Computational Complexity Examples

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Exercise 1
For the sequence $f(n) = (n^2 + 1)(2n^4 + 3n - 8)$ find the smallest k such that $f(n) = O(n^k)$

Solution We first recall the definition that $f(n) \in O(g(n))$ if

- $\exists_C |f(n)| \le C|g(n)|$ for n large enough
- or equivalently $\limsup_{n\to\infty} |f(n)|/|g(n)| < \infty$.

Here we have $f(n) = 2n^6 + 2n^4 + 3n^3 - 8n^2 + 3n - 8$ and let us take $k = 6$. Then,

$$n^{-6} f(n) = 2 + 2n^{-2} + 3n^{-3} - 8n^{-4} + 3n^{-5} - 8n^{-6} \to 2 < \infty.$$

Exercise 2
For what "simple" $g(n)$ does $f(n) = (4n \log_2 n + 1)^2 \in O(g(n))$?

Solution We have $f(n) = 16n^2 \log_2^2 n + 8n \log_2 n + 1$. Take $g(n) = n^2 \log_2^2 n$ and then

$$\frac{f(n)}{n^2 \log_2^2 n} = 16 + 8n^{-1} \log_2^{-1} n + n^{-2} \log_2^{-2} n \to 16 < \infty.$$

Exercise 3
True or False: $\log_2 n^{73} = O(\log_2 n)$

Solution Notice that $\log_2 n^{73} = 73 \log_2 n$ and hence **TRUE** as

$$\frac{73 \log_2 n}{\log_2 n} = 73 \to 73 < \infty.$$

Exercise 4

Show that $A^n \in O(n!)$ for $A > 0$
Solution Take any (i.e. all) $n > A + 1$. Then,

$$n! = n(n-1) \ldots \cdot 1 > n(n-1) \ldots (A+1) > A \cdot \ldots \cdot A = A^{n-\lceil A \rceil} = A^n A^{-\lceil A \rceil}$$

and

$$\frac{A^n}{n!} < \frac{A^n}{A^n A^{-\lceil A \rceil}} = A^{\lceil A \rceil} \to A^{\lceil A \rceil} < \infty.$$

Exercise 5
Show that $A^n < n!$ for n large enough.

Solution Take any (i.e. all) $n > 2A + 1$. Then,

$$n! = n(n-1) \ldots \cdot 1 > n(n-1) \ldots (2A+1) > (2A) \cdot \ldots \cdot (2A) = (2A)^{n-\lceil 2A \rceil} = A^n 2^n (2A)^{-\lceil 2A \rceil}.$$

Now take $n > 2A + 1$ large enough so that $2^n (2A)^{-\lceil 2A \rceil} > 1$, i.e. $2^n > (2A)^{\lceil 2A \rceil}$. Such an n is found as

$$n > \log_2 (2A)^{\lceil 2A \rceil} = \lceil 2A \rceil (\log_2 2 + \log_2 A) = \lceil 2A \rceil (1 + \log_2 A).$$

Taking any $n > \max(2A + 1, \lceil 2A \rceil (1 + \log_2 A))$ we get $A^n < n!$.

Exercise 6

Let $n \in \mathbb{N}$ and define $s_n = \sum_{k=1}^n k^{-2}$. Show that $s_n \in O(1)$.

Solution It is known that $s_n \to \pi^2/6$ so this immediately gives us the answer. But the problem can also be attacked directly. Notice that for $k \geq 2$

$$\frac{1}{k^2} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}.$$

Now we use the *telescoping sum trick* (a series whose partial sums will only have a finite amount of terms after cancellation `https://en.wikipedia.org/wiki/Telescoping_series`)

$$\begin{aligned} s_n &= \sum_{k=1}^n \tfrac{1}{k^2} < 1 + \sum_{k=2}^n (\tfrac{1}{k-1} - \tfrac{1}{k}) = 1 + \sum_{k=2}^n \tfrac{1}{k-1} - \sum_{k=2}^n \tfrac{1}{k} = 1 + \sum_{k=1}^{n-1} \tfrac{1}{k} - \sum_{k=2}^n \tfrac{1}{k} \\ &= 1 + 1 + \sum_{k=2}^{n-1} \tfrac{1}{k} - \sum_{k=2}^{n-1} \tfrac{1}{k} - \tfrac{1}{n} = 2 - \tfrac{1}{n}, \end{aligned}$$

and then $s_n/1 < (2 - n^{-1})/1 = 2 - n^{-1} \to 2 < \infty$.

Exercise 7

Take $t_n = \sum_{k=1}^{n} k$. Show that $t_n \in O(n^2)$.
Solution We recall

$$t_n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$$

and then $n^{-2} t_n = \frac{1}{2} + \frac{1}{2} n^{-1} \to \frac{1}{2} < \infty$.