UPPSALA UNIVERSITET

Bayesian Statistics and Data Analysis
Lecture 2

Måns Magnusson
Department of Statistics, Uppsala University
Thanks to Aki Vehtari, Aalto University

UPPSALA
UNIVERSITET

Assignment 1

- "The formatting was quite time-consuming and could be improved."

UPPSALA UNIVERSITET

Assignment 1

- "The formatting was quite time-consuming and could be improved."
- "Maybe the first exercise could be slightly adjusted to be a bit more interesting and fun to work with."

UPPSALA UNIVERSITET

Assignment 1

- "The formatting was quite time-consuming and could be improved."
- "Maybe the first exercise could be slightly adjusted to be a bit more interesting and fun to work with."
- "I would appreciate exploring the concepts of aleatoric and epsitemic uncertainity in greater length. Maybe one of the exercises could be about writing about these concepts more thoroughly."

UPPSALA UNIVERSITET

Assignment 1

- "The formatting was quite time-consuming and could be improved."
- "Maybe the first exercise could be slightly adjusted to be a bit more interesting and fun to work with."
- "I would appreciate exploring the concepts of aleatoric and epsitemic uncertainity in greater length. Maybe one of the exercises could be about writing about these concepts more thoroughly."
- "A more detailed, question-specific example for using markmyassignment would be very helpful. "

UPPSALA UNIVERSITET

Master theses using Stan

- Joint supervision and work with researchers and students (PhD and masters)
- For students that want to work with research

UPPSALA UNIVERSITET

Master theses using Stan

- Joint supervision and work with researchers and students (PhD and masters)
- For students that want to work with research
- Multiple project (two from stats and three from IT) on Stan and Bayes:

UPPSALA
UNIVERSITET

Master theses using Stan

- Joint supervision and work with researchers and students (PhD and masters)
- For students that want to work with research
- Multiple project (two from stats and three from IT) on Stan and Bayes:
- Projects
  - Modelling of pollen data in Ancient greece (with Anton Bonnier)
  - Modelling proteins in pharmacometrics

UPPSALA UNIVERSITET

Master theses using Stan

- Joint supervision and work with researchers and students (PhD and masters)
- For students that want to work with research
- Multiple project (two from stats and three from IT) on Stan and Bayes:
- Projects
    - Modelling of pollen data in Ancient greece (with Anton Bonnier)
    - Modelling proteins in pharmacometrics
- Deadline to show interest this *Sunday at 23.59*.

Binomial: known θ

- Probability of event 1 in trial is θ

Binomial: known θ

- Probability of event 1 in trial is θ
- Probability of event 2 in trial is $1 - \theta$

Binomial: known θ

- Probability of event 1 in trial is θ
- Probability of event 2 in trial is $1 - \theta$
- Probability of several events in independent trials is e.g. $\theta\theta(1-\theta)\theta(1-\theta)(1-\theta)\ldots$

UPPSALA UNIVERSITET

UPPSALA UNIVERSITET

Binomial: known θ

- Probability of event 1 in trial is θ
- Probability of event 2 in trial is $1-\theta$
- Probability of several events in independent trials is e.g. $\theta\theta(1-\theta)\theta(1-\theta)(1-\theta)\ldots$
- If there are n trials and we don't care about the order of the events, then the probability that event 1 happens y times is

$$p(y|\theta, n) = \binom{n}{y}\theta^{y}(1-\theta)^{n-y}$$

Binomial: known θ

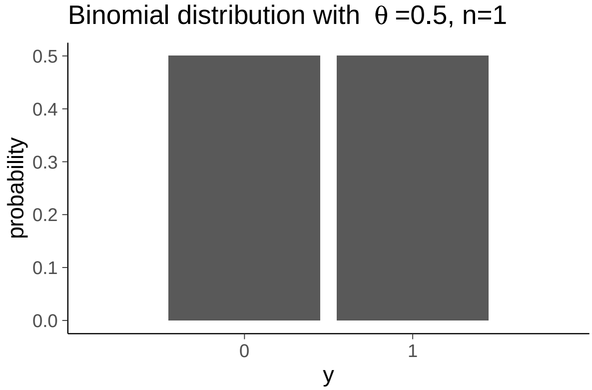
- Observation model (function of y, discrete)

$$p(y|\theta, n) = \binom{n}{y}\theta^{y}(1-\theta)^{n-y}$$

Binomial: known θ

- Observation model (function of y, discrete)

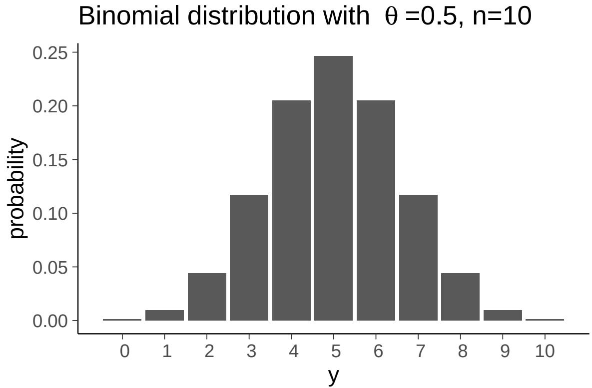
$$p(y|\theta, n) = \binom{n}{y}\theta^y(1-\theta)^{n-y}$$



Binomial: known θ

- Observation model (function of y, discrete)

$$p(y|\theta, n) = \binom{n}{y} \theta^y (1-\theta)^{n-y}$$



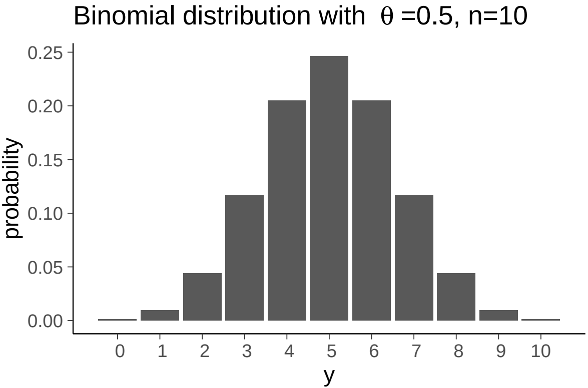
UPPSALA UNIVERSITET

- Posterior distributions
- Posterior distributions
- Predictive distributions
- Prior distributions
- Demo
- The Normal model

Binomial: known θ

- Observation model (function of y, discrete)

$$p(y|\theta, n) = \binom{n}{y}\theta^y(1-\theta)^{n-y}$$

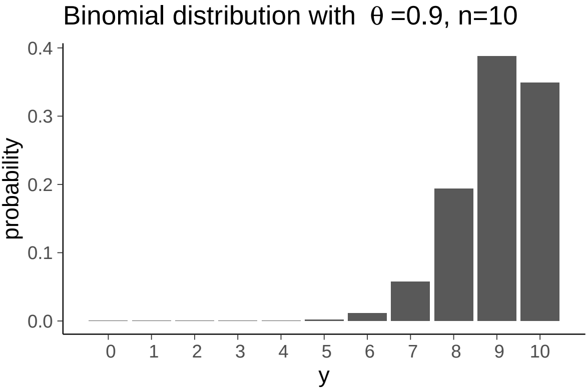


$p(y|n=10, \theta=0.5)$: 0.00 0.01 0.04 0.12 0.21 0.25 0.21 0.12 0.04 0.01 0.00

Binomial: known θ

- Observation model (function of y, discrete)

$$p(y|\theta, n) = \binom{n}{y} \theta^{y} (1-\theta)^{n-y}$$



$p(y|n=10, \theta=0.9)$: 0.00 0.00 0.00 0.00 0.00 0.00 0.01 0.06 0.19 0.39 0.35

UPPSALA UNIVERSITET

- Posterior distributions
- Posterior distributions
- Predictive distributions
- Prior distributions
- Demo
- The Normal model

Binomial: unknown θ

- Posterior with Bayes rule (function of θ, continuous)

$$p(\theta|y, n, M) = \frac{p(y|\theta, n, M)p(\theta|n, M)}{p(y|n, M)}$$

UPPSALA UNIVERSITET

Binomial: unknown θ

- Posterior with Bayes rule (function of θ, continuous)

$$p(\theta|y, n, M) = \frac{p(y|\theta, n, M)p(\theta|n, M)}{p(y|n, M)}$$

where $p(y|n, M) = \int p(y|\theta, n, M)p(\theta|n, M)d\theta$

Binomial: unknown θ

- Posterior with Bayes rule (function of θ, continuous)

$$p(\theta|y,n,M) = \frac{p(y|\theta,n,M)p(\theta|n,M)}{p(y|n,M)}$$

where $p(y|n,M) = \int p(y|\theta,n,M)p(\theta|n,M)d\theta$

- Start with uniform prior

$$p(\theta|n,M) = p(\theta|M) = 1, \text{ when } 0 \leq \theta \leq 1$$

UPPSALA UNIVERSITET

- Posterior distributions
- Posterior distributions
- Predictive distributions
- Prior distributions
- Demo
- The Normal model

Binomial: unknown θ

- Posterior with Bayes rule (function of θ, continuous)

$$p(\theta|y,n,M) = \frac{p(y|\theta,n,M)p(\theta|n,M)}{p(y|n,M)}$$

where $p(y|n,M) = \int p(y|\theta,n,M)p(\theta|n,M)d\theta$

- Start with uniform prior

$$p(\theta|n,M) = p(\theta|M) = 1, \text{ when } 0 \leq \theta \leq 1$$

- Then

$$p(\theta|y,n,M) = \frac{p(y|\theta,n,M)}{p(y|n,M)} = \frac{\binom{n}{y}\theta^y(1-\theta)^{n-y}}{\int_0^1 \binom{n}{y}\theta^y(1-\theta)^{n-y}d\theta}$$
$$= \frac{1}{Z}\theta^y(1-\theta)^{n-y}$$

$\propto \theta^y(1-\theta)^{n-y}$

Binomial: unknown θ

- Normalization term Z (constant given y)

$$Z = \int_0^1 \theta^y (1-\theta)^{n-y} d\theta = \frac{\Gamma(y+1)\Gamma(n-y+1)}{\Gamma(n+2)}$$

- Normalisation term has *Beta* function form
  - when integrated over $(0, 1)$ the result can presented with Gamma functions
  - with integers $\Gamma(n) = (n-1)!$
  - for large integers even this is challenging and usually $\log \Gamma(\cdot)$ is computed instead of $\Gamma(\cdot)$

Binomial: unknown θ

- Posterior is

$$p(\theta|y, n, M) = \frac{\Gamma(n+2)}{\Gamma(y+1)\Gamma(n-y+1)}\theta^{y}(1-\theta)^{n-y},$$

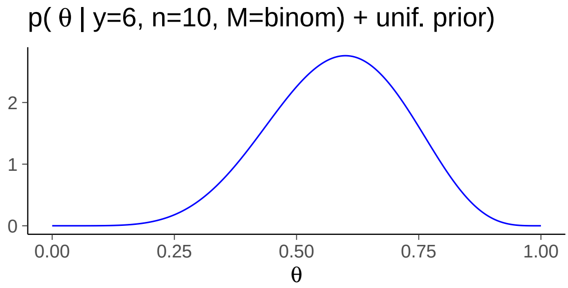
Binomial: unknown θ

- Posterior is

$$p(\theta|y, n, M) = \frac{\Gamma(n+2)}{\Gamma(y+1)\Gamma(n-y+1)}\theta^y(1-\theta)^{n-y},$$

which is called Beta distribution

$$\theta|y, n \sim \text{Beta}(y+1, n-y+1)$$



Binomial: computation

- R
  - density `dbeta`
  - CDF `pbeta`
  - quantile `qbeta`
  - random number `rbeta`
- Python
  - `from scipy.stats import beta`
  - density `beta.pdf`
  - CDF `beta.cdf`
  - prctile `beta.ppf`
  - random number `beta.rvs`

Binomial: computation*

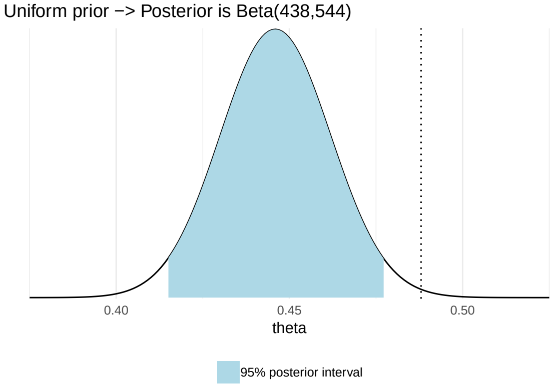
- Beta CDF not trivial to compute
- For example, `pbeta` in R uses a continued fraction with weighting factors and asymptotic expansion
- Laplace developed normal approximation (Laplace approximation), because he didn't know how to compute Beta CDF

Placenta previa

- Probability of a girl birth given placenta previa (BDA3 p. 37)
  - 437 girls and 543 boys have been observed
  - is the ratio 0.445 different from the population average 0.485?

Placenta previia

- Probability of a girl birth given placenta previa (BDA3 p. 37)
  - 437 girls and 543 boys have been observed
  - is the ratio 0.445 different from the population average 0.485?



UPPSALA UNIVERSITET

Predictive distribution – Effect of integration

- Predictive distribution for new $\tilde{y}$ (discrete)

$$p(\tilde{y} = 1|\theta, y, n, M)$$

Predictive distribution – Effect of integration

- Predictive distribution for new $\tilde{y}$ (discrete)

$$p(\tilde{y} = 1|y, n, M) = \int_0^1 p(\tilde{y} = 1|\theta, y, n, M)p(\theta|y, n, M)d\theta$$

Predictive distribution – Effect of integration

- Predictive distribution for new $\tilde{y}$ (discrete)

$$
\begin{aligned}
p(\tilde{y} = 1 | y, n, M) &= \int_0^1 p(\tilde{y} = 1 | \theta, y, n, M) p(\theta | y, n, M) d\theta \\
&= \int_0^1 \theta p(\theta | y, n, M) d\theta
\end{aligned}
$$

UPPSALA UNIVERSITET

UPPSALA UNIVERSITET

Predictive distribution – Effect of integration

- Predictive distribution for new $\tilde{y}$ (discrete)

$$
\begin{aligned}
p(\tilde{y} = 1|y, n, M) &= \int_0^1 p(\tilde{y} = 1|\theta, y, n, M)p(\theta|y, n, M)d\theta \\
&= \int_0^1 \theta p(\theta|y, n, M)d\theta \\
&= \mathrm{E}[\theta|y]
\end{aligned}
$$

Predictive distribution – Effect of integration

- Predictive distribution for new $\tilde{y}$ (discrete)

$$
\begin{aligned}
p(\tilde{y} = 1 | y, n, M) &= \int_0^1 p(\tilde{y} = 1 | \theta, y, n, M) p(\theta | y, n, M) d\theta \\
&= \int_0^1 \theta p(\theta | y, n, M) d\theta \\
&= \mathrm{E}[\theta | y]
\end{aligned}
$$

- With uniform prior

$$
\mathrm{E}[\theta | y] = \frac{y+1}{n+2}
$$

Predictive distribution – Effect of integration

- Predictive distribution for new $\tilde{y}$ (discrete)

$$\begin{aligned} p(\tilde{y}=1|y,n,M) &= \int_0^1 p(\tilde{y}=1|\theta,y,n,M)p(\theta|y,n,M)d\theta \\ &= \int_0^1 \theta p(\theta|y,n,M)d\theta \\ &= \mathrm{E}[\theta|y] \end{aligned}$$

- With uniform prior

$$\mathrm{E}[\theta|y] = \frac{y+1}{n+2}$$

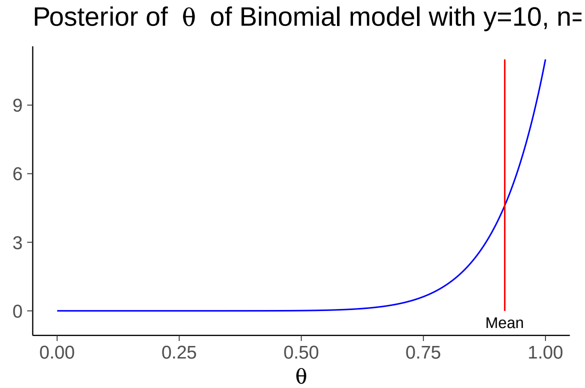
- Extreme cases

$$\begin{aligned} p(\tilde{y}=1|y=0,n,M) &= \frac{1}{n+2} \\ p(\tilde{y}=1|y=n,n,M) &= \frac{n+1}{n+2} \end{aligned}$$

- cf. maximum likelihood

Benefits of integration

Example: $n = 10, y = 10$



Predictive distribution

- Prior predictive distribution for new $\tilde{y}$ (discrete)

$$p(\tilde{y} = 1|M) = \int_0^1 p(\tilde{y} = 1|\theta, y, n, M)p(\theta|M)d\theta$$

- Posterior predictive distribution for new $\tilde{y}$ (discrete)

$$p(\tilde{y} = 1|y, n, M) = \int_0^1 p(\tilde{y} = 1|\theta, y, n, M)p(\theta|y, n, M)d\theta$$

Justification for uniform prior

- $p(\theta|M) = 1$ if
  1) we want the prior predictive distribution to be uniform

$$p(\tilde{y} = 1|n = 0, M) = \frac{1}{2}$$

  - nice justification as it is based on observables y and n

Justification for uniform prior

- $p(\theta|M) = 1$ if
  1) we want the prior predictive distribution to be uniform

$$p(\tilde{y} = 1|n = 0, M) = \frac{1}{2}$$

  - nice justification as it is based on observables y and n

  2) we think all values of θ are equally likely

Priors

- Conjugate prior (BDA3 p. 35)
- Noninformative prior (BDA3 p. 51)
- Proper and improper prior (BDA3 p. 52)
- Weakly informative prior (BDA3 p. 55)
- Informative prior (BDA3 p. 55)
- Prior sensitivity (BDA3 p. 38)

Conjugate prior

- Prior and posterior have the same form
  - only for exponential family distributions (plus for some irregular cases)

Conjugate prior

- Prior and posterior have the same form
  - only for exponential family distributions (plus for some irregular cases)
- Used to be important for computational reasons
- Still used for special models to allow partial analytic marginalization (Ch 3)
  - with dynamic Hamiltonian Monte Carlo used e.g. in Stan no any computational benefit

Beta prior for Binomial model

- Prior

$$\text{Beta}(\theta|\alpha, \beta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

- Posterior

$$p(\theta|y, n, M) \propto \theta^{y}(1-\theta)^{n-y}\theta^{\alpha-1}(1-\theta)^{\beta-1}$$

UPPSALA UNIVERSITET

Beta prior for Binomial model

- Prior

$$\text{Beta}(\theta|\alpha,\beta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

- Posterior

$$\begin{aligned} p(\theta|y,n,M) &\propto \theta^{y}(1-\theta)^{n-y}\theta^{\alpha-1}(1-\theta)^{\beta-1} \\ &\propto \theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1} \end{aligned}$$

Beta prior for Binomial model

- Prior

$$\text{Beta}(\theta|\alpha,\beta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

- Posterior

$$\begin{aligned} p(\theta|y,n,M) &\propto \theta^{y}(1-\theta)^{n-y}\theta^{\alpha-1}(1-\theta)^{\beta-1} \\ &\propto \theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1} \end{aligned}$$

after normalization

$$p(\theta|y,n,M) = \text{Beta}(\theta|\alpha+y,\beta+n-y)$$

Beta prior for Binomial model

- Prior

$$\text{Beta}(\theta|\alpha,\beta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

- Posterior

$$\begin{aligned} p(\theta|y,n,M) &\propto \theta^{y}(1-\theta)^{n-y}\theta^{\alpha-1}(1-\theta)^{\beta-1} \\ &\propto \theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1} \end{aligned}$$

  after normalization

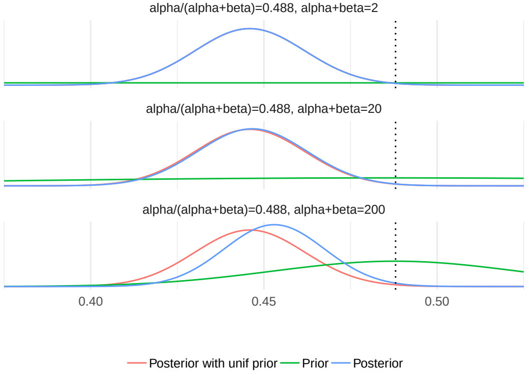
$$p(\theta|y,n,M) = \text{Beta}(\theta|\alpha+y,\beta+n-y)$$

- α and β can considered to be number of prior observations
- Uniform prior when $\alpha = 1$ and $\beta = 1$

UPPSALA UNIVERSITET

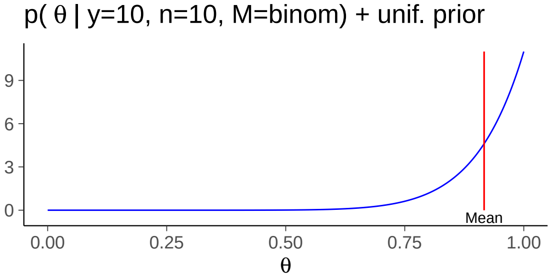
Placenta previia

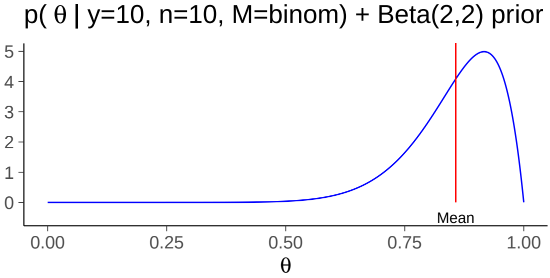
- Beta prior centered on population average 0.485



Benefits of integration and prior

Example: $n = 10, y = 10$ - uniform vs Beta(2,2) prior





UPPSALA UNIVERSITET

Beta prior for Binomial model

- Posterior

$$p(\theta|y, n, M) = \text{Beta}(\theta|\alpha + y, \beta + n - y)$$

- Posterior mean

$$\mathrm{E}[\theta|y] = \frac{\alpha + y}{\alpha + \beta + n}$$

  - combination prior and likelihood information
  - when $n \to \infty$, $\mathrm{E}[\theta|y] \to y/n$

Beta prior for Binomial model

- Posterior

$$p(\theta|y,n,M) = \mathrm{Beta}(\theta|\alpha+y,\beta+n-y)$$

- Posterior mean

$$\mathrm{E}[\theta|y] = \frac{\alpha+y}{\alpha+\beta+n}$$

  - combination prior and likelihood information
  - when $n \to \infty$, $\mathrm{E}[\theta|y] \to y/n$

- Posterior variance

$$\mathrm{var}[\theta|y] = \frac{\mathrm{E}[\theta|y](1-\mathrm{E}[\theta|y])}{\alpha+\beta+n+1}$$

  - decreases when n increases
  - when $n \to \infty$, $\mathrm{var}[\theta|y] \to 0$

Noninformative prior

- Vague, flat, diffuse, or noninformative
  - try to "to let the data speak for themselves"
  - flat is not non-informative
  - flat can be stupid
  - making prior flat somewhere can make it non-flat somewhere else

Proper and improper prior

- Proper prior has $\int p(\theta) = 1$
- Improper prior density doesn't have a finite integral
  - the posterior can still sometimes be proper

Proper and improper prior

- Proper prior has $\int p(\theta) = 1$
- Improper prior density doesn't have a finite integral
  - the posterior can still sometimes be proper
- Example: Binomial model
  - Beta(0,0) prior is improper
  - If $y \neq 0$ and $y \neq n$, the posterior is proper
- *Be careful with improper priors!*

Weakly informative priors

- Weakly informative priors produce computationally better behaving posteriors
  - If we want to model IQ in children, how to construct a prior?

UPPSALA UNIVERSITET

Weakly informative priors

- Weakly informative priors produce computationally better behaving posteriors
  - If we want to model IQ in children, how to construct a prior?
  - often there's some knowledge about the scale
  - Using the prior predictive distribution

$$p(\tilde{y}|M) = \int p(\tilde{y}|\theta, M)p(\theta|M)d\theta\,,$$

    we can simulate data from the model:
    Does it look (remotely) reasonable?
  - useful if there's more information from previous observations - not certain how well that information is applicable in a new case

Construction of weakly informative priors

- Prior prediction checks!
- Start with some version of a noninformative prior, then add information until reasonable.
- Start with a strong prior, then broaden it to account for uncertainty

Construction of weakly informative priors

- Prior prediction checks!
- Start with some version of a noninformative prior, then add information until reasonable.
- Start with a strong prior, then broaden it to account for uncertainty
- Stan team prior choice recommendations https://github.com/stan-dev/stan/wiki/Prior-Choice-Recommendations

Example of informative prior

- The percentage of girl births is remarkably stable at about 48.8% girls, rarely varying by more than 0.5% from this rate

UPPSALA UNIVERSITET

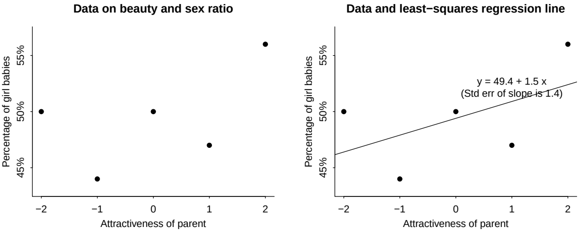
Example of informative prior

- The percentage of girl births is remarkably stable at about 48.8% girls, rarely varying by more than 0.5% from this rate
- There was a study on the percentage of girl births among parents in attractiveness categories 1–5 (assessed by interviewers in a face-to-face survey)

UPPSALA UNIVERSITET

Example of informative prior

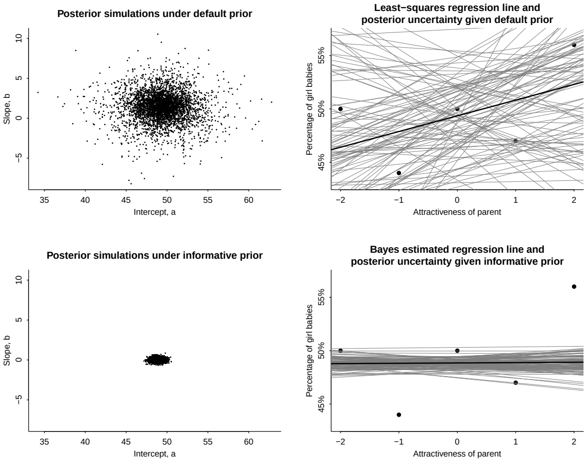
- The percentage of girl births is remarkably stable at about 48.8% girls, rarely varying by more than 0.5% from this rate
- There was a study on the percentage of girl births among parents in attractiveness categories 1–5 (assessed by interviewers in a face-to-face survey)



UPPSALA UNIVERSITET

Example of informative prior

- The percentage of girl births is remarkably stable at about 48.8% girls, rarely varying by more than 0.5% from this rate



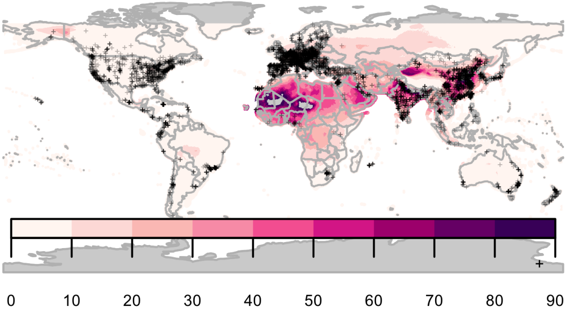
UPPSALA UNIVERSITET

Example of weakly informative prior

- Gabry et al (2019). Visualization in Bayesian workflow.
  - Estimation of human exposure to air pollution from particulate matter measuring less than 2.5 microns in diameter ($PM_{2.5}$)
  - A recent report estimated that $PM_{2.5}$ is responsible for three million deaths worldwide each year (Shaddick et al, 2017)

Example of weakly informative prior

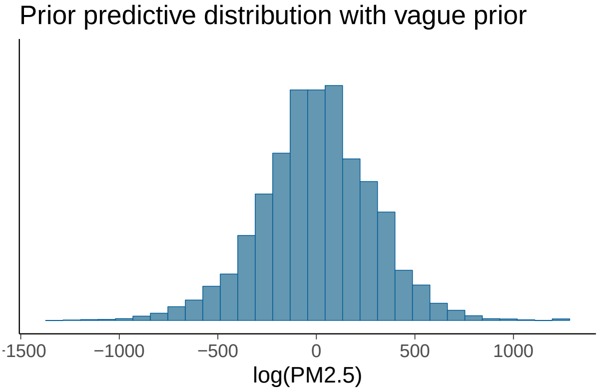
- Gabry et al (2019). Visualization in Bayesian workflow.



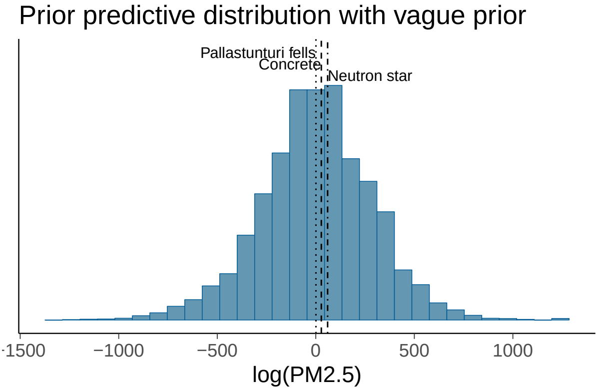
Satellite estimates and ground monitor locations

UPPSALA UNIVERSITET

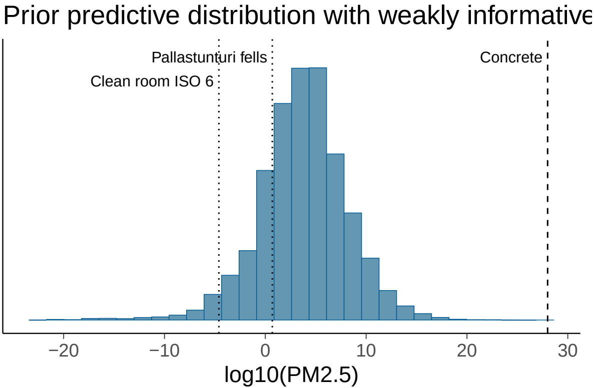
Example of weakly informative prior



Example of weakly informative prior



Example of weakly informative prior



UPPSALA UNIVERSITET

Effect of incorrect priors?

- Introduce bias, but often still produce smaller estimation error because the variance is reduced
  - bias-variance tradeoff

Sufficient statistics

UPPSALA UNIVERSITET

- The function $t(y)$ of data y is said to be a *sufficient statistic* for θ if the likelihood for θ depends on the data y only through the value of $t(y)$.
- Example: Binomial model (with known n, and $y_i \in \{0, 1\}$)

$$p(\theta|y) \propto p(\theta) \prod^{n} p(y_i|\theta)$$

Sufficient statistics

- The function $t(y)$ of data y is said to be a *sufficient statistic* for θ if the likelihood for θ depends on the data y only through the value of $t(y)$.
- Example: Binomial model (with known n, and $y_i \in \{0, 1\}$)

$$
\begin{aligned}
p(\theta|y) &\propto p(\theta) \prod^{n} p(y_i|\theta) \\
&\propto \theta^{\alpha-1}(1-\theta)^{\beta-1} \prod^{n} \theta_i^{y}(1-\theta)^{1-y_i}
\end{aligned}
$$

Sufficient statistics

- The function $t(y)$ of data y is said to be a *sufficient statistic* for θ if the likelihood for θ depends on the data y only through the value of $t(y)$.
- Example: Binomial model (with known n, and $y_i \in \{0, 1\}$)

$$
\begin{aligned}
p(\theta|y) &\propto p(\theta) \prod^{n} p(y_i|\theta) \\
&\propto \theta^{\alpha-1}(1-\theta)^{\beta-1} \prod^{n} \theta_i^{y}(1-\theta)^{1-y_i} \\
&\propto \theta^{\alpha-1}(1-\theta)^{\beta-1} \theta^{\sum y_i}(1-\theta)^{n-\sum y_i}
\end{aligned}
$$

Sufficient statistics

- The function $t(y)$ of data y is said to be a *sufficient statistic* for θ if the likelihood for θ depends on the data y only through the value of $t(y)$.
- Example: Binomial model (with known n, and $y_i \in \{0, 1\}$)

$$
\begin{aligned}
p(\theta|y) &\propto p(\theta) \prod^{n} p(y_i|\theta) \\
&\propto \theta^{\alpha-1}(1-\theta)^{\beta-1} \prod^{n} \theta_i^{y}(1-\theta)^{1-y_i} \\
&\propto \theta^{\alpha-1}(1-\theta)^{\beta-1}\theta^{\sum y_i}(1-\theta)^{n-\sum y_i} \\
&\propto \theta^{\sum y_i+\alpha-1}(1-\theta)^{n-\sum y_i+\beta-1}
\end{aligned}
$$

Sufficient statistics

- The function $t(y)$ of data y is said to be a *sufficient statistic* for θ if the likelihood for θ depends on the data y only through the value of $t(y)$.
- Example: Binomial model (with known n, and $y_i \in \{0,1\}$)

$$
\begin{aligned}
p(\theta|y) &\propto p(\theta)\prod^{n} p(y_i|\theta) \\
&\propto \theta^{\alpha-1}(1-\theta)^{\beta-1}\prod^{n}\theta_i^{y}(1-\theta)^{1-y_i} \\
&\propto \theta^{\alpha-1}(1-\theta)^{\beta-1}\theta^{\sum y_i}(1-\theta)^{n-\sum y_i} \\
&\propto \theta^{\sum y_i+\alpha-1}(1-\theta)^{n-\sum y_i+\beta-1}
\end{aligned}
$$

after normalization

$$
p(\theta|y,n,M) = \mathrm{Beta}(\theta|\alpha+\sum^{n} y_i, \beta+n-\sum^{n} y_i)
$$

Sufficient statistics

- The function $t(y)$ of data y is said to be a *sufficient statistic* for θ if the likelihood for θ depends on the data y only through the value of $t(y)$.
- Example: Binomial model (with known n, and $y_i \in \{0, 1\}$)

$$
\begin{aligned}
p(\theta|y) &\propto p(\theta) \prod^{n} p(y_i|\theta) \\
&\propto \theta^{\alpha-1}(1-\theta)^{\beta-1} \prod^{n} \theta_i^{y}(1-\theta)^{1-y_i} \\
&\propto \theta^{\alpha-1}(1-\theta)^{\beta-1}\theta^{\sum y_i}(1-\theta)^{n-\sum y_i} \\
&\propto \theta^{\sum y_i+\alpha-1}(1-\theta)^{n-\sum y_i+\beta-1}
\end{aligned}
$$

after normalization

$$p(\theta|y, n, M) = \text{Beta}(\theta|\alpha + \sum^{n} y_i, \beta + n - \sum^{n} y_i)$$

Hence, $\sum y_i$ is a sufficient statistic for θ in this model.

UPPSALA UNIVERSITET

Demo in R

- L2demo.R

UPPSALA UNIVERSITET

Algae

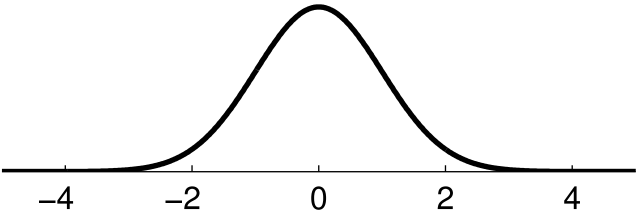
Algae status is monitored in 274 sites at Finnish lakes and rivers. The observations for the 2008 algae status at each site are presented in file *algae.mat* ('0': no algae, '1': algae present). Let π be the probability of a monitoring site having detectable blue-green algae levels.

- Use a binomial model for observations and a *beta*(2,10) prior.
- What can you say about the value of the unknown π?
- Experiment how the result changes if you change the prior.

Normal / Gaussian

- Observations $y \in \mathcal{R}$ (real valued)
- Mean θ and variance σ^2 (or deviation σ)
- For now: assume σ^2 is known

$$p(y|\theta) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right)$$
$$y \sim \mathcal{N}(\theta, \sigma^2)$$



Reasons to use Normal distribution

- Normal distribution often justified based on central limit theorem
- More often used due to the computational convenience or tradition

Central limit theorem (recap)

- De Moivre, Laplace, Gauss, Chebysev, Liapounov, Markov, et al.
- Given certain conditions, sums (and means) of random variables approach Gaussian distribution as $n \to \infty$
- Problems
  - does not hold for all distributions, e.g., Cauchy
  - may require large n,
    e.g. Binomial, when θ close to 0 or 1

UPPSALA UNIVERSITET

Normal distribution - conjugate prior for θ

- Assume σ^2 known

Likelihood $$p(y|\theta) \propto \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right)$$

Prior $$p(\theta) \propto \exp\left(-\frac{1}{2\tau_0^2}(\theta-\mu_0)^2\right)$$

Normal distribution - conjugate prior for θ

- Assume σ^2 known

Likelihood $$p(y|\theta) \propto \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right)$$

Prior $$p(\theta) \propto \exp\left(-\frac{1}{2\tau_0^2}(\theta-\mu_0)^2\right)$$

$$\exp(a)\exp(b) = \exp(a+b)$$

Normal distribution - conjugate prior for θ

- Assume σ^2 known

Likelihood
$$p(y|\theta) \propto \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right)$$

Prior
$$p(\theta) \propto \exp\left(-\frac{1}{2\tau_0^2}(\theta-\mu_0)^2\right)$$

$$\exp(a)\exp(b) = \exp(a+b)$$

Posterior

$$p(\theta|y) \propto \exp\left(-\frac{1}{2}\left[\frac{(y-\theta)^2}{\sigma^2} + \frac{(\theta-\mu_0)^2}{\tau_0^2}\right]\right)$$

Normal distribution - conjugate prior for θ

- Posterior (see ex 2.14a)

$$p(\theta|y) \propto \exp\left(-\frac{1}{2}\left[\frac{(y-\theta)^2}{\sigma^2}+\frac{(\theta-\mu_0)^2}{\tau_0^2}\right]\right)$$

$$\propto \exp\left(-\frac{1}{2\tau_1^2}(\theta-\mu_1)^2\right)$$

$$\theta|y \sim \mathcal{N}(\mu_1, \tau_1^2), \text{ where}$$

$$\mu_1 = \frac{\frac{1}{\tau_0^2}\mu_0 + \frac{1}{\sigma^2}y}{\frac{1}{\tau_0^2}+\frac{1}{\sigma^2}} \text{and} \frac{1}{\tau_1^2} = \frac{1}{\tau_0^2}+\frac{1}{\sigma^2}$$

Normal distribution - conjugate prior for θ

- Posterior (see ex 2.14a)

$$p(\theta|y) \propto \exp\left(-\frac{1}{2}\left[\frac{(y-\theta)^2}{\sigma^2} + \frac{(\theta-\mu_0)^2}{\tau_0^2}\right]\right)$$
$$\propto \exp\left(-\frac{1}{2\tau_1^2}(\theta-\mu_1)^2\right)$$

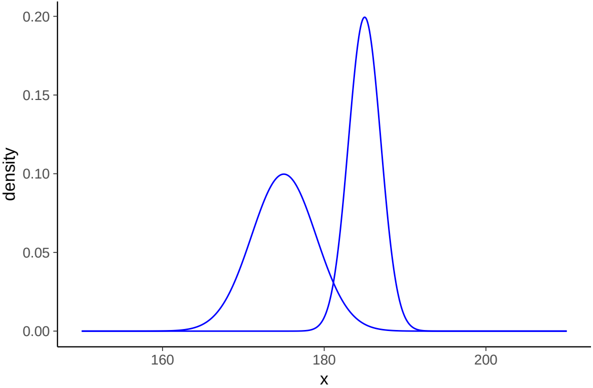
$$\theta|y \sim \mathcal{N}(\mu_1, \tau_1^2), \text{ where}$$

$$\mu_1 = \frac{\frac{1}{\tau_0^2}\mu_0 + \frac{1}{\sigma^2}y}{\frac{1}{\tau_0^2} + \frac{1}{\sigma^2}} \text{and} \frac{1}{\tau_1^2} = \frac{1}{\tau_0^2} + \frac{1}{\sigma^2}$$

- $1/\text{variance} = \text{precision}$
- Posterior precision = prior precision + data precision
- Posterior mean is precision weighted mean

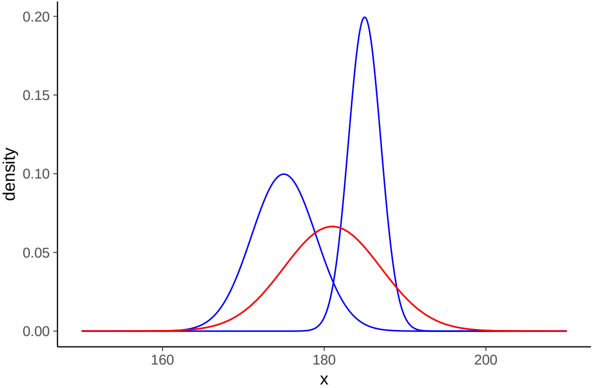
UPPSALA UNIVERSITET

Normal distribution - example



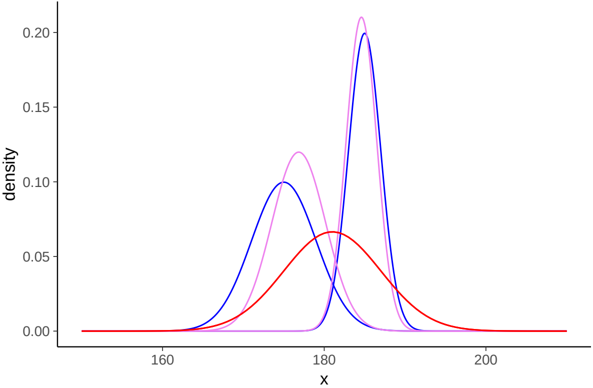
Blue: data point(s), Red: Prior, Purple: Posterior

Normal distribution - example



Blue: data point(s), Red: Prior, Purple: Posterior

Normal distribution - example



Blue: data point(s), Red: Prior, Purple: Posterior

Normal distribution - conjugate prior for θ

Posterior (several observations $y = (y_1, \ldots, y_n)$)

$p(\theta|y) \propto p(\theta)p(y|\theta)$

Normal distribution - conjugate prior for θ

Posterior (several observations $y = (y_1, \ldots, y_n)$)

$$
\begin{aligned}
p(\theta|y) &\propto p(\theta)p(y|\theta) \\
&= p(\theta)\prod^{n} p(y_i|\theta)
\end{aligned}
$$

Normal distribution - conjugate prior for θ

Posterior (several observations $y = (y_1, \dots, y_n)$)

$$
\begin{aligned}
p(\theta|y) &\propto p(\theta)p(y|\theta) \\
&= p(\theta)\prod^{n} p(y_i|\theta) \\
&\propto \exp\left(-\frac{1}{2}\left[\frac{\sum^{n}(y_i-\theta)^2}{\sigma^2} + \frac{(\theta-\mu_0)^2}{\tau_0^2}\right]\right)
\end{aligned}
$$

UPPSALA UNIVERSITET

Normal distribution - conjugate prior for θ

Posterior (several observations $y = (y_1, \ldots, y_n)$)

$$
\begin{aligned}
p(\theta|y) &\propto p(\theta)p(y|\theta) \\
&= p(\theta)\prod^{n} p(y_i|\theta) \\
&\propto \exp\left(-\frac{1}{2}\left[\frac{\sum^n (y_i-\theta)^2}{\sigma^2} + \frac{(\theta-\mu_0)^2}{\tau_0^2}\right]\right) \\
&= \exp\left(-\frac{1}{2}\left[\frac{n(\bar{y}-\theta)^2 + \sum^n (y_i-\bar{y})^2}{\sigma^2} + \frac{(\theta-\mu_0)^2}{\tau_0^2}\right]\right) \\
&\propto \exp\left(-\frac{1}{2}\left[\frac{n(\bar{y}-\theta)^2}{\sigma^2} + \frac{(\theta-\mu_0)^2}{\tau_0^2}\right]\right)
\end{aligned}
$$

Normal distribution - conjugate prior for θ

- Several observations $y = (y_1, \ldots, y_n)$

$$p(\theta|y) = \mathcal{N}(\theta|\mu_n, \tau_n^2)$$

$$\text{where} \quad \mu_n = \frac{\frac{1}{\tau_0^2}\mu_0 + \frac{n}{\sigma^2}\bar{y}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}} \quad \text{and} \quad \frac{1}{\tau_n^2} = \frac{1}{\tau_0^2} + \frac{n}{\sigma^2}$$

- If $\tau_0^2 = \sigma^2$, prior corresponds to one virtual observation with value μ_0

Normal distribution - conjugate prior for θ

UPPSALA UNIVERSITET

- Several observations $y = (y_1, \ldots, y_n)$

$$p(\theta|y) = \mathcal{N}(\theta|\mu_n, \tau_n^2)$$

$$\text{where} \quad \mu_n = \frac{\frac{1}{\tau_0^2}\mu_0 + \frac{n}{\sigma^2}\bar{y}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}} \quad \text{and} \quad \frac{1}{\tau_n^2} = \frac{1}{\tau_0^2} + \frac{n}{\sigma^2}$$

- If $\tau_0^2 = \sigma^2$, prior corresponds to one virtual observation with value μ_0
- If $\tau_0 \to \infty$ when n fixed
  or if $n \to \infty$ when τ_0 fixed

$$p(\theta|y) \to^d \mathcal{N}(\theta|\bar{y}, \sigma^2/n)$$

Normal distribution - conjugate prior for θ

UPPSALA UNIVERSITET

- Several observations $y = (y_1, \ldots, y_n)$

$$p(\theta|y) = \mathcal{N}(\theta|\mu_n, \tau_n^2)$$

$$\text{where} \quad \mu_n = \frac{\frac{1}{\tau_0^2}\mu_0 + \frac{n}{\sigma^2}\bar{y}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}} \quad \text{and} \quad \frac{1}{\tau_n^2} = \frac{1}{\tau_0^2} + \frac{n}{\sigma^2}$$

- If $\tau_0^2 = \sigma^2$, prior corresponds to one virtual observation with value μ_0
- If $\tau_0 \to \infty$ when n fixed
  or if $n \to \infty$ when τ_0 fixed

$$p(\theta|y) \to^d \mathcal{N}(\theta|\bar{y}, \sigma^2/n)$$

- Find the sufficient statistic for θ!

Normal distribution - conjugate prior for θ

- Posterior predictive distribution

$$p(\tilde{y}|y) = \int p(\tilde{y}|\theta)p(\theta|y)d\theta$$

$$p(\tilde{y}|y) \propto \int \exp\left(-\frac{1}{2\sigma^2}(\tilde{y}-\theta)^2\right)\exp\left(-\frac{1}{2\tau_1^2}(\theta-\mu_1)^2\right)d\theta$$

$$\tilde{y}|y \sim \mathcal{N}(\mu_1, \sigma^2 + \tau_1^2)$$

Normal distribution - conjugate prior for θ

- Posterior predictive distribution

$$p(\tilde{y}|y) = \int p(\tilde{y}|\theta)p(\theta|y)d\theta$$

$$p(\tilde{y}|y) \propto \int \exp\left(-\frac{1}{2\sigma^2}(\tilde{y}-\theta)^2\right)\exp\left(-\frac{1}{2\tau_1^2}(\theta-\mu_1)^2\right)d\theta$$

$$\tilde{y}|y \sim \mathcal{N}(\mu_1, \sigma^2 + \tau_1^2)$$

- Can be derived in multiple ways
  1. integrate
  2. $p(\tilde{y}, \theta)$ is a bivariate normal - marginalize out θ
- Predictive variance
  1. observation model variance σ^2
  2. posterior variance τ_1^2

UPPSALA UNIVERSITET

Normal distribution - conjugate prior for θ

- Posterior predictive distribution

$$p(\tilde{y}|y) = \int p(\tilde{y}|\theta)p(\theta|y)d\theta$$

$$p(\tilde{y}|y) \propto \int \exp\left(-\frac{1}{2\sigma^2}(\tilde{y}-\theta)^2\right)\exp\left(-\frac{1}{2\tau_1^2}(\theta-\mu_1)^2\right)d\theta$$

$$\tilde{y}|y \sim \mathcal{N}(\mu_1, \sigma^2 + \tau_1^2)$$

- Can be derived in multiple ways
  1. integrate
  2. $p(\tilde{y}, \theta)$ is a bivariate normal - marginalize out θ
- Predictive variance
  1. observation model variance σ^2
  2. posterior variance τ_1^2
- Aleatoric and epistemic uncertainty?

UPPSALA UNIVERSITET

Poisson model

- Poisson likelihood/model: $Po(\lambda)$
- Used for count data

UPPSALA UNIVERSITET

Poisson model

- Poisson likelihood/model: $Po(\lambda)$
- Used for count data
- Let $X \sim Po(\lambda)$, then

$$E(X) = V(X) = \lambda$$

UPPSALA UNIVERSITET

Poisson model

- Poisson likelihood/model: $Po(\lambda)$
- Used for count data
- Let $X \sim Po(\lambda)$, then

$$E(X) = V(X) = \lambda$$

- The gamma distribution is a conjugate prior - what is the posterior?

Poisson model

- Poisson likelihood/model: $Po(\lambda)$
- Used for count data
- Let $X \sim Po(\lambda)$, then

$$E(X) = V(X) = \lambda$$

- The gamma distribution is a conjugate prior - what is the posterior?
- Today: Mostly the negative binomial is used for count data