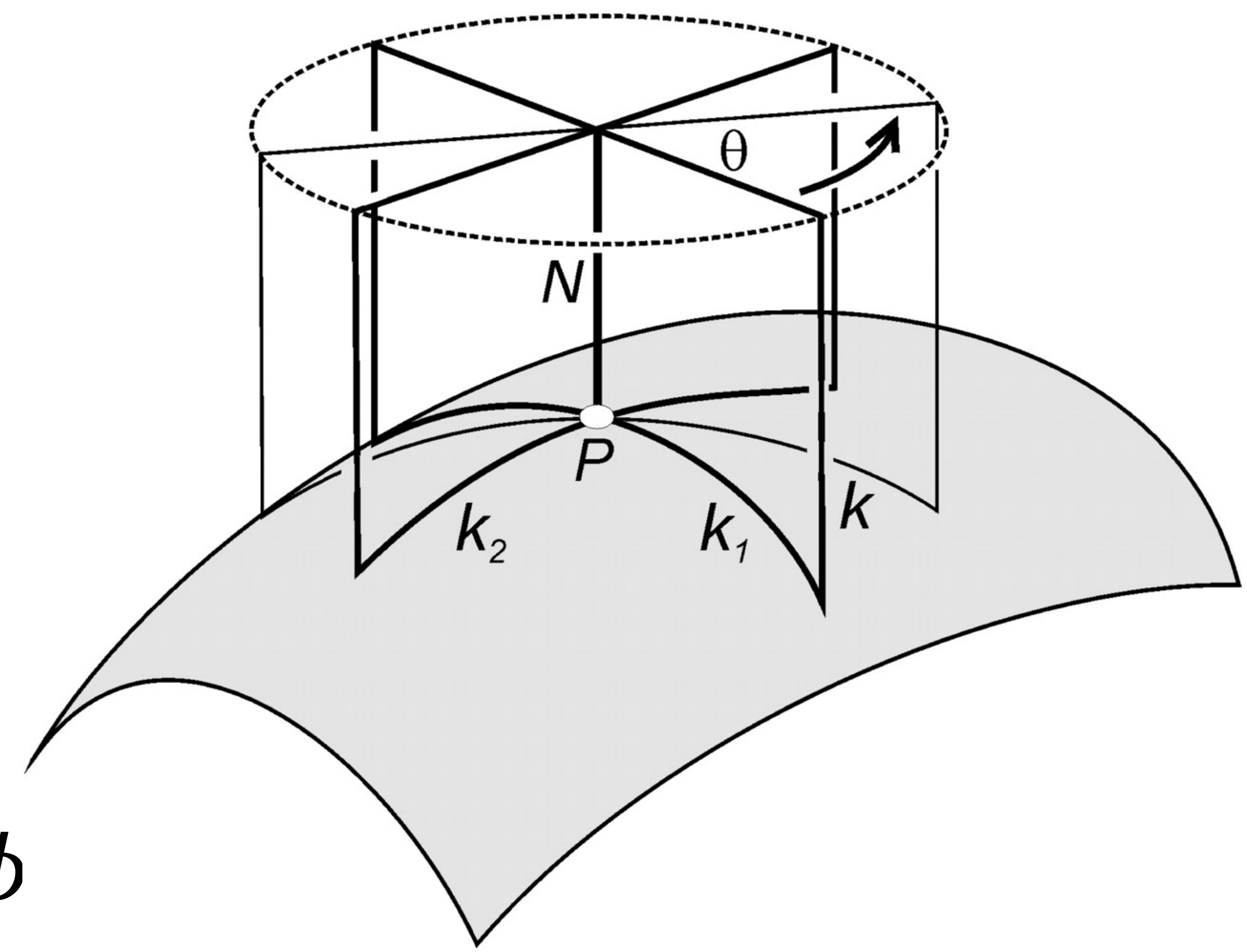
$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

$$\mathbf{t} = \cos\phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin\phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

Normal Curvature

$\kappa_n(\phi)$ - Curvature in direction $\mathbf{t} = \cos\phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin\phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$

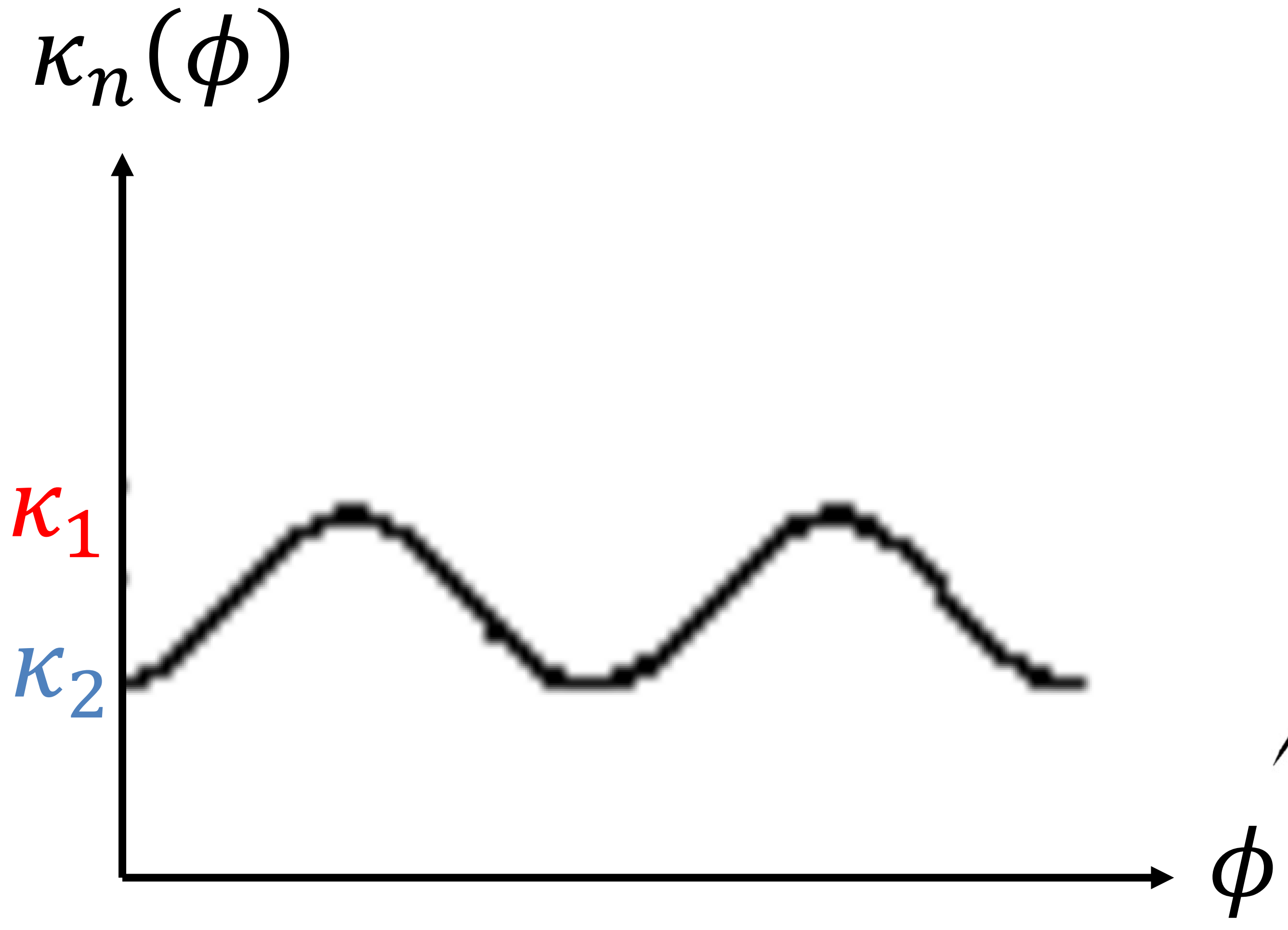


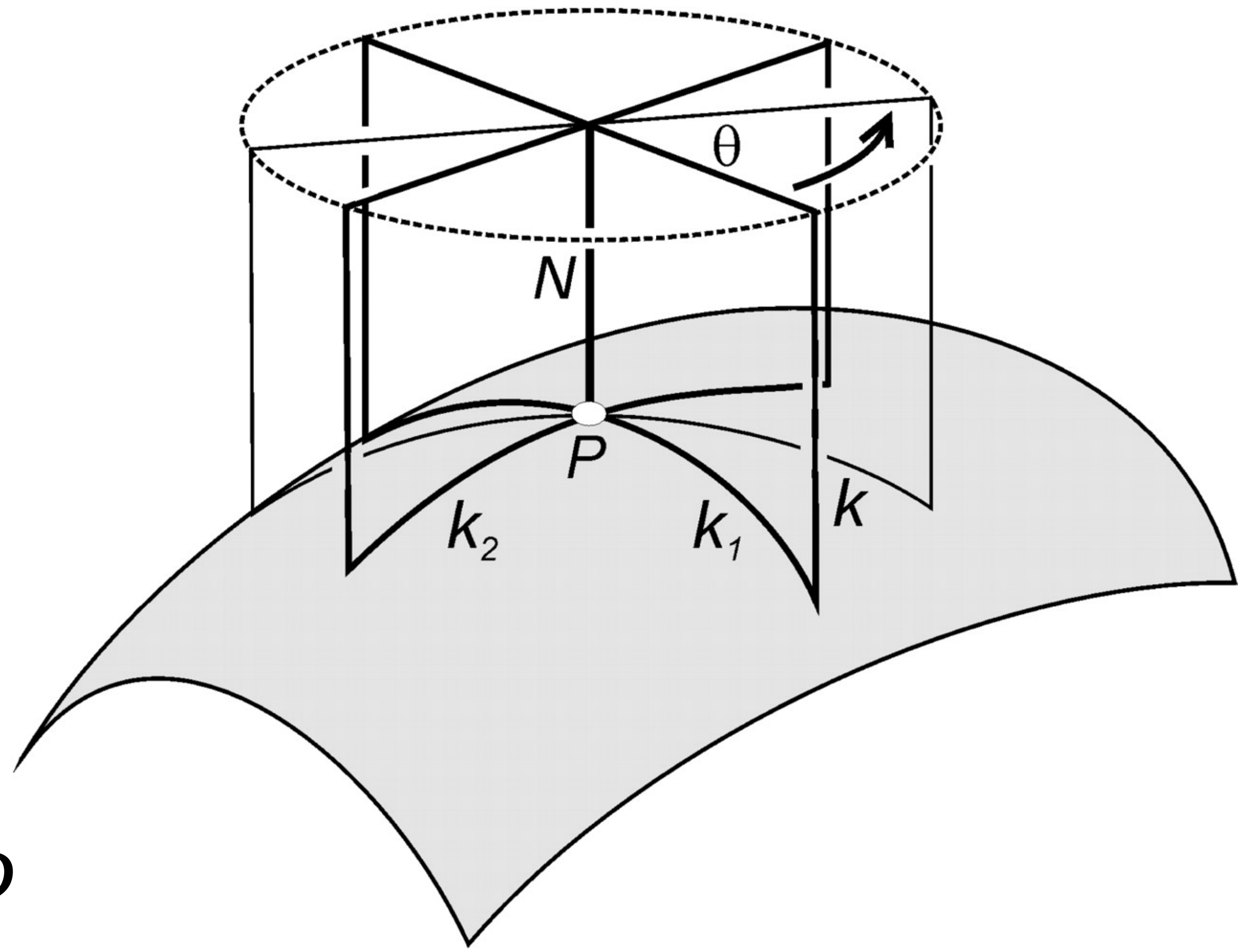


Principal Curvatures

maximum curvature $\kappa_1 = \max_{\phi} \kappa_n(\phi)$

minimum curvature $\kappa_2 = \min_{\phi} \kappa_n(\phi)$

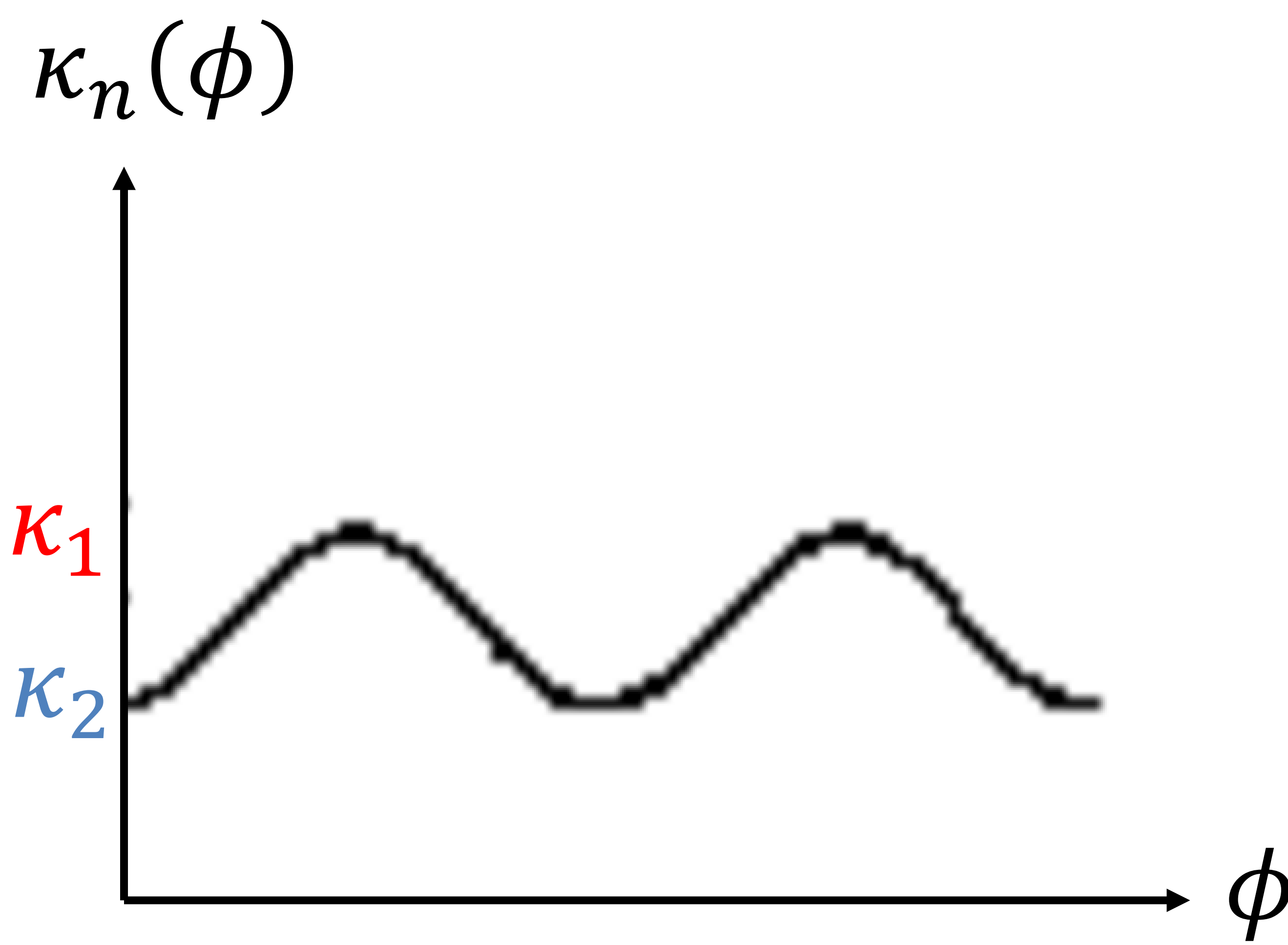


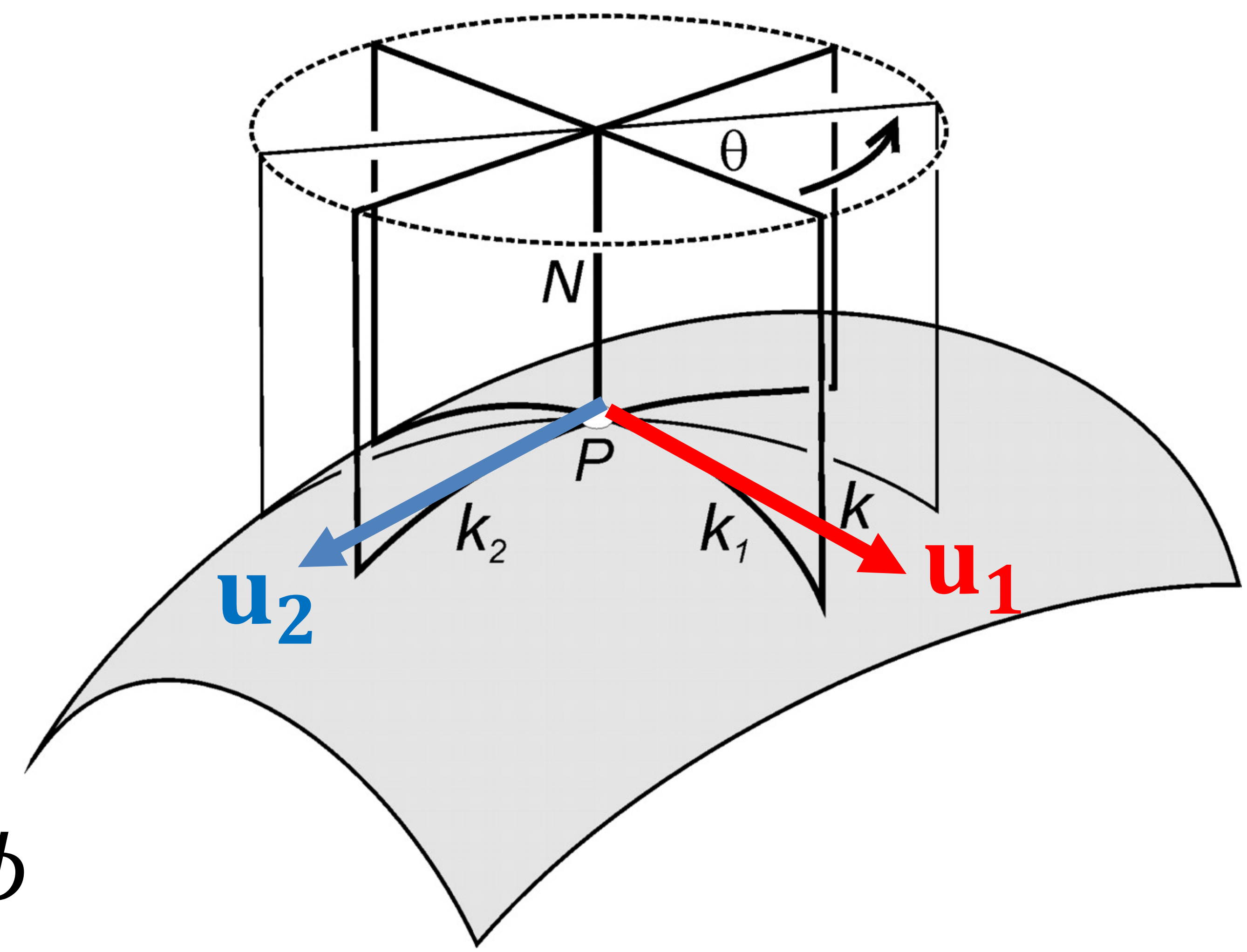


Principal Curvature Directions

maximum curvature *direction* $\mathbf{u_1}$

minimum curvature *direction* $\mathbf{u_2}$



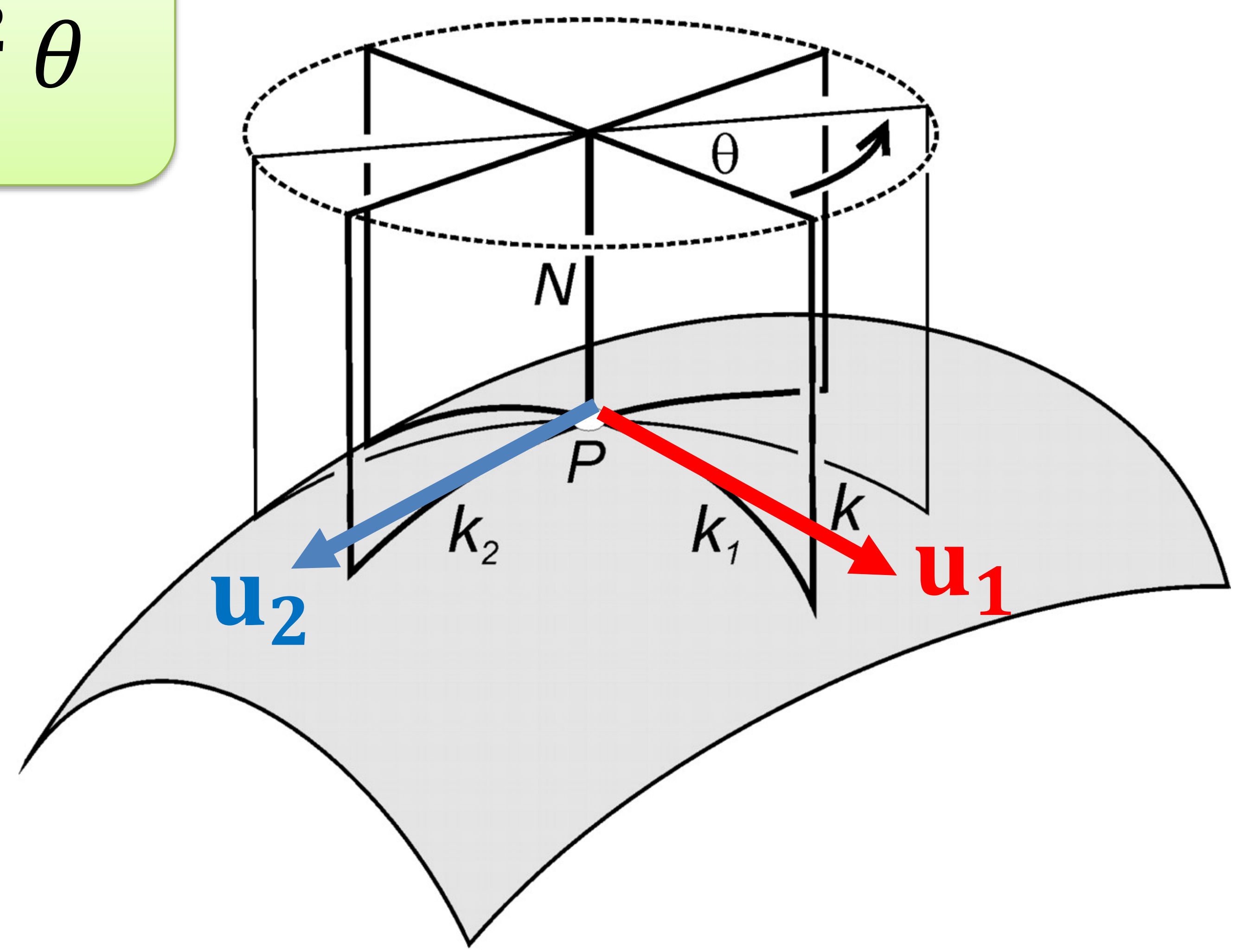


Principal Curvature Directions

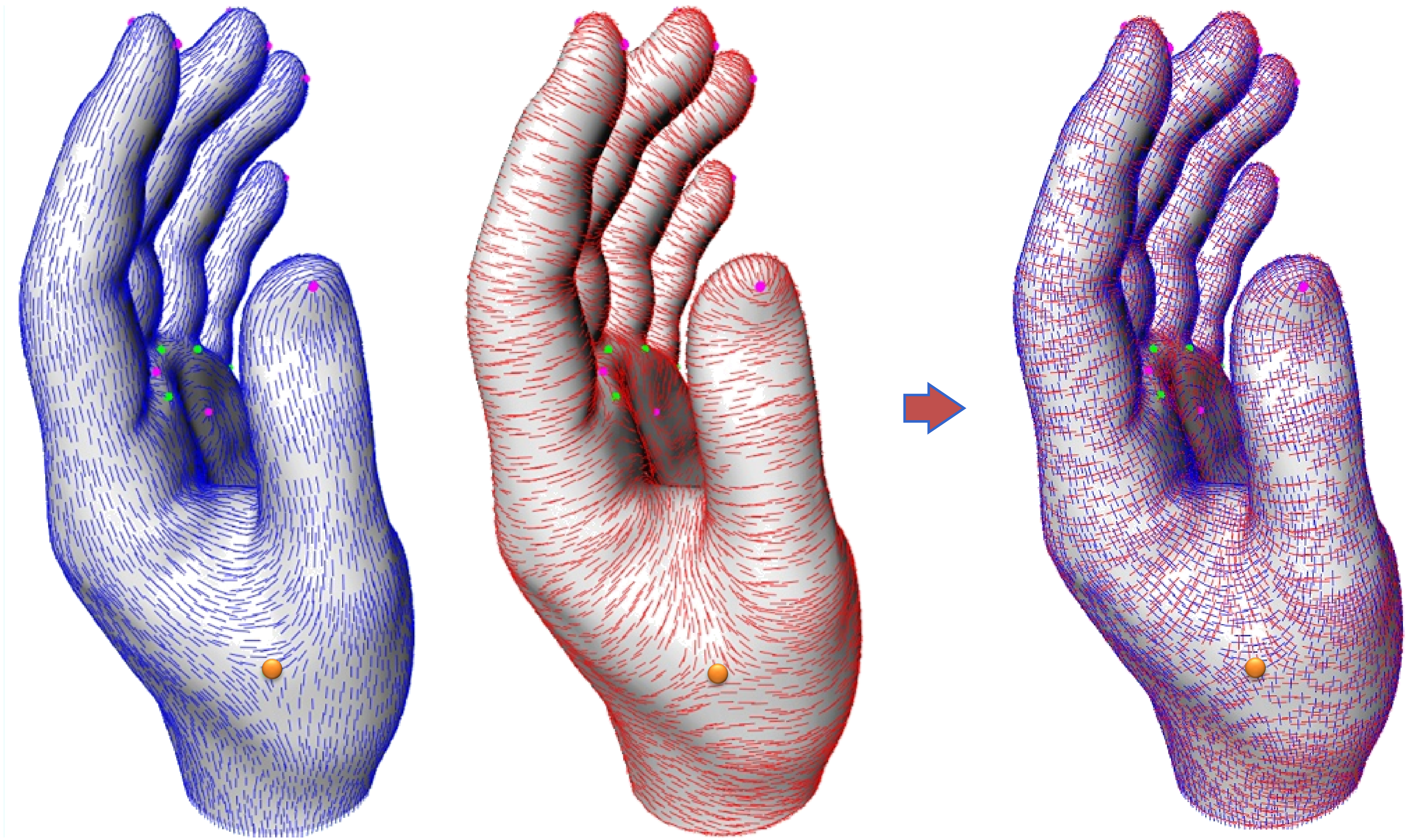
Euler's Theorem: Planes of principal curvature are **orthogonal** and independent of parameterization.

$$\kappa_n(\theta) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$$

θ = angle with $\mathbf{u_1}$

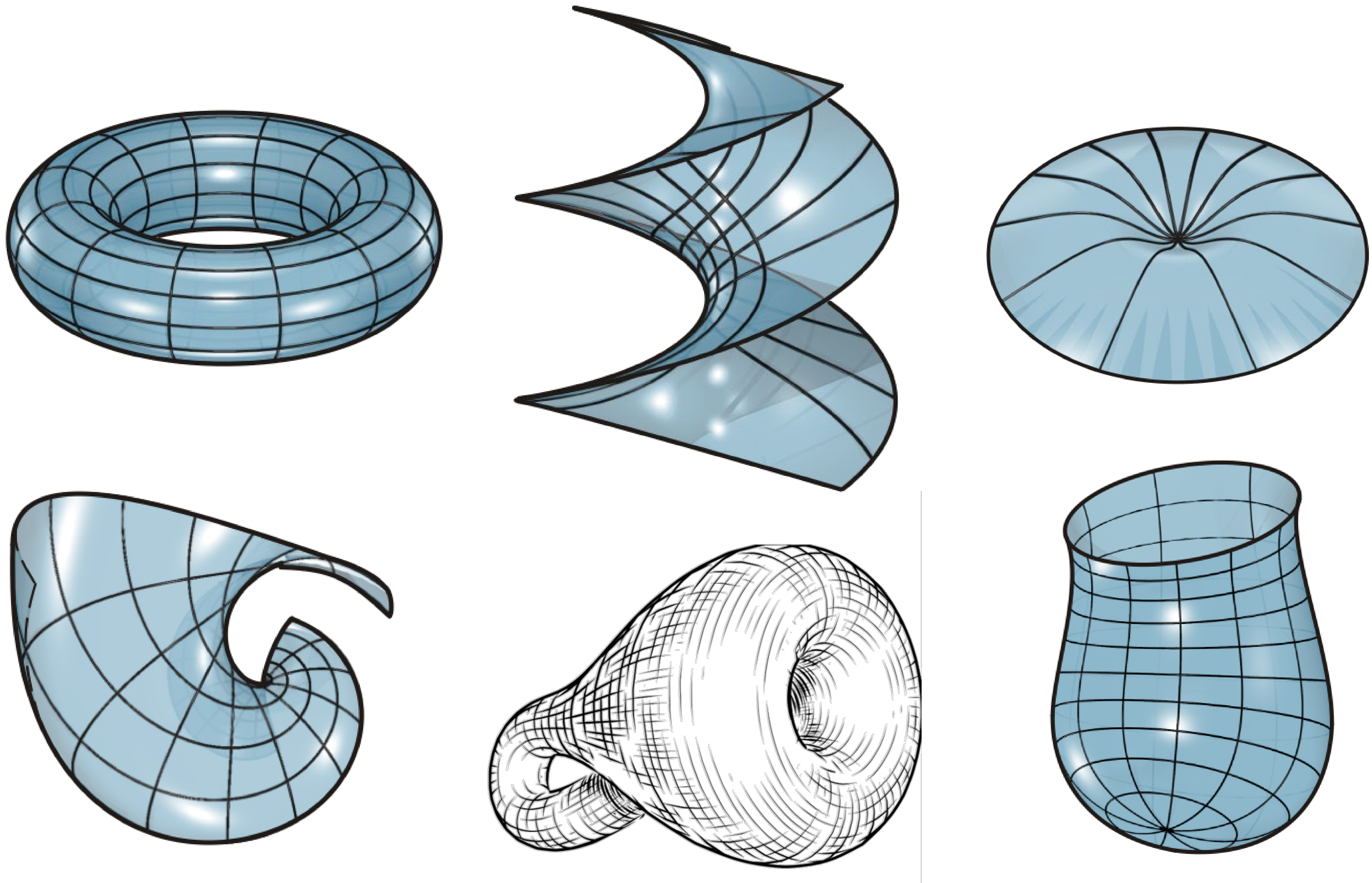


Principal Curvature Directions



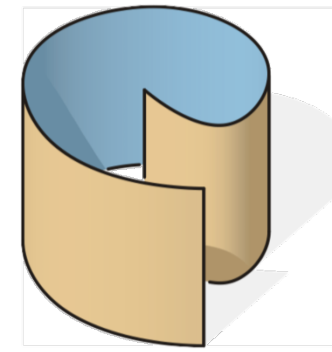
$\mathbf{u}_2$ $\mathbf{u}_1$

Principal Curvature Lines

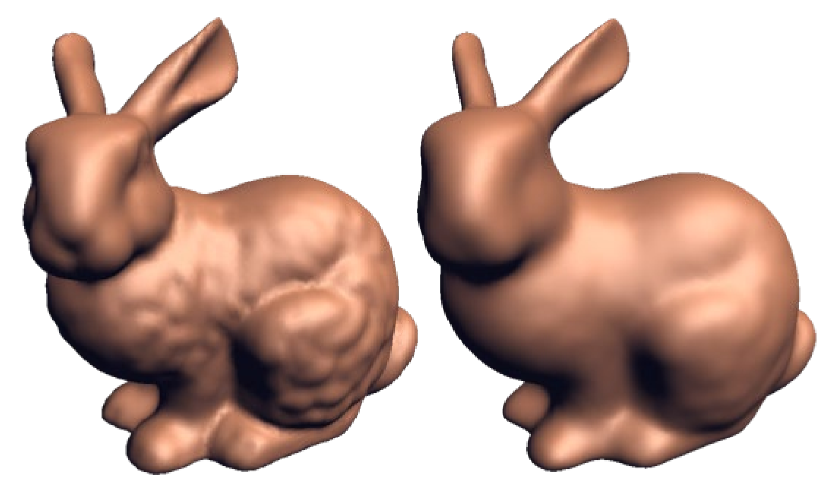


How to use the curvatures?

- Are κ_1, κ_2 *invariant* to surface change?



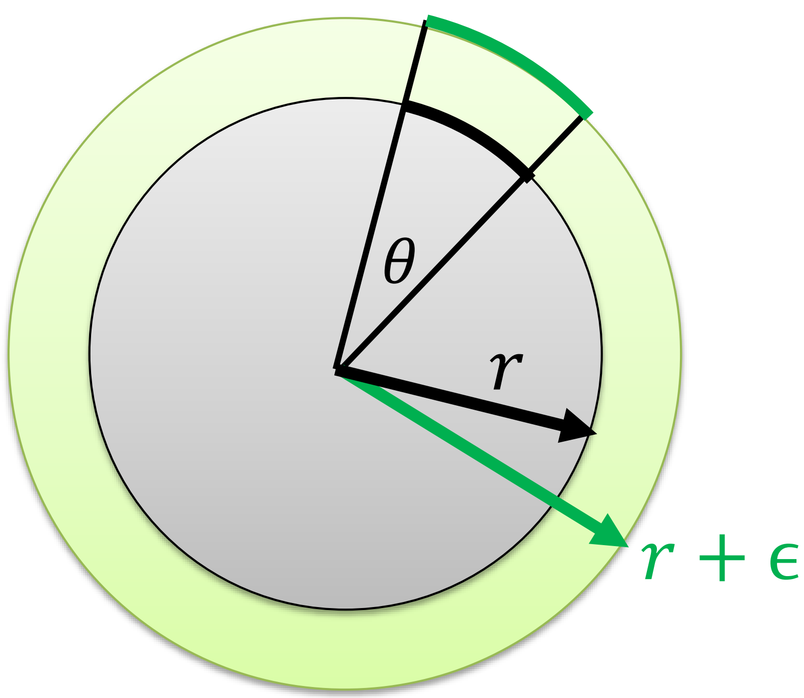
- Optimize for κ_1, κ_2 to improve surface properties?

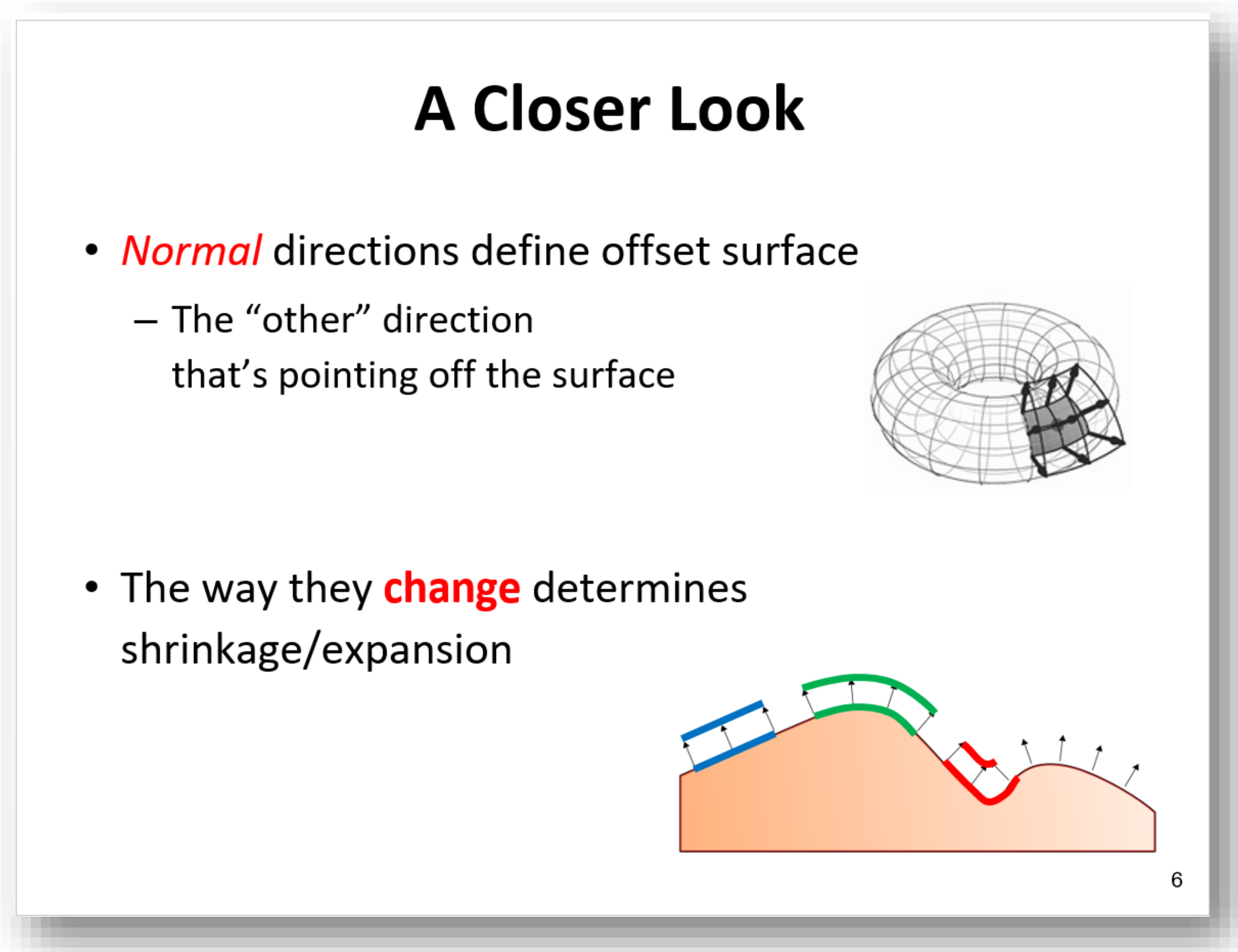


- Fully represent the surface?

One Curvature to Rule Them All?

How does length change?



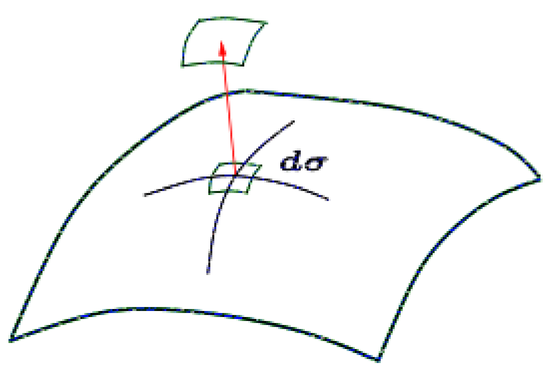


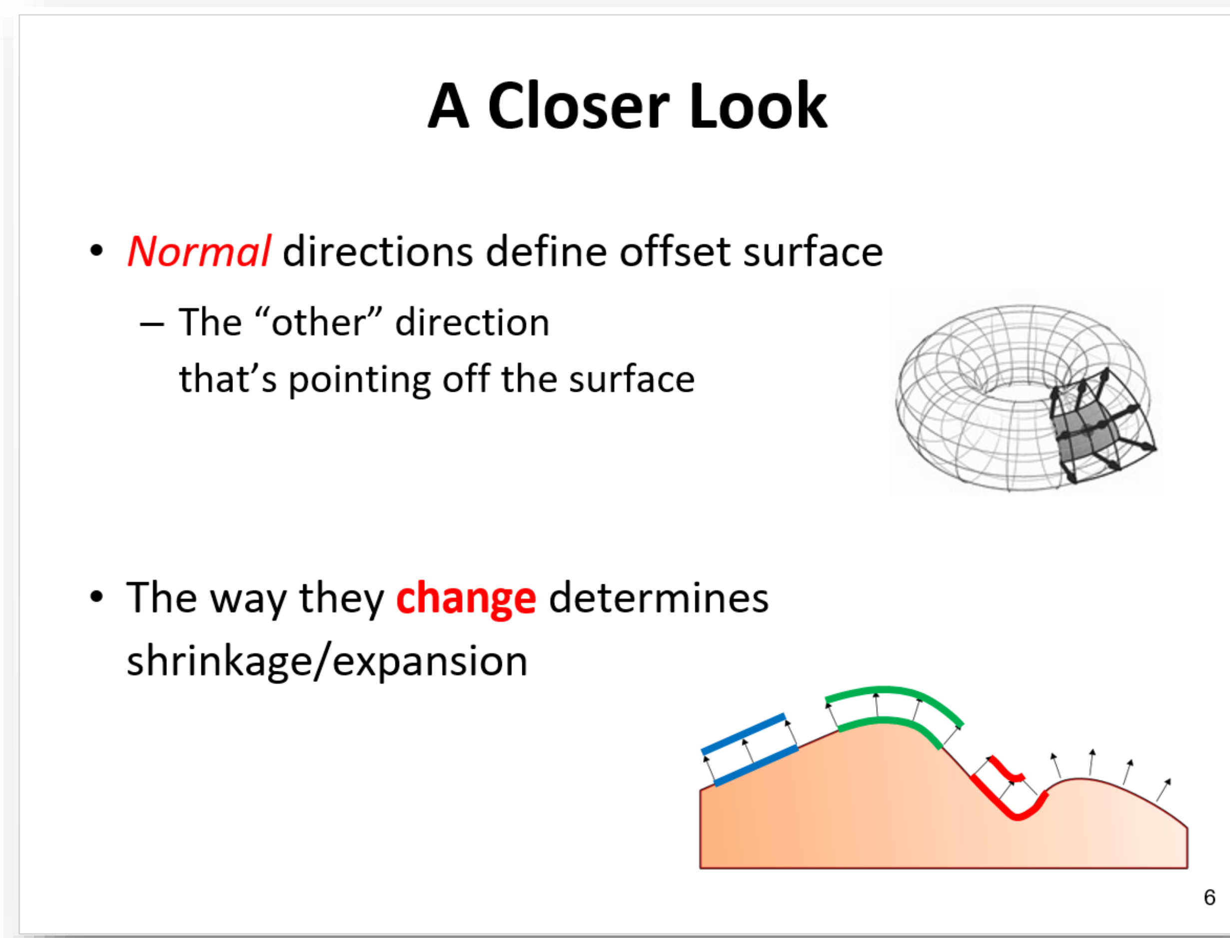
$$\text{length ratio} = \frac{\theta(r+\epsilon)}{\theta r} = 1 + \epsilon\kappa$$

One Curvature to Rule Them All?

How does area change?

$$length \text{ ratio} = 1 + \epsilon\kappa$$





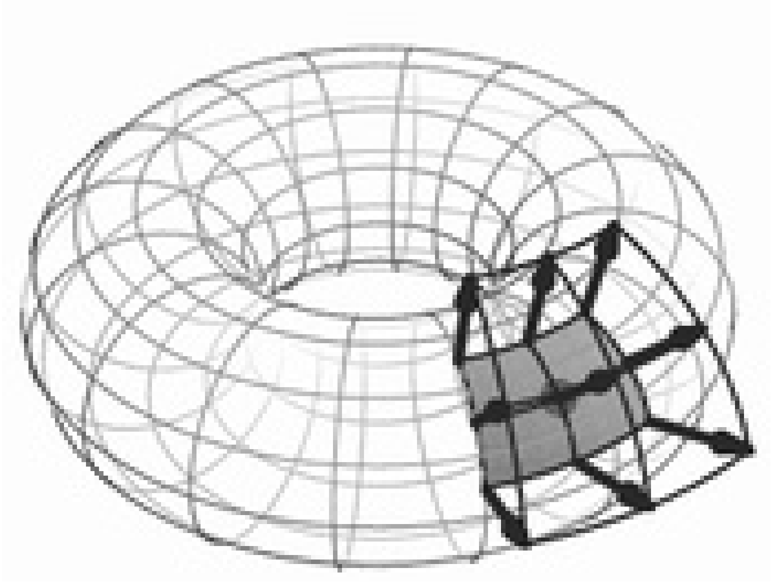
$$area \text{ ratio} = (1 + \epsilon\kappa_1)(1 + \epsilon\kappa_2)$$

$$= 1 + \epsilon(\kappa_1 + \kappa_2) + \epsilon^2\kappa_1\kappa_2$$

Two Curvatures to Rule Them All!

$$\mathit{area}\ \text{ratio} =$$

$$1 + \epsilon\underbrace{(\kappa_1 + \kappa_2)}_{\substack{\mathit{mean} \\ \text{curvature}}} + \epsilon^2\underbrace{\kappa_1\kappa_2}_{\substack{\mathit{Gaussian} \\ \text{curvature}}}$$

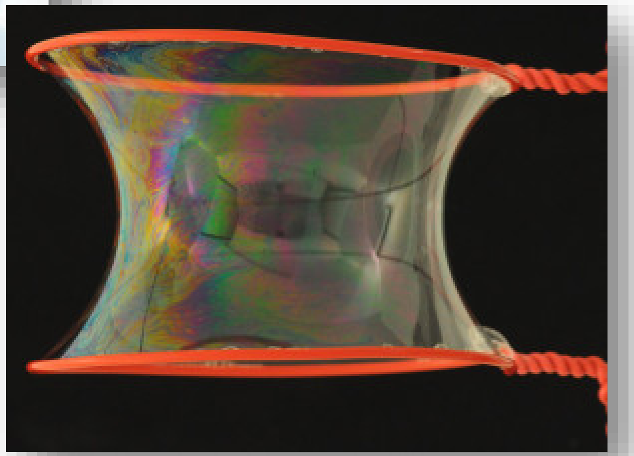


Mean Curvature

$$H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2\pi}\int_0^{2\pi} \kappa_n(\phi)d\phi$$

Minimize *bending*

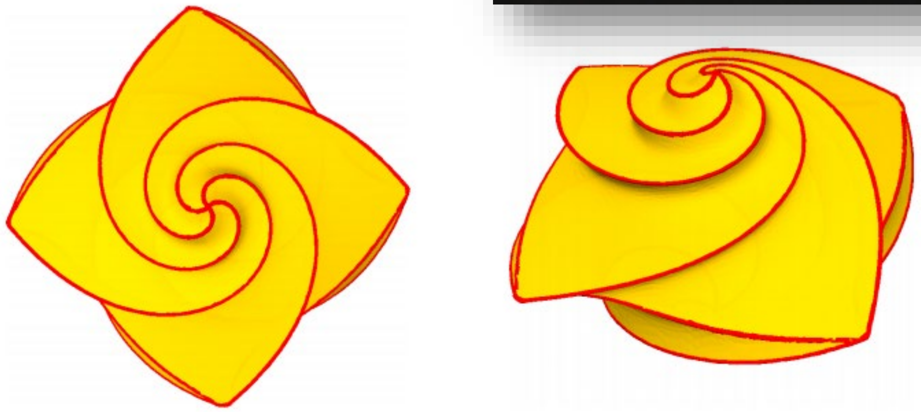
["Discrete Quadratic Curvature Energies", Wardetzky et al., 2007]



Minimal surface, H=0

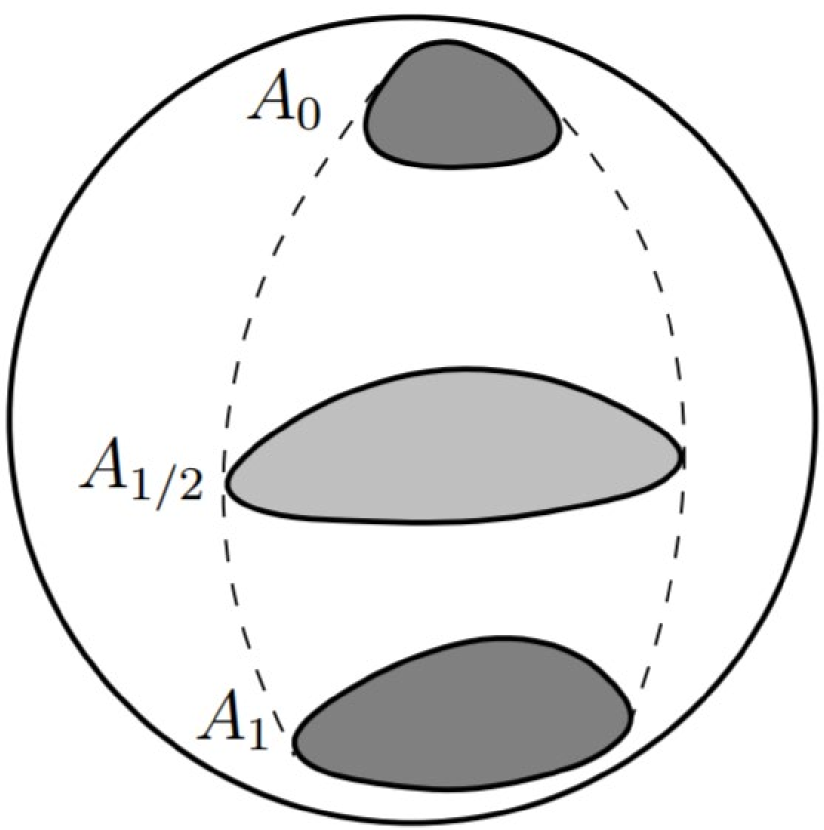
As a descriptor for features

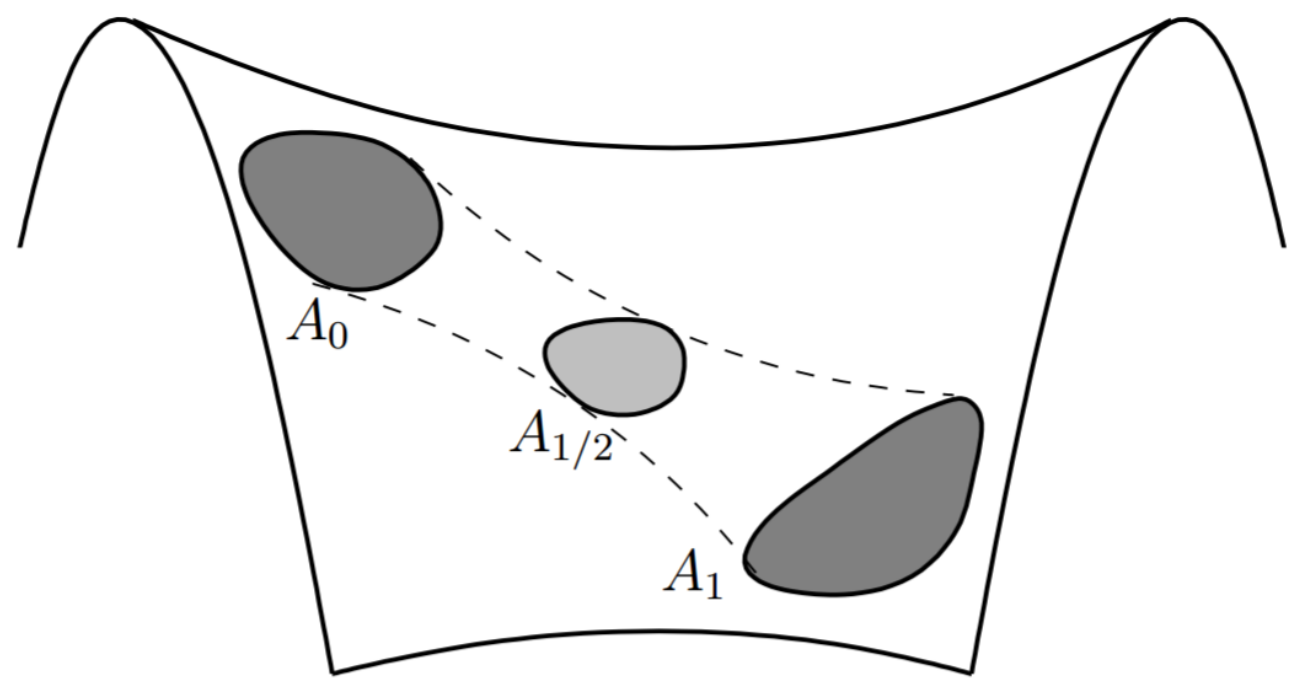
["Robust Voronoi-based Curvature and Feature Estimation", Mérigot et al., 2009]



Gaussian Curvature

$$K = \kappa_1 \cdot \kappa_2$$

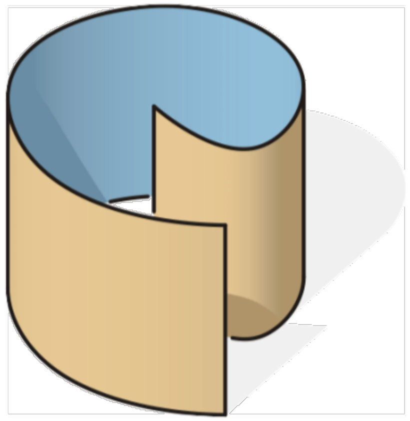




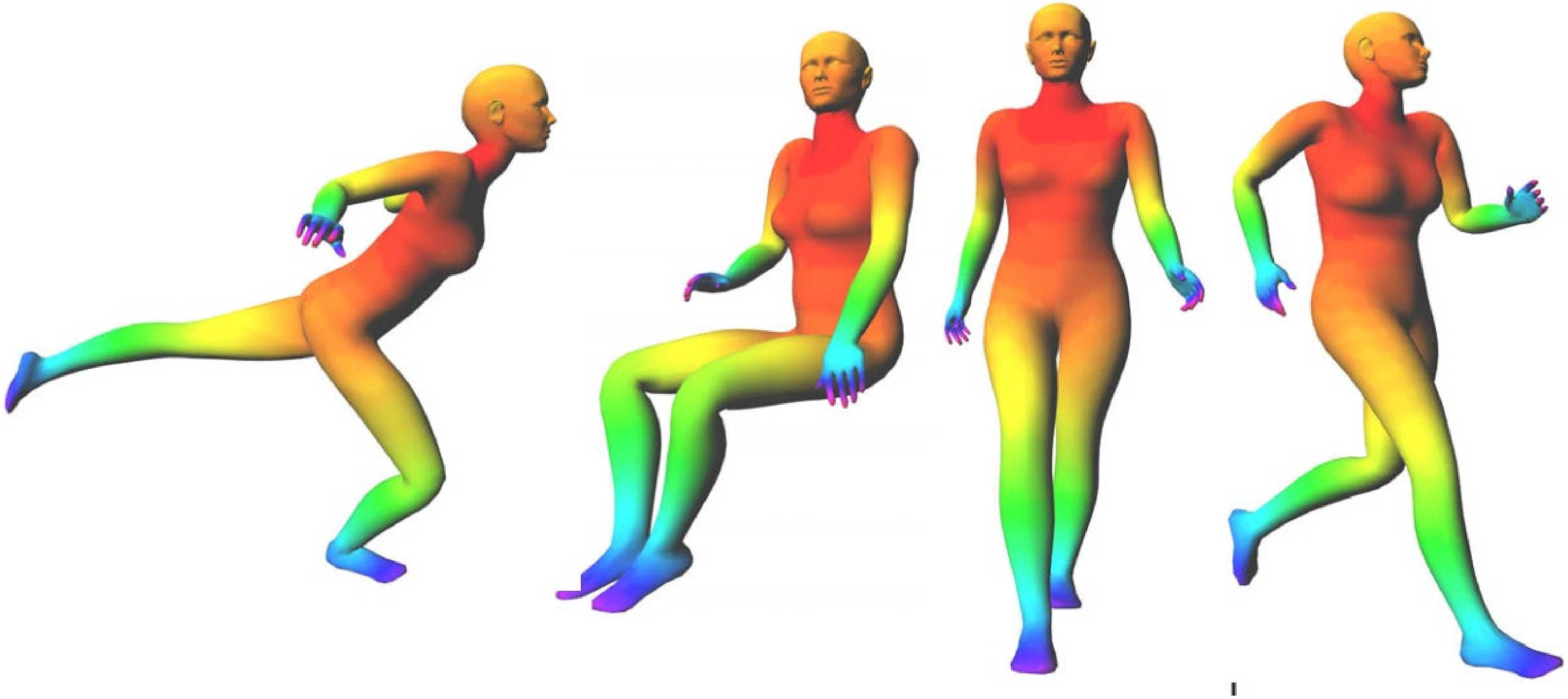
http://www.yann-ollivier.org/rech/publs/visualcurvature.pdf

Gaussian Curvature

- Intrinsic – invariant to bending



- As a descriptor for shape similarity

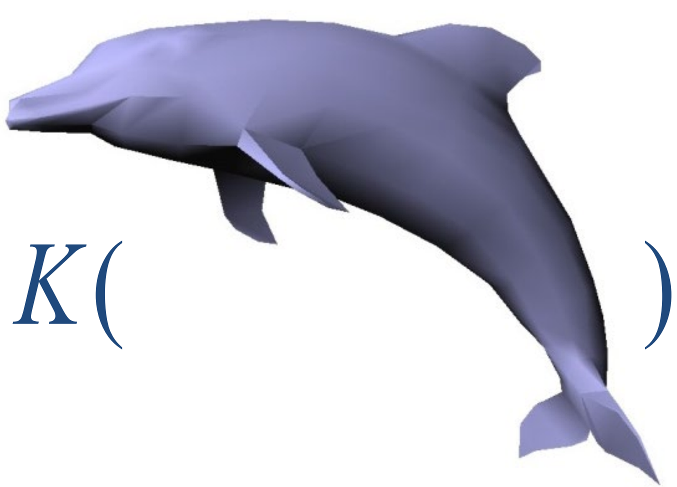
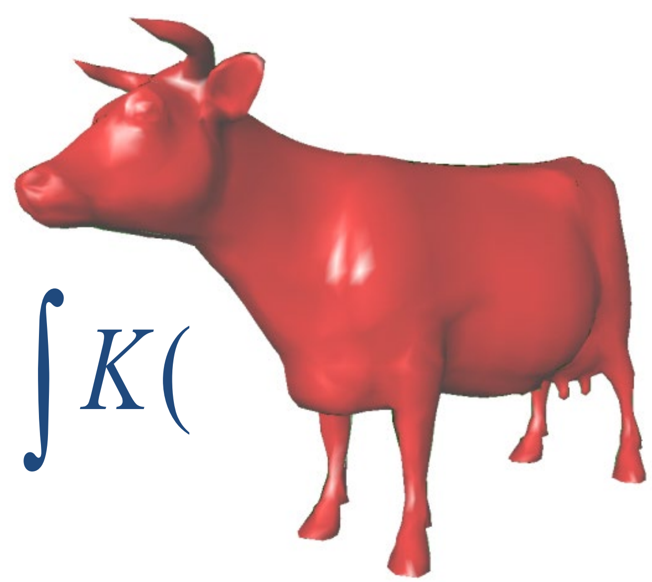
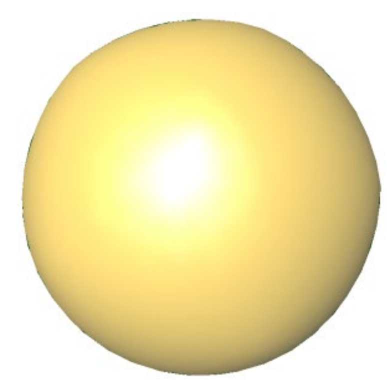


["Characterizing Shape Using Conformal Factors", Ben Chen et al., 2008]

Gauss-Bonnet Theorem

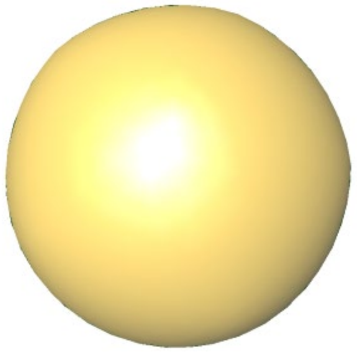
For ANY closed manifold surface with Euler number $\chi=2-2g$:

$$\int K = 2\pi\chi$$

$\int K($  $) = \int K($  $) = \int K($  $) = 4\pi$

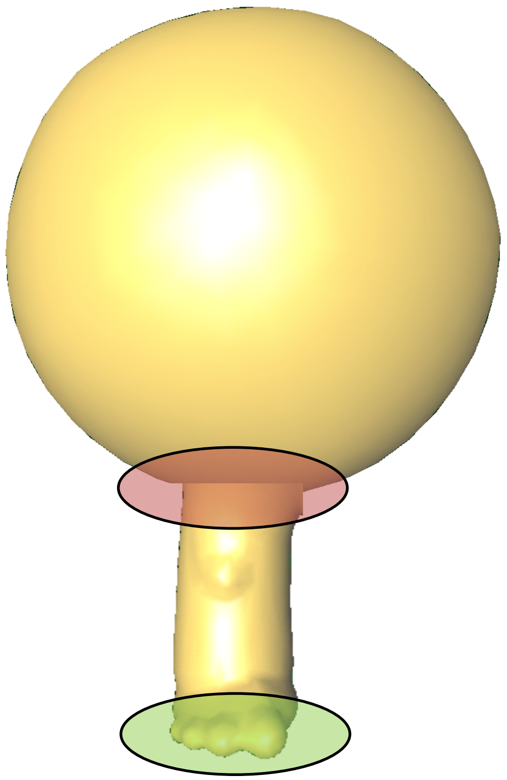
Gauss-Bonnet Theorem
Example

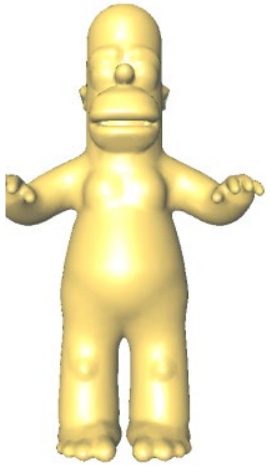
- Sphere
  - $k_1 = k_2 = 1/r$
  - $K = k_1 k_2 = 1/r^2$



$$\int K = 4\pi r^2 \cdot \frac{1}{r^2} = 4\pi$$

- Manipulate sphere
  - New positive + negative curvature
  - Cancel out!

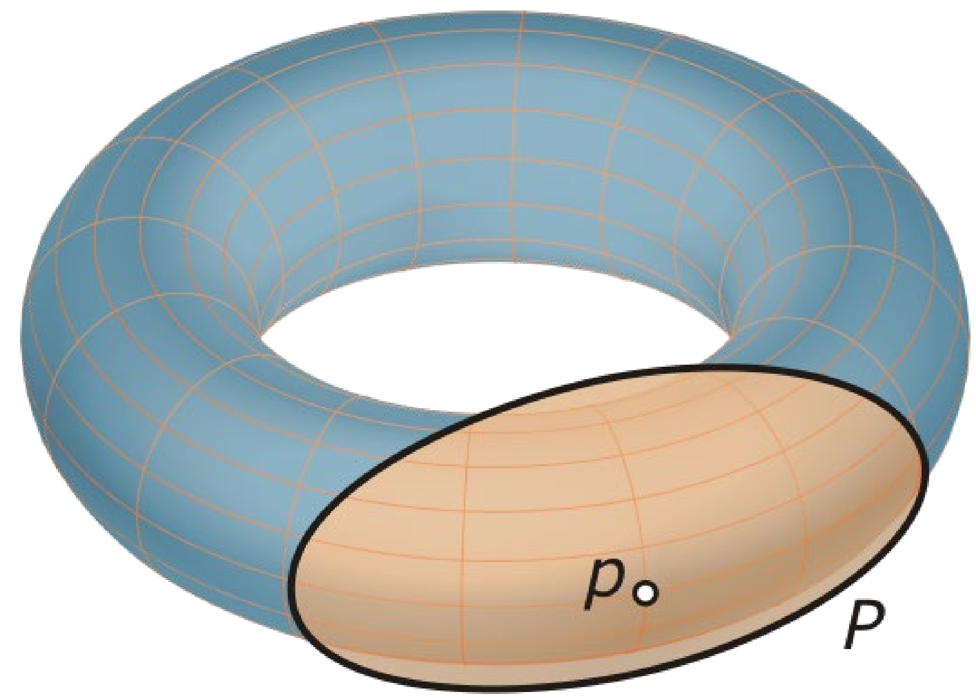




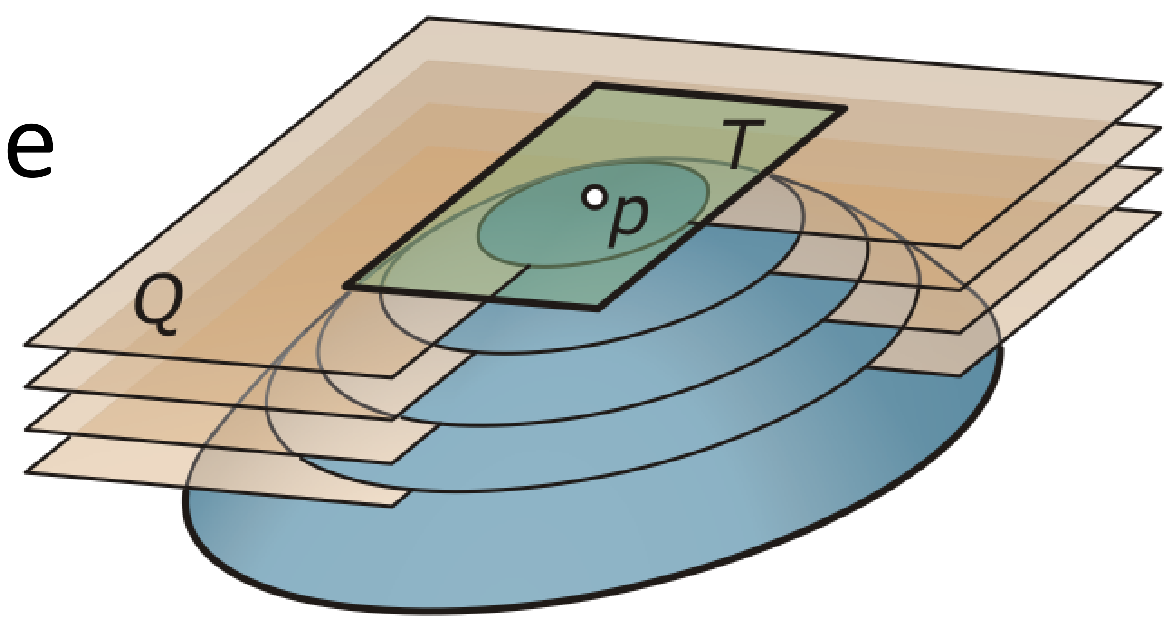
Surface Classification

Elliptic Points

$$\kappa_1 \cdot \kappa_2 > 0$$



Surface lies locally on the same side of the tangent plane

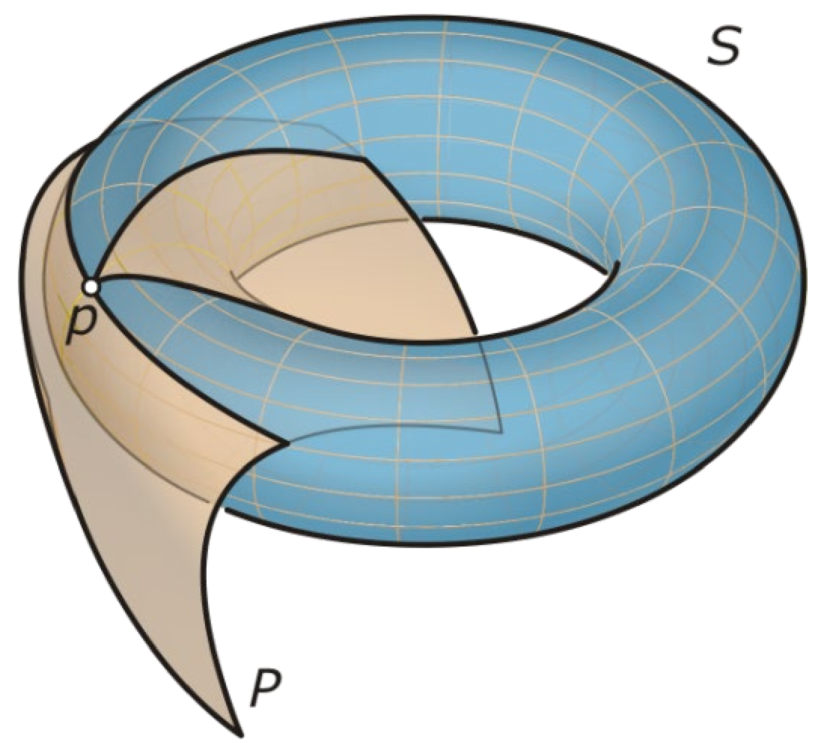


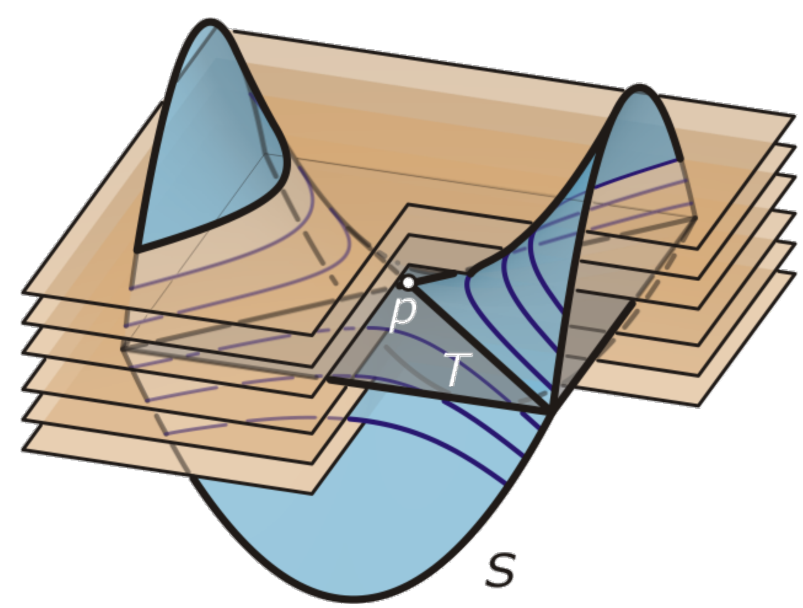
Surface Classification

Hyperbolic Points

$$\kappa_1 \cdot \kappa_2 < 0$$

Surface lies locally on both sides of the tangent plane





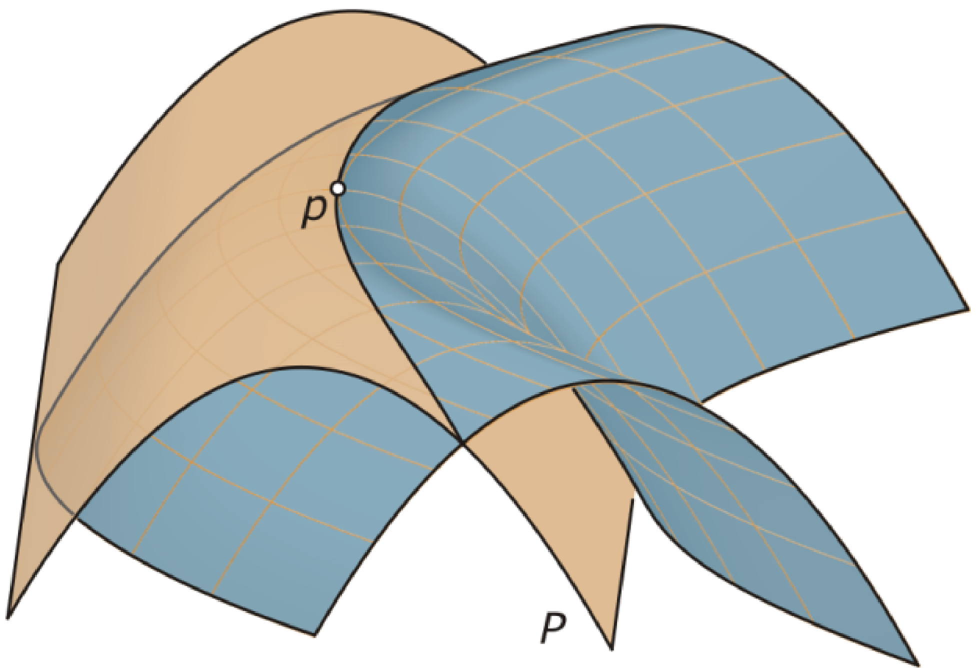
Surface Classification

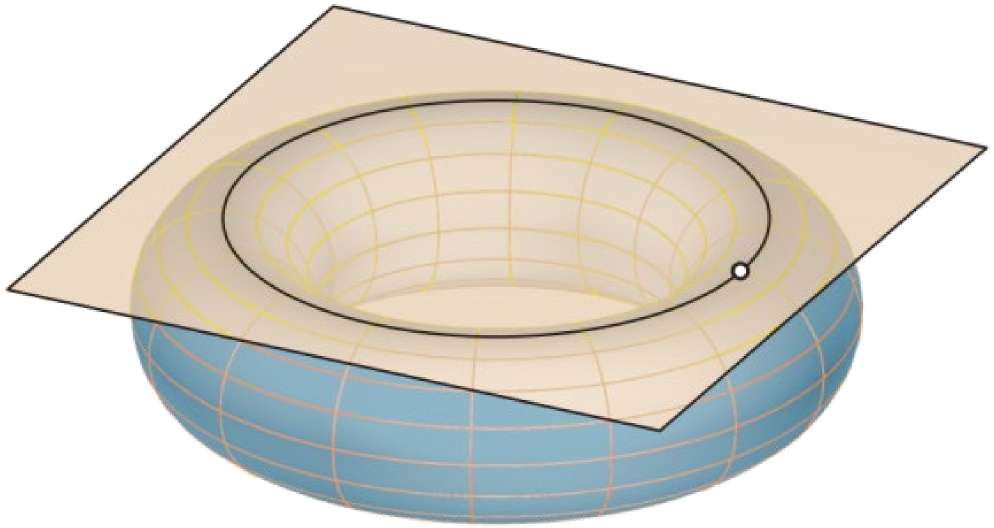
Parabolic Points

$$\kappa_1 = 0$$

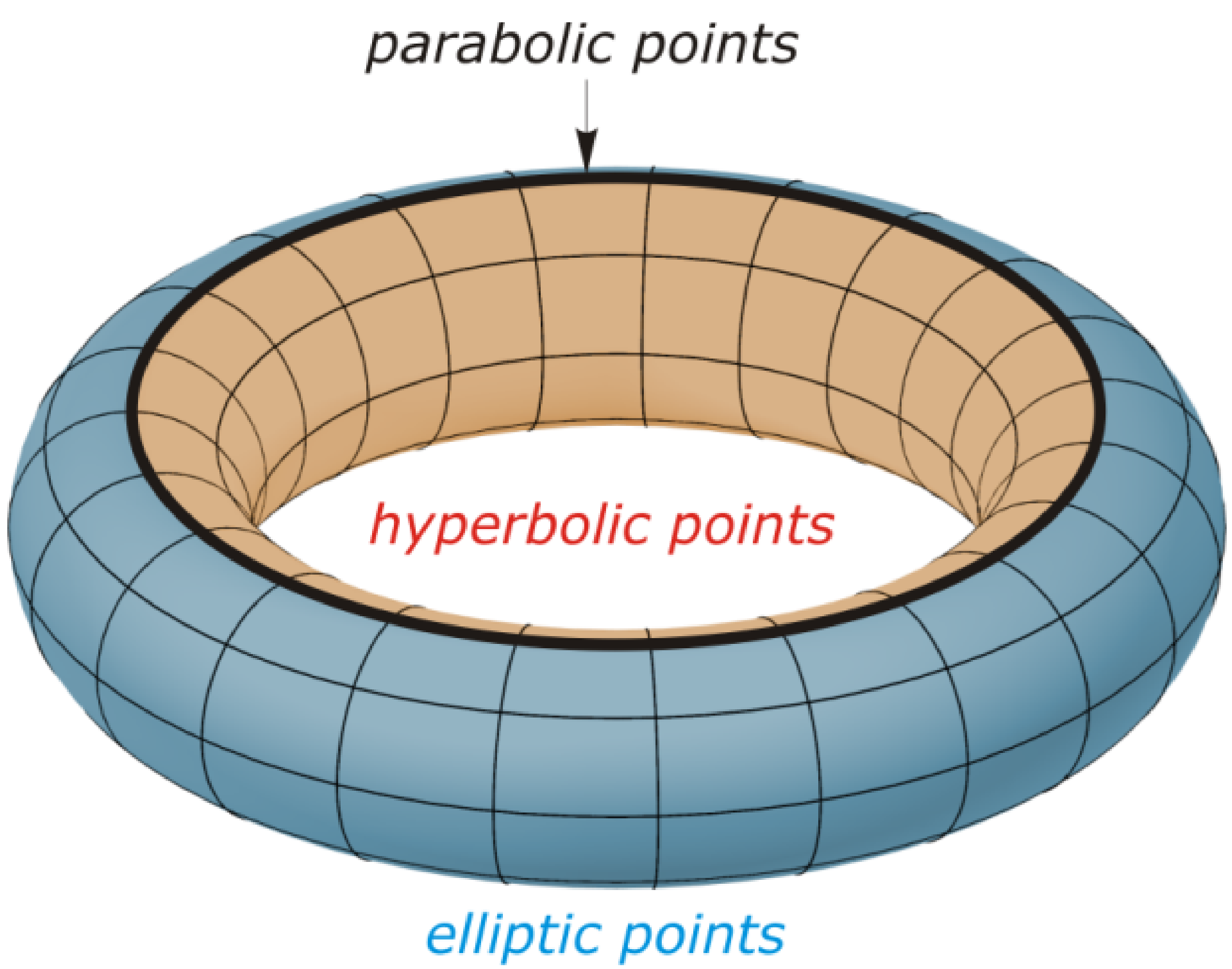
$$or$$

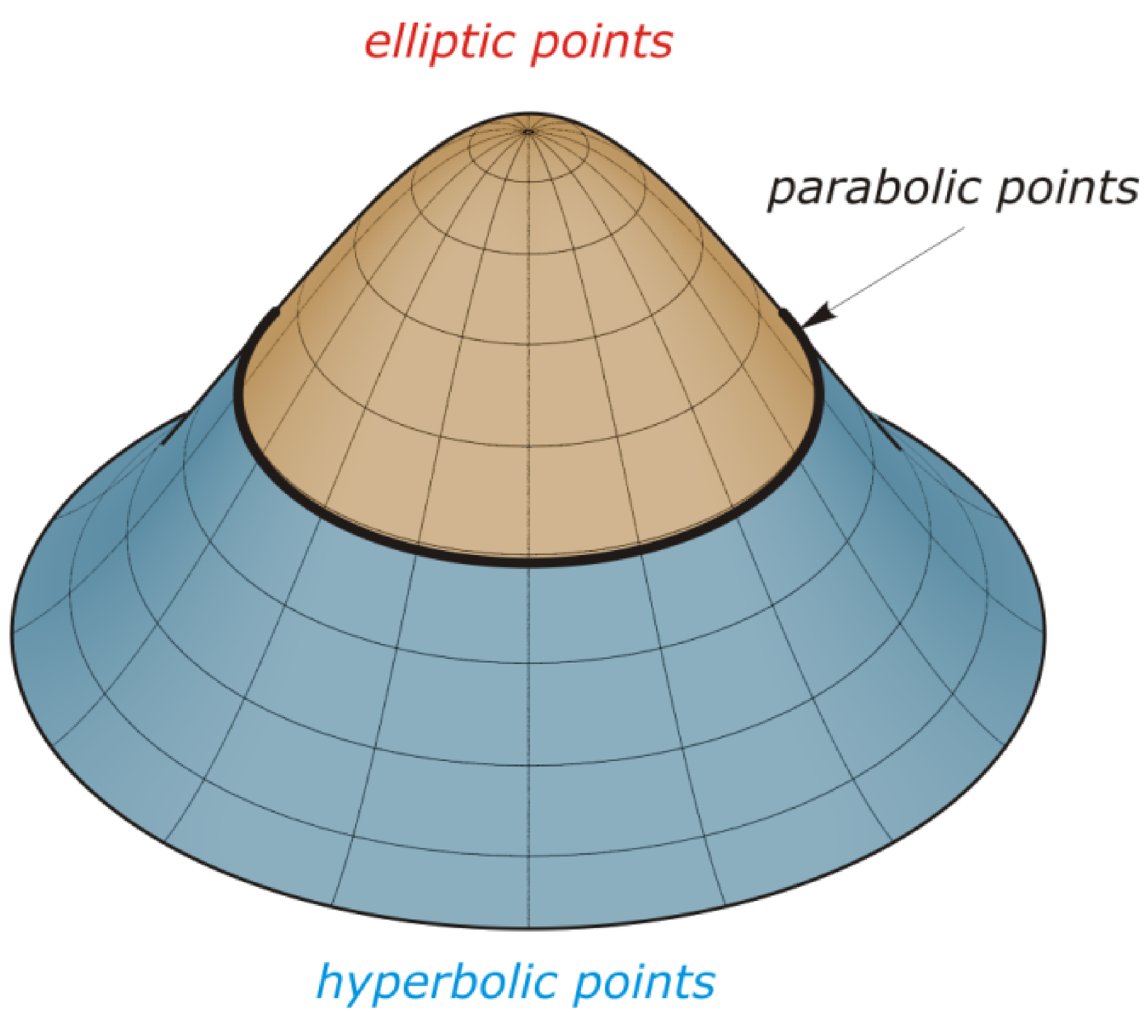
$$\kappa_2 = 0$$





Examples

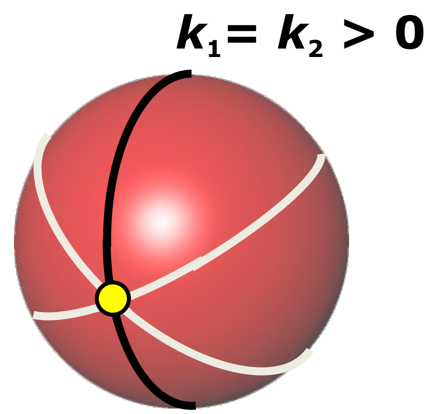

parabolic points
hyperbolic points
elliptic points


elliptic points
parabolic points
hyperbolic points

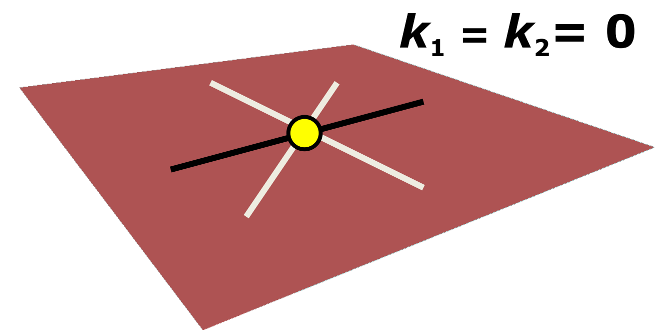
Surface Classification

Isotropic

Equal in all directions



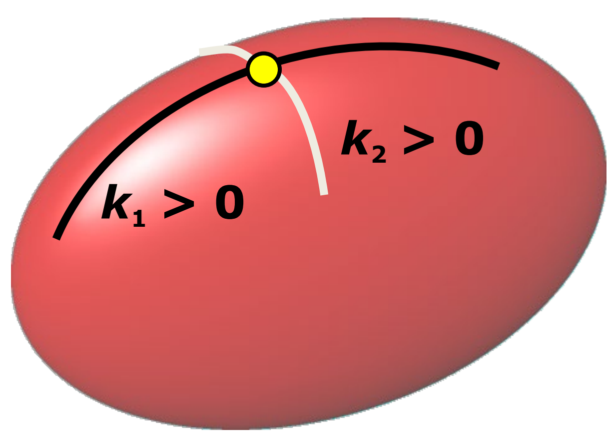
spherical



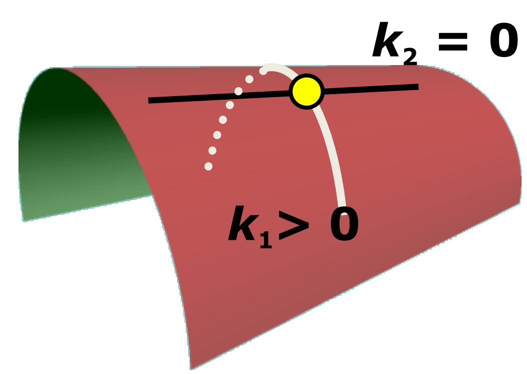
planar

Anisotropic

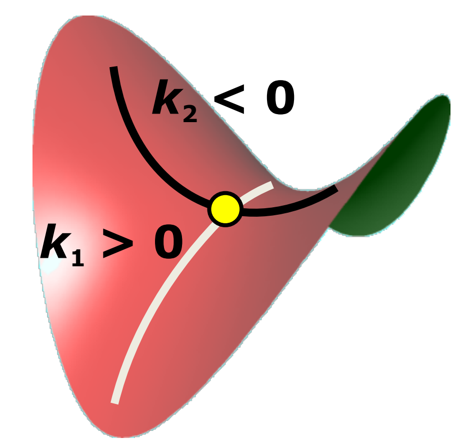
Distinct principal directions



elliptic
$K > 0$



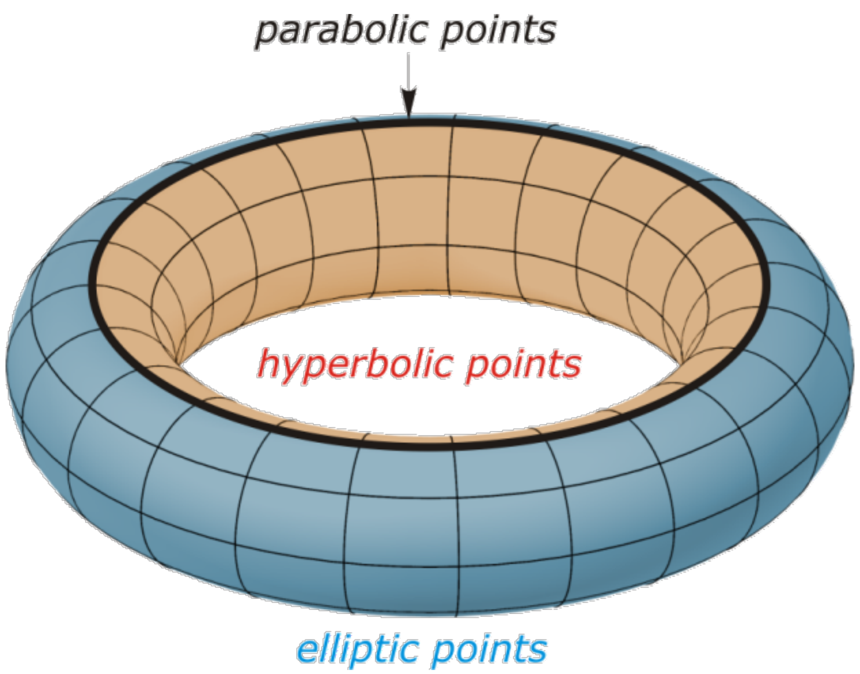
parabolic
$K = 0$
developable

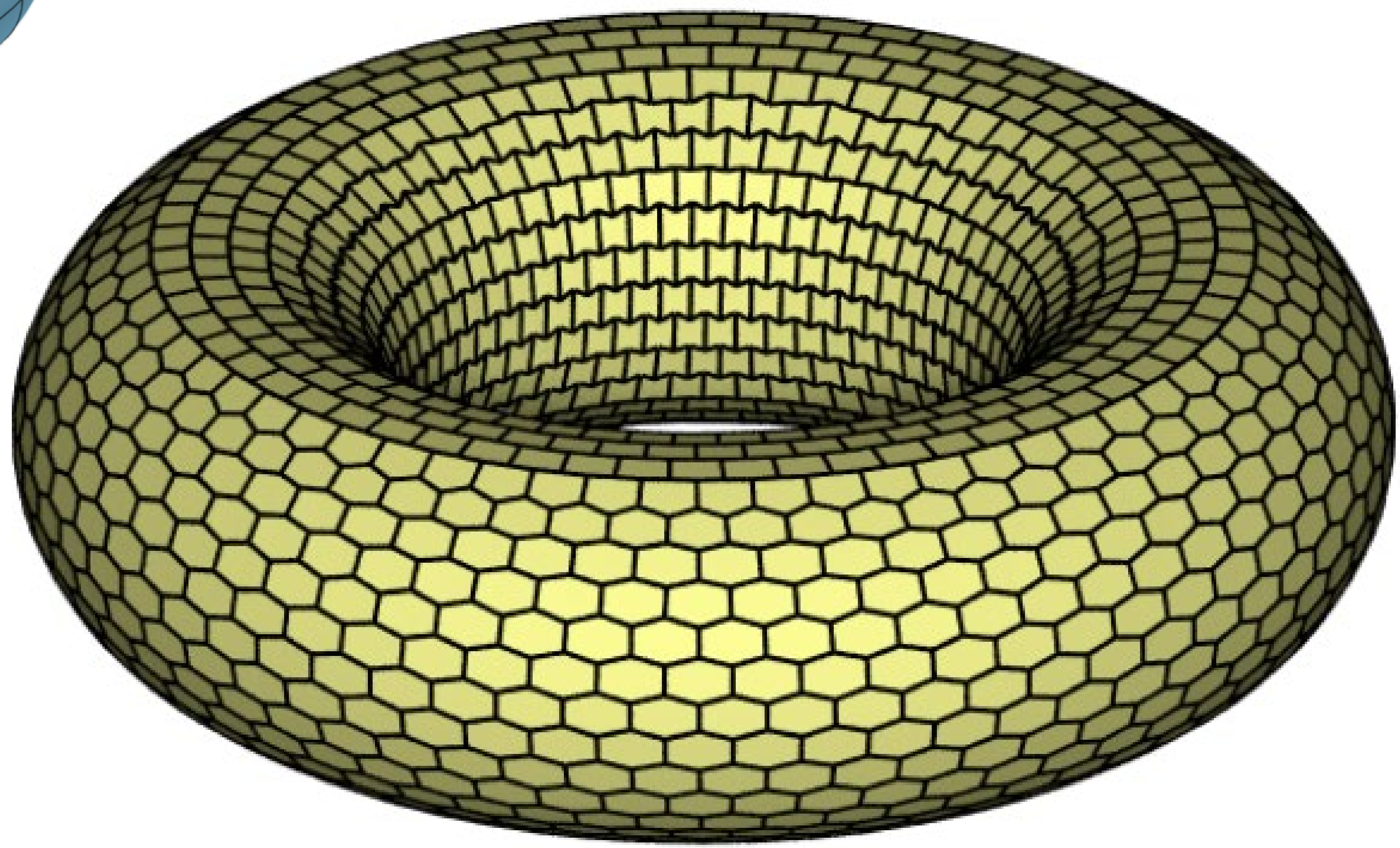


hyperbolic
$K < 0$

[Demo]

Classification for Planar Hexagonal Meshing





[“Dupin Meshing: A Parameterization Approach to Planar Hex-Dominant Meshing”, Vaxman et al., 2015]

Fundamental Forms

- First fundamental form

$$\mathbf{I} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} := \begin{bmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{bmatrix}$$

- Second fundamental form

$$\mathbf{II} = \begin{bmatrix} e & f \\ f & g \end{bmatrix} := \begin{bmatrix} \mathbf{x}_{uu}^T \mathbf{n} & \mathbf{x}_{uv}^T \mathbf{n} \\ \mathbf{x}_{uv}^T \mathbf{n} & \mathbf{x}_{vv}^T \mathbf{n} \end{bmatrix}$$

Fundamental Forms

- $\mathbf{I}$ and $\mathbf{II}$ allow to measure
  - length, angles, area, curvature
  - arc element

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

  - area element

$$dA = \sqrt{EG - F^2}dudv$$

Intrinsic Geometry

- Properties of the surface that only depend on the first fundamental form
  - length
  - angles
  - Gaussian curvature (Theorema Egregium)

- First + second FF enough to reconstruct up to rigid motion

Interesting applications of curvature

Bonus Task

- Film a short video demonstrating Theorema Egregium
  - E.g., the pizza experiment, corrugated paper, etc.

- Be creative!

- We'll vote on the best one
  - Winner gets negatively curved potatoes

No Parametric Surface?

- Input is triangle mesh and not parametric surface
- How to compute curvature (and everything else…)?
- Variational approach
  - Define an energy and minimize it
  - Need ***differential operators***

Differential Operators

Gradient ∇

- **Input:** scalar function

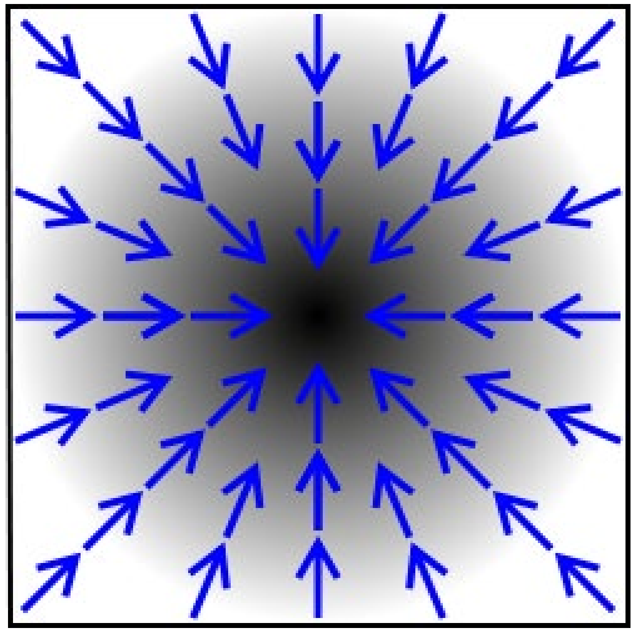
$$f: \mathbb{R}^n \rightarrow \mathbb{R}$$

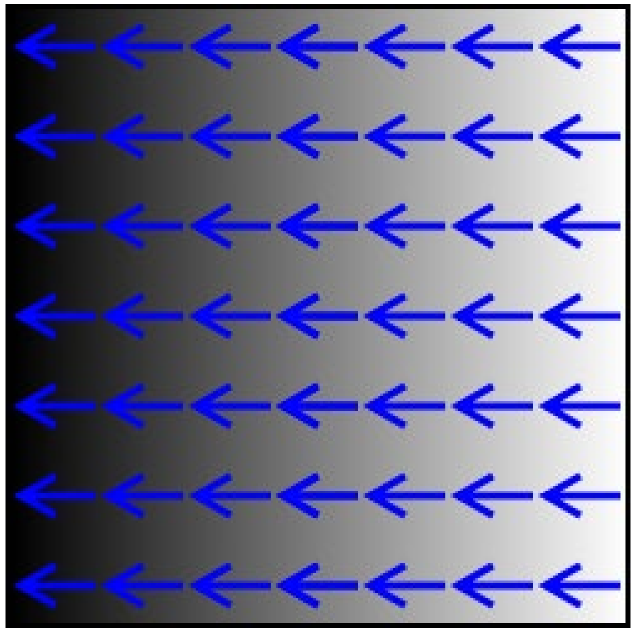
- **Output:** vector field

$$\nabla f: \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$\operatorname{grad} f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

- **Intuition:** slope





Differential Operators

Divergence $\nabla \cdot$

- **Input:** vector field

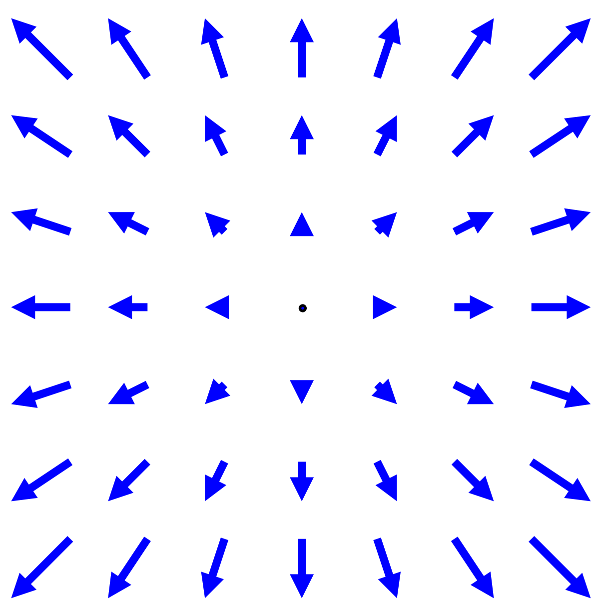
$$F: \mathbb{R}^n \to \mathbb{R}^n$$

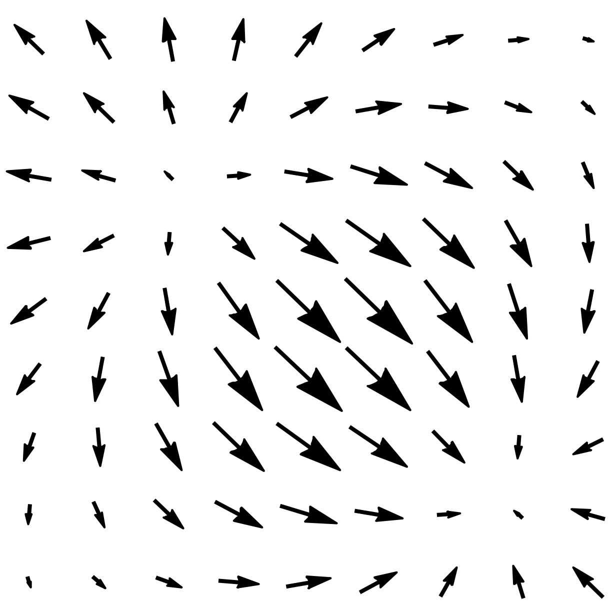
$$\text{div}\, \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

- **Output:** scalar function

$$\nabla \cdot F: \mathbb{R}^n \to \mathbb{R}$$

- **Intuition:** sources/sinks (think Jacuzzi…)





Intuition video

Differential Operators

Laplacian ∇^2, Δ

- **Input:** scalar function

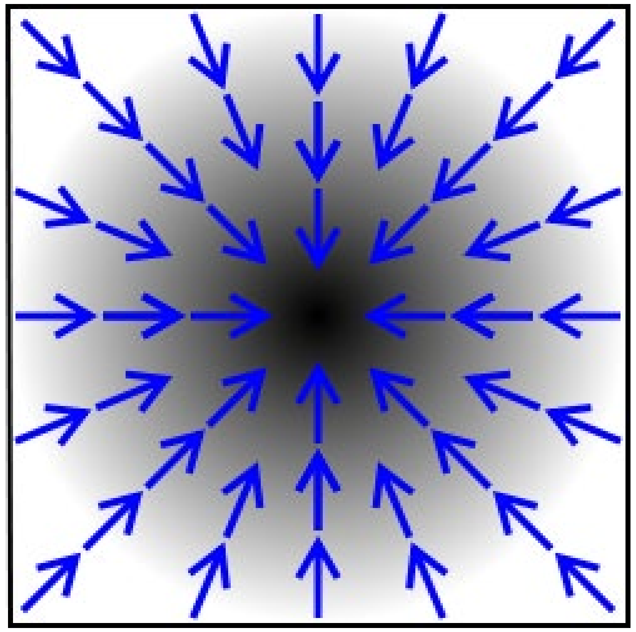
$$f: \mathbb{R}^n \to \mathbb{R}$$

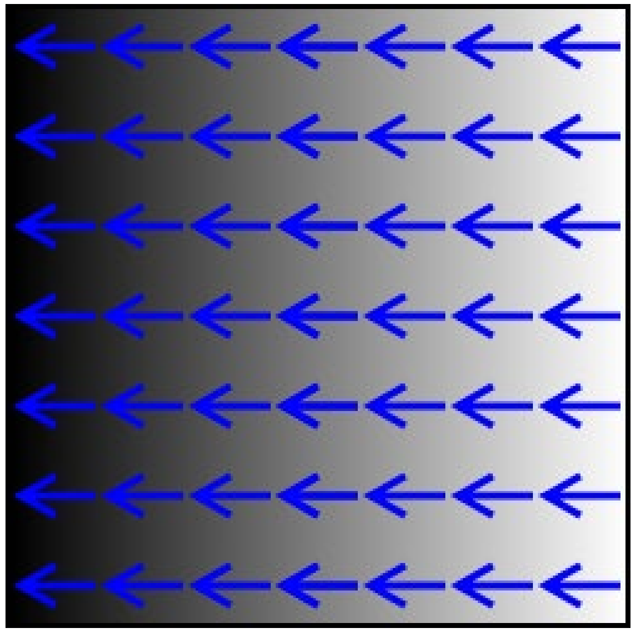
- **Output:** scalar function

$$\Delta F: \mathbb{R}^n \to \mathbb{R}$$

$$\Delta f = \mathrm{div} \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$$

- **Intuition:** smoothness, deviation from average





[Intuition video]

Laplace Operator

$$f : R^3 \to R \qquad \Delta f : R^3 \to R$$

Laplace operator

gradient operator

2nd partial derivatives

$$\Delta f = \mathrm{div} \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2} \in R$$

scalar function in Euclidean space

divergence operator

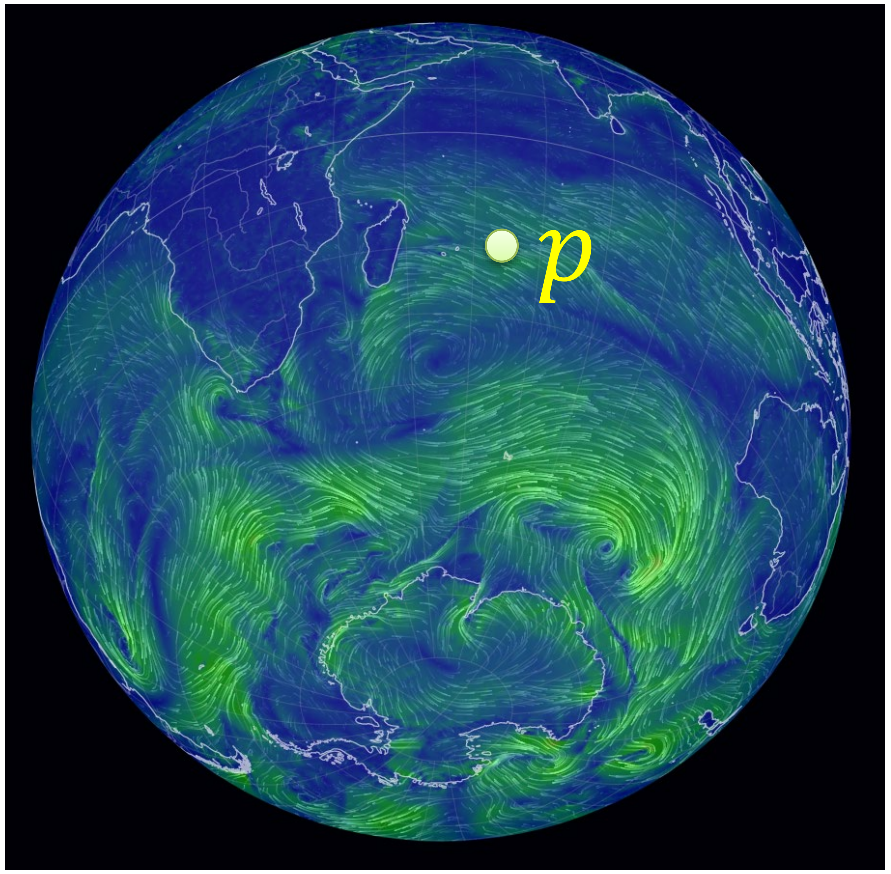
Cartesian coordinates

$$\mathrm{grad}\, f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$\mathrm{div}\, \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Laplace-Beltrami Operator

- Extension to functions on manifolds S, $p \in S$



$f(p) \in \mathbb{R}$

$F(p) \in \underbrace{T_p S}$

Tangent plane at p

Laplace-Beltrami Operator

- Extension to functions on manifolds

$$f : S \to \mathbb{R}$$

Laplace-Beltrami

gradient operator

$$\Delta_{\mathcal{S}} f = \mathrm{div}_{\mathcal{S}} \nabla_{\mathcal{S}} f \quad \in \mathbb{R}$$

scalar function on manifold $\mathcal{S}$

divergence operator

Laplace-Beltrami Operator (LBO)

- For coordinate function(s)

$$f(x, y, z) = \mathbf{x}$$

Laplace-Beltrami

gradient operator

mean curvature

$$\Delta_{\mathcal{S}} \boxed{\mathbf{x}} = \mathrm{div}_{\mathcal{S}}\, \nabla_{\mathcal{S}} \boxed{\mathbf{x}} = -2H\mathbf{n} \in \mathbb{R}^3$$

coordinate function

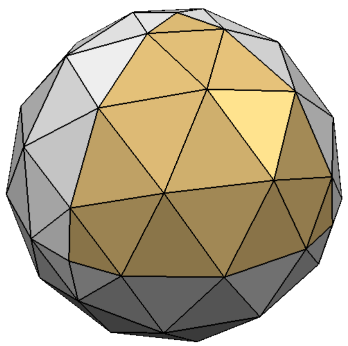
divergence operator

surface normal

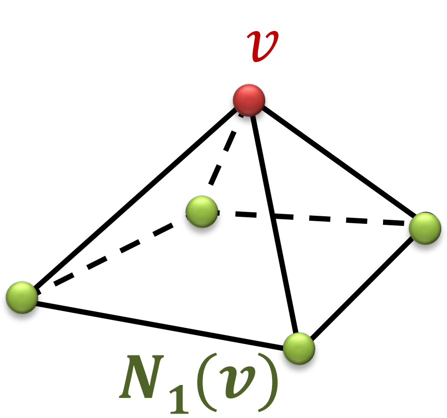
LBO is a second derivative

LBO of coordinates → normal direction times curvature, similar to curves!

Discrete Differential Operators



- **Assumption:** Meshes are piecewise linear approximations of smooth surfaces
- **Approach:** Approximate differential properties at point v as finite differences over local mesh neighborhood $N(v)$
  - v = mesh vertex
  - $N_d(v) = d$-ring neighborhood

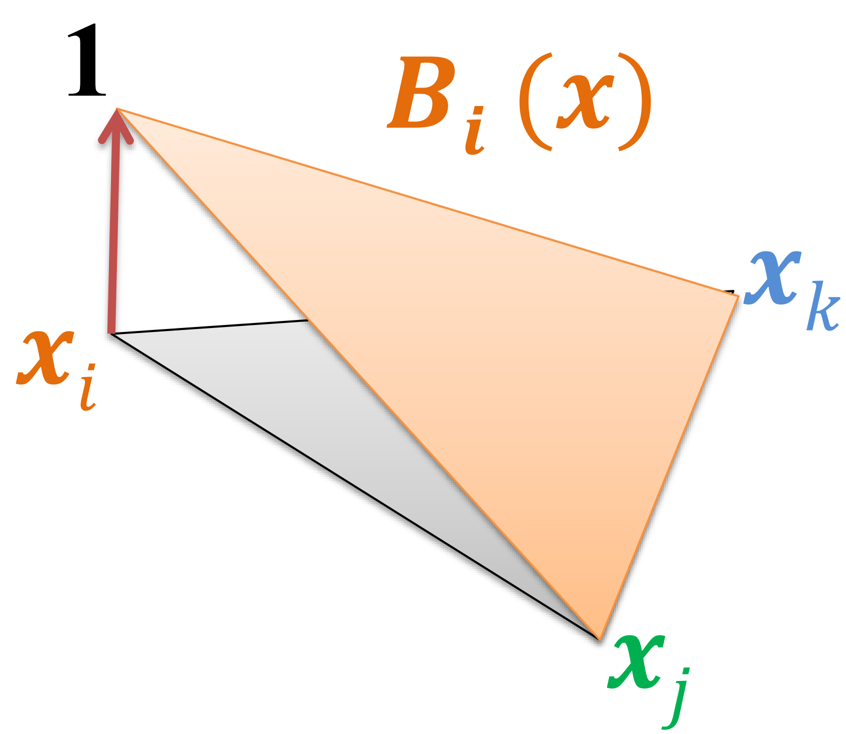


- **Disclaimer:** many possible discretizations, none is “perfect”

Functions on Meshes

- Function f given at mesh vertices $f(vi) = f(\boldsymbol{x}_i) = f_i$
- Linear interpolation to triangle $\mathbf{x} \in \left(\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k\right)$

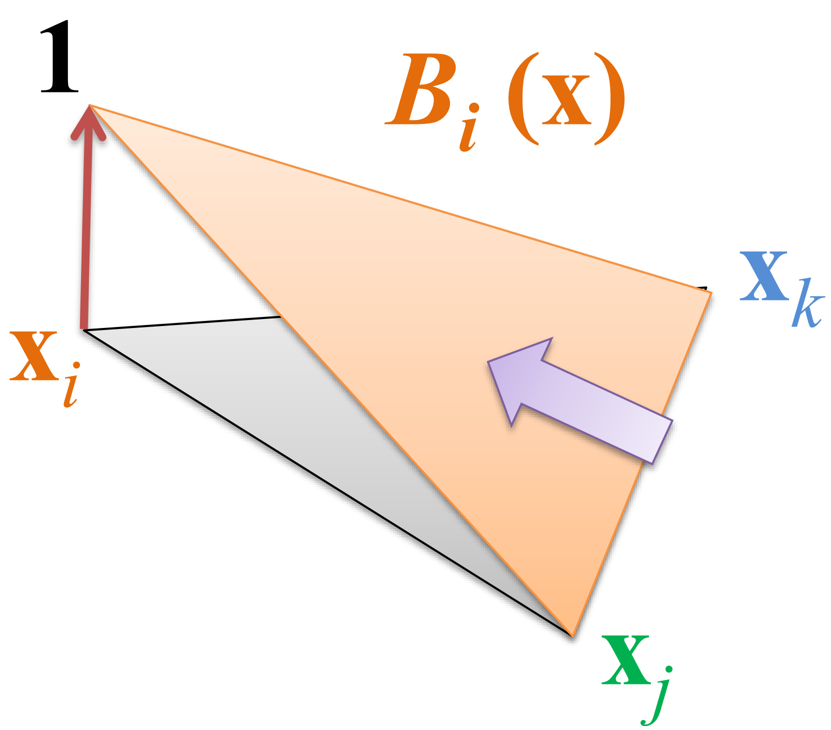
$$f(\mathbf{x}) = f_i B_i(\mathbf{x}) + f_j B_j(\mathbf{x}) + f_k B_k(\mathbf{x})$$



Gradient of a Function

$$f(\mathbf{x}) = f_i B_i(\mathbf{x}) + f_j B_j(\mathbf{x}) + f_k B_k(\mathbf{x})$$

$$\nabla f(\mathbf{x}) = f_i \nabla B_i(\mathbf{x}) + f_j \nabla B_j(\mathbf{x}) + f_k \nabla B_k(\mathbf{x})$$



Steepest ascent direction perpendicular to opposite edge

$$\nabla B_i(\mathbf{x}) = \nabla B_i = \frac{(\mathbf{x}_k - \mathbf{x}_j)^{\perp}}{2A_T}$$

Constant in the triangle

Gradient of a Function

$$B_i(\mathbf{x}) + B_j(\mathbf{x}) + B_k(\mathbf{x}) = 1$$

$$\nabla B_i + \nabla B_j + \nabla B_k = 0$$

$$\nabla f(\mathbf{x}) = \left(f_j - f_i\right)\nabla B_j(\mathbf{x}) + \left(f_k - f_i\right)\nabla B_k(\mathbf{x})$$

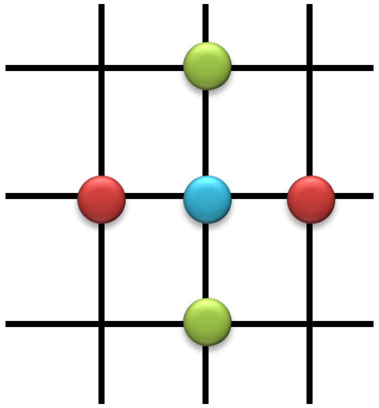
$$\nabla f(\mathbf{x}) = \left(f_j - f_i\right)\frac{\left(\mathbf{x}_i - \mathbf{x}_k\right)^{\perp}}{2A_T} + \left(f_k - f_i\right)\frac{\left(\mathbf{x}_j - \mathbf{x}_i\right)^{\perp}}{2A_T}$$

Discrete Laplace-Beltrami
First Approach

- Laplace operator: $\Delta f = \mathrm{div} \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$

- In 2D: $\Delta f = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

- On a grid – finite differences discretization:

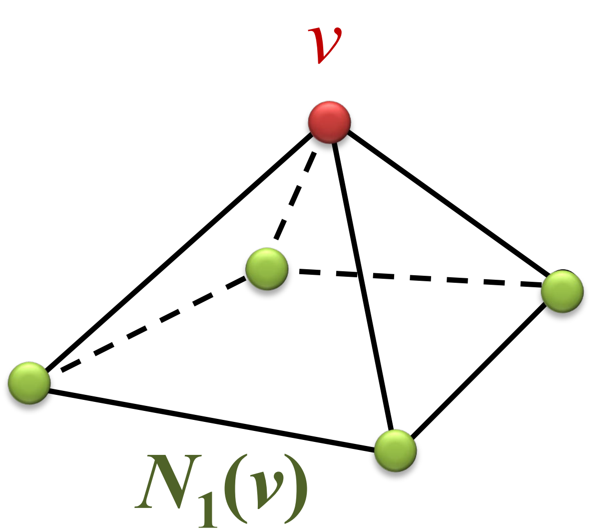


$$\Delta f(x_i, y_i) = \frac{f(x_{i+1}, y_i) - f(x_i, y_i)}{h^2} + \frac{f(x_{i-1}, y_i) - f(x_i, y_i)}{h^2} + \\ + \frac{f(x_i, y_{i+1}) - f(x_i, y_i)}{h^2} + \frac{f(x_i, y_{i-1}) - f(x_i, y_i)}{h^2}$$

Discrete Laplace-Beltrami
Uniform Discretization

$$\Delta f = \sum_{v_i \in N_1(v)} (f(v) - f(v_i)) =$$

$$= |N_1(v)| f(v) - \sum_{v_i \in N_1(v)} f(v_i)$$

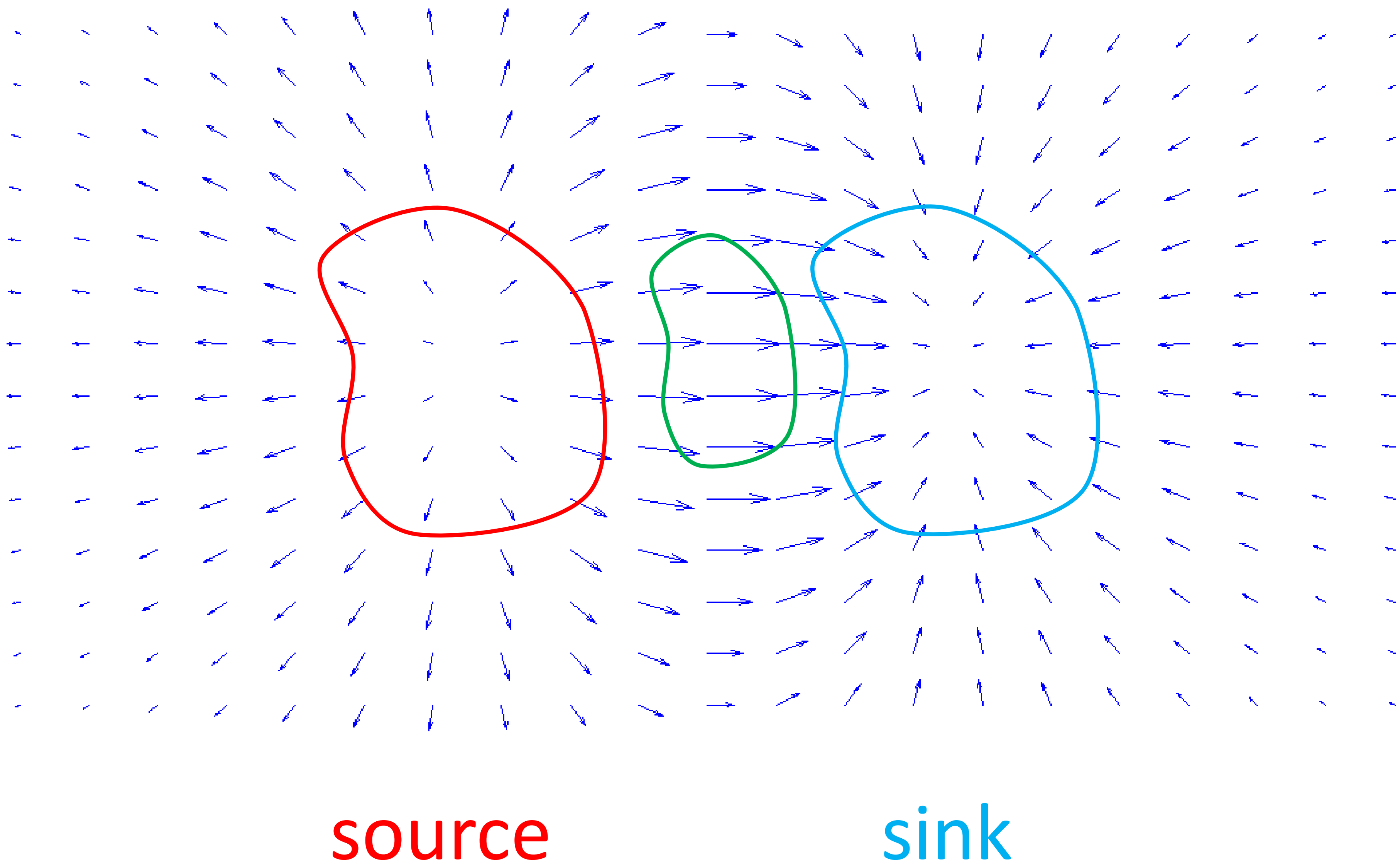


Normalized: $$\Delta f = \frac{1}{|N_1(v)|} \sum_{v_i \in N_1(v)} (f(v) - f(v_i)) =$$

$$= \boxed{f(v) - \frac{1}{|N_1(v)|} \sum_{v_i \in N_1(v)} f(v_i)}$$

Divergence

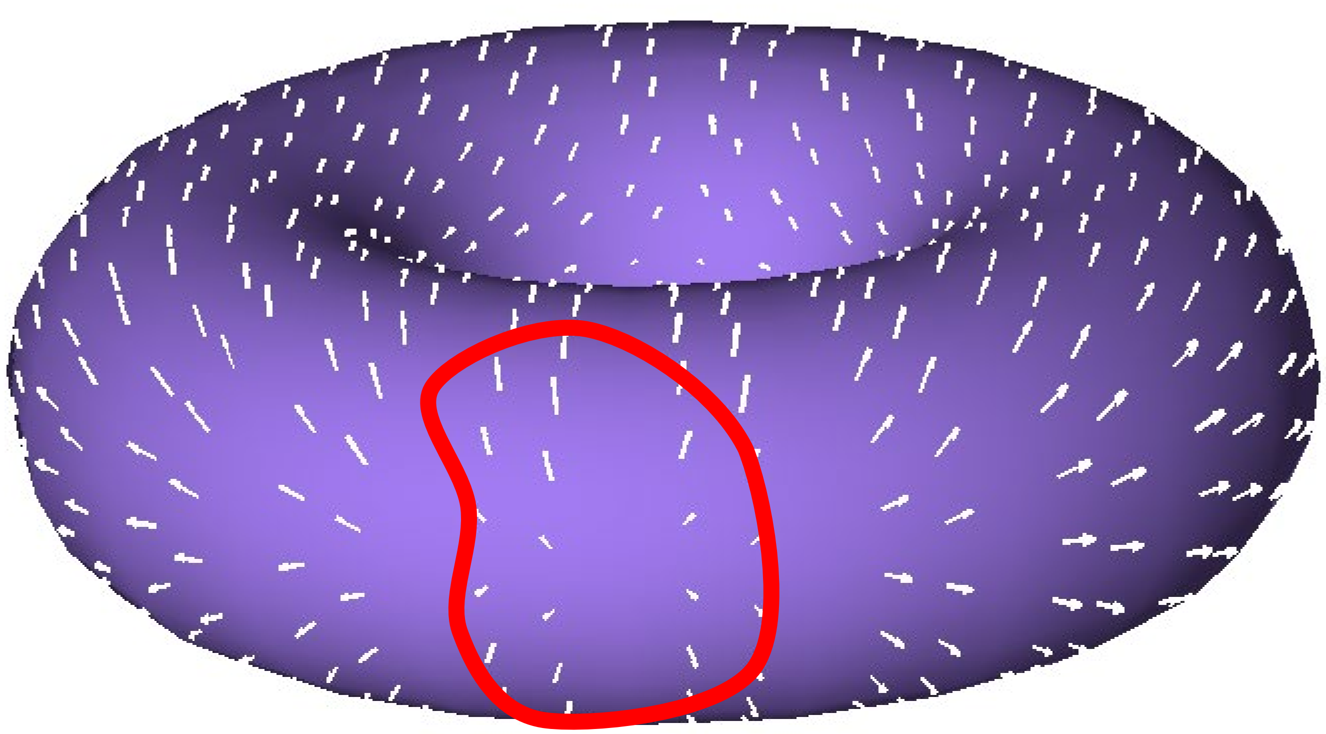
- How much goes out – how much comes in

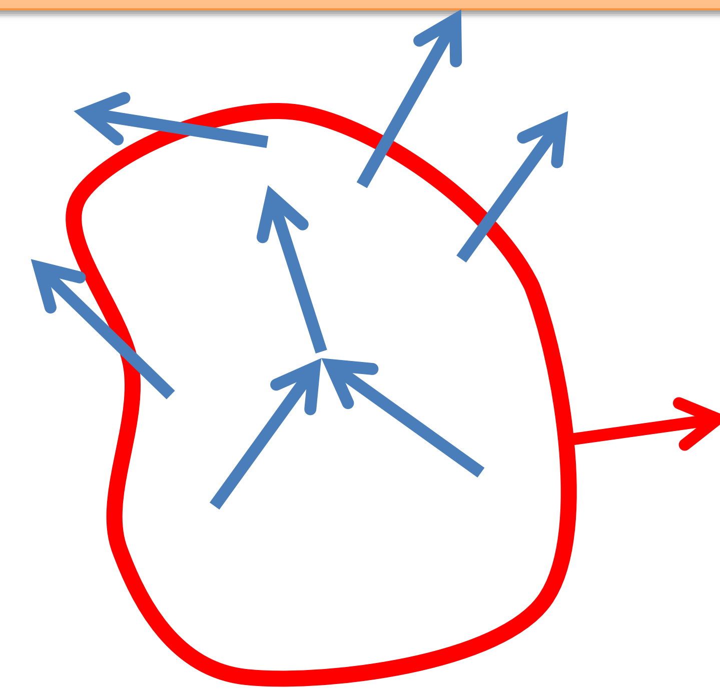


Divergence

- How to measure "all the stuff that comes in – stuff that goes out"?
  - Only the boundary matters!

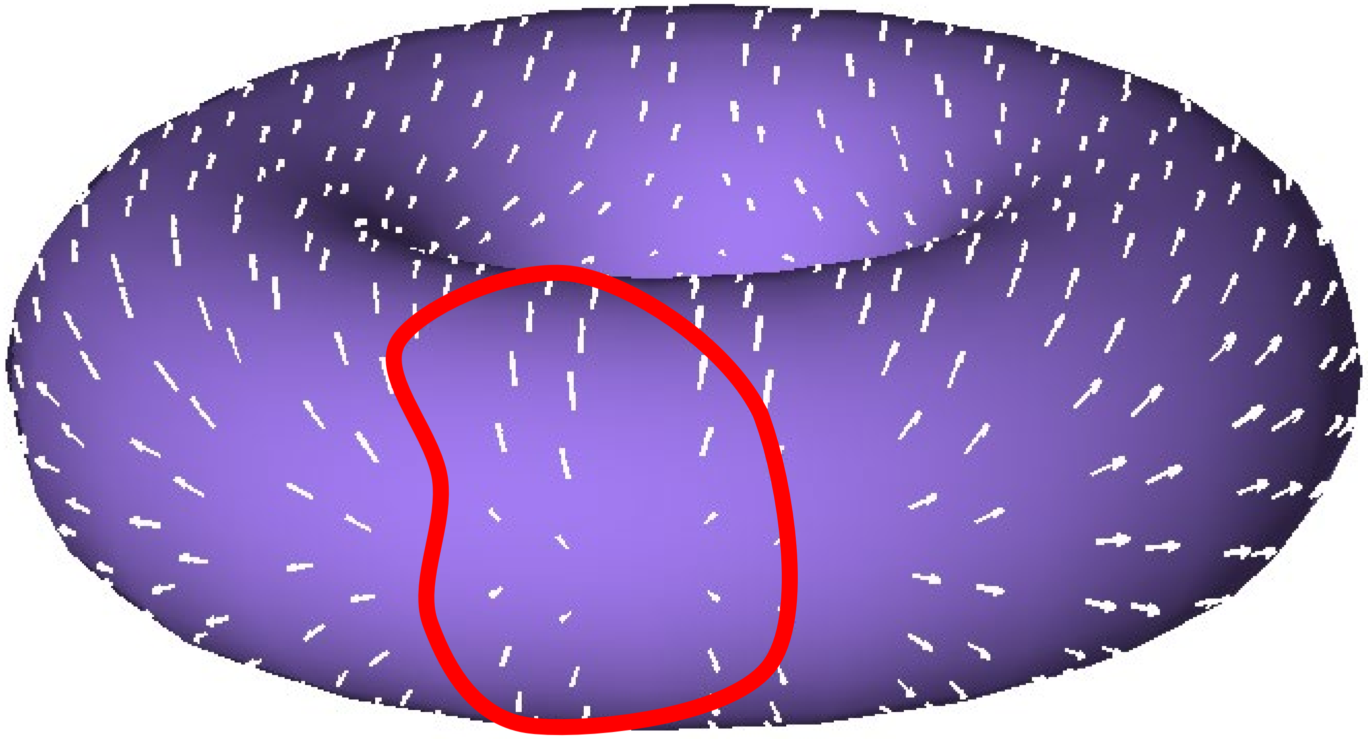
Only **normal** to the curve

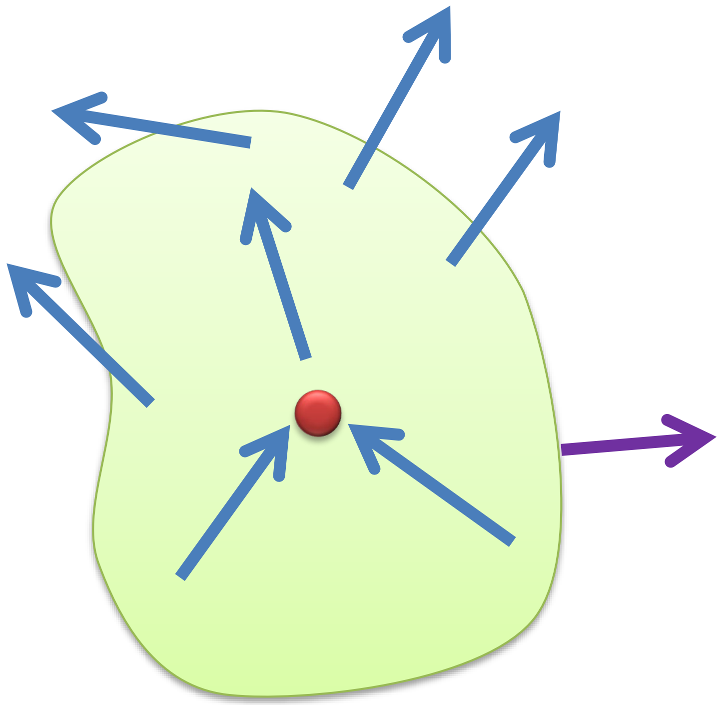




Divergence Theorem

$$\nabla \cdot \boldsymbol{V}(\boldsymbol{p}) = \qquad\qquad \boldsymbol{V} \cdot \boldsymbol{N}$$



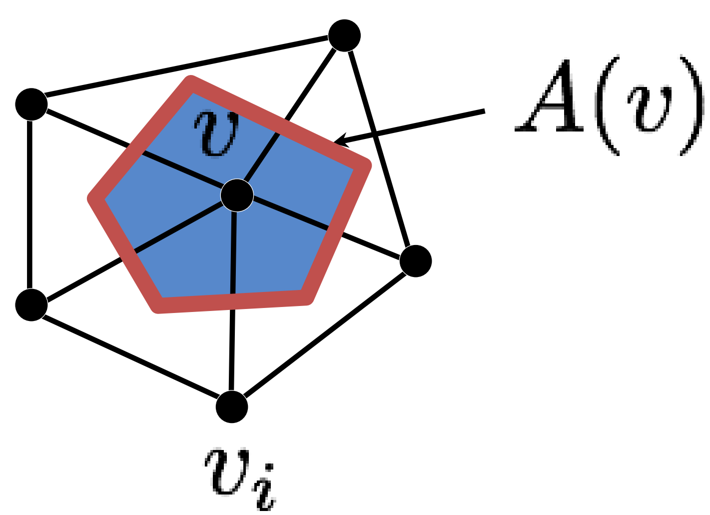


Discrete Laplace-Beltrami
Second Approach

- Laplace-Beltrami operator: $\Delta_{\mathcal{S}} f = \text{div}_{\mathcal{S}} \nabla_{\mathcal{S}} f$
- Compute integral around vertex

Divergence theorem

$$\int_{A(v)} \Delta f(\mathbf{u}) dA$$

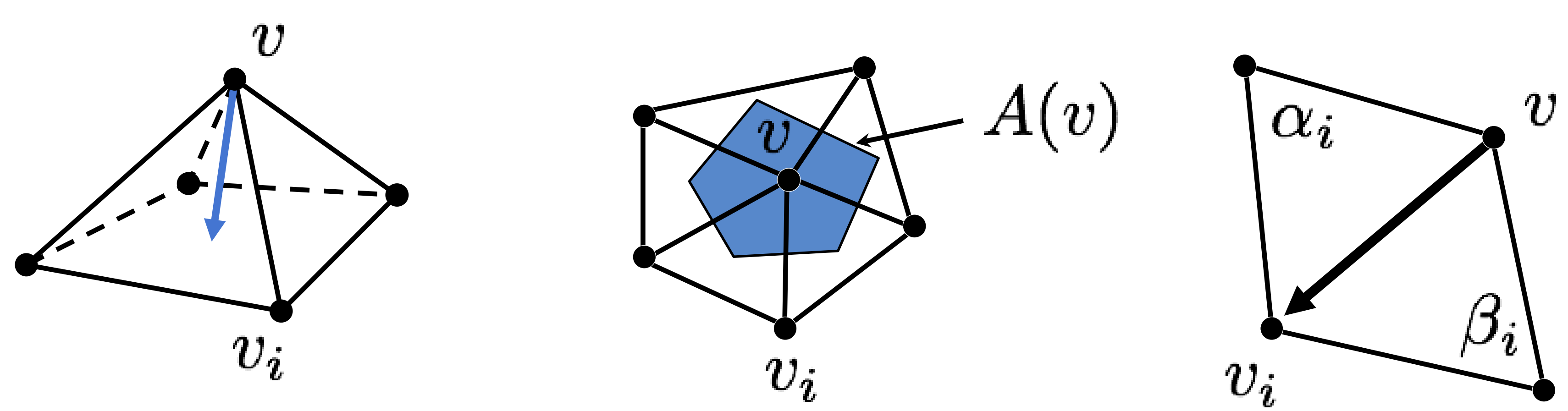


Discrete Laplace-Beltrami
Cotangent Formula

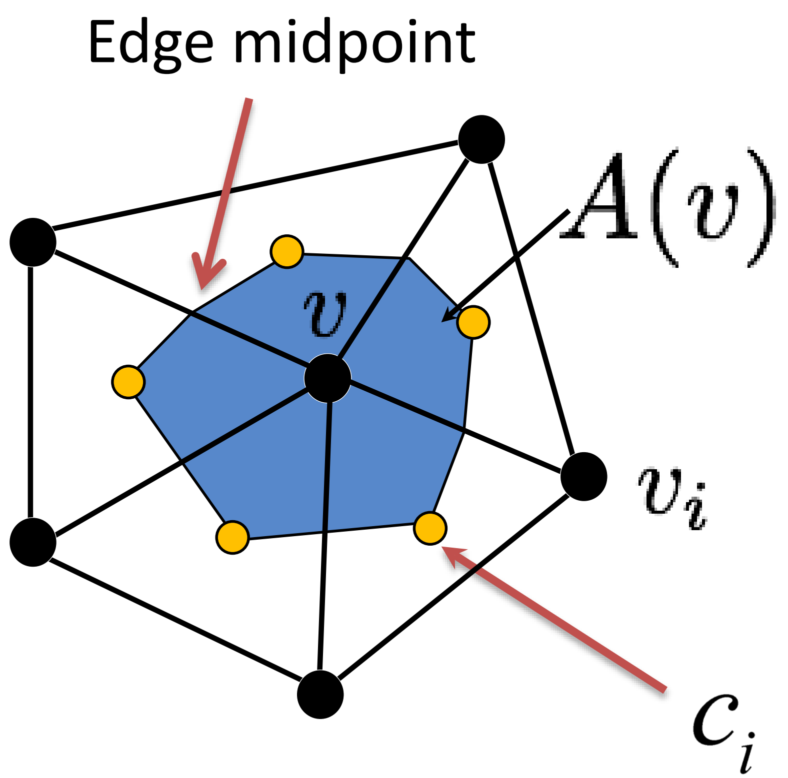
Plugging in expression for gradients gives:

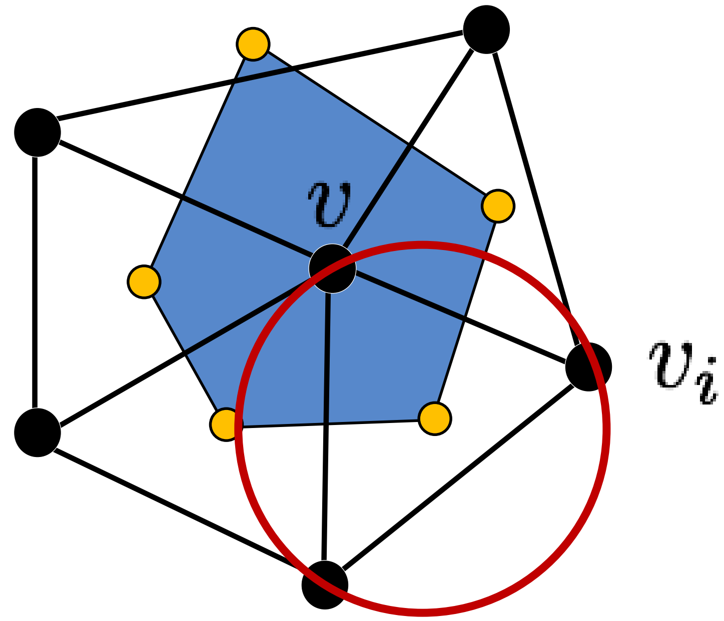
$$\Delta f(v) = \sum_{v_i \in N_1(v)} w_i (f(v_i) - f(v))$$

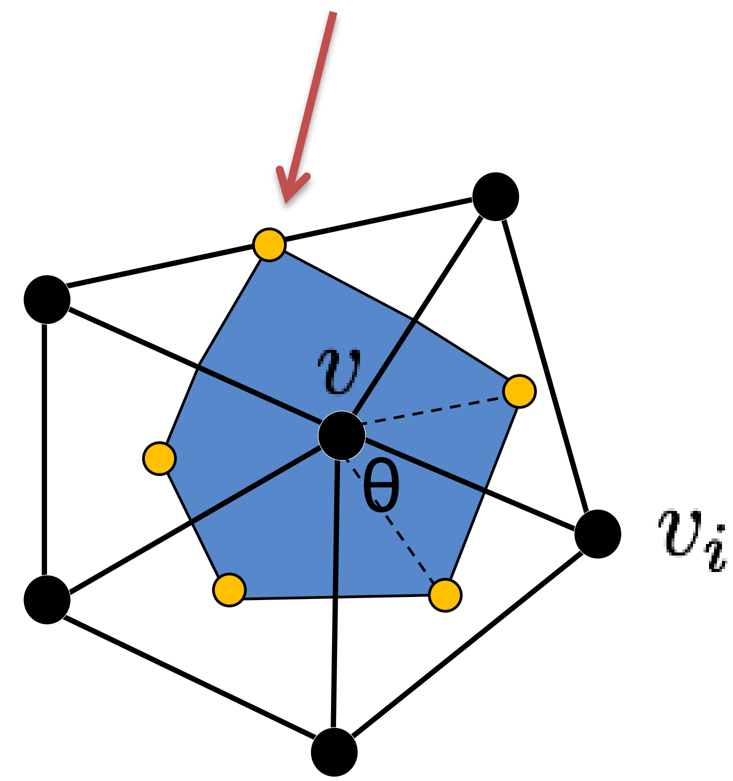
$$= \boxed{\frac{1}{2A(v)}} \sum_{v_i \in \mathcal{N}_1(v)} \boxed{(\cot \alpha_i + \cot \beta_i)} (f(v_i) - f(v))$$



The Averaging Region







Barycentric cell

c_i = barycenter of triangle

Voronoi cell

c_i = circumcenter of triangle

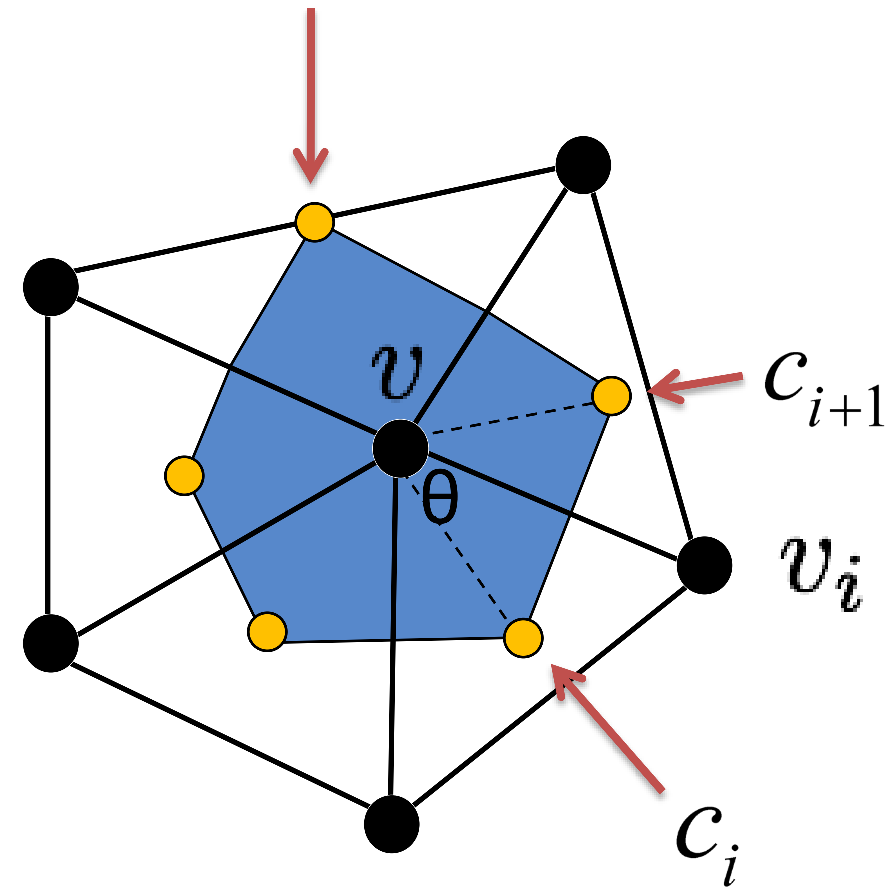
Mixed cell

The Averaging Region

Mixed Cell

If $\theta < \pi/2$, c_i is the circumcenter of the triangle $(v_i\ , v, v_{i+1}\ )$

If $\theta \geq \pi/2$, c_i is the midpoint of the edge $(v_i\ , v_{i+1}\ )$



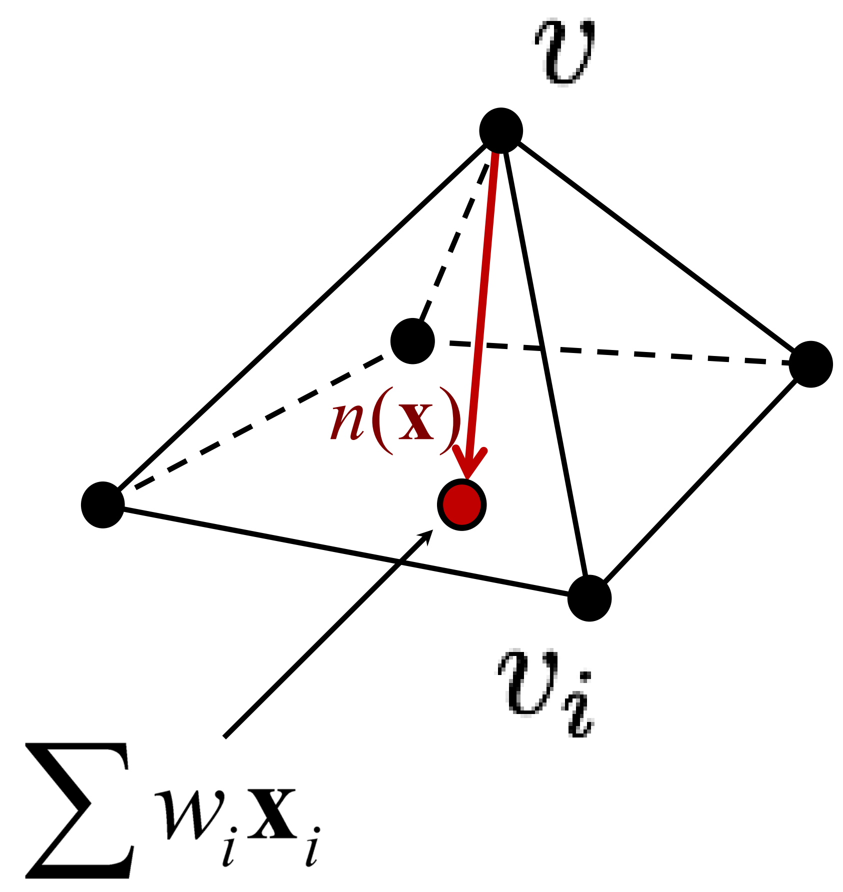
$$A(v) = \sum_{v_i \in \mathcal{N}(v)} \left(Area(c_i, v, (v+v_i)/2) + Area(c_{i+1}, v, (v+v_i)/2) \right)$$

Discrete Normal

$$\Delta_{\mathcal{S}}\mathbf{x} = \mathrm{div}_{\mathcal{S}}\,\nabla_{\mathcal{S}}\mathbf{x} = -2H\mathbf{n}$$

$$n(\mathbf{x}) = \sum_{v_i \in N_1(v)} w_i(\mathbf{x}_i - \mathbf{x}) \qquad \sum_i w_i = 1$$

$$= \left(\sum_{\mathbf{x}_i \in N_1(\mathbf{x})} w_i \mathbf{x}_i\right) - \mathbf{x}$$



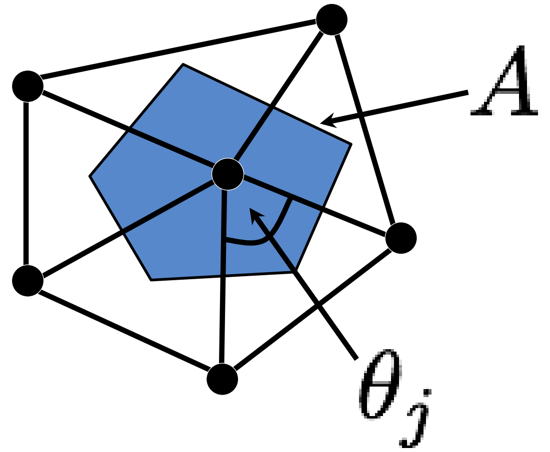
Discrete Curvatures

- Mean curvature

$$H = \|\Delta_{\mathcal{S}} \mathbf{x}\|$$

- Gaussian curvature

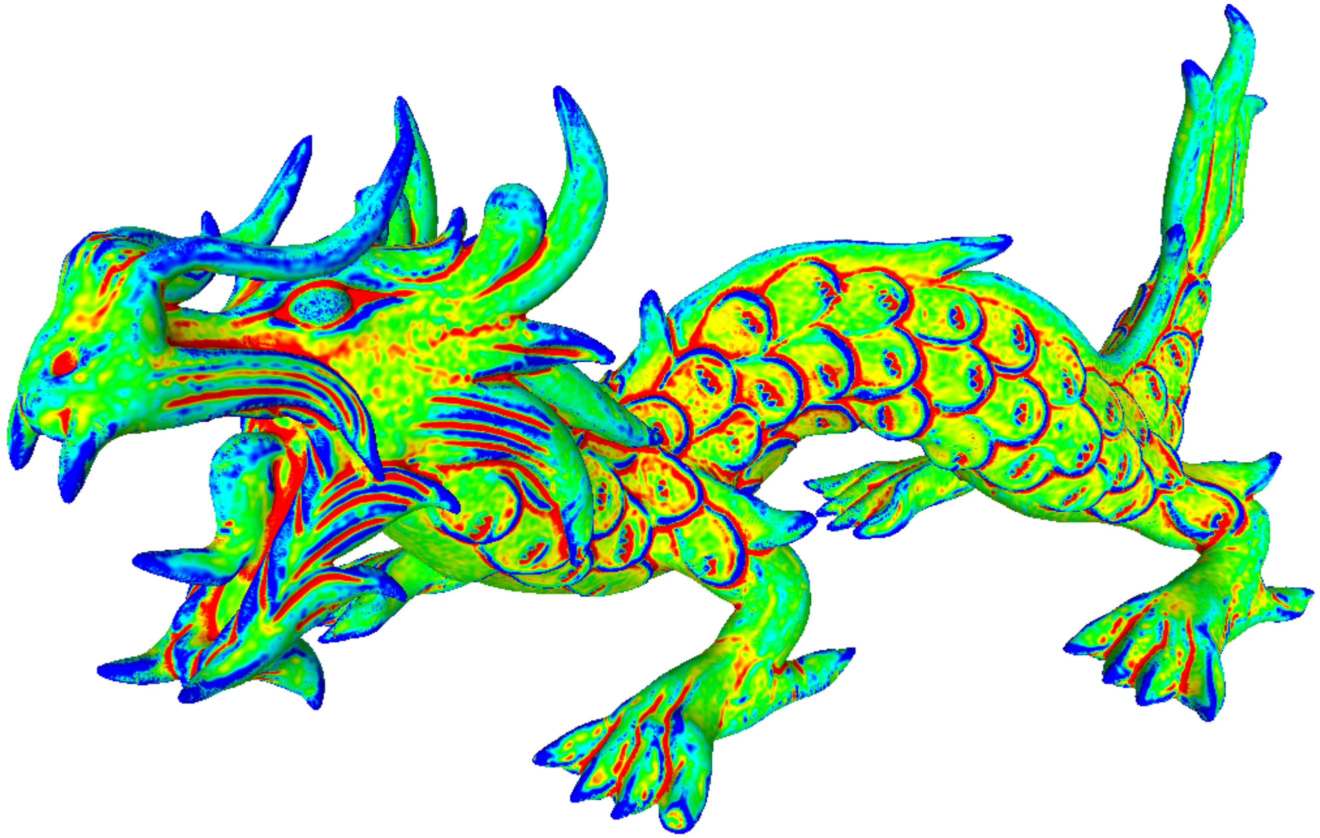
$$G = (2\pi - \sum_j \theta_j)/A$$



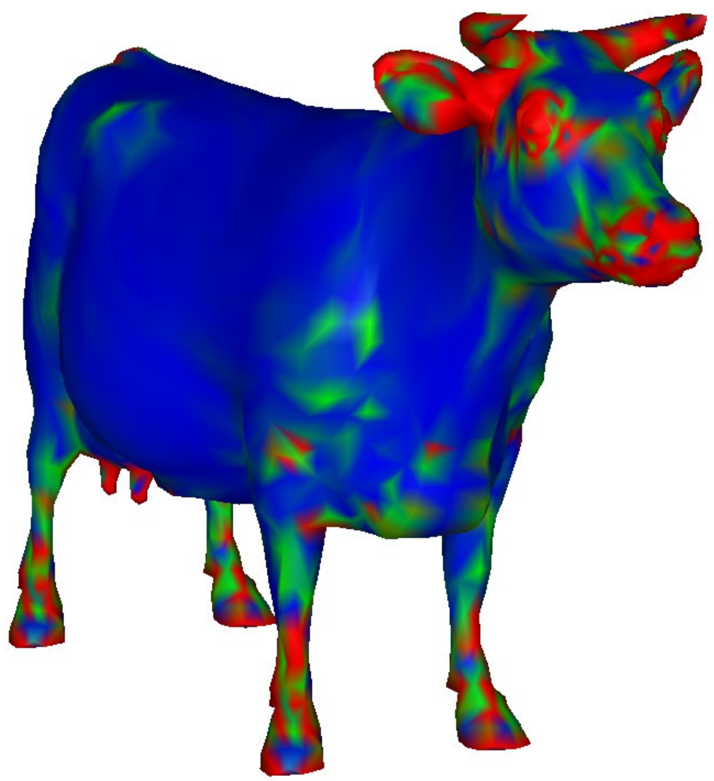
- Principal curvatures

$$\kappa_1 = H + \sqrt{H^2 - G} \qquad \kappa_2 = H - \sqrt{H^2 - G}$$

Example: Mean Curvature



Example: Gaussian Curvature



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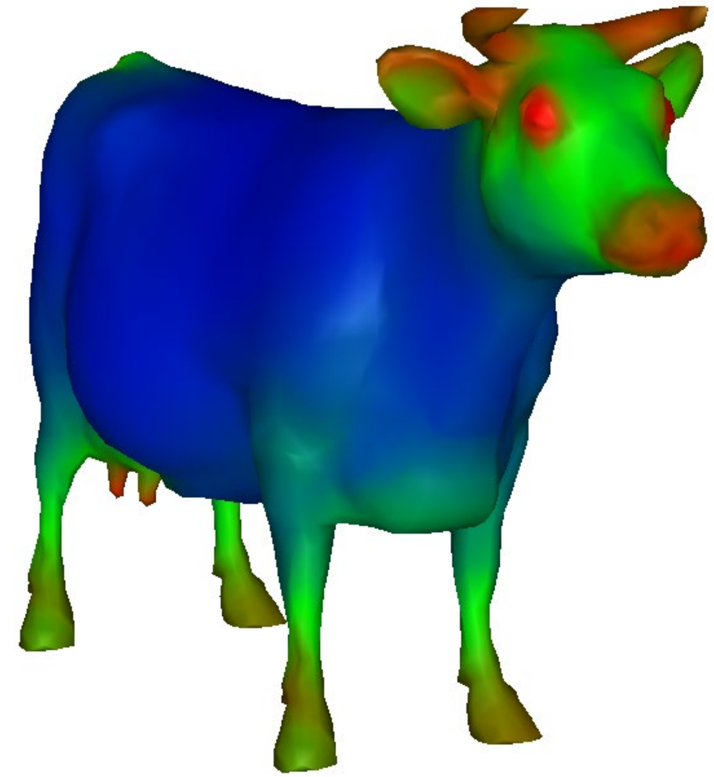

smoothed

Discrete Gauss-Bonnet (Descartes) theorem:

$$\sum_v K_v = \sum_v \left[2\pi - \sum_i \theta_i \right] = 2\pi\chi$$

References

- “Discrete Differential-Geometry Operators for Triangulated 2-Manifolds”, Meyer et al., ’02
- “Restricted Delaunay triangulations and normal cycle”, Cohen-Steiner et al., SoCG ‘03
- “On the convergence of metric and geometric properties of polyhedral surfaces”, Hildebrandt et al., ’06
- “Discrete Laplace operators: No free lunch”, Wardetzky et al., SGP ‘07