



UPPSALA
UNIVERSITET

Bayesian Statistics and Data Analysis

Lecture 4

Måns Magnusson

Department of Statistics, Uppsala University
Thanks to Aki Vehtari, Aalto University

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling



UPPSALA
UNIVERSITET

Notation

- **Introduction**

- Computational aspects

- **Bayesian Computation**

- Numerical integration

- **Monte Carlo Methods**

- **Direct sampling**

- **Indirect sampling**

- Rejection sampling
- Importance sampling
- Pareto-Smoothed
Importance Sampling

- In this chapter, generic $p(\theta)$ is used instead of $p(\theta|y)$



- **Introduction**

- Computational aspects

- Bayesian Computation

- Numerical integration

- Monte Carlo Methods

- Direct sampling

- Indirect sampling

- Rejection sampling
- Importance sampling
- Pareto-Smoothed
Importance Sampling

- In this chapter, generic $p(\theta)$ is used instead of $p(\theta|y)$
- **Unnormalized** distribution is denoted by $q(\cdot)$
 - $\int q(\theta)d\theta \neq 1$, but finite (i.e. $\int q(\theta)d\theta \leq \infty$)
 - $q(\cdot) \propto p(\cdot)$



- **Introduction**

- Computational aspects

- **Bayesian Computation**

- Numerical integration

- **Monte Carlo Methods**

- **Direct sampling**

- **Indirect sampling**

- Rejection sampling
- Importance sampling
- Pareto-Smoothed
Importance Sampling

- In this chapter, generic $p(\theta)$ is used instead of $p(\theta|y)$
- **Unnormalized** distribution is denoted by $q(\cdot)$
 - $\int q(\theta)d\theta \neq 1$, but finite (i.e. $\int q(\theta)d\theta \leq \infty$)
 - $q(\cdot) \propto p(\cdot)$
- **Proposal** distribution is denoted by $g(\cdot)$



- Floating point presentation of numbers. e.g. with 64bits
 - closest value to zero is $\approx 2.2 \cdot 10^{-308}$
 - generate sample of 600 from normal distribution:
`qr=rnorm(600)`
 - calculate joint density given normal:
`prod(dnorm(qr))` $\rightarrow 0$ (underflow)

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling



- Floating point presentation of numbers. e.g. with 64bits
 - closest value to zero is $\approx 2.2 \cdot 10^{-308}$
 - generate sample of 600 from normal distribution:
`qr=rnorm(600)`
 - calculate joint density given normal:
`prod(dnorm(qr))` $\rightarrow 0$ (underflow)
 - see log densities in the next slide

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling



- Floating point presentation of numbers. e.g. with 64bits
 - closest value to zero is $\approx 2.2 \cdot 10^{-308}$
 - generate sample of 600 from normal distribution:
`qr=rnorm(600)`
 - calculate joint density given normal:
`prod(dnorm(qr))` $\rightarrow 0$ (underflow)
 - see log densities in the next slide
 - the smallest distinguishable difference from 1 is about
 $\approx 1 \pm 2.2 \cdot 10^{-16}$
 - Ratio of girl and boy babies
 - `pbeta(0.5, 241945, 251527)` $\rightarrow 1$ (rounding)

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling



- Floating point presentation of numbers. e.g. with 64bits
 - closest value to zero is $\approx 2.2 \cdot 10^{-308}$
 - generate sample of 600 from normal distribution:
`qr=rnorm(600)`
 - calculate joint density given normal:
`prod(dnorm(qr))` $\rightarrow 0$ (underflow)
 - see log densities in the next slide
 - the smallest distinguishable difference from 1 is about
 $\approx 1 \pm 2.2 \cdot 10^{-16}$
 - Ratio of girl and boy babies
 - `pbeta(0.5, 241945, 251527)` $\rightarrow 1$ (rounding)
 - `pbeta(0.5, 241945, 251527, lower.tail=FALSE)`
 $\approx 1.15 \cdot 10^{-42}$
there is more accuracy near 0

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling



- Floating point presentation of numbers. e.g. with 64bits
 - closest value to zero is $\approx 2.2 \cdot 10^{-308}$
 - generate sample of 600 from normal distribution:
`qr=rnorm(600)`
 - calculate joint density given normal:
`prod(dnorm(qr))` $\rightarrow 0$ (underflow)
 - see log densities in the next slide
 - the smallest distinguishable difference from 1 is about
 $\approx 1 \pm 2.2 \cdot 10^{-16}$
 - Ratio of girl and boy babies
 - `pbeta(0.5, 241945, 251527)` $\rightarrow 1$ (rounding)
 - `pbeta(0.5, 241945, 251527, lower.tail=FALSE)`
 $\approx 1.15 \cdot 10^{-42}$
there is more accuracy near 0
- DEMO in R!



- Log densities

- use log densities to avoid over- and underflows in floating point presentation
 - `prod(dnorm(qr))` $\rightarrow 0$ (underflow)
 - `sum(dnorm(qr,log=TRUE))` $\rightarrow -847.3$

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling



- Log densities
 - use log densities to avoid over- and underflows in floating point presentation
 - `prod(dnorm(qr))` $\rightarrow 0$ (underflow)
 - `sum(dnorm(qr,log=TRUE))` $\rightarrow -847.3$
 - How many observations we can now handle?
 $\approx 10^{308}$

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling



- Log densities
 - use log densities to avoid over- and underflows in floating point presentation
 - `prod(dnorm(qr))` $\rightarrow 0$ (underflow)
 - `sum(dnorm(qr, log=TRUE))` $\rightarrow -847.3$
 - How many observations we can now handle?
 $\approx 10^{308}$
 - Compute exp as late as possible

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling



- Log densities
 - use log densities to avoid over- and underflows in floating point presentation
 - `prod(dnorm(qr))` $\rightarrow 0$ (underflow)
 - `sum(dnorm(qr, log=TRUE))` $\rightarrow -847.3$
 - How many observations we can now handle?
 $\approx 10^{308}$
 - Compute exp as late as possible
 - e.g. for $a > b$, compute
 $\log(\exp(a) + \exp(b)) = a + \log(1 + \exp(b - a))$

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling



- Log densities

- use log densities to avoid over- and underflows in floating point presentation
 - `prod(dnorm(qr))` $\rightarrow 0$ (underflow)
 - `sum(dnorm(qr, log=TRUE))` $\rightarrow -847.3$
 - How many observations we can now handle?
 $\approx 10^{308}$
- Compute exp as late as possible
 - e.g. for $a > b$, compute
 $\log(\exp(a) + \exp(b)) = a + \log(1 + \exp(b - a))$
e.g. `log(exp(800) + exp(800))` $\rightarrow \text{Inf}$

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling



- Log densities

- use log densities to avoid over- and underflows in floating point presentation

- `prod(dnorm(qr))` $\rightarrow 0$ (underflow)
- `sum(dnorm(qr, log=TRUE))` $\rightarrow -847.3$
- How many observations we can now handle?
 $\approx 10^{308}$

- Compute exp as late as possible

- e.g. for $a > b$, compute
 $\log(\exp(a) + \exp(b)) = a + \log(1 + \exp(b - a))$
e.g. `log(exp(800) + exp(800))` $\rightarrow \text{Inf}$
but `800 + log(1 + exp(800 - 800))` ≈ 800.69

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling



- Log densities

- use log densities to avoid over- and underflows in floating point presentation

- `prod(dnorm(qr))` $\rightarrow 0$ (underflow)
- `sum(dnorm(qr, log=TRUE))` $\rightarrow -847.3$
- How many observations we can now handle?
 $\approx 10^{308}$

- Compute exp as late as possible

- e.g. for $a > b$, compute
 $\log(\exp(a) + \exp(b)) = a + \log(1 + \exp(b - a))$
e.g. `log(exp(800) + exp(800))` $\rightarrow \text{Inf}$
but `800 + log(1 + exp(800 - 800))` ≈ 800.69
- e.g. in Metropolis-algorithm (next week) compute the log of ratio of densities using the identity
 $\log(a/b) = \log(a) - \log(b)$



UPPSALA
UNIVERSITET

- Introduction
 - Computational aspects
- **Bayesian Computation**
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Section 2

Bayesian Computation



It's all about expectations

$$E_{p(\theta|y)}[f(\theta)] = \int f(\theta) p(\theta|y) d\theta,$$

$$\text{where } p(\theta|y) = \frac{p(y|\theta)p(\theta)}{\int p(y|\theta)p(\theta)d\theta}$$

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

It's all about expectations

$$E_{p(\theta|y)}[f(\theta)] = \int f(\theta)p(\theta|y)d\theta,$$

$$\text{where } p(\theta|y) = \frac{p(y|\theta)p(\theta)}{\int p(y|\theta)p(\theta)d\theta}$$

We can easily evaluate $p(y|\theta)p(\theta)$ for any θ , but the integral $\int p(y|\theta)p(\theta)d\theta$ is usually difficult.



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

It's all about expectations

$$E_{p(\theta|y)}[f(\theta)] = \int f(\theta)p(\theta|y)d\theta,$$

$$\text{where } p(\theta|y) = \frac{p(y|\theta)p(\theta)}{\int p(y|\theta)p(\theta)d\theta}$$

We can easily evaluate $p(y|\theta)p(\theta)$ for any θ , but the integral $\int p(y|\theta)p(\theta)d\theta$ is usually difficult.

We can use the unnormalized posterior $q(\theta|y) = p(y|\theta)p(\theta)$, for example, in



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

It's all about expectations

$$E_{p(\theta|y)}[f(\theta)] = \int f(\theta)p(\theta|y)d\theta,$$

$$\text{where } p(\theta|y) = \frac{p(y|\theta)p(\theta)}{\int p(y|\theta)p(\theta)d\theta}$$

We can easily evaluate $p(y|\theta)p(\theta)$ for any θ , but the integral $\int p(y|\theta)p(\theta)d\theta$ is usually difficult.

We can use the unnormalized posterior $q(\theta|y) = p(y|\theta)p(\theta)$, for example, in

- Grid (equal spacing) evaluation with self-normalization

$$E_{p(\theta|y)}[f(\theta)] \approx \frac{\sum_{s=1}^S [f(\theta^{(s)})q(\theta^{(s)}|y)]}{\sum_{s=1}^S q(\theta^{(s)}|y)}$$



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

It's all about expectations

$$E_{p(\theta|y)}[f(\theta)] = \int f(\theta)p(\theta|y)d\theta,$$

$$\text{where } p(\theta|y) = \frac{p(y|\theta)p(\theta)}{\int p(y|\theta)p(\theta)d\theta}$$

We can easily evaluate $p(y|\theta)p(\theta)$ for any θ , but the integral $\int p(y|\theta)p(\theta)d\theta$ is usually difficult.

We can use the unnormalized posterior $q(\theta|y) = p(y|\theta)p(\theta)$, for example, in

- Grid (equal spacing) evaluation with self-normalization

$$E_{p(\theta|y)}[f(\theta)] \approx \frac{\sum_{s=1}^S [f(\theta^{(s)})q(\theta^{(s)}|y)]}{\sum_{s=1}^S q(\theta^{(s)}|y)}$$

- Monte Carlo methods which can sample from $p(\theta^{(s)}|y)$ using only $q(\theta^{(s)}|y)$

$$E_{p(\theta|y)}[f(\theta)] \approx \frac{1}{S} \sum_{s=1}^S f(\theta^{(s)})$$



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

$$E_{p(\theta|y)}[f(\theta)] = \int f(\theta)p(\theta|y)d\theta$$

- Multiple approaches to compute $E_{p(\theta|y)}[f(\theta)]$
 - Conjugate priors and analytic solutions (Ch 1-5)



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

$$E_{p(\theta|y)}[f(\theta)] = \int f(\theta)p(\theta|y)d\theta$$

- Multiple approaches to compute $E_{p(\theta|y)}[f(\theta)]$
 - Conjugate priors and analytic solutions (Ch 1-5)
 - Numerical (deterministic) integration (grid and quadrature, Ch 3, 10)



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

$$E_{p(\theta|y)}[f(\theta)] = \int f(\theta)p(\theta|y)d\theta$$

- Multiple approaches to compute $E_{p(\theta|y)}[f(\theta)]$
 - Conjugate priors and analytic solutions (Ch 1-5)
 - Numerical (deterministic) integration (grid and quadrature, Ch 3, 10)
 - Independent Monte Carlo, rejection and importance sampling (Ch 10)



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

$$E_{p(\theta|y)}[f(\theta)] = \int f(\theta)p(\theta|y)d\theta$$

- Multiple approaches to compute $E_{p(\theta|y)}[f(\theta)]$
 - Conjugate priors and analytic solutions (Ch 1-5)
 - Numerical (deterministic) integration (grid and quadrature, Ch 3, 10)
 - Independent Monte Carlo, rejection and importance sampling (Ch 10)
 - Markov Chain Monte Carlo (Ch 11-12)



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

$$E_{p(\theta|y)}[f(\theta)] = \int f(\theta)p(\theta|y)d\theta$$

- Multiple approaches to compute $E_{p(\theta|y)}[f(\theta)]$
 - Conjugate priors and analytic solutions (Ch 1-5)
 - Numerical (deterministic) integration (grid and quadrature, Ch 3, 10)
 - Independent Monte Carlo, rejection and importance sampling (Ch 10)
 - Markov Chain Monte Carlo (Ch 11-12)
 - (Distributional approximations, Ch 4, 13)



UPPSALA
UNIVERSITET

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Subsection 1

Numerical integration



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Riemann recap

Remember the Riemann sum (where x_s^* :

$$I_a^b = \sum_{s=1}^S f(x_s^*) \Delta x_s,$$

where

$$\Delta x = \frac{b-a}{S} \text{ and } x_s^* \in [x_{i-s}, x_s]$$



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Riemann recap

Remember the Riemann sum (where x_s^* :

$$I_a^b = \sum_{s=1}^S f(x_s^*) \Delta x_s,$$

where

$$\Delta x = \frac{b-a}{S} \text{ and } x_s^* \in [x_{i-s}, x_s]$$

And the Riemann integral:

$$\int_a^b f(x) dx = \lim_{\|\Delta x\| \rightarrow 0} \sum_{s=1}^S f(x_i^*) \Delta x_i,$$



UPPSALA
UNIVERSITET

Numerical integration

- Introduction
 - Computational aspects
 - Bayesian Computation
 - Numerical integration
 - Monte Carlo Methods
 - Direct sampling
 - Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling
- Now lets use the Riemann sum to approximate the Riemann integral!

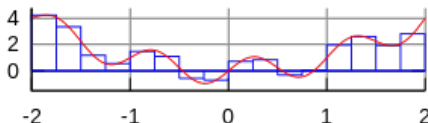


- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Numerical integration

- The simplest quadrature integration is grid integration
 - Evaluate function in a grid and compute

$$E[-\alpha/\beta] \approx \sum_{t=1}^T w_{\text{cell}}^{(t)} \frac{\alpha^{(t)}}{\beta^{(t)}},$$



where $w_{\text{cell}}^{(t)}$ is the normalized probability of a grid cell t , and $\alpha^{(t)}$ and $\beta^{(t)}$ are center locations of grid cells

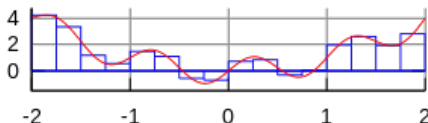


- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Numerical integration

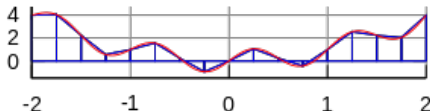
- The simplest quadrature integration is grid integration
 - Evaluate function in a grid and compute

$$E[-\alpha/\beta] \approx \sum_{t=1}^T w_{\text{cell}}^{(t)} \frac{\alpha^{(t)}}{\beta^{(t)}},$$



where $w_{\text{cell}}^{(t)}$ is the normalized probability of a grid cell t , and $\alpha^{(t)}$ and $\beta^{(t)}$ are center locations of grid cells

- In 1D further variations with smaller error, e.g. trapezoid



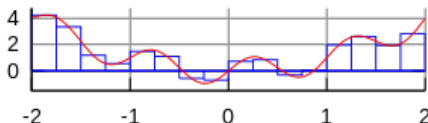


- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Numerical integration

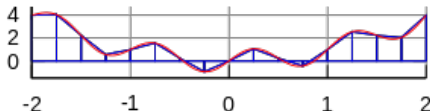
- The simplest quadrature integration is grid integration
 - Evaluate function in a grid and compute

$$E[-\alpha/\beta] \approx \sum_{t=1}^T w_{\text{cell}}^{(t)} \frac{\alpha^{(t)}}{\beta^{(t)}},$$



where $w_{\text{cell}}^{(t)}$ is the normalized probability of a grid cell t , and $\alpha^{(t)}$ and $\beta^{(t)}$ are center locations of grid cells

- In 1D further variations with smaller error, e.g. trapezoid



- In 2D and higher
 - nested quadrature, product rules
 - but theres a curse of dimensionality...



UPPSALA
UNIVERSITET

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Section 3

Monte Carlo Methods



- In Gelman et al (2013) notation and for a posteriors $p(\theta|y)$

$$E_{p(\theta|y)}(h(\theta)) = \int h(\theta)p(\theta|y)d\theta \approx \sum_s^S h(\theta_s)p(\theta_s|y)\frac{w_s}{S}$$

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Monte Carlo integration/method

- In Gelman et al (2013) notation and for a posteriors $p(\theta|y)$

$$E_{p(\theta|y)}(h(\theta)) = \int h(\theta)p(\theta|y)d\theta \approx \sum_s^S h(\theta_s)p(\theta_s|y)\frac{w_s}{S}$$

- If we have samples $\theta_s \sim p(\theta|y)$ we can approximate

$$E_{p(\theta|y)}(h(\theta)) = \int h(\theta)p(\theta|y)d\theta \approx \frac{1}{S} \sum_s^S h(\theta_s)$$



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Monte Carlo integration/method

- In Gelman et al (2013) notation and for a posteriors $p(\theta|y)$

$$E_{p(\theta|y)}(h(\theta)) = \int h(\theta)p(\theta|y)d\theta \approx \sum_s^S h(\theta_s)p(\theta_s|y) \frac{w_s}{S}$$

- If we have samples $\theta_s \sim p(\theta|y)$ we can approximate

$$E_{p(\theta|y)}(h(\theta)) = \int h(\theta)p(\theta|y)d\theta \approx \frac{1}{S} \sum_s^S h(\theta_s)$$

$$\begin{aligned} E_{p(\theta|y)} \left(\frac{1}{S} \sum_s^S h(\theta_s) \right) &= \frac{1}{S} \sum_s^S E_{p(\theta|y)} (h(\theta_s)) \\ &= \int h(\theta)p(\theta|y)d\theta \end{aligned}$$



- Introduction
 - Computational aspects
 - Bayesian Computation
 - Numerical integration
 - Monte Carlo Methods
 - Direct sampling
 - Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling
- Used already before computers
 - Buffon (18th century; needles)
 - De Forest, Darwin, Galton (19th century)
 - Pearson (19th century; roulette)
 - Gosset (Student, 1908; hat)



- Introduction
 - Computational aspects
 - Bayesian Computation
 - Numerical integration
 - Monte Carlo Methods
 - Direct sampling
 - Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling
- Used already before computers
 - Buffon (18th century; needles)
 - De Forest, Darwin, Galton (19th century)
 - Pearson (19th century; roulette)
 - Gosset (Student, 1908; hat)
 - "Monte Carlo method" term was proposed by Metropolis, von Neumann or Ulam in the end of 1940s
 - they worked together in atomic bomb project
 - Metropolis and Ulam, "The Monte Carlo Method", 1949



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

- Used already before computers
 - Buffon (18th century; needles)
 - De Forest, Darwin, Galton (19th century)
 - Pearson (19th century; roulette)
 - Gosset (Student, 1908; hat)
- "Monte Carlo method" term was proposed by Metropolis, von Neumann or Ulam in the end of 1940s
 - they worked together in atomic bomb project
 - Metropolis and Ulam, "The Monte Carlo Method", 1949
- Bayesians started to have enough cheap computation time in 1990s
 - BUGS project started 1989 (last OpenBUGS release 2014)
 - Gelfand & Smith, 1990
 - Stan initial release 2012



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling
- Simulate draws from the target distribution $p(\theta|y)$
 - these draws can be treated as any observations
 - a collection of draws is a sample of size S
- Use these draws, for example,
 - to compute means, deviations, quantiles
 - to draw histograms
 - to marginalize
 - etc.



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Monte Carlo vs. Deterministic Methods

- Monte Carlo (approximation) error is $\propto S^{-1/2}$
- Midpoint rule error is $\propto S^{-2}$
- Trapezoidal rule error is $\propto S^{-2}$
- Simpson rule error is $\propto S^{-4}$
- Monte Carlo is bad (even worse than midpoint approximation) **Why use Monte Carlo integration?**



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Monte Carlo vs. Deterministic Methods

- Monte Carlo (approximation) error is $\propto S^{-1/2}$
- Midpoint rule error is $\propto S^{-2}$
- Trapezoidal rule error is $\propto S^{-2}$
- Simpson rule error is $\propto S^{-4}$
- Monte Carlo is bad (even worse than midpoint approximation)
- Monte Carlo has the same error irrespective of dimension D , i.e. $S_D = S$
- Numerical methods create a grid with $S_D = S^D$ When is Monte Carlo a better approach than Simpsons?



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Monte Carlo vs. Deterministic Methods

- Monte Carlo (approximation) error is $\propto S^{-1/2}$
- Midpoint rule error is $\propto S^{-2}$
- Trapezoidal rule error is $\propto S^{-2}$
- Simpson rule error is $\propto S^{-4}$
- Monte Carlo is bad (even worse than midpoint approximation)
- Monte Carlo has the same error irrespective of dimension D , i.e. $S_D = S$
- Numerical methods create a grid with $S_D = S^D$

$$(S_D^{\frac{1}{D}})^{-4} = S_D^{-\frac{1}{2}},$$

i.e. for $d > 8$ Monte Carlo is better.



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Grid sampling and curse of dimensionality

- 10 parameters
- if we don't know beforehand where the posterior mass is
 - need to choose wide box for the grid
 - need to have enough grid points to get some of them where essential mass is

Can we do this?

- e.g. 50 or 1000 grid points per dimension
 - $50^{10} \approx 1e17$ grid points
 - $1000^{10} \approx 1e30$ grid points



Grid sampling and curse of dimensionality

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

- 10 parameters
- if we don't know beforehand where the posterior mass is
 - need to choose wide box for the grid
 - need to have enough grid points to get some of them where essential mass is
- e.g. 50 or 1000 grid points per dimension
 - $50^{10} \approx 1e17$ grid points
 - $1000^{10} \approx 1e30$ grid points
- R and my current laptop can compute density of normal distribution about 20 million times per second
 - evaluation in $1e17$ grid points would take 150 years
 - evaluation in $1e30$ grid points would take 1 500 billion years



How many simulation draws are needed?

- Introduction
 - Computational aspects
 - Bayesian Computation
 - Numerical integration
 - Monte Carlo Methods
 - Direct sampling
 - Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling
- How many draws or how big sample size S ?
 - If draws are independent
 - usual methods to estimate the uncertainty due to a finite number of observations (finite sample size)
 - Markov chain Monte Carlo produces dependent draws (next week)
 - requires additional work to estimate the *effective sample size*



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

How many simulation draws are needed?

- Expectation of unknown quantity

$$E(\theta) \approx \frac{1}{S} \sum_{s=1}^S \theta^{(s)}$$

if S is big and $\theta^{(s)}$ are independent, way may assume that the distribution of the expectation approaches normal distribution (see Ch 4) with variance σ_{θ}^2/S (asymptotic normality)

- this variance is independent on dimensionality of θ (!)



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

How many simulation draws are needed?

- Expectation of unknown quantity

$$E(\theta) \approx \frac{1}{S} \sum_{s=1}^S \theta^{(s)}$$

if S is big and $\theta^{(s)}$ are independent, way may assume that the distribution of the expectation approaches normal distribution (see Ch 4) with variance σ_{θ}^2/S (asymptotic normality)

- this variance is independent on dimensionality of θ (!)
- total variance is sum of the epistemic uncertainty in the posterior and the uncertainty due to using finite number of Monte Carlo draws

$$\sigma_{\theta}^2 + \sigma_{\theta}^2/S$$



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

How many simulation draws are needed?

- Expectation of unknown quantity

$$E(\theta) \approx \frac{1}{S} \sum_{s=1}^S \theta^{(s)}$$

if S is big and $\theta^{(s)}$ are independent, way may assume that the distribution of the expectation approaches normal distribution (see Ch 4) with variance σ_{θ}^2/S (asymptotic normality)

- this variance is independent on dimensionality of θ (!)
- total variance is sum of the epistemic uncertainty in the posterior and the uncertainty due to using finite number of Monte Carlo draws

$$\sigma_{\theta}^2 + \sigma_{\theta}^2/S = \sigma_{\theta}^2(1 + 1/S)$$



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

How many simulation draws are needed?

- Expectation of unknown quantity

$$E(\theta) \approx \frac{1}{S} \sum_{s=1}^S \theta^{(s)}$$

if S is big and $\theta^{(s)}$ are independent, way may assume that the distribution of the expectation approaches normal distribution (see Ch 4) with variance σ_{θ}^2/S (asymptotic normality)

- this variance is independent on dimensionality of θ (!)
- total variance is sum of the epistemic uncertainty in the posterior and the uncertainty due to using finite number of Monte Carlo draws

$$\sigma_{\theta}^2 + \sigma_{\theta}^2/S = \sigma_{\theta}^2(1 + 1/S)$$

- e.g. if $S = 100$, deviation increases by $\sqrt{1 + 1/S} = 1.005$
i.e. Monte Carlo error is very small (for the expectation)



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

How many simulation draws are needed?

- Expectation of unknown quantity

$$E(\theta) \approx \frac{1}{S} \sum_{s=1}^S \theta^{(s)}$$

if S is big and $\theta^{(s)}$ are independent, way may assume that the distribution of the expectation approaches normal distribution (see Ch 4) with variance σ_{θ}^2/S (asymptotic normality)

- this variance is independent on dimensionality of θ (!)
- total variance is sum of the epistemic uncertainty in the posterior and the uncertainty due to using finite number of Monte Carlo draws

$$\sigma_{\theta}^2 + \sigma_{\theta}^2/S = \sigma_{\theta}^2(1 + 1/S)$$

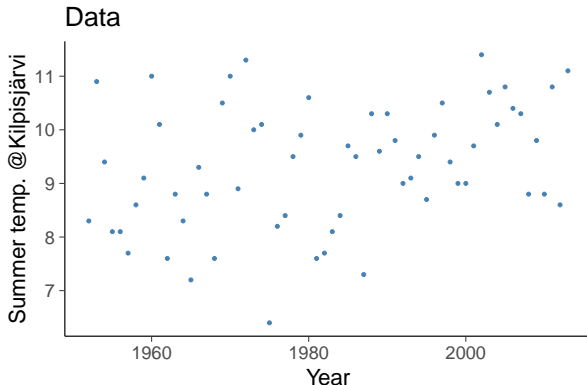
- e.g. if $S = 100$, deviation increases by $\sqrt{1 + 1/S} = 1.005$ i.e. Monte Carlo error is very small (for the expectation)
- See Ch 4 for counter-examples for asymptotic normality



Example: Kilpisjärvi summer temperature

Average temperature in June, July, and August at Kilpisjärvi, Finland

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

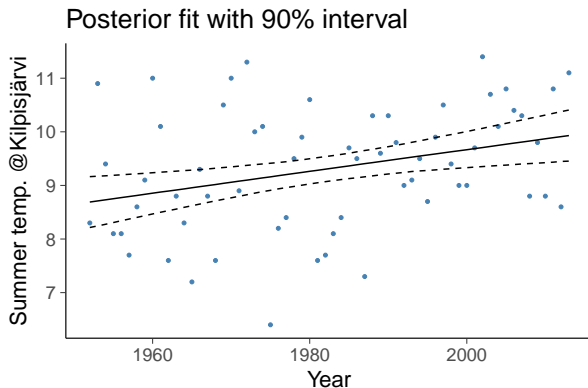




- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Example: Kilpisjärvi summer temperature

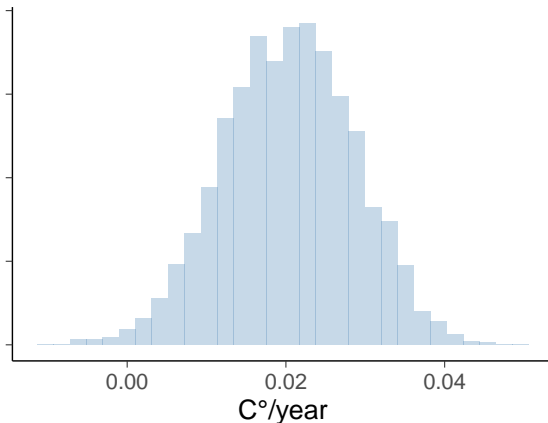
Average temperature in June, July, and August at Kilpisjärvi, Finland





Example: Kilpisjärvi summer temperature

Posterior of temperature change

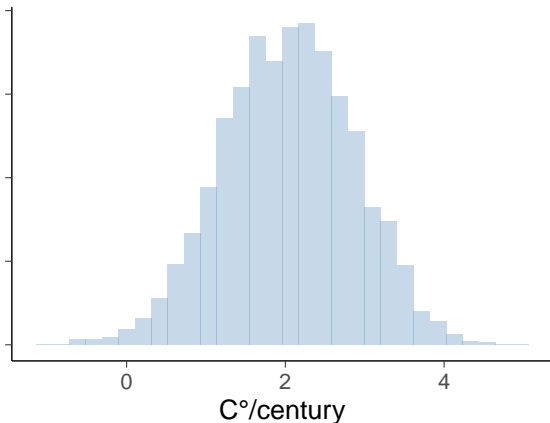


- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling



Example: Kilpisjärvi summer temperature

Posterior of temperature change



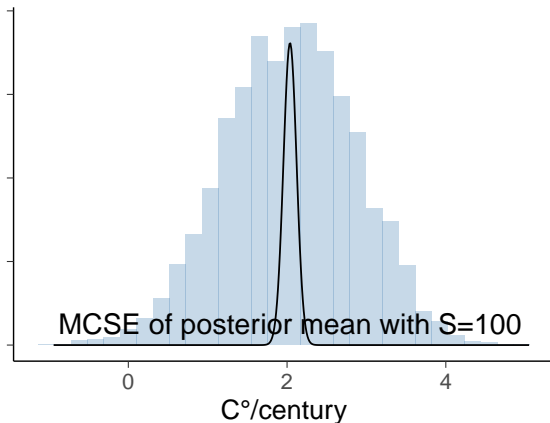
- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Example: Kilpisjärvi summer temperature

Posterior of temperature change



$\sigma_{\theta} \approx 0.827$, $\text{MCSE} \approx 0.0827$, total deviation ≈ 0.831

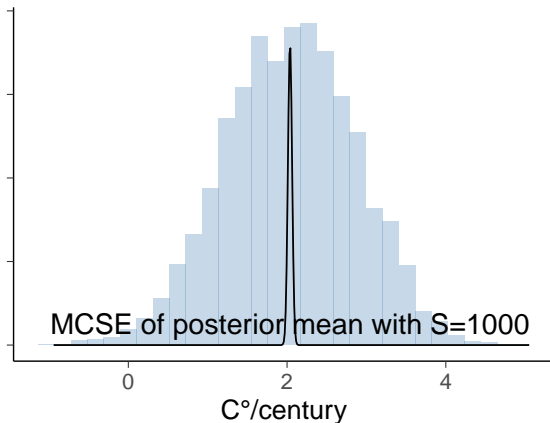
$$\text{total deviation}^2 = \sigma_{\theta}^2 + \text{MCSE}^2$$



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Example: Kilpisjärvi summer temperature

Posterior of temperature change



$\sigma_{\theta} \approx 0.827$, $\text{MCSE} \approx 0.0261$, total deviation ≈ 0.827

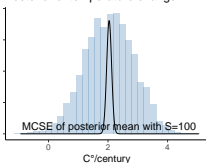
$$\text{total deviation}^2 = \sigma_{\theta}^2 + \text{MCSE}^2$$



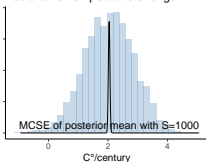
Example: Kilpisjärvi summer temperature

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Posterior of temperature change



Posterior of temperature change





Example: Kilpisjärvi summer temperature

- Introduction

- Computational aspects

- Bayesian Computation

- Numerical integration

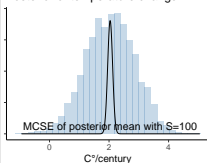
- Monte Carlo Methods

- Direct sampling

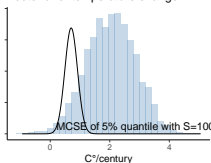
- Indirect sampling

- Rejection sampling
- Importance sampling
- Pareto-Smoothed Importance Sampling

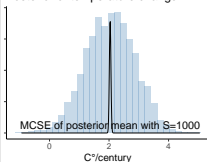
Posterior of temperature change



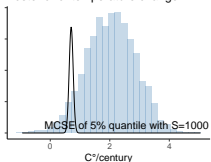
Posterior of temperature change



Posterior of temperature change



Posterior of temperature change





Example: Kilpisjärvi summer temperature

- Introduction

- Computational aspects

- Bayesian Computation

- Numerical integration

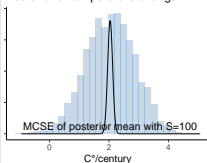
- Monte Carlo Methods

- Direct sampling

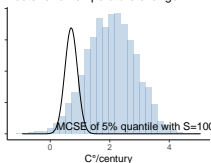
- Indirect sampling

- Rejection sampling
- Importance sampling
- Pareto-Smoothed Importance Sampling

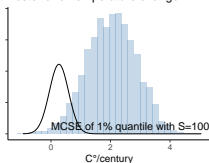
Posterior of temperature change



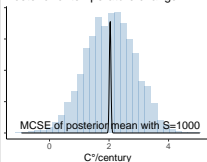
Posterior of temperature change



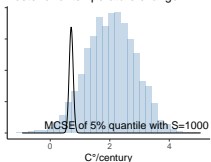
Posterior of temperature change



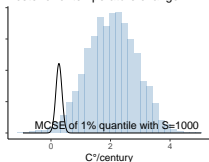
Posterior of temperature change



Posterior of temperature change



Posterior of temperature change





Example: Kilpisjärvi summer temperature

- Introduction

- Computational aspects

- Bayesian Computation

- Numerical integration

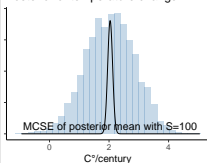
- Monte Carlo Methods

- Direct sampling

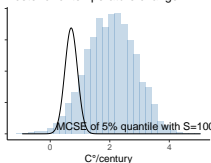
- Indirect sampling

- Rejection sampling
- Importance sampling
- Pareto-Smoothed Importance Sampling

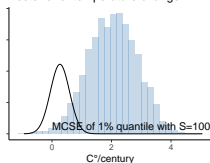
Posterior of temperature change



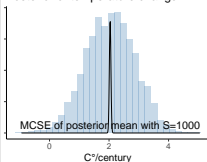
Posterior of temperature change



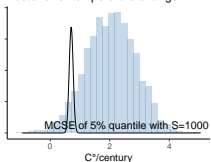
Posterior of temperature change



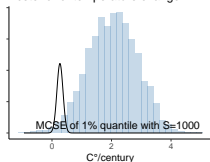
Posterior of temperature change



Posterior of temperature change



Posterior of temperature change



Tail quantiles are more difficult to estimate



How many simulation draws are needed?

- Posterior probability

$$p(\theta \in A) \approx \frac{1}{S} \sum_l I(\theta^{(s)} \in A)$$

where $I(\theta^{(s)} \in A) = 1$ if $\theta^{(s)} \in A$

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

How many simulation draws are needed?

- Posterior probability

$$p(\theta \in A) \approx \frac{1}{S} \sum_I I(\theta^{(s)} \in A)$$

where $I(\theta^{(s)} \in A) = 1$ if $\theta^{(s)} \in A$

- $I(\cdot)$ is binomially distributed as $p(\theta \in A)$
 - $\text{var}(I(\cdot)) = p(1 - p)$ (Appendix A, p. 579)
 - standard deviation of p is $\approx \sqrt{p(1 - p)/S}$



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

How many simulation draws are needed?

- Posterior probability

$$p(\theta \in A) \approx \frac{1}{S} \sum_l I(\theta^{(s)} \in A)$$

where $I(\theta^{(s)} \in A) = 1$ if $\theta^{(s)} \in A$

- $I(\cdot)$ is binomially distributed as $p(\theta \in A)$
 - $\text{var}(I(\cdot)) = p(1 - p)$ (Appendix A, p. 579)
 - standard deviation of p is $\approx \sqrt{p(1 - p)/S}$
- if $S = 100$ and $p \approx 0.5$, $\sqrt{p(1 - p)/S} = 0.05$
i.e. accuracy is about 5% units



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

How many simulation draws are needed?

- Posterior probability

$$p(\theta \in A) \approx \frac{1}{S} \sum_l I(\theta^{(s)} \in A)$$

where $I(\theta^{(s)} \in A) = 1$ if $\theta^{(s)} \in A$

- $I(\cdot)$ is binomially distributed as $p(\theta \in A)$
 - $\text{var}(I(\cdot)) = p(1-p)$ (Appendix A, p. 579)
 - standard deviation of p is $\approx \sqrt{p(1-p)/S}$
- if $S = 100$ and $p \approx 0.5$, $\sqrt{p(1-p)/S} = 0.05$
i.e. accuracy is about 5% units
- $S = 2500$ draws needed for 1% unit accuracy



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

How many simulation draws are needed?

- Posterior probability

$$p(\theta \in A) \approx \frac{1}{S} \sum_l I(\theta^{(s)} \in A)$$

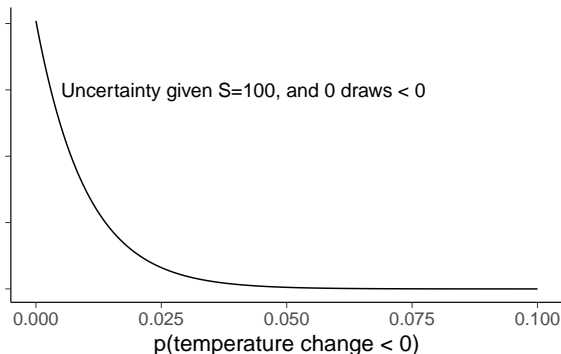
where $I(\theta^{(s)} \in A) = 1$ if $\theta^{(s)} \in A$

- $I(\cdot)$ is binomially distributed as $p(\theta \in A)$
 - $\text{var}(I(\cdot)) = p(1-p)$ (Appendix A, p. 579)
 - standard deviation of p is $\approx \sqrt{p(1-p)/S}$
- if $S = 100$ and $p \approx 0.5$, $\sqrt{p(1-p)/S} = 0.05$
i.e. accuracy is about 5% units
- $S = 2500$ draws needed for 1% unit accuracy
- To estimate small probabilities, a large number of draws is needed
 - to be able to estimate p , need to get draws with $\theta^{(l)} \in A$, which in expectation requires $S \gg 1/p$



Example: Kilpisjärvi summer temperature

Posterior uncertainty $p(\text{temperature change} < 0)$

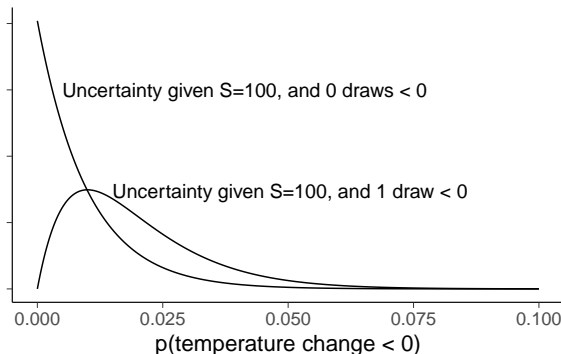


- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling



Example: Kilpisjärvi summer temperature

Posterior uncertainty $p(\text{temperature change} < 0)$

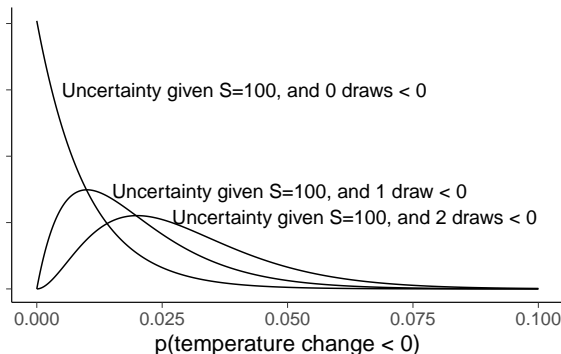


- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling



Example: Kilpisjärvi summer temperature

Posterior uncertainty $p(\text{temperature change} < 0)$

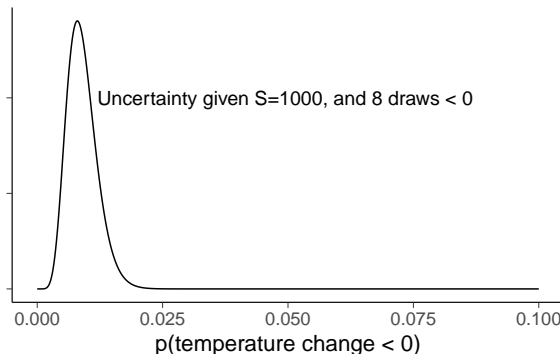


- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling



Example: Kilpisjärvi summer temperature

Posterior uncertainty $p(\text{temperature change} < 0)$

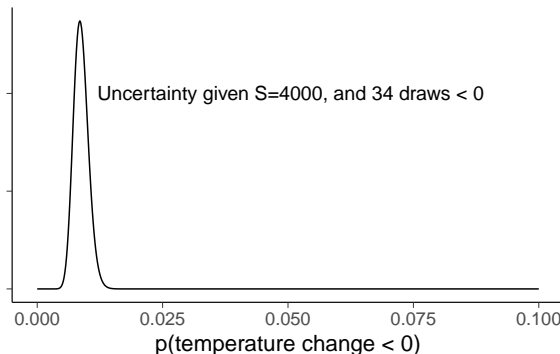


- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling



Example: Kilpisjärvi summer temperature

Posterior uncertainty $p(\text{temperature change} < 0)$



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling



How many digits to show in reports?

- Too many digits make reading of the results slower and give false impression of the accuracy

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling



How many digits to show in reports?

- Too many digits make reading of the results slower and give false impression of the accuracy
- Don't show digits which are just random noise
 - check what is the Monte Carlo standard error

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling



How many digits to show in reports?

- Too many digits make reading of the results slower and give false impression of the accuracy
- Don't show digits which are just random noise
 - check what is the Monte Carlo standard error
- Show meaningful digits given the posterior uncertainty

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

How many digits to show in reports?

- Too many digits make reading of the results slower and give false impression of the accuracy
- Don't show digits which are just random noise
 - check what is the Monte Carlo standard error
- Show meaningful digits given the posterior uncertainty
- Example: The mean and 90% central posterior interval for temperature increase C° /century based on posterior draws



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

How many digits to show in reports?

- Too many digits make reading of the results slower and give false impression of the accuracy
- Don't show digits which are just random noise
 - check what is the Monte Carlo standard error
- Show meaningful digits given the posterior uncertainty
- Example: The mean and 90% central posterior interval for temperature increase $^{\circ}\text{C}/\text{century}$ based on posterior draws
 - 2.050774 and [0.7472868 3.3017524] (NO!)



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

How many digits to show in reports?

- Too many digits make reading of the results slower and give false impression of the accuracy
- Don't show digits which are just random noise
 - check what is the Monte Carlo standard error
- Show meaningful digits given the posterior uncertainty
- Example: The mean and 90% central posterior interval for temperature increase $^{\circ}\text{C}/\text{century}$ based on posterior draws
 - 2.050774 and [0.7472868 3.3017524] (NO!)
 - 2.1 and [0.7 3.3]



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

How many digits to show in reports?

- Too many digits make reading of the results slower and give false impression of the accuracy
- Don't show digits which are just random noise
 - check what is the Monte Carlo standard error
- Show meaningful digits given the posterior uncertainty
- Example: The mean and 90% central posterior interval for temperature increase C° /century based on posterior draws
 - 2.050774 and [0.7472868 3.3017524] (NO!)
 - 2.1 and [0.7 3.3]
 - 2 and [1 3] (depends on the context)



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

How many digits to show in reports?

- Too many digits make reading of the results slower and give false impression of the accuracy
- Don't show digits which are just random noise
 - check what is the Monte Carlo standard error
- Show meaningful digits given the posterior uncertainty
- Example: The mean and 90% central posterior interval for temperature increase $^{\circ}\text{C}/\text{century}$ based on posterior draws
 - 2.050774 and [0.7472868 3.3017524] (NO!)
 - 2.1 and [0.7 3.3]
 - 2 and [1 3] (depends on the context)
- Example: The probability that temp increase is positive



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

How many digits to show in reports?

- Too many digits make reading of the results slower and give false impression of the accuracy
- Don't show digits which are just random noise
 - check what is the Monte Carlo standard error
- Show meaningful digits given the posterior uncertainty
- Example: The mean and 90% central posterior interval for temperature increase C° /century based on posterior draws
 - 2.050774 and [0.7472868 3.3017524] (NO!)
 - 2.1 and [0.7 3.3]
 - 2 and [1 3] (depends on the context)
- Example: The probability that temp increase is positive
 - 0.9960000 (NO!)



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

How many digits to show in reports?

- Too many digits make reading of the results slower and give false impression of the accuracy
- Don't show digits which are just random noise
 - check what is the Monte Carlo standard error
- Show meaningful digits given the posterior uncertainty
- Example: The mean and 90% central posterior interval for temperature increase $^{\circ}\text{C}/\text{century}$ based on posterior draws
 - 2.050774 and [0.7472868 3.3017524] (NO!)
 - 2.1 and [0.7 3.3]
 - 2 and [1 3] (depends on the context)
- Example: The probability that temp increase is positive
 - 0.9960000 (NO!)
 - 1.00 (depends on the context)



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

How many digits to show in reports?

- Too many digits make reading of the results slower and give false impression of the accuracy
- Don't show digits which are just random noise
 - check what is the Monte Carlo standard error
- Show meaningful digits given the posterior uncertainty
- Example: The mean and 90% central posterior interval for temperature increase C° /century based on posterior draws
 - 2.050774 and [0.7472868 3.3017524] (NO!)
 - 2.1 and [0.7 3.3]
 - 2 and [1 3] (depends on the context)
- Example: The probability that temp increase is positive
 - 0.9960000 (NO!)
 - 1.00 (depends on the context)
 - With 4000 draws MCSE ≈ 0.002 . We could report that probability is **very likely larger than 0.99**, or sample more to justify reporting three digits



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

How many digits to show in reports?

- Too many digits make reading of the results slower and give false impression of the accuracy
- Don't show digits which are just random noise
 - check what is the Monte Carlo standard error
- Show meaningful digits given the posterior uncertainty
- Example: The mean and 90% central posterior interval for temperature increase $^{\circ}\text{C}/\text{century}$ based on posterior draws
 - 2.050774 and [0.7472868 3.3017524] (NO!)
 - 2.1 and [0.7 3.3]
 - 2 and [1 3] (depends on the context)
- Example: The probability that temp increase is positive
 - 0.9960000 (NO!)
 - 1.00 (depends on the context)
 - With 4000 draws $\text{MCSE} \approx 0.002$. We could report that probability is **very likely larger than 0.99**, or sample more to justify reporting three digits
 - For probabilities close to 0 or 1, consider also when the model assumption justify certain accuracy
- For your project: Think for each reported value how many digits is sensible.



How many simulation draws are needed?

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling
- Less draws needed with
 - deterministic methods
 - marginalization (Rao-Blackwellization)
 - variance reduction methods, such, control variates



How many simulation draws are needed?

- Introduction
 - Computational aspects
 - Bayesian Computation
 - Numerical integration
 - Monte Carlo Methods
 - Direct sampling
 - Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling
- Number of independent draws needed doesn't depend on the number of dimensions
 - but it may be difficult to obtain independent draws in high dimensional case



UPPSALA
UNIVERSITET

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- **Direct sampling**
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Section 4

Direct sampling



- Introduction
 - Computational aspects
 - Bayesian Computation
 - Numerical integration
 - Monte Carlo Methods
 - **Direct sampling**
 - Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling
- Direct simulation from known pdf/pmf, e.g. $p(\theta|y)$ in conjugate case
 - Produces independent draws
 - Using analytic transformations of uniform random numbers (e.g. appendix A)
 - factorization
 - numerical inverse-CDF
 - **Problem:** restricted to limited set of models



UPPSALA
UNIVERSITET

Random number generators

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

- How to sample from a pdf?



- Introduction
 - Computational aspects
 - Bayesian Computation
 - Numerical integration
 - Monte Carlo Methods
 - **Direct sampling**
 - Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling
- **How to sample from a pdf?**
 - Good **pseudo** random number generators are sufficient for Bayesian inference
 - pseudo random generator uses deterministic algorithm to produce a sequence which is difficult to make difference from truly random sequence
 - modern software used for statistical analysis have good pseudo RNGs



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- **Direct sampling**
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

- Box-Muller -method:

If U_1 and U_2 are independent draws from distribution $\mathcal{U}(0, 1)$, and

$$X_1 = \sqrt{-2 \log(U_1)} \cos(2\pi U_2)$$

$$X_2 = \sqrt{-2 \log(U_1)} \sin(2\pi U_2)$$

then X_1 and X_2 are independent draws from the distribution $\mathcal{N}(0, 1)$



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- **Direct sampling**
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Direct simulation: Example

- Box-Muller -method:

If U_1 and U_2 are independent draws from distribution $\mathcal{U}(0, 1)$, and

$$X_1 = \sqrt{-2 \log(U_1)} \cos(2\pi U_2)$$

$$X_2 = \sqrt{-2 \log(U_1)} \sin(2\pi U_2)$$

then X_1 and X_2 are independent draws from the distribution $\mathcal{N}(0, 1)$

- not the fastest method due to trigonometric computations
- for normal distribution more than ten different methods
- e.g. R uses inverse-CDF



UPPSALA
UNIVERSITET

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Section 5

Indirect sampling



UPPSALA
UNIVERSITET

Indirect sampling

- Introduction
 - Computational aspects
 - Bayesian Computation
 - Numerical integration
 - Monte Carlo Methods
 - Direct sampling
 - Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling
- Rejection sampling
 - Importance sampling
 - Markov chain Monte Carlo (next week)



- Introduction
 - Computational aspects
 - Bayesian Computation
 - Numerical integration
 - Monte Carlo Methods
 - Direct sampling
 - Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling
- Efficient sampling size S_{eff} the number of samples using direct methods
 - Common with **weighted** or **correlated** samples



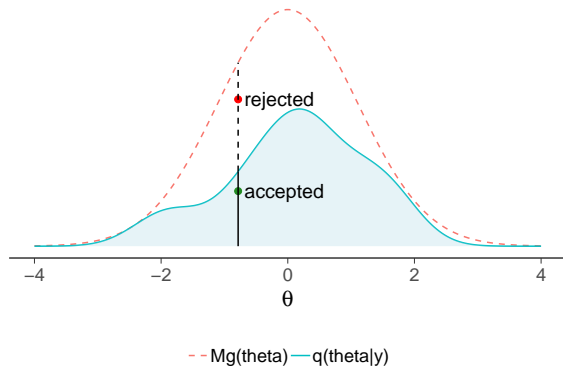
- Introduction
 - Computational aspects
 - Bayesian Computation
 - Numerical integration
 - Monte Carlo Methods
 - Direct sampling
 - Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling
- Efficient sampling size S_{eff} the number of samples using direct methods
 - Common with **weighted** or **correlated** samples
 - Indirect methods usually have an $S_{\text{eff}} < S$
 - Informally an indication of performance of method



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Rejection sampling

- Proposal ($g(\theta)$) forms envelope over the target distribution $q(\theta|y)/Mg(\theta) \leq 1$
- Draw from the proposal and accept with probability $q(\theta|y)/Mg(\theta)$

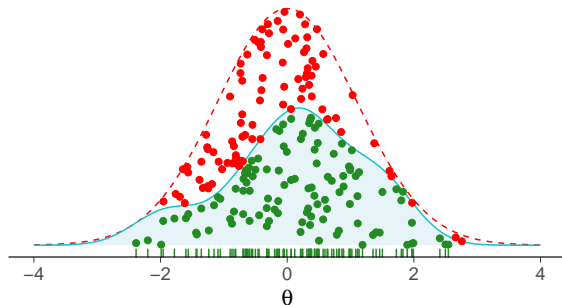




- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Rejection sampling

- Proposal ($g(\theta)$) forms envelope over the target distribution $q(\theta|y)/Mg(\theta) \leq 1$
- Draw from the proposal and accept with probability $q(\theta|y)/Mg(\theta)$



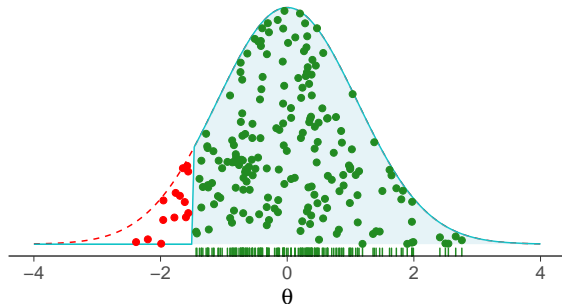
• Accepted • Rejected - - $Mg(\theta)$ — $q(\theta|y)$



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Rejection sampling

- Proposal ($g(\theta)$) forms envelope over the target distribution $q(\theta|y)/Mg(\theta) \leq 1$
- Draw from the proposal and accept with probability $q(\theta|y)/Mg(\theta)$
- Common for truncated distributions



● Accepted ● Rejected - - Mg(theta) — q(theta|y)



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

- The number of accepted draws is the effective sample size S_{eff}
When will this be work/not work (i.e. give high/low S_{eff})?



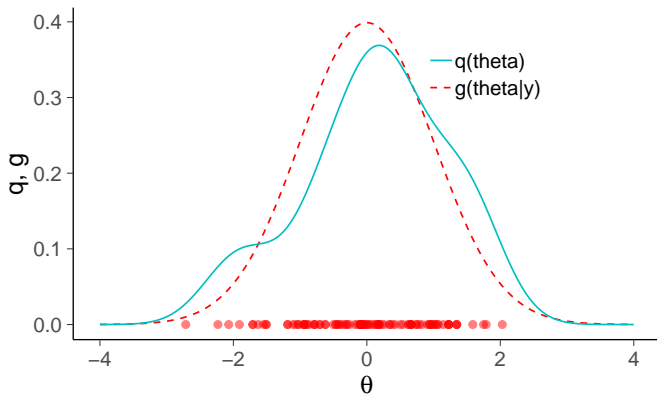
- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

- The number of accepted draws is the effective sample size S_{eff}
 - with bad proposal distribution may require a lot of trials
 - selection of good proposal gets very difficult when the number of dimensions increase



- Proposal does not need to have a higher value everywhere

Target, proposal, and draws



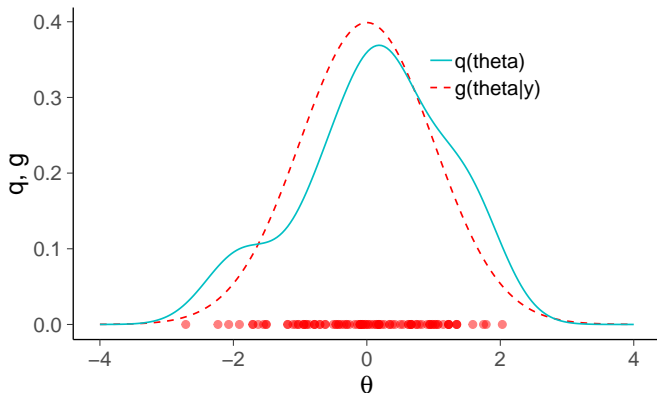


- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Importance sampling

- Proposal does not need to have a higher value everywhere

Target, proposal, and draws



$$E[f(\theta)] \approx \frac{\sum_s w_s f(\theta^{(s)})}{\sum_s w_s}, \quad \text{where} \quad w_s = \frac{q(\theta^{(s)})}{g(\theta^{(s)})}$$

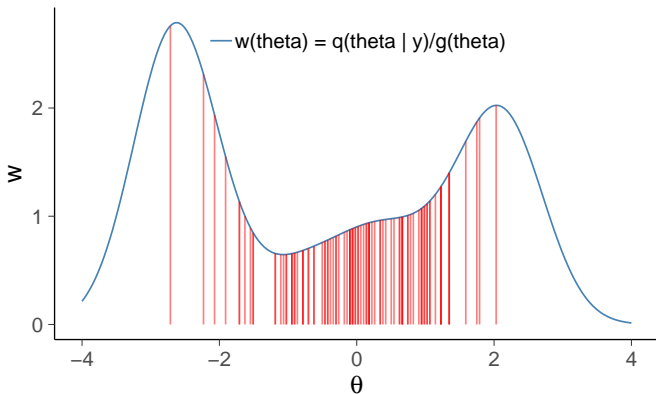


- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Importance sampling

- Proposal does not need to have a higher value everywhere

Draws and importance weights



$$E[f(\theta)] \approx \frac{\sum_s w_s f(\theta^{(s)})}{\sum_s w_s}, \quad \text{where} \quad w_s = \frac{q(\theta^{(s)})}{g(\theta^{(s)})}$$



- Introduction
 - Computational aspects
 - Bayesian Computation
 - Numerical integration
 - Monte Carlo Methods
 - Direct sampling
 - Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling
- Resampling using normalized importance weights can be used to pick a smaller number of draws with uniform weights



- Introduction
 - Computational aspects
 - Bayesian Computation
 - Numerical integration
 - Monte Carlo Methods
 - Direct sampling
 - Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling
- Resampling using normalized importance weights can be used to pick a smaller number of draws with uniform weights
 - Selection of good proposal gets more difficult when the number of dimensions increase



- Introduction

- Computational aspects

- Bayesian Computation

- Numerical integration

- Monte Carlo Methods

- Direct sampling

- Indirect sampling

- Rejection sampling

- Importance sampling

- Pareto-Smoothed
Importance Sampling

- Resampling using normalized importance weights can be used to pick a smaller number of draws with uniform weights
- Selection of good proposal gets more difficult when the number of dimensions increase
- Often used to correct distributional approximations and leave-one-out cross-validation



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

- Variation of the weights affect the **effective sample size**
 - if single weight dominates, we have effectively one sample
 - if all weights are equal, we have effectively S draws

What does this mean? What is a good proposal $g(\theta)$?



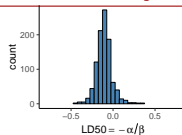
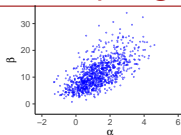
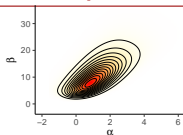
- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

- Variation of the weights affect the **effective sample size**
 - if single weight dominates, we have effectively one sample
 - if all weights are equal, we have effectively S draws
- Central limit theorem holds only if variance of the weight distribution is finite



Example: Importance sampling in Bioassay

Grid



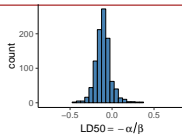
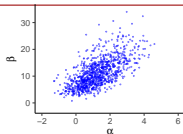
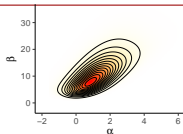
- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling



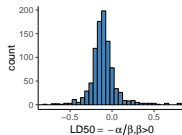
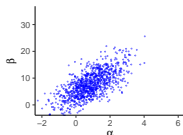
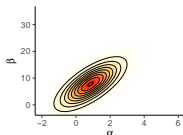
- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Example: Importance sampling in Bioassay

Grid



Normal



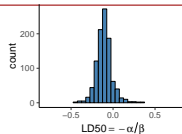
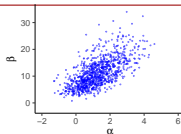
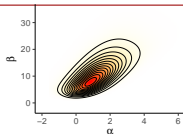
Normal approximation is discussed more in BDA3 Ch 4



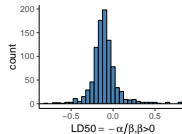
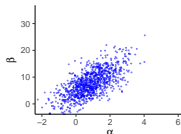
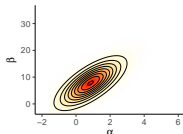
- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Example: Importance sampling in Bioassay

Grid



Normal



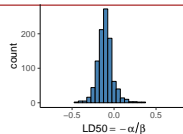
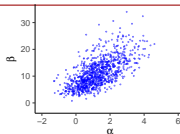
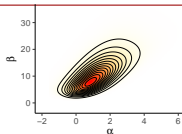
Normal approximation is discussed more in BDA3 Ch 4
But the normal approximation is not that good here:
 $\text{Grid } \text{sd}(LD50) \approx 0.1$, $\text{Normal } \text{sd}(LD50) \approx .75!$



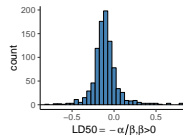
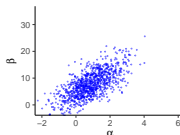
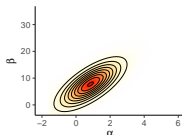
- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Example: Importance sampling in Bioassay

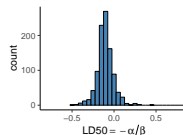
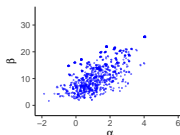
Grid



Normal



IR

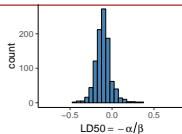
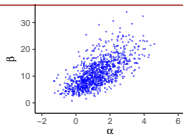
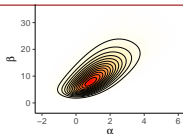




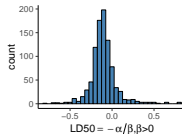
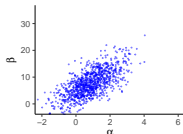
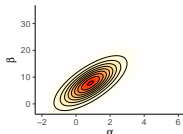
- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Example: Importance sampling in Bioassay

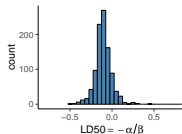
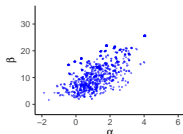
Grid



Normal



IR



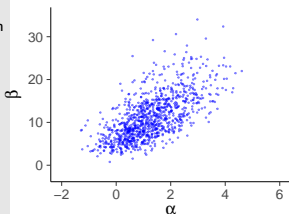
Grid $\text{sd}(\text{LD50}) \approx 0.1$, IR $\text{sd}(\text{LD50}) \approx 0.1$



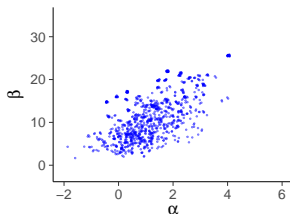
Example: Importance sampling in Bioassay

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Grid



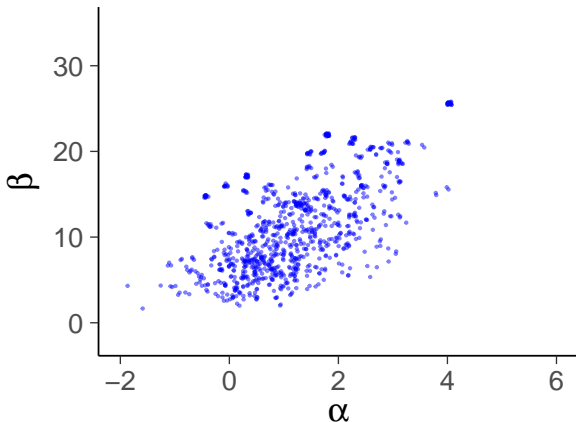
IR





Example: Importance sampling in Bioassay

IR



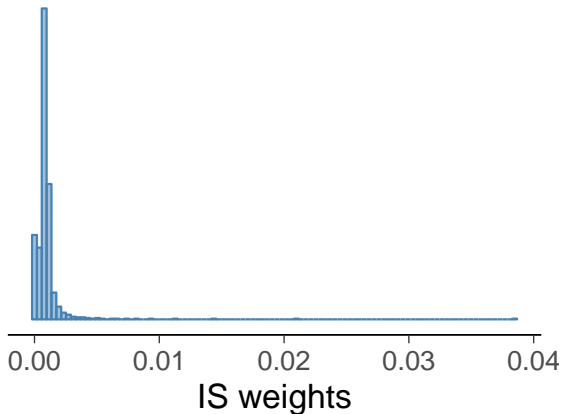
- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling



UPPSALA
UNIVERSITET

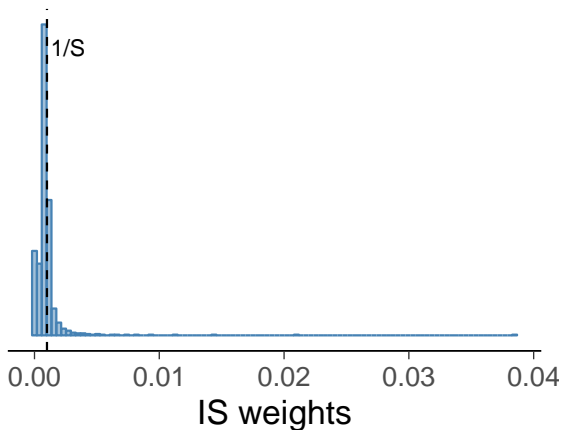
Example: Importance sampling in Bioassay

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling





Example: Importance sampling in Bioassay

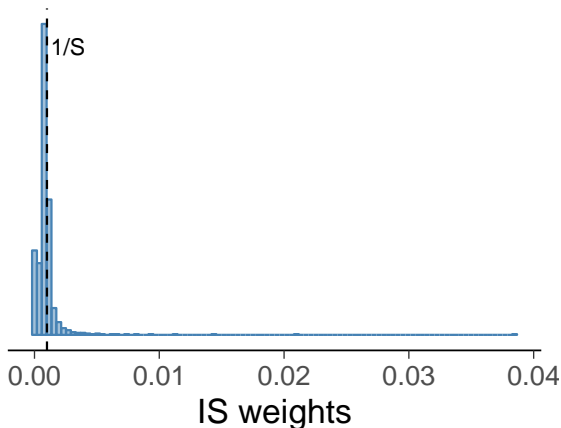


- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Example: Importance sampling in Bioassay

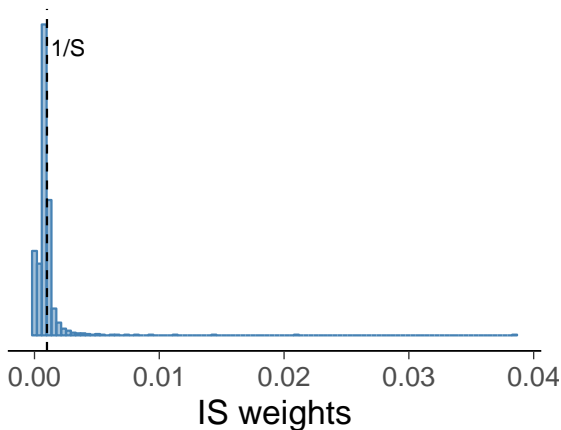


$$S_{\text{eff}} = \frac{1}{\sum_{s=1}^S (\tilde{w}(\theta^s))^2}, \quad \text{where } \tilde{w}(\theta^s) = w(\theta^s) / \sum_{s'=1}^S w(\theta^{s'})$$



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Example: Importance sampling in Bioassay



$$S_{\text{eff}} = \frac{1}{\sum_{s=1}^S (\tilde{w}(\theta^s))^2}, \quad \text{where } \tilde{w}(\theta^s) = w(\theta^s) / \sum_{s'=1}^S w(\theta^{s'})$$

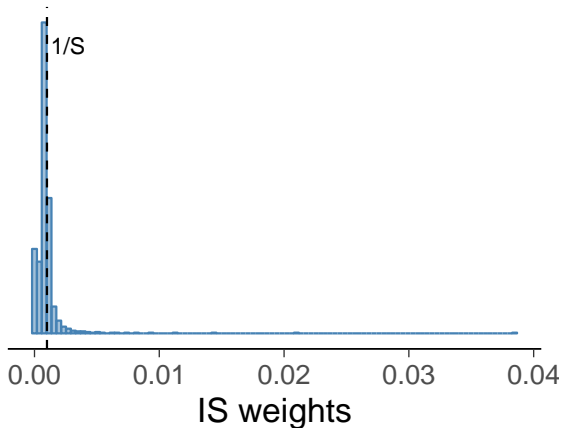
BDA3 1st (2013) and 2nd (2014) printing have an error for $\tilde{w}(\theta^s)$. The normalized weights equation should not have the multiplier S (the normalized weights should sum to one). Errata for the book

http://www.stat.columbia.edu/~gelman/book/errata_bda3.txt



- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

Example: Importance sampling in Bioassay



$$S_{\text{eff}} = \frac{1}{\sum_{s=1}^S (\tilde{w}(\theta^s))^2}, \quad \text{where } \tilde{w}(\theta^s) = w(\theta^s) / \sum_{s'=1}^S w(\theta^{s'})$$
$$S_{\text{eff}} \approx 270$$



- Introduction
 - Computational aspects
 - Bayesian Computation
 - Numerical integration
 - Monte Carlo Methods
 - Direct sampling
 - Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling
- Pareto-Smoothed Importance sampling smooth the weights according to a Generalized Pareto(k) distribution
 - Pareto- k diagnostic estimate the number of existing moments ($\lfloor 1/k \rfloor$)



- Introduction
 - Computational aspects
 - Bayesian Computation
 - Numerical integration
 - Monte Carlo Methods
 - Direct sampling
 - Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling
- Pareto-Smoothed Importance sampling smooth the weights according to a Generalized Pareto(k) distribution
 - Pareto- k diagnostic estimate the number of existing moments ($\lfloor 1/k \rfloor$)
 - Finite variance and central limit theorem for $k < 1/2$



- Introduction
 - Computational aspects
 - Bayesian Computation
 - Numerical integration
 - Monte Carlo Methods
 - Direct sampling
 - Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling
- Pareto-Smoothed Importance sampling smooth the weights according to a Generalized Pareto(k) distribution
 - Pareto- k diagnostic estimate the number of existing moments ($\lfloor 1/k \rfloor$)
 - Finite variance and central limit theorem for $k < 1/2$
 - Finite mean and generalized central limit theorem for $k < 1$, but pre-asymptotic constant grows impractically large for $k > 0.7$



Importance sampling leave-one-out cross-validation

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

- Later in the course you will learn how $p(\theta|y)$ can be used as a proposal distribution for $p(\theta|y_{-i})$
 - which allows fast computation of leave-one-out cross-validation

$$p(y_i|y_{-i}) = \int p(y_i|\theta)p(\theta|y_{-i})d\theta$$



Next week: Markov chain Monte Carlo (MCMC)

- Introduction
 - Computational aspects
 - Bayesian Computation
 - Numerical integration
 - Monte Carlo Methods
 - Direct sampling
 - Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling
- Pros
 - Markov chain goes where most of the posterior mass is
 - Certain MCMC methods scale well to high dimensions
 - Cons
 - Draws are dependent (affects how many draws are needed)
 - Convergence in practical time is not guaranteed



Next week: Markov chain Monte Carlo (MCMC)

- Introduction
 - Computational aspects
- Bayesian Computation
 - Numerical integration
- Monte Carlo Methods
- Direct sampling
- Indirect sampling
 - Rejection sampling
 - Importance sampling
 - Pareto-Smoothed Importance Sampling

- Pros
 - Markov chain goes where most of the posterior mass is
 - Certain MCMC methods scale well to high dimensions
- Cons
 - Draws are dependent (affects how many draws are needed)
 - Convergence in practical time is not guaranteed
- MCMC methods in this course
 - Gibbs sampling: “iterative conditional sampling”
 - Metropolis: “random walk in joint distribution”
 - Dynamic Hamiltonian Monte Carlo: “state-of-the-art” used in Stan