Assignment AE4136, CFD II, 2021-2022

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Abstract In this assignment you will be asked to write a Navier-Stokes solver based on incidence matrices and Hodge matrices. Note that this assignment deviates from the assignment for this course in previous years.

1 Introduction

We want to solve the Navier-Stokes equations on a unit square. These equations are given by conservation of mass

$$\nabla \cdot \vec{u} = 0 \,,$$

and conservation of momentum

$$\frac{\partial \vec{u}}{\partial t} + \left(\vec{u}, \vec{\nabla}\right) \vec{u} + \vec{\nabla} p = \frac{1}{Re} \Delta \vec{u} \,,$$

where $\vec{u}$ is the velocity field, p is the pressure and Re is the Reynolds number defined by

$$Re = \frac{\rho U L}{\mu} \,.$$

In this equation ρ is the constant fluid density, μ is the fluid viscosity, U is a characteristic velocity of the flow and L is a characteristic length scale.

We are going to rewrite this equation slightly by using the following identities

$$\Delta \vec{u} = \nabla \left(\nabla \cdot \vec{u} \right) - \nabla \times \left(\nabla \times \vec{u} \right) \,,$$

and

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$$(\vec{u},\nabla)\,\vec{u} = \nabla\left(\frac{1}{2}\|\vec{u}\|^2\right) - \vec{u}\times(\nabla\times\vec{u})\ .$$

In both these identities, the factor $\nabla\times\vec{u}$ appears and we abbreviate this expression by

$$\vec{\xi} = \nabla\times\vec{u}\ .$$

With these expressions we can write the momentum equation as

$$\frac{\partial\vec{u}}{\partial t} - \vec{u}\times\vec{\xi} + \nabla P = -\frac{1}{Re}\nabla\times\vec{\xi}\ ,$$

where

$$P = p + \frac{1}{2}\|\vec{u}\|^2\ .$$

The final system of equations is then given by

$$\nabla\cdot\vec{u} = 0\ , \tag{1a}$$

$$\xi = \nabla\times\vec{u}\ , \tag{1b}$$

and

$$\frac{\partial\vec{u}}{\partial t} - \vec{u}\times\vec{\xi} + \nabla P + \frac{1}{Re}\nabla\times\vec{\xi} = 0\ . \tag{1c}$$

These are the equations that were presented in Lecture 1 and which have further been used in Lecture 6. Please consult the videos of these lectures for further clarification.

2 Structure of the Navier-Stokes equations

We know from the lectures that we need to determine how to represent the unknowns which appear in the equations on a grid. Furthermore, some unknowns live on the primal grid, while others are represented on the dual grid. The vector operations

$$\text{div}\equiv\nabla\cdot\ ,\quad \text{curl}\equiv\nabla\times\quad\text{and}\ \ \text{grad}\equiv\nabla\ ,$$

will be represented by incidence matrices. Switching from a representation form one grid to its dual will be performed through the Hodge matrix, as discussed in the lectures.

2.1 Conservation of mass and pressure gradient

Consider the case $n = 2$, i.e. a two-dimensional flow problem. The location of these fluxes is indicated in Figure 1. The red arrows in this figure denote the default orientation, so if fluid flows from left to right or upwards, then the flux is considered to be positive. The orientation of the outer-oriented 2-cells is source-like. So when some-

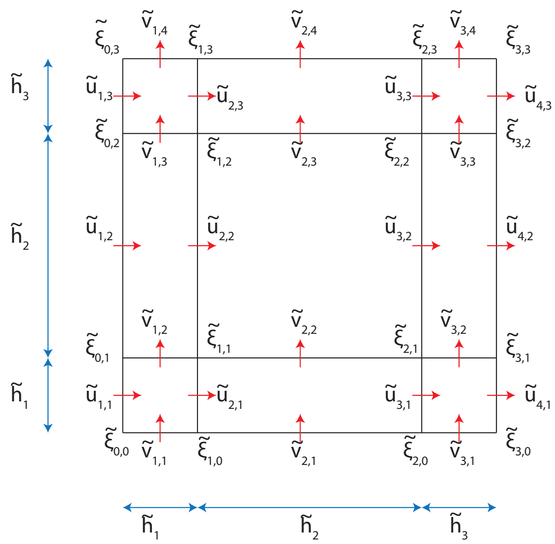

Fig. 1 Location of the velocity flux and the vorticity on the outer-oriented grid

thing flows out of a 2-cell it is considered positive, while when something flows into the 2-cell it is considered negative. Note that although the grid is orthogonal, it is not uniform. The length of the line segments are denoted by $\tilde{h}_i$ in the figure. In Figure 1 the number of cells in the x- and the y-direction is denoted by N, so in the figure $N = 3$.

The unknowns $\tilde{u}_{i,j}$ and $\tilde{v}_{i,j}$ are defined as

$$\tilde{u}_{i,j} := \int_{y_{i,j-1}}^{y_{i,j}} \vec{u} \cdot \vec{n}|_{x=x_{i,j}} \, \mathrm{d}y \quad \text{and} \quad \tilde{v}_{i,j} := \int_{x_{i-1,j}}^{x_{i,j}} \vec{u} \cdot \vec{n}|_{y=y_{i,j}} \, \mathrm{d}x \,.$$

Question Set up the incidence matrix $\tilde{\mathbb{E}}^{(2,1)}$ for this 3×3 grid. We do this by augmenting the mesh slightly as shown in Figure 2. As discussed in the lectures, we add an edge along the outer boundary which has a prescribed values. This value needs to be coupled to the edge on the boundary of the mesh. So we add the boundary conditions as extra equations. These extra equations can be included in the incidence matrix $\tilde{\mathbb{E}}^{(2,1)}$ by treating the additional edges as part of ”volumes” along the boundary where there is in- and/or outflow perpendicular to the boundary, but the tangential velocity is not taken into account. Effectively, this means that we do not take any in- or outflow over the dashed line segments into account. These edges are completely ignored.Note that more ”volumes” also implies more pressure un-

knowns, so the final solution will also give the pressure along the boundary.
Extend this to $N \times N$. This general incidence matrix needs to implemented in the program $\square$

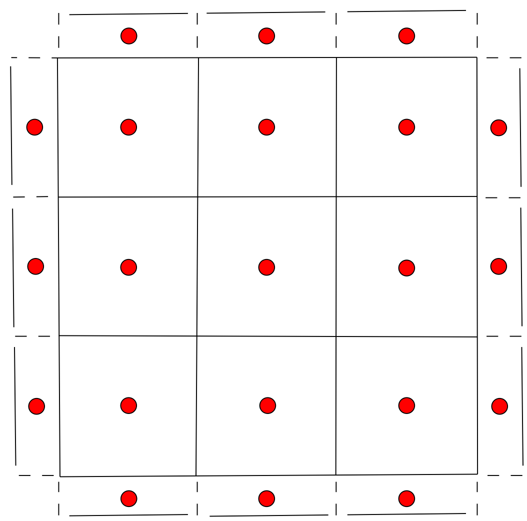

Fig. 2 Outer-oriented mesh plus extra "volumes" along the boundary where we impose the boundary conditions

In order to efficiently use memory in your program use the `sparse`-matrix option in Matlab, `S = sparse(i,j,s)`. See the help function in Matlab how this is used.

Conservation of mass then reads

$$\tilde{\mathbb{E}}^{(2,1)}\tilde{\tilde{u}} = 0\,.$$

Along the newly added edges in our mesh, see Figure 2, the velocity flux is prescribed, these prescribed values can be transferred to the right hand side of the equation.

If the full $\tilde{\mathbb{E}}^{(2,1)}$ matrix is set up, the we can indicate those edges where the prescribed value is given. In the matrix vector system below, the prescribed values of the flux $\tilde{u}$ are shown in red and the corresponding rows in the matrix are also shown in red.

$$
\begin{pmatrix}
\tilde{e}_{1,1} & \dots & \tilde{e}_{1,i} & \tilde{e}_{1,i+1} & \tilde{e}_{1,i+2} & \dots & \tilde{e}_{1,j} & \tilde{e}_{1,j+1} & \tilde{e}_{1,i+2} & \dots & \tilde{e}_{1,L} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\tilde{e}_{k,1} & \dots & \tilde{e}_{k,i} & \tilde{e}_{k,i+1} & \tilde{e}_{k,i+2} & \dots & \tilde{e}_{k,j} & \tilde{e}_{k,j+1} & \tilde{e}_{k,j+2} & \dots & \tilde{e}_{k,L} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\tilde{e}_{K,1} & \dots & \tilde{e}_{K,i} & \tilde{e}_{K,i+1} & \tilde{e}_{K,i+2} & \dots & \tilde{e}_{K,j} & \tilde{e}_{K,j+1} & \tilde{e}_{K,j+2} & \dots & \tilde{e}_{K,L}
\end{pmatrix}
\begin{pmatrix}
\tilde{u}_1 \\ \vdots \\ \tilde{u}_i \\ \tilde{u}_{i+1} \\ \tilde{u}_{i+2} \\ \vdots \\ \tilde{u}_j \\ \tilde{u}_{j+1} \\ \tilde{u}_{j+2} \\ \vdots \\ \tilde{u}_L
\end{pmatrix} = 0\,. \tag{2}
$$

With the fluxes indicated in red we can split this equation as

$$
\begin{pmatrix}
\tilde{e}_{1,1} & \dots & \tilde{e}_{1,i} & \tilde{e}_{1,i+2} & \dots & \tilde{e}_{1,j} & \tilde{e}_{1,i+2} & \dots & \tilde{e}_{1,L} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\tilde{e}_{k,1} & \dots & \tilde{e}_{k,i} & \tilde{e}_{k,i+2} & \dots & \tilde{e}_{k,j} & \tilde{e}_{k,j+2} & \dots & \tilde{e}_{k,L} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\tilde{e}_{K,1} & \dots & \tilde{e}_{K,i} & \tilde{e}_{K,i+2} & \dots & \tilde{e}_{K,j} & \tilde{e}_{K,j+2} & \dots & \tilde{e}_{K,L}
\end{pmatrix}
\begin{pmatrix}
\tilde{u}_1 \\ \vdots \\ \tilde{u}_i \\ \tilde{u}_{i+2} \\ \vdots \\ \tilde{u}_j \\ \tilde{u}_{j+2} \\ \vdots \\ \tilde{u}_L
\end{pmatrix}
+
\begin{pmatrix}
u_{norm,1} \\ \vdots \\ u_{norm,k} \\ \vdots \\ u_{norm,K}
\end{pmatrix} = 0\,,
$$

where the vector $\vec{u}_{norm}$ is given by

$$
\vec{u}_{norm} =
\begin{pmatrix}
\tilde{e}_{1,i+1} & \dots & \tilde{e}_{1,j+1} \\
\vdots & & \vdots \\
\tilde{e}_{k,i+1} & \dots & \tilde{e}_{k,j+1} \\
\vdots & & \vdots \\
\tilde{e}_{K,i+1} & \dots & \tilde{e}_{K,j+1}
\end{pmatrix}
\begin{pmatrix}
\tilde{u}_{i+1} \\ \vdots \\ \tilde{u}_{j+1}
\end{pmatrix}\,.
$$

The newly introduced 1-cells corresponding to prescribed velocity fluxes can effectively be removed from the grid and after elimination of these edge we have (once again) the mesh shown in Figure 1.

In equation (2) we see that all fluxes are stored in a vector. It is important to first list all the fluxes in the x-direction and then the fluxes in the y-direction. Also, start the numbering in the lower left corner of your grid and then move from left to right and then upward. So, for the grid shown in Figure 1, the vector of fluxes would look something like

$$\begin{pmatrix} \tilde{u}_{1,1} \\ \tilde{u}_{2,1} \\ \tilde{u}_{3,1} \\ \tilde{u}_{4,1} \\ \tilde{u}_{1,2} \\ \vdots \tilde{v}_{1,1} \\ \tilde{u}_{2,1} \\ \tilde{u}_{3,1} \\ \tilde{u}_{1,2} \\ \vdots \end{pmatrix} .$$

If you store the fluxes in this way, it is compatible with the convective terms in the program which have been programmed already.

After we have inserted the normal boundary conditions and eliminated the corresponding columns from the incidence matrix $\tilde{\mathbb{E}}^{(2,1)}$ we still refer to this matrix as $\tilde{\mathbb{E}}^{(2,1)}$. In the code this matrix is denoted by `tE21`.

With the grid where the extra boundary faces have been removed, see Figure **??** we are going to associate a dual grid, shown in red in Figure 3. The dual points (red

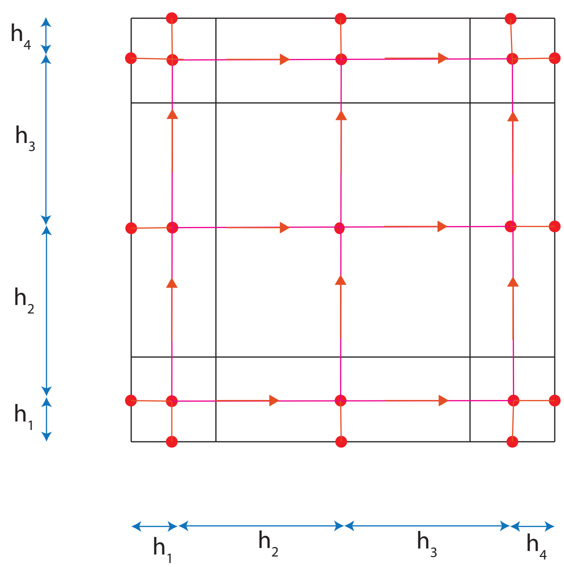

Fig. 3 The dual grid associated with the grid shown in Figure 2

dots in Figure 3) are precisely in the middle of the outer-oriented 2-cells and along the boundary the dual points are located at the midpoint of the boundary edges. In the dual points the pressure is defined. Make sure that you label the pressure points in the same way as the 2-cells on your primal mesh. Also label the dual edges the same as the primal edges. The dual points are considered as sinks while the orientation of the dual edge is indicated in Figure 3.

Question Set up the incidence matrix $\mathbb{E}^{(1,0)}$ between the points and the line segments of the dual grid and show that this matrix is either plus or minus the transpose of $\tilde{\mathbb{E}}^{(2,1)}$ on the primal grid from which the prescribed normal boundary conditions have been removed $\square$

Once we established this relation, it follows that we only have to set up one incidence matrix (for conservation of mass, for example). The incidence matrix for the pressure gradient is then simply its transpose.

3 Vorticity and diffusion

Consider the primal cell complex with its dual grid as shown in Figure 4. On the

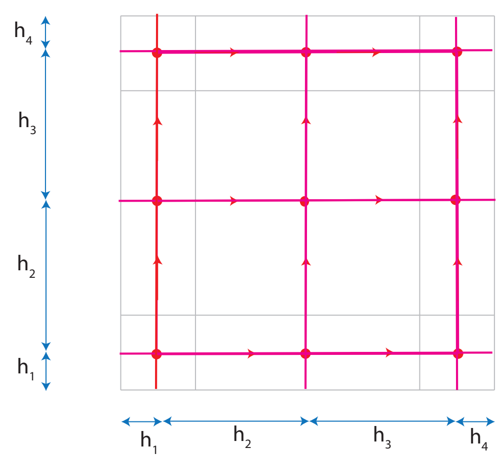

Fig. 4 Original primal complex where the edge. The red grid is its associated dual.

dual grid we represent the inner-oriented velocity *along* the dual edge as indicated in the figure. The positive orientation is indicated by the red arrows. The integrated velocity along the dual edges will be called the *circulation*. These circulations are defined by

$$u_{i,j} := \int_{x_{i-1/2}}^{x_{i+1/2}} \vec{u}\cdot\vec{t}\big|_{y=y_{j-1/2}}\,\mathrm{d}x \quad \text{and} \quad v_{i,j} := \int_{y_{j-1/2}}^{y_{j+1/2}} \vec{u}\cdot\vec{t}\big|_{x=x_{i-1/2}}\,\mathrm{d}x\,,$$

where $\vec{t}$ is a tangent vector along the red grid lines.

Note that the circulation *along* the red edges in Figure 4 is related by a Hodge matrix to the fluxes on on the outer oriented grid.

$$\vec{u} = \mathbb{H}^{1,\tilde{1}}\tilde{\vec{u}}\,,$$

where $\vec{u}$ denotes the circulation along the dual (red) edges and $\tilde{\vec{u}}$ denotes the flux through the black edge on the primal grid. Assume that the velocity for the flux is constant over an edge and that the velocity for the circulation is constant along an edge. Use this assumption to set up the Hodge matrices $\mathbb{H}^{1,\tilde{1}}$ and $\mathbb{H}^{\tilde{1},1}$. These Hodge matrices will be a diagonal matrices.

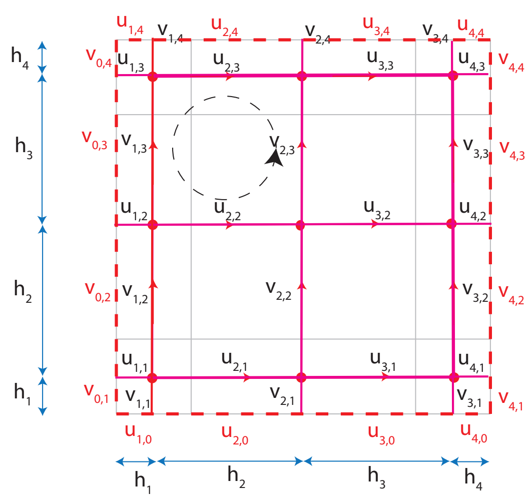

Fig. 5 The dual grid from Figure 4 supplemented with edges (red dashed lines) to close the grid and the additional degrees of freedom associated with the newly created edges.

The dual grid (red grid) is *not* a cell complex. In order to remedy this, we add nodes and edges to 'close the grid'. These additional 0- and 1-cells are represented by the red dashed lines in Figure 5 at the outer boundary. By adding additional edges in the dual grid, we also need to associate with these edges degrees of freedom. These extra tangential velocities along these edges are indicated in red dashed line segments in Figure 5. Note that if the tangential velocity is prescribed at the outer boundary (no slip condition), we know what these extra degrees of freedom are. They are not real unknowns.

We can set up the incidence matrix $\mathbb{E}^{(2,1)}$ which models the circulation around a closed surface on the (red) dual grid and call the variable associated to the surfaces on the dual grid $\xi^{(2)}$.

$$\xi^{(2)} = \mathbb{E}^{(2,1)} u^{(1)} .$$

Because some of the 1-cochains along the boundary are known (boundary conditions) we can remove them and the corresponding columns of $\mathbb{E}^{(2,1)}$ which yields

$$\xi^{(2)} = \mathbb{E}^{(2,1)} u^{(1)} + u_{prescr} . \tag{3}$$

After elimination of the columns from $\mathbb{E}^{(2,1)}$ we still refer to this incidence matrix in the program as $\mathbb{E}^{(2,1)}$.

Question What does the variable $\xi^{(2)}$ physically represent? □

For the viscous contribution in the Navier-Stokes equations, we are going to apply the Hodge matrix $\mathbb{H}^{\tilde{0},2}$ to convert $\xi^{(2)}$ associated with surfaces on the dual (red) mesh, to points on the primal (black) mesh. We will call these variables on the primal mesh $\tilde{\xi}^{(0)}$. Then we apply the incidence matrix $\tilde{\mathbb{E}}^{(1,0)}$ to convert the nodal values $\tilde{\xi}^{(0)}$ to variables over the edges of the primal grid. Then we apply the Hodge matrix $\mathbb{H}^{1,\tilde{1}}$ to bring the final result back to the line segments on the dual (red) mesh. So we started with circulations along the dual edge and after several operations, the results is again represented on the dual (red) edges.

Question Make the simple assumption that $\tilde{\xi}^{(0)}$ is constant over over the dual 2-cells to construct the Hodge matrix. This will result in a diagonal Hodge matrix □

Question Show that the first $\nabla\times$-operator in $\nabla \times \nabla \times \vec{u}$ is represented by $\tilde{\mathbb{E}}^{(1,0)}$ which coincides with the transpose of $\mathbb{E}^{(2,1)}$. Once, this is established, we see that no extra work needs to be done when implementing this operation □

4 Final Discrete system

With all the matrices constructed, the discrete system becomes: Conservation of mass

$$\tilde{\mathbb{E}}^{(2,1)} H^{\tilde{1},1} \vec{u}^{n+1} + \vec{u}_{norm} = 0 .$$

Velocity-vorticity relation

$$\xi^n = \mathbb{E}^{(2,1)} \vec{u}^n .$$

and the momentum equation

$$\frac{\vec{u}^{n+1} - \vec{u}^n}{\Delta t} + \text{convective}^n + \mathbb{E}^{(1,0)} P^{n+1} + \frac{1}{Re} H^{1,\tilde{1}} \tilde{\mathbb{E}}^{(1,0)} H^{\tilde{0},2} (\xi^n + \xi_{prescr}) = 0 .$$

Here the superscripts n and $(n+1)$ refer to the time level at which these variables are evaluated, see Lecture 6 of this course. We assume that we know the variables with superscript n. They have already been computed or are given as initial conditions for $n=0$. So the real unknowns are the ones with the superscript $n+1$.

In order to ensure that new velocity field $\vec{u}^{n+1}$ will satisfy conservation of mass, we apply $\tilde{\mathbb{E}}^{(2,1)}H^{\tilde{1},1}$ to the momentum equation and add $\vec{u}_{norm}$ and set this result to zero. In fact, we eliminate $\vec{u}^{n+1}$ from conservation of mass and momentum. We end up with an equation for the pressure

$$A\vec{P} = f \; ,$$

where

$$A = -\tilde{\mathbb{E}}^{(2,1)}H^{\tilde{1},1}\mathbb{E}^{(1,0)} \; ,$$

and

$$f = \tilde{\mathbb{E}}^{(2,1)}H^{\tilde{1},1}\left[\frac{\vec{u}^n}{dt} - \text{convective}^n - \frac{1}{Re}H^{1,\tilde{1}}\tilde{\mathbb{E}}^{(1,0)}H^{\tilde{0},2}(\xi^n + \xi_{pres})\right] + \frac{\vec{u}_{norm}}{\Delta t} \; .$$

Derive this equation. Once you have solved for P^{n+1} you insert this in the momentum equation and update $\vec{u}^{n+1}$. By construction you satisfy the momentum equation. The only thing that remains to be checked is whether you satisfy conservation of mass with this velocity field.

5 A note on the vectors

In all figures we refer to $u_{i,j}$ and $v_{i,j}$, but in the the code we store this in one large vector

$$\vec{u} = \begin{pmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ \vdots \\ u_{N-1,N} \\ v_{1,1} \\ v_{2,1} \\ \vdots \\ v_{N,N-1} \end{pmatrix} \; .$$

So $u_{i,j}$ can be found in `u(i + (j-1)*(N+1))` for $1 \le i \le N+1$ and $1 \le j \le N$ and $v_{i,j}$ at `u(N*(N+1) + i + (j-1)*N)` for $1 \le i \le N$ and $1 \le j \le N+1$. Note that in Python the counters are slightly different, because arrays in Python start with the index ”0”, while Matlab starts with ”1”.

6 Test case

The test case we will consider is the lid-driven cavity problem as described in the reference by Botella and Peyret. The domain is a unit square $[0,1] \times [0,1]$. Along all boundaries the velocity is set to zero, except the top boundary which moves with a unit velocity to the *left*.

Complete the code by implementing the required pieces of code. If you are developing a code, start with a small number of cells, say $N = 3$.

If done correctly, your pressure matrix will have row sum equal to zero, i.e. if A is the pressure matrix, then

$$\sum_{j=1}^{L} A_{ij} = 0$$

and this system is symmetric, i.e. $A_{ij} = A_{ji}$. Check this. This also explains why the pressure matrix is singular (why?) What does this singularity physically represent? In order to understand what the singular mode (the eigenvector associated with the zero eigenvalue) means, compute this eigenmode (either analytically or let Matlab/Python evaluate this mode for you).

Try to determine a suitable time step. You may consider to use

$$\Delta t = \min\left(h_{min}, \frac{1}{2} Re\, h_{min}^2\right) ,$$

where h_{min} is the smallest edge length in your mesh. This is a conservative estimate, so see if you can take a larger time step.

Also determine a suitable stopping criterion. What is the danger of stopping too early? What if you wait too long? You will be asked to motivate your stopping criterion in the oral examination, so not only choose the time step, but make sure you can justify your choice.

After the simulation has completed, make contour plots for the stream function, the vorticity field and the pressure field and compare the plots with results from Botella and Peyret, for $N = 15, 31, 47, 55, 63$. Make sure you use the same isolines as used by Botella and Peyret. For the pressure plots, only use the presure solution in the domain and not the pressure values along the boundary of the domain.

Also plot pressure, vorticity, x- and y-component of the velocity along the lines $x = 0.5$ and $y = 0.5$ for the various values of N. Also plot the reference values from Botella and Peyret. For these cross section plots you can also use the pressure solution at the boundary.

Question Determine the integrated vorticity over the entire domain as a function of N. How does the integrated vorticity depend on the Reynolds number. Explain this. $\square$.

Write a report in which you explain your code and in which you present and discuss your results. Motivate all choices you make in setting up the code. Try to hand in your assignment before May 1, 2022.

Good luck!