# Comparison of Alternative Linear Approximations of Optimal Power Flow Problem

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# Motivation

- The Optimal Power Flow (OPF) problem plays a critical role in ensuring **efficient**, **cost-effective**, and **secure operation** of power systems.
- The **AC-OPF model** provides **high accuracy but is nonlinear** and **computationally demanding**, making it less suitable for large-scale or real-time use.
- Linear OPF models offer **faster and simpler solutions**, but often at the expense of **reduced accuracy**.
- This thesis **introduces a linear OPF** model and evaluates its performance against classical linear approximations (Bθ and Decoupled).
- The goal is to investigate whether the proposed linear model can serve as a practical alternative for Distribution System Operators (DSOs), enabling **faster and simpler decision-making** while still capturing the essential behaviour of the power system**.**

# General structure of the model

$$minimize_{p,q,f,fq,u,\theta,rp,rq} \text{ Generation Cost}$$

$$s.t.$$

$$Linear\ power\ flow\ equations$$

$$Generators\ technical\ limits$$

$$Lines\ technical\ flow\ limits$$

$$Nodal\ power\ balance$$

$$Flows\ calculation$$

$$p \geq 0$$

# Objective function

The objective of the Optimal Power Flow (OPF) problem is to minimize the total generation cost.

This is formulated through the objective function:

$$min_{p,q,f,fq,u,\theta,rp,rq} \sum_{g \in G} MC_g \cdot p_g$$

**Notation**

- $MC_g$: Marginal cost of generator g
- $p_g$: Active power production of generator g
- $G$: Set with generators' names

# Linear power flow equations

- At the core of the power flow problem lie the power flow equations.
- The linearized formulation used in this thesis is based on the work of Saverio Bolognani and Sandro Zampieri: *"On the existence and linear approximation of the power flow solution in power distribution networks"*.
- These equations provide a simplified yet effective description of the system, capturing the **linear relationships between voltages and active, reactive power injections.**

- **Voltage magnitude:**

$$u_J = \mathbf{1}V_0 + \frac{1}{V_0} Re\left(Z\bar{s}_J\right) \quad (1)$$

- **Voltage angle:**

$$\theta_J = \mathbf{1}\Theta_0 + \frac{1}{V_0^2} \mathrm{Im}\left(Z\bar{s}_J\right) \quad (2)$$

**Notation**
J: Set of all nodes except slack node
$V_0$: Voltage magnitude of slack node
$\Theta_0$: Voltage angle of slack node
$Z$ : Inverse of admittance matrix Y
$\bar{s}_J$: Vector of complex power injection

# Linear power flow equations

Understanding how the previous equations are incorporated into the model formulation is crucial.

This requires a clear grasp of how the impendance matrix $Z$ and power injections vector $\bar{s}_J$ are defined and constructed.

- For $Z$ matrix:

$$Z = \begin{bmatrix} Z_{KK} & Z_{KL} \\ Z_{LK} & Z_{LL} \end{bmatrix}$$

- For $\bar{s}_J$ :

$$\bar{s}_J = \begin{bmatrix} s_K \\ s_L \end{bmatrix}$$

In both cases, a remapping is necessary to position the elements of the matrix and vector correctly. In the new indexing, **generator nodes** are assigned the **initial positions**, followed by the non-generator nodes.

**Notation**

- $K$: Set of generator nodes, excluding the slack node
- $L$: Set of non-generator nodes

# Linear power flow equations

The equations for voltage magnitudes and angles, using the previously defined $Z$ matrix and $\bar{s}_J$ vector, are expressed in the form of equations (3) and (4).

- **Voltage magnitude**

$$u_j = V_0 + \frac{1}{V_0} \cdot \sum_{i \in K} (Re[Z_{j,i}] \cdot rp_i + Im[Z_{j,i}] \cdot rq_i) + \frac{1}{V_0} \cdot \sum_{i \in L} (Re[Z_{j,i}] \cdot rp_i + Im[Z_{j,i}] \cdot rq_i), \ j \epsilon J \quad (3)$$

- **Voltage angle**

$$\theta_j = \Theta_0 + \frac{1}{V_0^2} \cdot \sum_{i \in K} (-Re[Z_{j,i}] \cdot rq_i + Im[Z_{j,i}] \cdot rp_i) + \frac{1}{V_0^2} \cdot \sum_{i \in L} (-Re[Z_{j,i}] \cdot rq_i + Im[Z_{j,i}] \cdot rp_i), j \epsilon J \quad (4)$$

**Notation**

- $J$: Set of all nodes except slack node
- $rp_j$: Active power injection at node j
- $rq_j$: Reactive power injection at node j

# Voltages at nodes with generators

- **Voltage magnitudes at generator nodes:**

$$u_K = \mathbf{1} \quad (5)$$

- Fixing the voltage magnitudes at generator buses to a specified value is a necessary constraint. It ensures that, for a given active power injection vector, the model will yield unique solution vectors for voltage magnitudes ($u$), angles ($\theta$) and reactive power ($rq$) variables.

- If equation (5) were omitted, then for a given active power injection vector, the system would admit **multiple solutions for the vectors of $\boldsymbol{u, \theta}$ *and* $\boldsymbol{rq}$.**

- The value to which we assume the voltages are equal is not immediately obvious, but a common assumption is that they are set to 1.

**Notation**

- $K$: Set of generator nodes, excluding the slack node

# Generators technical limits, voltage magnitude limits and power injections

- **Maximum and minimum generation capability:**

$$P_{min_g} \leq p_g \leq P_{max_g}, g \in G$$

$$Q_{min_g} \leq q_g \leq Q_{max_g}, g \in G$$

- **Nodal voltage magnitude tolerance:**

$$V_{min} \leq u_j \leq V_{max}, j \in J$$

**Notation**
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- **Active power injection**

$$(\rho_b): \ rp_b = D_b^p + \sum_{g \in G} p_g, b \ \in B$$

$$(\rho_l): \ rp_l = D_l^p, l \in L$$

- **Reactive power injection**

$$rq_b = D_b^q + \sum_{g \in G} q_g, b \ \in B$$

$$rq_l = D_l^q, l \in L$$

**Notation**

- $J$: Set of all nodes except slack node
- $D_n^p$: Active power load at node $n$
- $D_n^q$: Reactive power load at node $n$
- $B$: Set of all generator nodes
- $L$: Set of non-generator nodes
- $N$: Set of all nodes, $\mathrm{N} = B \cup L$

**Dual variables $\rho$**

- The constraints of active power injection are important, as their dual variables have economic interpretation.
- $\rho_n$ is the **nodal price** of node $n \ \in N$.

# Power Flows Calculation and Nodal Power Balance

To determine the line flows, a Taylor series approximation is employed to linearize the nonlinear AC flow equations:

- **Active power flows**

$$f_m = \frac{R_{ij}\left(u_i - u_j\right) + X_{ij}(\theta_i - \theta_j)}{R_{ij}^2 + X_{ij}^2}, m \in M$$

- **Reactive power flows**

$$fq_m = \frac{X_{ij}\left(u_i - u_j\right) - R_{ij}(\theta_i - \theta_j)}{R_{ij}^2 + X_{ij}^2}, m \in M$$

**Notation**

- $f_m$: Active power flow in line $m \in M$
- $fq_m$: Reactive power flow in line $m \in M$
- $S_m$: Technical power flow limit of line m (positive value)
- $M$: Set of all lines
- $m = (i, j)$: line from node i to node j

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At each node, power balance must be maintained between the net injections and the incoming and outgoing flows. This condition is enforced through the following constraints:

- **Active power balance**

$$rp_n = \sum_{m=(n,:)} f_m - \sum_{m=(:,n)} f_m , n \in N$$

- **Reactive power balance**

$$rq_n = \sum_{m=(n,:)} fq_m - \sum_{m=(:,n)} fq_m , n \in N$$

# Technical Limits of Transimission Lines

- **Constraints for maximum and minimum flow limits**

$$-S_m \leq f_m \leq S_m, m \in \mathrm{M}$$

$$-S_m \leq fq_m \leq S_m, m \in M$$

○ The above constraints represent a simple linearization of the nonlinear maximum flow constraint: $f_m^2 + fq_m^2 \leq \ S_m^2$

○ The set of feasible values for the active and reactive power flows in the nonlinear (AC) constraint is a subset of feasible values in the linear constraint.

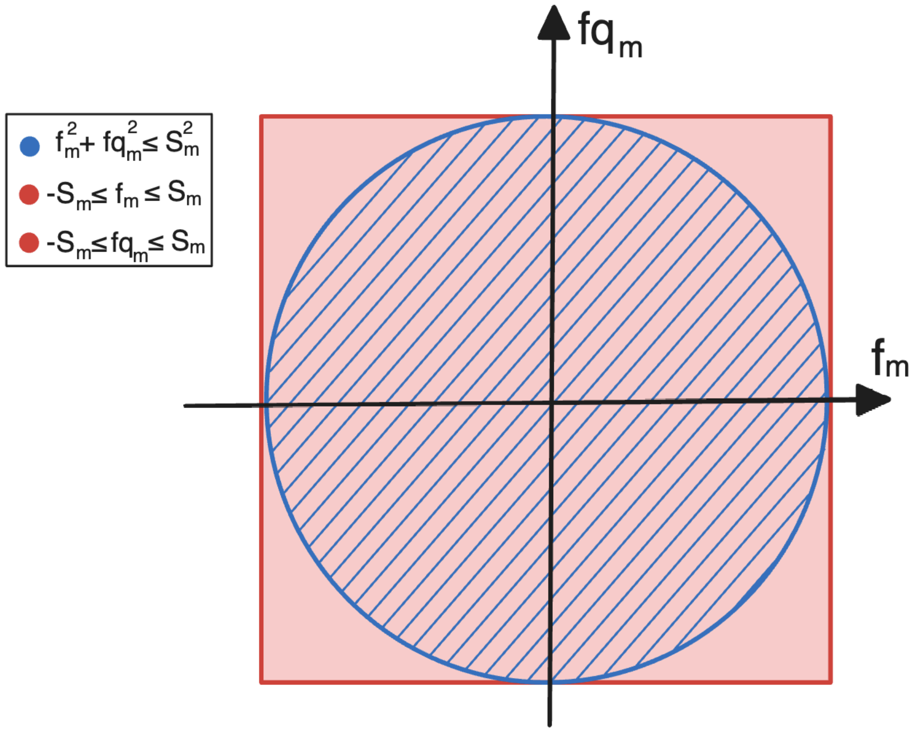


*Figure: Comparison between linear and nonlinear constraint*

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Proposed model description

**Accuracy of the Thesis model**

Case study

Conclusions

Extra material

# Accuracy of Thesis OPF

- **Inaccurate approximation of active power** production compared to the AC-OPF

**Root cause**

**Linear model ignores losses.**

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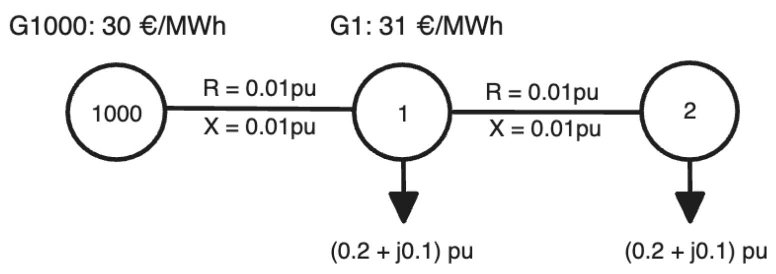



| Nodes | Active production [p.u.] **ACOPF** | Active production [p.u.] **Linear** |
|---|---|---|
| 1000 | 0.402 | 0.4 |
| 1 | 0 | 0 |

**Conclusions**

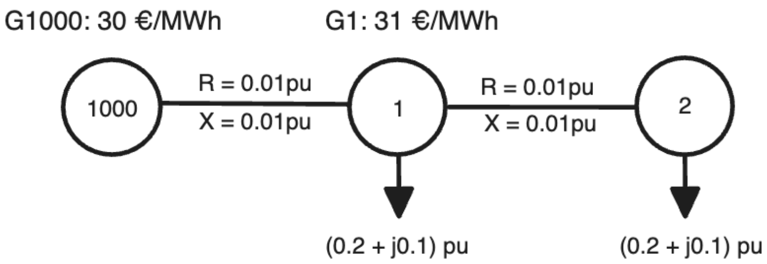
- Linear model results in lower total active power production.

# Accuracy of Thesis OPF

- **Inaccurate approximation of active power** production compared to the AC-OPF

**Root cause** → **Linear model ignores losses.**

Resistance Increase

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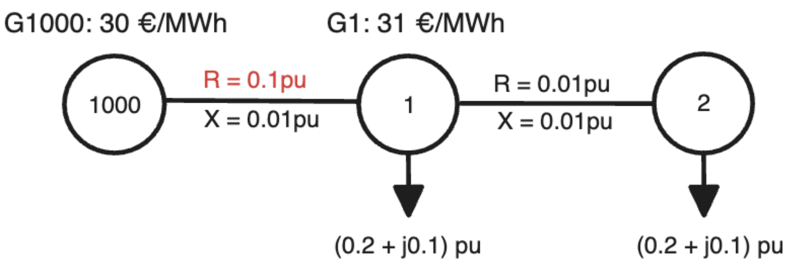



| Nodes | Active production [p.u.] **ACOPF** | Active production [p.u.] **Linear** |
|---|---|---|
| 1000 | 0.161 | 0.4 |
| 1 | 0.242 | 0 |

**Conclusions**

- Linear model results in lower total active power production.
- Linear model is unaffected by increases in line losses, unlike the AC model, which adjusts its production decisions in order to avoid the lines with more losses.

# Accuracy of the Thesis OPF

- **Weak approximation of reactive power production**, compared to the AC-OPF

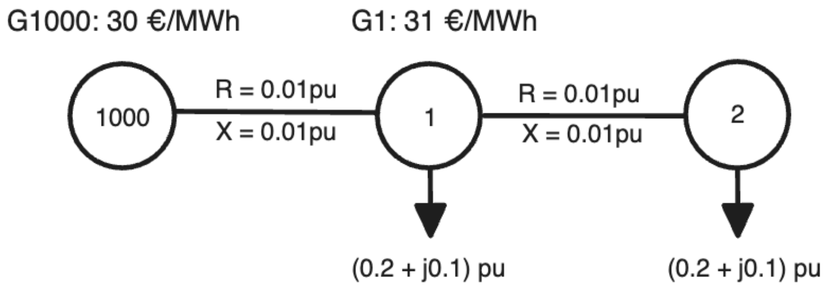
**Root causes**

- Fixing the voltage magnitude at generator nodes to a specific value.
- Inaccurate approximation of active power production.

- To examine these two root causes within the previously introduced 3-node systems, a modified version of the ACOPF model will be employed.
- From this point onward, ***"ACOPF modified"*** refers to the AC model with **voltage magnitudes of generator nodes fixed to 1 per unit.**
- We will observe that the **linear model attempts to approximate the results of the *ACOPF modified*** model, which is assumed to produce outcomes reasonably close to those of the ACOPF model.

# Accuracy of the Thesis OPF

- **Weak approximation of reactive power production**, compared to the AC-OPF

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|---|---|---|---|
| 1000 | 0.4021 | 0.4037 | 0.4 |
| 1 | 0 | 0 | 0 |

| Nodes | Reactive power production [p.u.] **ACOPF** | Reactive power production [p.u.] **ACOPF modified** | Reactive power production [p.u.] **Linear** |
|---|---|---|---|
| 1000 | -0.0005 | -0.4005 | -0.4 |
| 1 | 0.2016 | 0.6030 | 0.6 |

**Root causes**

**Fixing the voltage magnitude at generator nodes to a specific value.**

Inaccurate approximation of active power production.

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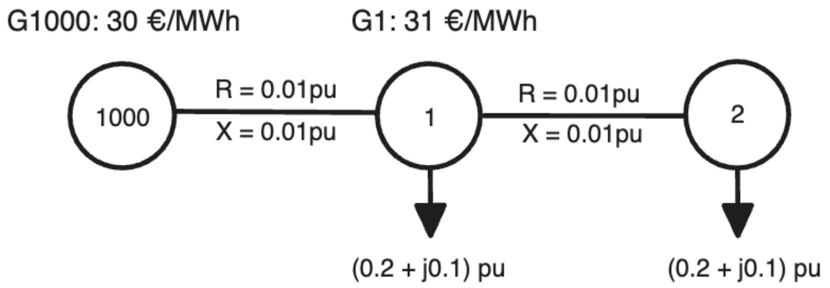


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## Observations

- The linear model accurately approximates the active power results of both the ACOPF and the ACOPF *modified*.

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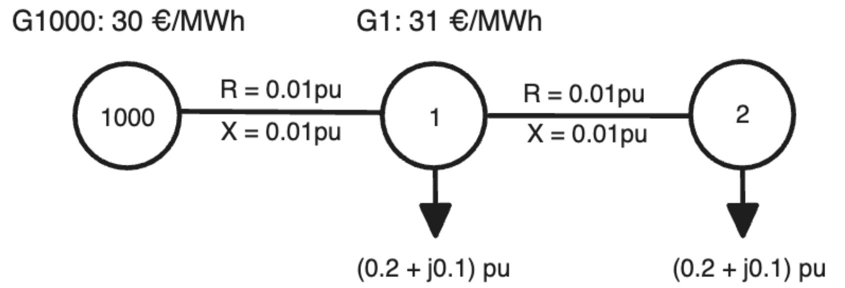


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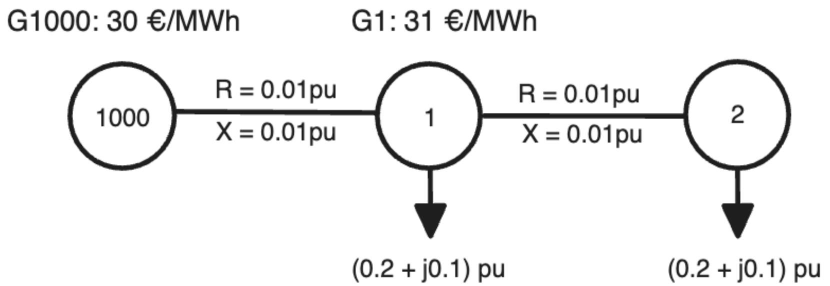
## Observations

- The linear model accurately approximates the active power results of both the ACOPF and the ACOPF *modified*.
- For reactive power, the linear model aligns closely **only** with the results of *ACOPF modified*.
- **The linear model aims to approximate the reactive power output of the *ACOPF modified* model ,** primarily because the voltage magnitudes at generator buses are set identically in both models.

# Accuracy of the Thesis OPF

- **Weak approximation of reactive power production**, compared to the AC-OPF

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- **The linear model aims to approximate the reactive power output of the *ACOPF modified* model ,** primarily because the voltage magnitudes at generator buses are set identically in both models.
- *A strong reactive power approximation between the linear model and the ACOPF modified requires a strong match in active power between the two models.*

# Accuracy of the Thesis OPF

- **Weak approximation of reactive power production**, compared to the AC-OPF

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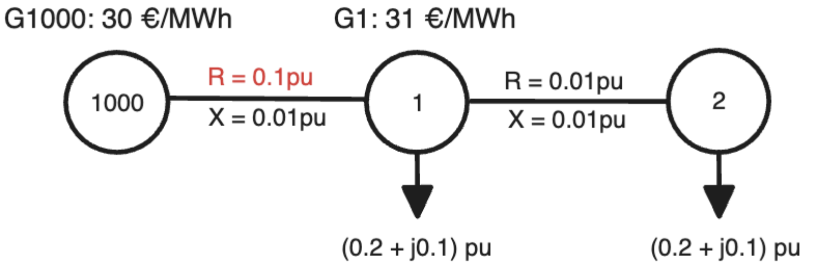


| **Nodes** | Active power production [p.u.] **ACOPF** | Active power production [p.u.] **ACOPF modified** | Active power production [p.u.] **Linear** |
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| 1000 | 0.1608 | 0.0016 | 0.4 |
| 1 | 0.2423 | 0.3989 | 0 |

| Nodes | Rective power production [p.u.] **ACOPF** | Rective power production [p.u.] **ACOPF modified** | Rective power production [p.u.] **Linear** |
|---|---|---|---|
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| 1 | 0.1998 | 0.2157 | 4.2 |

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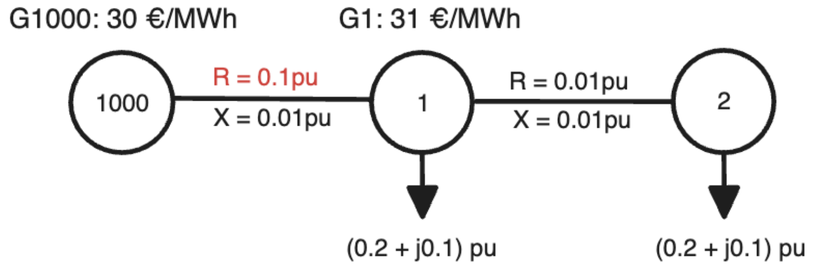
- **Weak approximation of reactive power production**, compared to the AC-OPF

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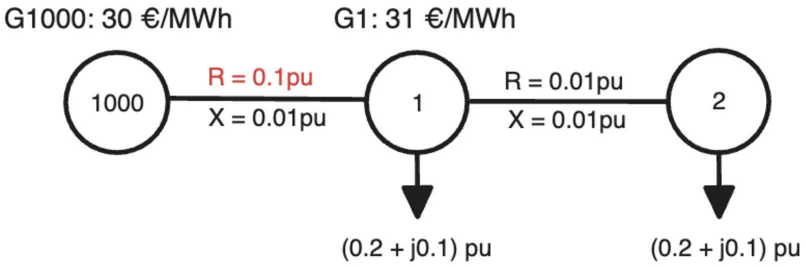
**Root causes**
- Fixing the voltage magnitude at generator nodes to a specific value.
- **Inaccurate approximation of active power production.**

## Observations

- In contrast to the previous case, the active power production between the linear model and the *ACOPF modified* is poorly approximated.

# Accuracy of the Thesis OPF

- **Weak approximation of reactive power production**, compared to the AC-OPF

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- This leads to a weak approximation of the reactive power.

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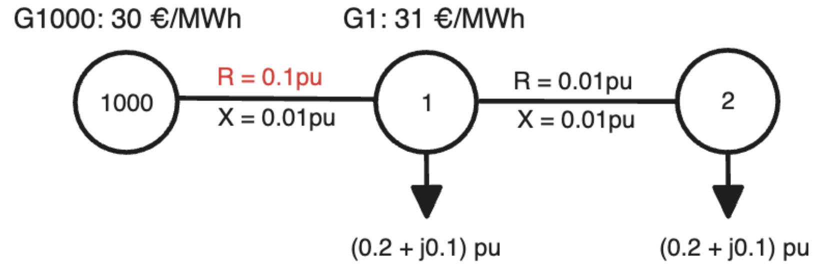


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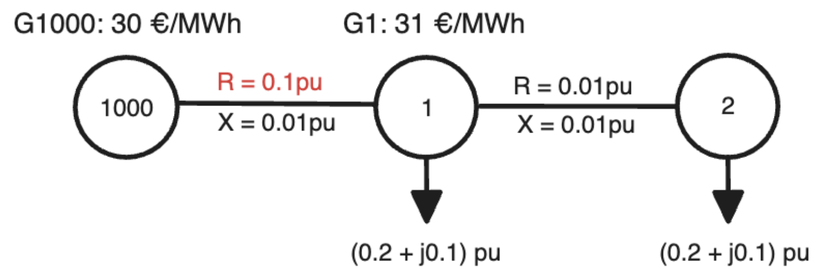
## Observations

- In contrast to the previous case, the active power production between the linear model and the *ACOPF modified* is poorly approximated.
- This leads to a weak approximation of the reactive power.
- As a result, accurate active power approximation plays a crucial role in achieving a reliable reactive power approximation.

# Accuracy of the Thesis OPF

- **Weak approximation of reactive power production**, compared to the AC-OPF

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**Root causes**

- Fixing the voltage magnitude at generator nodes to a specific value.
- **Inaccurate approximation of active power production.**

## Observations

- In contrast to the previous case, the active power production between the linear model and the *ACOPF modified* is poorly approximated.
- This leads to a weak approximation of the reactive power.
- As a result, accurate active power approximation plays a crucial role in achieving a reliable reactive power approximation.
- Also, the result is weak approximation of reactive power compared to ACOPF.

# Accuracy of the Thesis OPF

**Voltage magnitudes and angles approximation inherent the inaccuracy of active and reactive power approximation**

- The voltage magnitudes and angles are variables dependent on the active and reactive power production.
- As a result, errors in active and reactive power approximations propagate to voltage variable inaccuracies.
- **Voltage magnitudes and angles of the linear Thesis model will approximate the results of the *ACOPF modified model,*** which is assumed to produce outcomes reasonably close to those of the ACOPF model.

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- The voltage magnitudes and angles are variables dependent on the active and reactive power production.
- As a result, errors in active and reactive power approximations propagate to voltage variable inaccuracies.
- **Voltage magnitudes and angles of the linear Thesis model will approximate the results of the *ACOPF modified model,*** which is assumed to produce outcomes reasonably close to those of the ACOPF model.

**Taylor series approximation for power flows works only for voltage magnitude of slack bus equal to 1**

- Taylor series constraint was structured around a slack bus voltage magnitude of 1.
- In practice, if $V_{slack} \neq 1$, the model will decide an active power generation that differs form the active power demand.

# Alternative Linear OPF models

**Bθ**

$$min_{p,u,\theta} \sum_{g \in G} MC_g \cdot p_g$$

$$P_{min_g} \leq p_g \leq P_{max_g}, g \in G$$

$$u_n = 1, n \ \in N$$

$$p_{ij} = b_{ij}(\theta_i - \theta_j), i \ \in N \ and \ j \ \in M(i)$$

$$\sum_{j \in M(i)} p_{ij} = p_i - D_i^p, i \in N$$

$$-S_{ij} \leq p_{ij} \leq S_{ij}$$

$$p_g \geq 0$$

**Decoupled**

$$min_{p,u,\theta} \sum_{g \in G} MC_g \cdot p_g$$

$$P_{min_g} \leq p_g \leq P_{max_g}, g \in G$$

$$Q_{min_g} \leq q_g \leq Q_{max_g}, g \in G$$

$$p_{ij} = b_{ij}(\theta_i - \theta_j), i \ \in N \ and \ j \ \in M(i)$$

$$q_{ij} = b_{ij}(u_i - u_j), i \ \in N \ and \ j \ \in M(i)$$

$$\sum_{j \in M(i)} p_{ij} = p_i - D_i^p, i \in N$$

$$\sum_{j \in M(i)} q_{ij} = q_i - D_i^q, i \in N$$

$$-S_{ij} \leq p_{ij} \leq S_{ij}$$

$$-S_{ij} \leq q_{ij} \leq S_{ij}$$

$$p_g \geq 0$$

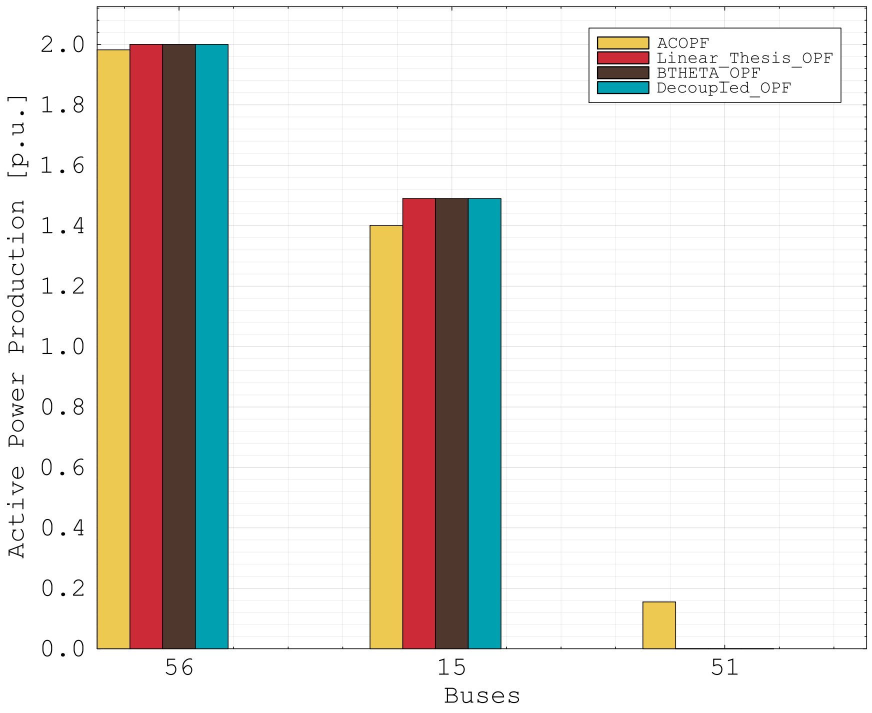
# Overview of the test system

The testbed used in this case study is a modified version of the IEEE 123 Test Feeder. It is the same system employed in the work by Saverio Bolognani and Sandro Zampieri, titled *“On the existence and linear approximation of the power flow solution in power distribution networks.”*

Brief system description

- 56 nodes in total.
- 1000 serves as the slack bus.
- 15 and 51 are the two additional generators of the system.
- Lines are represented as a resistance in series with inductive reactance.
- Generators Marginal Cost:
  - Generator 1000 = $29\ € \,/\, MWh$
  - Generator 15 = $30\ € \,/\, MWh$
  - Generator 51 = $30\ € \,/\, MWh$

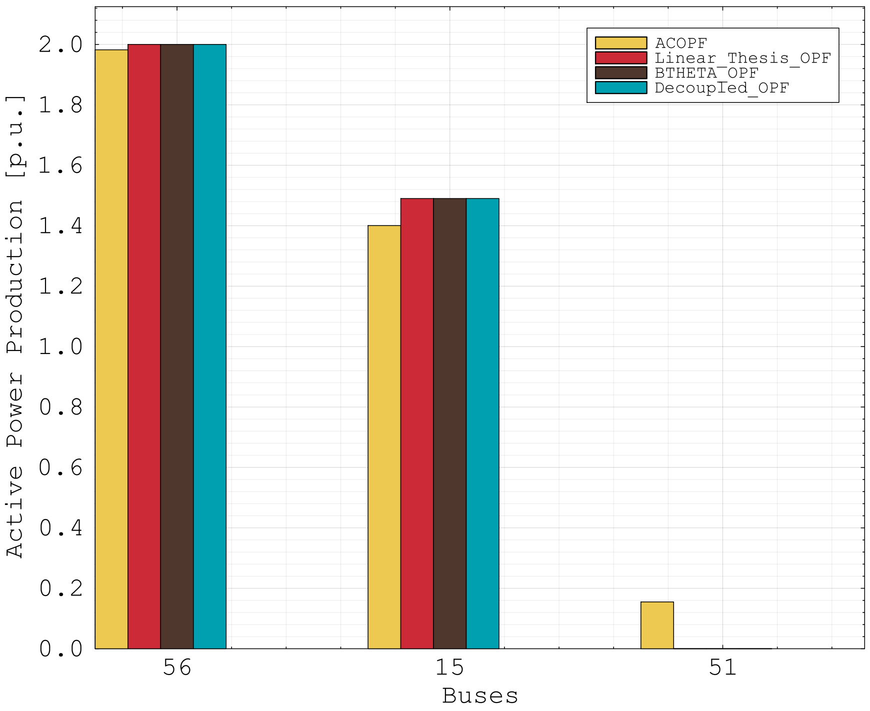
## Active power production




**Observations**

- Linear models decide less total active power because they ignore losses.
- All linear models have the exact same results.

## Active power production

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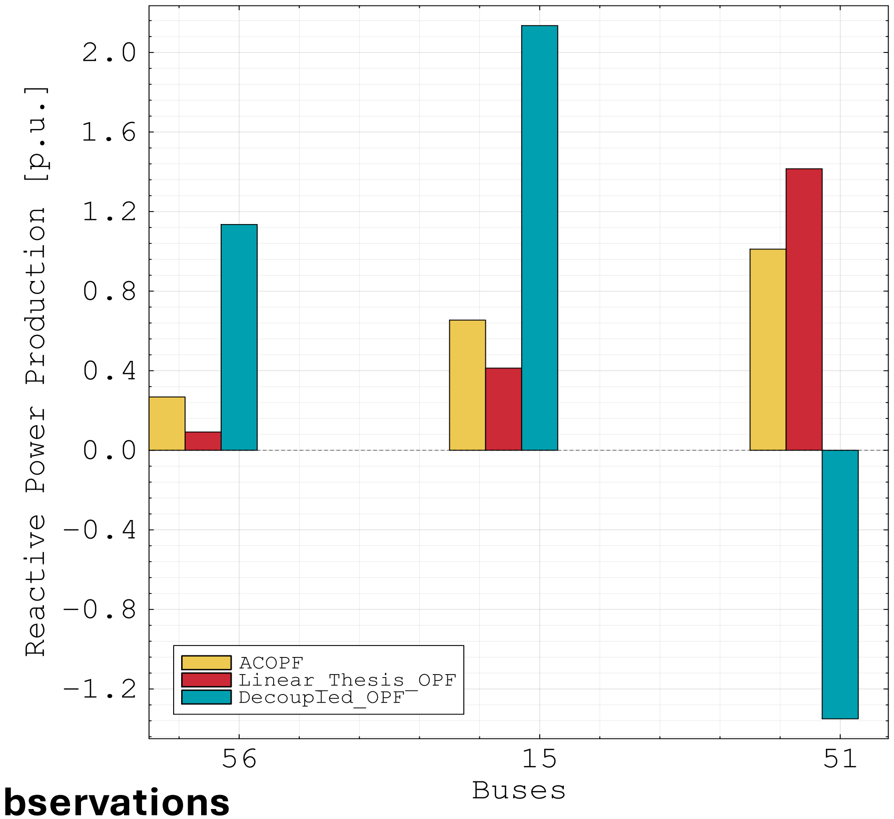
## Reactive power production



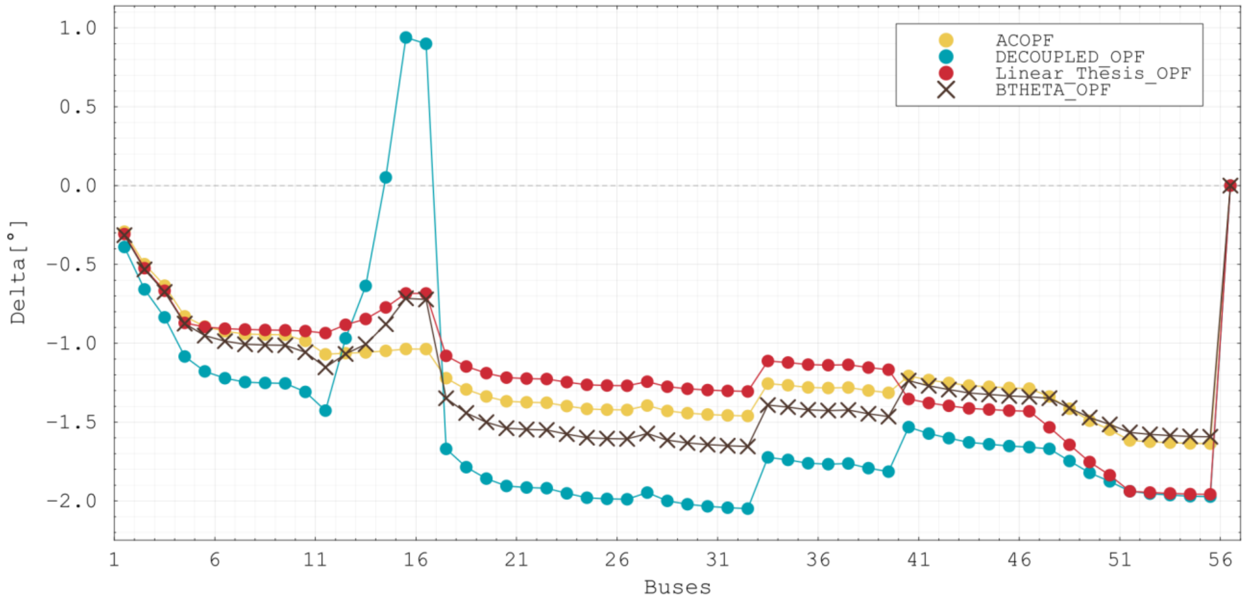

### Observations

- Bθ model ignores reactive power
- The Thesis OPF seems to have better approximation, for this case, than the Decoupled model.
- The thesis model shows a smoother and more balanced reactive power profile, avoiding the large swings in generation and absorption seen in the Decoupled model.

**Observations for voltage angles**

- The Bθ and Decoupled models produce **identical voltage angle results**.

- All three models appear to **follow the voltage angle pattern** of the ACOPF solution. However, the **Decoupled and Bθ models provide a more accurate approximation** of the voltage angles-both in terms of the overall pattern and the maximum and average error metrics.



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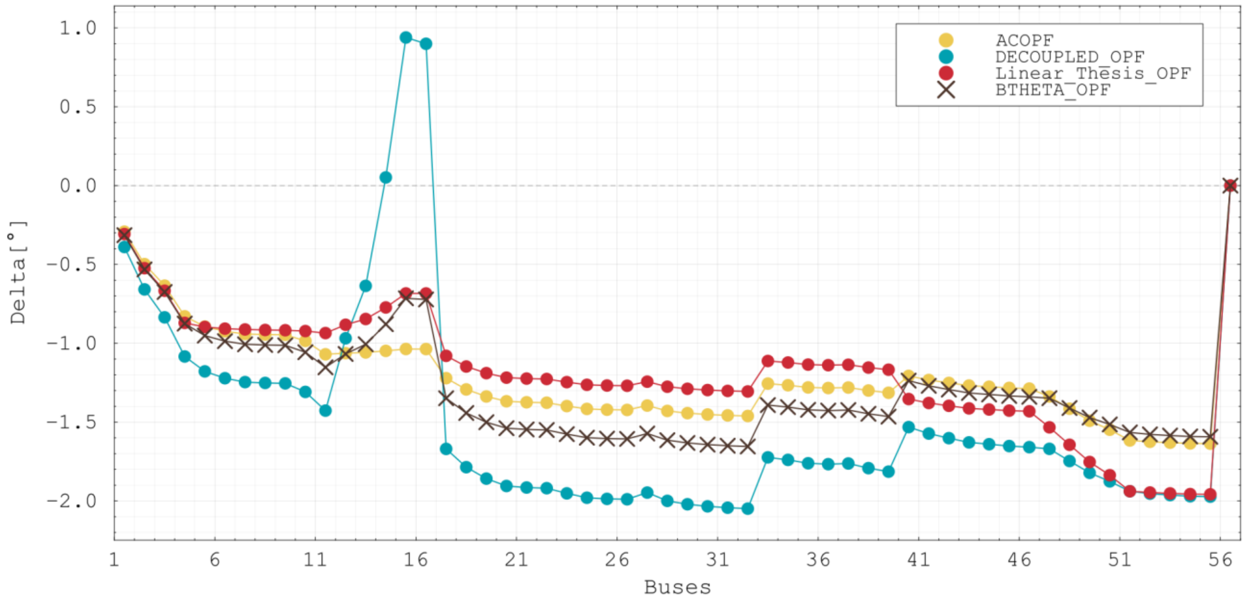


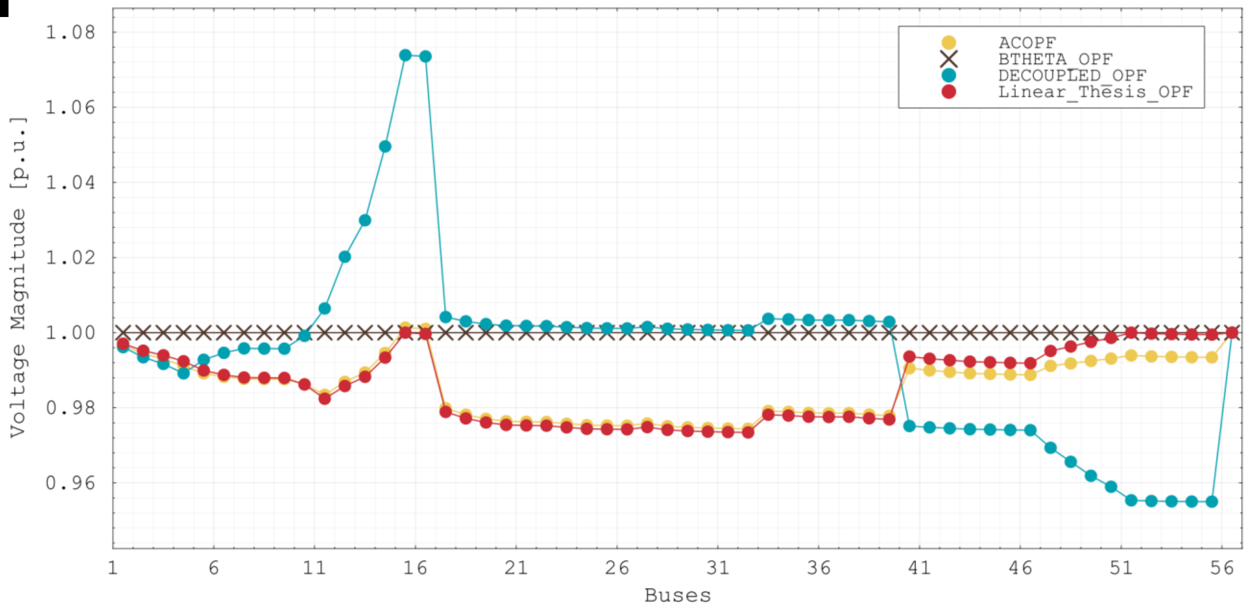


## Observations for voltage magnitudes

- Bθ model assumes that the voltage magnitudes are equal to 1 for all nodes.
- Both the **Decoupled and Thesis models replicate the general pattern of ACOPF voltage magnitudes.** However, in the region of the chart for buses 46 to 51, the Decoupled model diverges from the ACOPF trend, whereas the Thesis model maintains consistency across all nodes.
- Thesis OPF has **smaller maximum and average error** than the other linear models.

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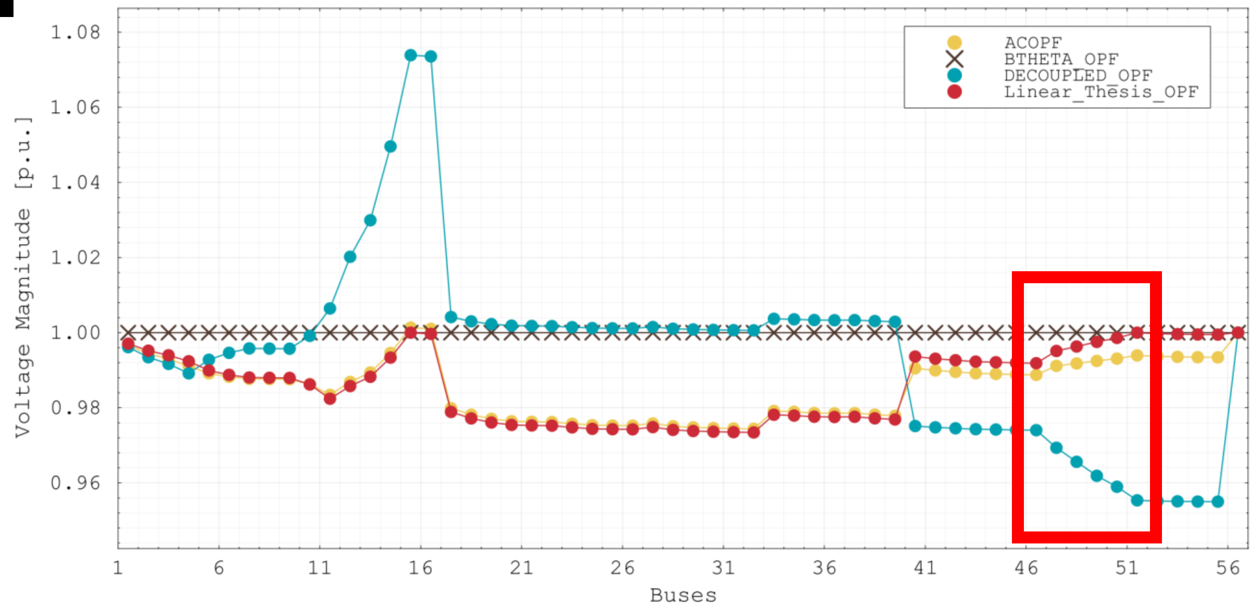


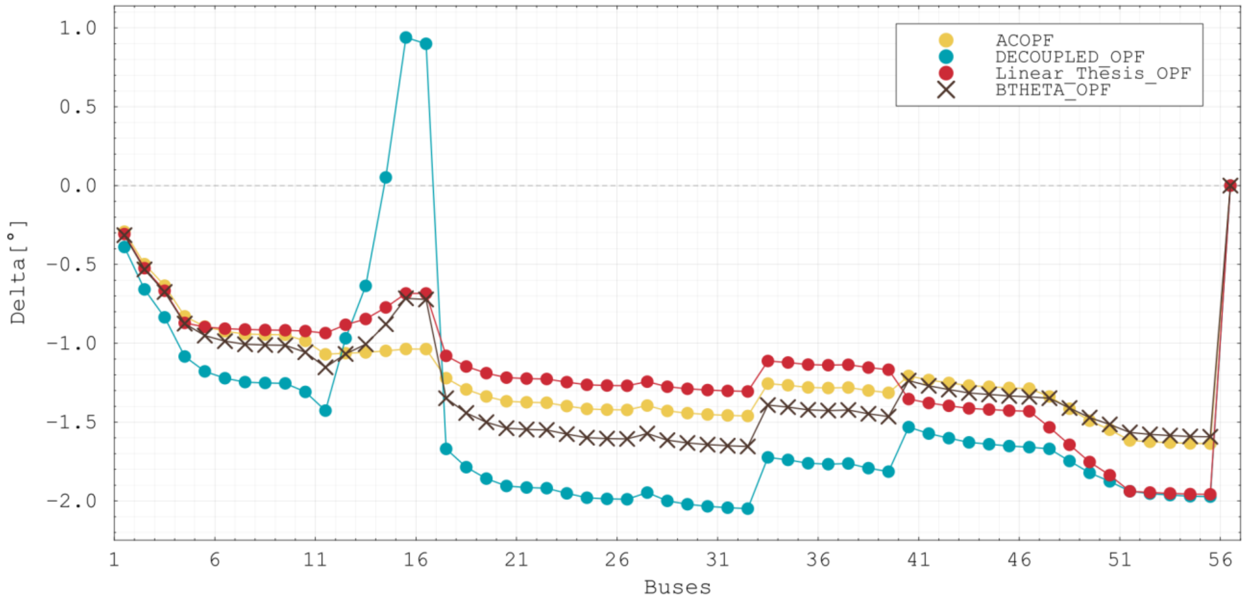


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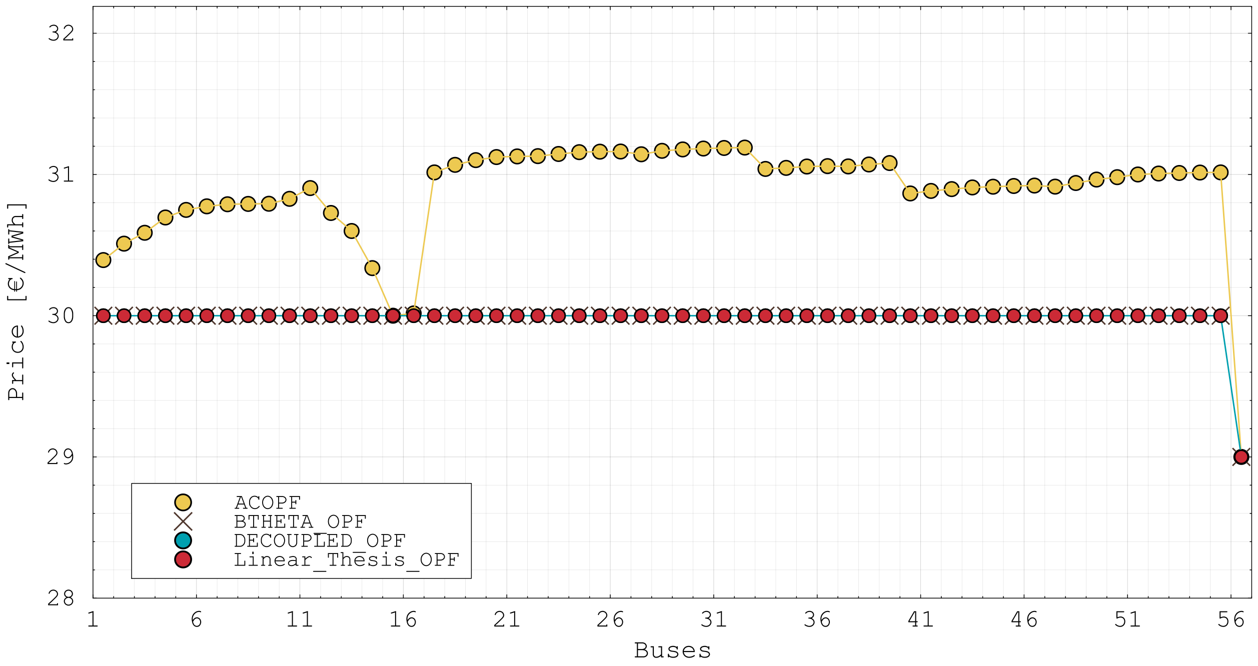
# Nodal Prices




## Observations

- **Linear models produce identical nodal prices,** as they make the same decisions for active power generation.
- Only node 56 has lower price. At node 56 the generator with the lowest marginal cost operates until the thermal limit of the transmission line is reached.
- Between neighboring nodes without congestion, nodal prices:
  - Are equal for the linear models.
  - Differ for the AC model as it accounts for line losses.

# Conclusions

- I propose a linear model for the Optimal Power Flow (OPF) problem.
- The conclusions from more case studies were:
  - All models share the same disadvantages in terms of **active power** production.
  - The thesis model demonstrates smoother behaviour for **reactive power** production, compared to the Decoupled model. However, it cannot be definitively stated that one model offers better approximations than the other.
  - All three models follow the ACOPF patterns for voltage magnitudes and angles. The thesis model shows slightly better accuracy for magnitudes, while the Decoupled and Bθ models perform better for angles in terms of maximum and average errors.
  - Linear models exhibit different behaviour for nodal prices, compared to ACOPF, with the root cause being the disregard of line losses.

# Thank You!

Email: alexandrakismanousos@gmail.com
Source code: https://github.com/ManousosAlexandrakis/Linear_Approximation_OPF.git
Web App code: https://github.com/ManousosAlexandrakis/OPF_app.git

# Importance of OPF problem

- **Purpose of Solving the Optimal Power Flow (OPF) Problem**
  - **Efficient and reliable energy dispatch**

    The OPF problem determines the most cost-effective generation schedule that meets system demand **while satisfying operational constraints**.
  - **Minimization of operating costs**

    Ensures that electricity is delivered to consumers at the **lowest possible cost**, considering generator costs, transmission limits and system reliability.
  - **Secure system operation**

    Maintains **voltage levels and power flows** within safe limits, promoting grid stability and resilience.
- Real – World Applications Examples
  - **Real-time market clearing**
  - **Day-ahead market clearing**
  - **Expansion planning**

# History of OPF

**1930-1950**

Energy system problems solved **by hand** or **analog circuits**.

**1950-1960**

**Digital** solution for Power Flow Problem.

**1962**

**Carpentier** formalizes OPF as a **constrained nonlinear optimization** problem.

**Mid 1960s**

Iterative **Newton-Raphson** method becomes popular and efficient handling of **sparse matrices** significantly improves performance.

**1991**

**Huneault & Galliana publish** extensive survey of optimal power flow literature up to 1991. They conclude that challenges with **convergence** and **speed** still remain.

**Late 1990s**

Development of **Security-Constrained OPF (SCOPF)** to include **contingencies** and system risks. Increased modeling realism but also **complexity**.

**2000 - Today**

No **universally adopted, fast, and reliable** OPF model yet. **Modern solutions** use **approximations** — practical, but not fully optimal.

# Linear power flow equations

At the core of the power flow problem lie the power flow equations. The linearized formulation used in this thesis is based on the work of Saverio Bolognani and Sandro Zampieri. These equations provide a simplified yet effective description of the system, capturing the **linear relationships between voltages and active, reactive power injections.**

- Voltage magnitude:

$$u_J = \mathbf{1}V_0 + \frac{1}{V_0} Re\left(Z\bar{s}_J\right) \quad (1)$$

- Voltage angle:

$$\theta_J = \mathbf{1}\Theta_0 + \frac{1}{V_0^2} \mathrm{Im}\left(Z\bar{s}_J\right) \quad (2)$$

- Voltage magnitudes at generator nodes:

$$u_K = \mathbf{1} \quad (3)$$

**Notation**
J: Set of all nodes except slack node
$V_0$: Voltage magnitude of slack node
$\Theta_0$: Voltage angle of slack node
$Z$ : Inverse of admittance matrix Y
$\bar{s}_J$: Vector of complex power injection

# Linear power flow equations

Understanding how the previous equations are incorporated into the model formulation is crucial. This requires a clear grasp of how the impendance matrix $Z$ and power injections vector $\bar{s}_J$ are defined and constructed.

- For $Z$ matrix:

$$Z = \begin{bmatrix} Z_{KK} & Z_{KL} \\ Z_{LK} & Z_{LL} \end{bmatrix}$$

- For $\bar{s}_J$ :

$$\bar{s}_J = \begin{bmatrix} s_K \\ s_L \end{bmatrix}$$

In both cases, a remapping is necessary to position the elements of the matrix and vector correctly. In the new indexing, **generator nodes** are assigned the **initial positions**, followed by the non-generator nodes.

**Notation**

- $K$: Set of generator nodes, excluding the slack node
- $L$: Set of non-generator nodes

# Linear power flow equations

The equations for voltage magnitude and angle, using the previously defined $Z$ matrix and $\bar{s}_J$ vector, are expressed in the form of equations (4) and (5).

- **Voltage magnitude**

$$u_j = V_0 + \frac{1}{V_0} \cdot \sum_{i \in K} (Re[Z_{j,i}] \cdot rp_i + Im[Z_{j,i}] \cdot rq_i) + \frac{1}{V_0} \cdot \sum_{i \in L} (Re[Z_{j,i}] \cdot rp_i + Im[Z_{j,i}] \cdot rq_i), \quad j \in J \quad (4)$$

- **Voltage angle**

$$\theta_j = \Theta_0 + \frac{1}{V_0^2} \cdot \sum_{i \in K} (-Re[Z_{j,i}] \cdot rq_i + Im[Z_{j,i}] \cdot rp_i) + \frac{1}{V_0^2} \cdot \sum_{i \in L} (-Re[Z_{j,i}] \cdot rq_i + Im[Z_{j,i}] \cdot rp_i), \quad j \in J \quad (5)$$

- **Reactive power production**

  - The **issue** that arises is that the equations (4) and (5), for a given value of vector $rp_K$, determine the values of three variables $u, \theta$ *and* $rq$.
  - As a result, for a given value of vector $rp_K$, the system admits **multiple solutions for the vectors of $\boldsymbol{u}, \boldsymbol{\theta}$ and $\boldsymbol{rq}$.**

**Notation**

- $\boldsymbol{rp_j}$: Active power injection at node j
- $\boldsymbol{rq_j}$: Reactive power injection at node j

# Linear power flow equations

By inverting the equation (4) and solving for $rq_K$,

we obtain an expression for the reactive power injection.

$$\boxed{\begin{aligned} rq_K = &-(Im[Z_{KK}])^{-1}(V_0(V_0\mathbf{1} - u_K) \\ &+Re[Z_{KK}]rp_K + Re[Z_{KL}]rp_L + Im[Z_{KL}]rq_L \quad (6) \end{aligned}}$$

This equation cannot be incorporated into the model, as it is identical to equation (4).

- By setting the vector $u_K$ to a specific value, we obtain:
  - A unique solution for the vector $rq_K$, as expressed in equation (6), for a given vector $rp_K$.

- As a result voltage magnitude and voltage angle equations have only two unknown variables.

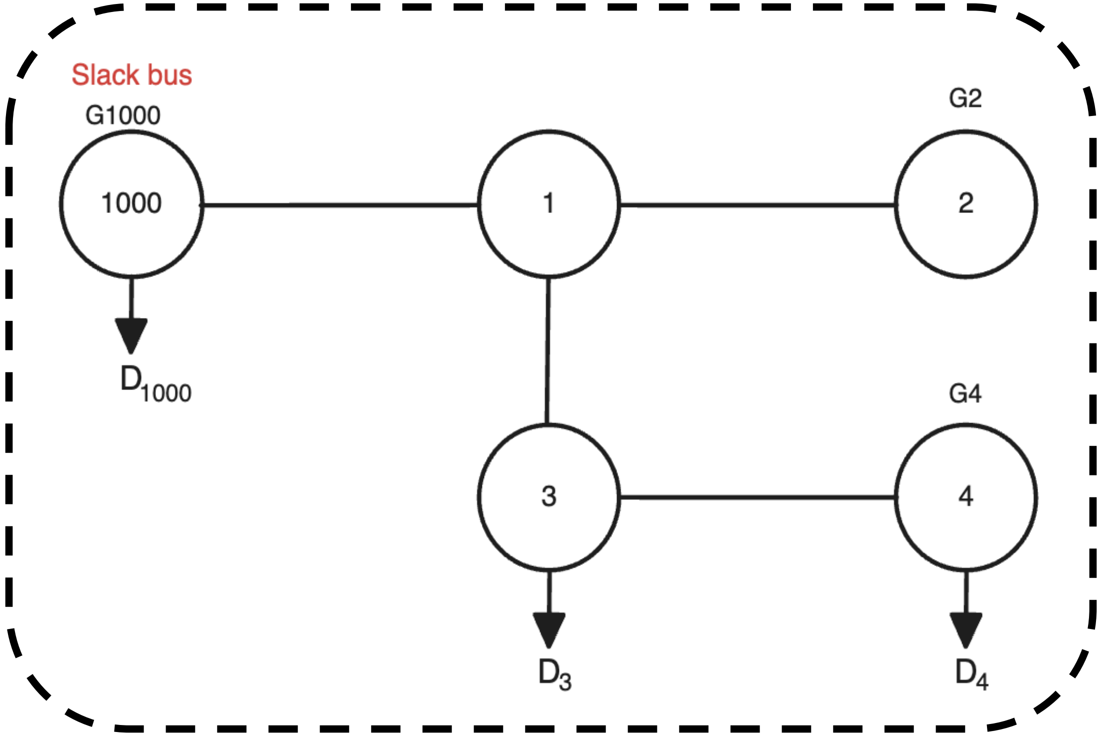
- ***Conclusion***: Fixing the voltage magnitude at generator nodes leads to unique solution vectors for $u_J, \theta_J$ *and* $rq_K$

and that is the reason why equation (3) must be incorporated into the model.

# Linear power flow equations

Example for the formulation of the $Z$ matrix and the $\bar{s}_J$ vector

## Example case




From the system topology: K = [2,4] and L = [1,3]

**$Z$ submatrices**

$$\begin{array}{cc} Z_{KK} & Z_{KL} \end{array}$$

$$\left[\begin{array}{cc|cc} Z_{22} & Z_{24} & Z_{21} & Z_{23} \\ Z_{42} & Z_{44} & Z_{41} & Z_{43} \\ \hline Z_{12} & Z_{14} & Z_{11} & Z_{13} \\ Z_{32} & Z_{34} & Z_{31} & Z_{33} \end{array}\right]$$

$$\begin{array}{cc} Z_{LK} & Z_{LL} \end{array}$$

**$\bar{s}_J$ subvectors**

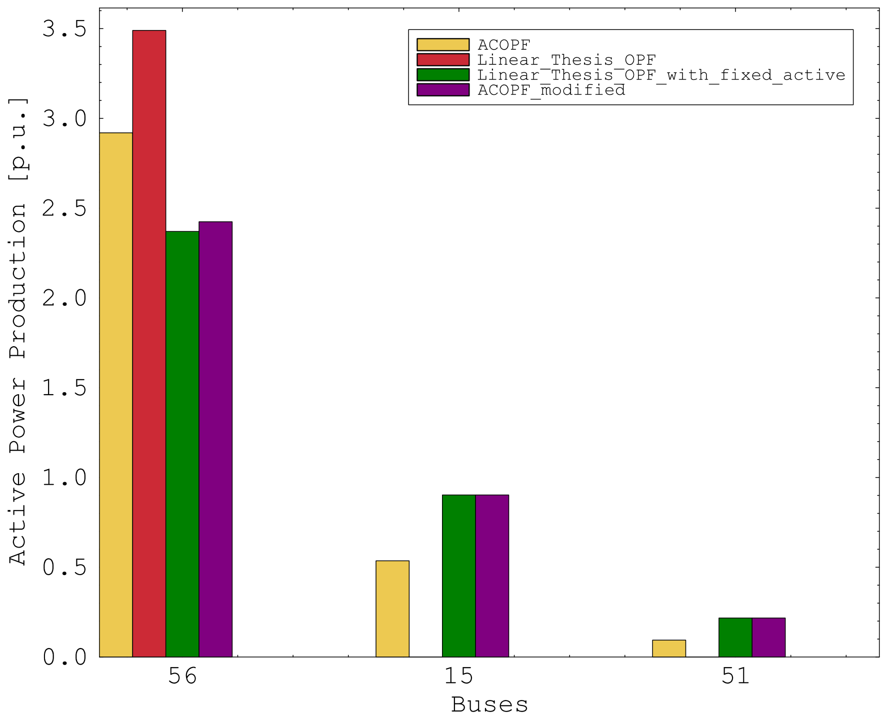
$$s_K$$

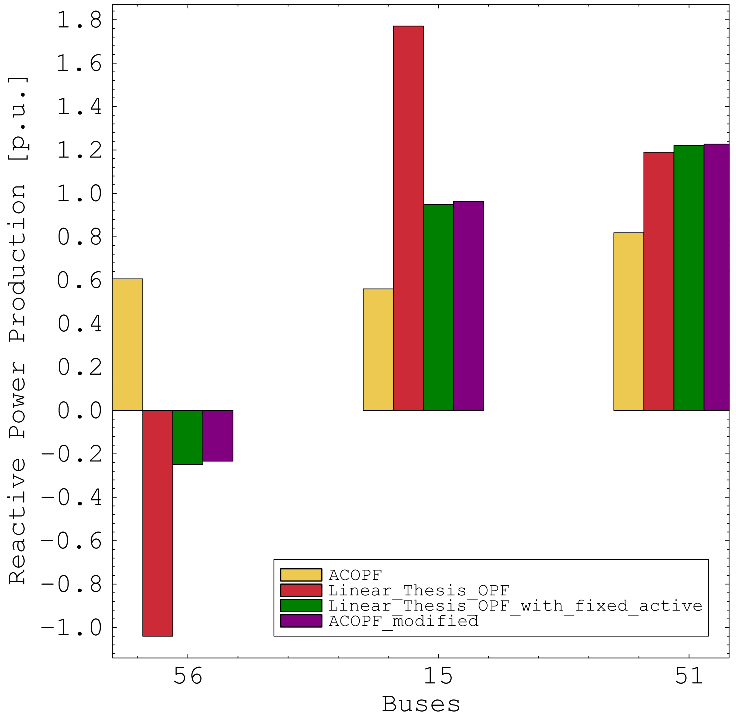
$$\left[\begin{array}{c} s_2 \\ s_4 \\ \hline s_1 \\ s_3 \end{array}\right]$$

$$s_L$$

# Scenario

Running the Thesis model with active power injections fixed to the values obtained from the ACOPF modified solution.




- The linear model, after the fixing of the active power,

has a very accurate approximation for the *ACOPF modified* results.

- **Note:** If the active power of ACOPF is fed into the linear model

we will not observe the same behaviour.

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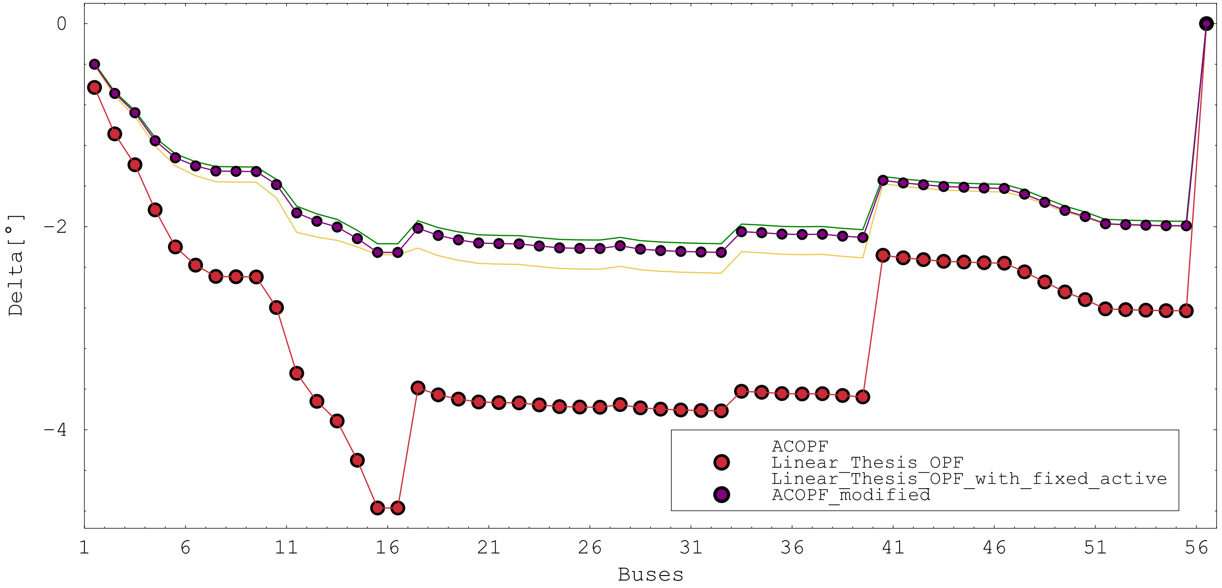


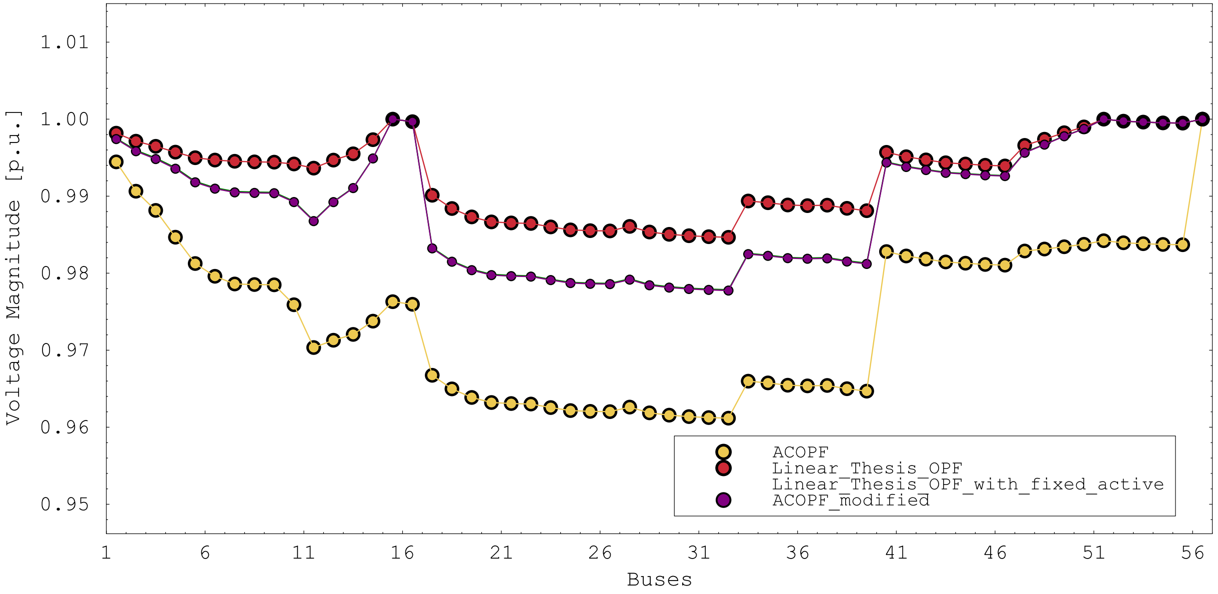


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