

# Computational Complexity Examples

KRZYSZTOF BARTOSZEK

732A94 Advanced R Programming

Department of Computer and Information Science, Linköping University

10, 11 X 2024 (FE249, I205)

## Exercise 1

For the sequence  $f(n) = (n^2 + 1)(2n^4 + 3n - 8)$  find the smallest  $k$  such that  $f(n) = O(n^k)$

*Solution* We first recall the definition that  $f(n) \in O(g(n))$  if

- $\exists_C |f(n)| \leq C|g(n)|$  for  $n$  large enough
- or equivalently  $\limsup_{n \rightarrow \infty} |f(n)|/|g(n)| < \infty$ .

Here we have  $f(n) = 2n^6 + 2n^4 + 3n^3 - 8n^2 + 3n - 8$  and let us take  $k = 6$ . Then,

$$n^{-6}f(n) = 2 + 2n^{-2} + 3n^{-3} - 8n^{-4} + 3n^{-5} - 8n^{-6} \rightarrow 2 < \infty.$$

## Exercise 2

For what “simple”  $g(n)$  does  $f(n) = (4n \log_2 n + 1)^2 \in O(g(n))$ ?

*Solution* We have  $f(n) = 16n^2 \log_2^2 n + 8n \log_2 n + 1$ . Take  $g(n) = n^2 \log_2^2 n$  and then

$$\frac{f(n)}{n^2 \log_2^2 n} = 16 + 8n^{-1} \log_2^{-1} n + n^{-2} \log_2^{-2} n \rightarrow 16 < \infty.$$

## Exercise 3

True or False:  $\log_2 n^{73} = O(\log_2 n)$

*Solution* Notice that  $\log_2 n^{73} = 73 \log_2 n$  and hence **TRUE** as

$$\frac{73 \log_2 n}{\log_2 n} = 73 \rightarrow 73 < \infty.$$

#### Exercise 4

Show that  $A^n \in O(n!)$  for  $A > 0$

*Solution* Take any (i.e. all)  $n > A + 1$ . Then,

$$n! = n(n-1) \dots \cdot 1 > n(n-1) \dots (A+1) > A \cdot \dots \cdot A = A^{n-\lceil A \rceil} = A^n A^{-\lceil A \rceil}$$

and

$$\frac{A^n}{n!} < \frac{A^n}{A^n A^{-\lceil A \rceil}} = A^{\lceil A \rceil} \rightarrow A^{\lceil A \rceil} < \infty.$$

#### Exercise 5

Show that  $A^n < n!$  for  $n$  large enough.

*Solution* Take any (i.e. all)  $n > 2A + 1$ . Then,

$$n! = n(n-1) \dots \cdot 1 > n(n-1) \dots (2A+1) > (2A) \cdot \dots \cdot (2A) = (2A)^{n-\lceil 2A \rceil} = A^n 2^n (2A)^{-\lceil 2A \rceil}.$$

Now take  $n > 2A + 1$  large enough so that  $2^n (2A)^{-\lceil 2A \rceil} > 1$ , i.e.  $2^n > (2A)^{\lceil 2A \rceil}$ . Such an  $n$  is found as

$$n > \log_2(2A)^{\lceil 2A \rceil} = \lceil 2A \rceil (\log_2 2 + \log_2 A) = \lceil 2A \rceil (1 + \log_2 A).$$

Taking any  $n > \max(2A + 1, \lceil 2A \rceil (1 + \log_2 A))$  we get  $A^n < n!$ .

#### Exercise 6

Let  $n \in \mathbb{N}$  and define  $s_n = \sum_{k=1}^n k^{-2}$ . Show that  $s_n \in O(1)$ .

*Solution* It is known that  $s_n \rightarrow \pi^2/6$  so this immediately gives us the answer. But the problem can also be attacked directly. Notice that for  $k \geq 2$

$$\frac{1}{k^2} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}.$$

Now we use the *telescoping sum trick* (a series whose partial sums will only have a finite amount of terms after cancellation [https://en.wikipedia.org/wiki/Telescoping\\_series](https://en.wikipedia.org/wiki/Telescoping_series))

$$\begin{aligned} s_n &= \sum_{k=1}^n \frac{1}{k^2} < 1 + \sum_{k=2}^n \left( \frac{1}{k-1} - \frac{1}{k} \right) = 1 + \sum_{k=2}^n \frac{1}{k-1} - \sum_{k=2}^n \frac{1}{k} = 1 + \sum_{k=1}^{n-1} \frac{1}{k} - \sum_{k=2}^n \frac{1}{k} \\ &= 1 + 1 + \sum_{k=2}^{n-1} \frac{1}{k} - \sum_{k=2}^{n-1} \frac{1}{k} - \frac{1}{n} = 2 - \frac{1}{n}, \end{aligned}$$

and then  $s_n/1 < (2 - n^{-1})/1 = 2 - n^{-1} \rightarrow 2 < \infty$ .

### Exercise 7

Take  $t_n = \sum_{k=1}^n k$ . Show that  $t_n \in O(n^2)$ .

*Solution* We recall

$$t_n = \sum_{k=1}^n k = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$$

and then  $n^{-2}t_n = \frac{1}{2} + \frac{1}{2}n^{-1} \rightarrow \frac{1}{2} < \infty$ .