Part 1

The transpose of the hat matrix $\boldsymbol{H} = \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T$, would be:

$$\boldsymbol{H}^T = [\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T]^T \tag{1}$$

Also, for an invertible matrix $\mathbf{A}$ the transpose and inverse operations are interchangeable :

$$(\boldsymbol{A}^{-1})^T = (\boldsymbol{A}^T)^{-1} \tag{2}$$

Since $\boldsymbol{X}^T\boldsymbol{X}$ is invertible *by hypothesis*, we reformulate (1) using the equality (2), along with the property of $(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$ as follows:

$$\begin{align}
\boldsymbol{H}^T &= [\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T]^T \tag{3}\\
\implies \boldsymbol{H}^T &= (\boldsymbol{X}^T)^T[(\boldsymbol{X}^T\boldsymbol{X})^{-1}]^T\boldsymbol{X}^T \tag{4}\\
\implies \boldsymbol{H}^T &= \boldsymbol{X}[(\boldsymbol{X}^T\boldsymbol{X})^T]^{-1}\boldsymbol{X}^T \tag{5}\\
\implies \boldsymbol{H}^T &= \boldsymbol{X}[\underbrace{(\boldsymbol{X}^T\boldsymbol{X})^T}]^{-1}\boldsymbol{X}^T \tag{6}\\
\implies \boldsymbol{H}^T &= \boldsymbol{X}[\underbrace{\boldsymbol{X}^T(\boldsymbol{X}^T)^T}]^{-1}\boldsymbol{X}^T \tag{7}\\
\implies \boldsymbol{H} &= \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T \tag{8}\\
\implies &\boxed{\boldsymbol{H}^T = \boldsymbol{H}} \quad \text{q.e.d.} \tag{9}
\end{align}$$

Part 2

If we assume that $\boldsymbol{H}$ is *idempotent* we can write :

$$\boldsymbol{H}^2 = \boldsymbol{H} \tag{10}$$

Then we would get :

$$\begin{align}
\boldsymbol{H}^2 = \boldsymbol{H}\boldsymbol{H} &= [\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T][\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T] \tag{11}\\
&= \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\underbrace{(\boldsymbol{X}^T\boldsymbol{X})}_{C}\underbrace{(\boldsymbol{X}^T\boldsymbol{X})^{-1}}_{C^{-1}}\boldsymbol{X}^T \tag{12}\\
&= \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{I}\boldsymbol{X}^T \tag{13}\\
&= \boldsymbol{H} \text{ which is TRUE} \tag{14}
\end{align}$$

$$\implies \boxed{\boldsymbol{H}^2 = \boldsymbol{H}\boldsymbol{H} = \boldsymbol{H}} \text{ q.e.d.} \tag{15}$$

Part 3

In regression we care more about the *"residual"* quantity $\boldsymbol{M} = \boldsymbol{I} - \boldsymbol{H}$. We prove that the *annihilator* matrix $\boldsymbol{M}$ is also *idempotent* :

$$\begin{aligned}
\boldsymbol{M}^2 = \boldsymbol{M}\boldsymbol{M} &= (\boldsymbol{I} - \boldsymbol{H})(\boldsymbol{I} - \boldsymbol{H}) && (16)\\
&= \boldsymbol{I}^2 - \boldsymbol{I}\boldsymbol{H} - \boldsymbol{H}\boldsymbol{I} + \boldsymbol{H}^2 && (17)\\
&\overset{\text{identity property}}{=\!=} \boldsymbol{I} - \boldsymbol{H} - \boldsymbol{H} + \boldsymbol{H}^2 && (18)\\
&= \boldsymbol{I} - 2\boldsymbol{H} + \boldsymbol{H}^2 && (19)\\
&\overset{\text{using (15)}}{=\!=} \boldsymbol{I} - 2\boldsymbol{H} + \boldsymbol{H} && (20)\\
&= \boldsymbol{I} - \boldsymbol{H} && (21)
\end{aligned}$$

$$\implies \boxed{\boldsymbol{M}^2 = \boldsymbol{M}} \text{ q.e.d.} \tag{22}$$