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Bayesian Statistics and Data Analysis Lecture 9

Måns Magnusson
Department of Statistics, Uppsala University
Thanks to Aki Vehtari, Aalto University

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Section 1

Bayesian/Statistical Decision Theory

Introduction to Decision Theory

- Statistical Decision Theory (SDT): Formalizing decision-making under uncertainty

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- "Decision theory concerns goal-directed behaviour in the presence of options" (and under uncertainty) (Hansson, 1994).

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  - d may be categorical, ordinal, real, scalar, vector, etc.
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- Probability distribution of outcome given decision $p(x|d)$
  - the decisions are controlled and thus $p(d)$ does not exist
  - Sometimes the decision has an effect in itself, hence $p(x|d)$
    e.g. x outcome in exam, d is whether to study or not
  - If outcome doesn't depend on the decision $p(x|d) = p(x)$

Bayesian decision theory

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  - e.g. euro or expected lifetime
  - instead of utility, sometimes cost or loss is defined as $-U(x, d)$
  - can be multiple types of values

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$$E[U(x, d)|d] = \int U(x, d)p(x|d)dx$$

- Optimal decision: d^*, which maximizes the expected utility (Maximal Expected Utility, MEU)

$$d^* = \arg\max_d E[U(x, d)|d]$$

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Subsection 1

A Trivial Example

Example of decision making: 2 choices

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- Helen measures that the length of the paw print is 14 cm
- Based on data she want to infer the probability of wolf

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- Bayesian/Statistical Decision Theory
  — A Trivial Example
- Integrating Inference and Decisions
- Course Evaluation

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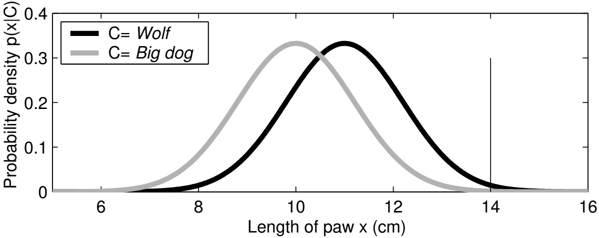

observed length has been marked with a horizontal line

- Likelihood of wolf is 0.92 (alternative being dog)

Example of decision making

- Helen also assumes that in her living area there are about one hundred times more free running dogs than wolves, i.e. *a priori* probability for wolf, before observation is 1%.
- Likelihood and posterior

Animal	Likelihood	Posterior probability
Wolf	0.92	0.10
Dog	0.08	0.90

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- Posterior probability of wolf is 10%

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- Expected utilities

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- Helen assigns negative utility -1000 for a event that she goes to the forest and wolf attacks
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Decision d	Wolf	Dog
Stay home	0	0
Go to the forest	-1000	1

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Action d	Conditional utility $E[U(x)\|d]$
Stay home	0
Go to the forest	-100+0.9 = -99.1

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- Maximum likelihood decision would be to assume that there is a wolf

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- Maximum likelihood decision would be to assume that there is a wolf
- Maximum posterior decision would be to assume that there is a dog
- Maximum Expected Utility (MEU) decision is to stay home, even if it is more likely that the animal is dog

- The uncertainties (probabilities) related to all consequences need to be take into account

Section 2

Integrating Inference and Decisions

Integrating inference and decisions

- To make an optimal decision we need $p(x|d)$

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  We can use $p(x|d, y)$ i.e. integrating previous data in decision making (some times referred to as Bayesianism)
- Formal Bayesian Decision making hence have two parts:
  1. Model $p(x|y, d)$ as good as possible

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- Formal Bayesian Decision making hence have two parts:
  1. Model $p(x|y, d)$ as good as possible
  2. Define $U(x, d)$

Challenges with the utility function

- Utility functions are rarely linear:
  How do we set it up?

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  How do we set it up?
- What is the cost of human life/illness? Can it be formulated to a utility function?
- Different parties might have different utilities (patient, physician, society)
- Personal vs institutional decisions
  - An individual have a subjective $p(x|d)$ and a subjective $U(x, d)$. Need for formal decision-making?
  - An institution might be better suited. Decision-recommendations.
- Decision theoretical approaches is better suited when $U(x, d)$ and $p(x|d)$ is well-defined.

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- If our approximation is bad - we will not necessarily make the optimal decision
- And we know that all models are wrong.
- $\hat{p}(x|y,d)$ is a statistical problem. Can we trust the model?
- But it can also be extremely hard to estimate, say:

$$P(\text{a new pandemic in 2025})$$

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Multi-stage decision making

- Slighly more complex: Multi-stage decision-making
- We need to take all uncertainties into account
- We can also condition after the decision is made

Multi-stage decision making example

- 95-year-old has a tumor that is malignant with 90% probability
- A priori knowledge
  - expected lifetime is 34.8 months if no cancer
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- Elaborated further in Bayesian Data Analysis

Section 3

Course Evaluation

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- What can be improved? What was annoying?

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- What was good? What was fun?

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Course Evaluation

- Did you get what you expected?
- What can be improved? What was annoying?
- What was good? What was fun?
- Anything else?