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# Bayesian Statistics and Data Analysis
# Lecture 8a


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Thanks to Aki Vehtari, Aalto University

# Mini-project comments

- Priors are part of model. Describe them only there.

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- For `posteriordb` students: (1) Please describe the data (or how you similate data) under data. Use the exact models that are in the papers. (2) Ignore prior recommendations - see it as replication study

# Section 1

## Model Checking and Assessment

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# The Box process: Probabilistic modeling

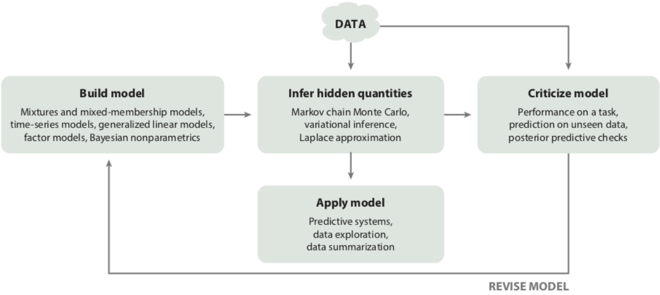


Figure: The Box approach (Box, 1976, Blei, 2014)

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# Model assessment

- Sensibility with respect to additional information not used in model
  - e.g., if posterior would claim that hazardous chemical decreases probability of death

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- External validation
  - compare predictions to completely new observations

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- Model Checking and Assessment
- Posterior predictive checking
  - Marginal Predictive Checking
  - Sensitivity analysis
  - Example

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- Sensibility with respect to additional information not used in model
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- External validation
  - compare predictions to completely new observations
- Internal validation
  - posterior predictive checking
  - cross-validation predictive checking

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# Section 2

# Posterior predictive checking

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- Model Checking and Assessment
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# Posterior predictive checking – example

- Newcombs speed of light measurements
  - model $y \sim \mathcal{N}(\mu, \sigma)$ with prior $(\mu, \log\sigma) \propto 1$

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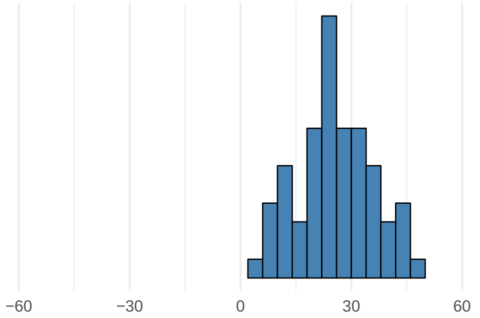
  - example $y^{\text{rep}}$:



# Replicates vs. predictive distributions



- Predictive $\tilde{y}$ is the next not yet observed possible observation.

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- Predictive $\tilde{y}$ is the next not yet observed possible observation.
- $y^{\mathrm{rep}}$ refers to replicating the whole experiment (potentially with same values of $x$)
  i.e. obtaining as many replicated observations as in the original data.

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# Posterior predictive checking – example

- Generate replicated datasets $y^{\mathrm{rep}}$

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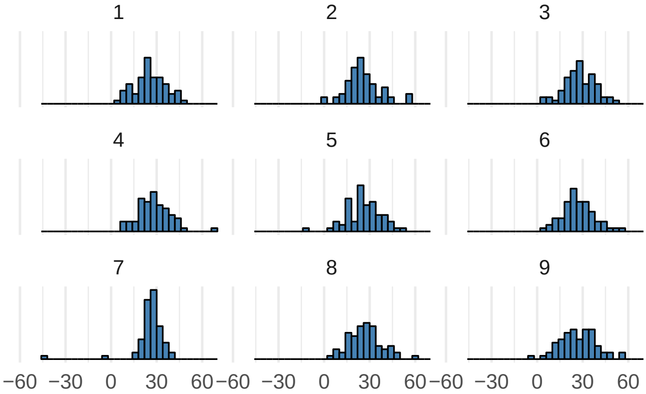

# Posterior predictive checking with test statistic

- Replicated data sets $y^{\mathrm{rep}}$
- Test quantity (or discrepancy measure) $T(y, \theta)$
  - summary quantity for the observed data $T(y, \theta)$
  - summary quantity for a replicated data $T(y^{\mathrm{rep}}, \theta)$
  - can be easier to compare summary quantities ($y^{\mathrm{rep}}$ statistics) than data sets

# Posterior predictive checking – example

- Compute test statistic for data $T(y, \theta) = \min(y)$

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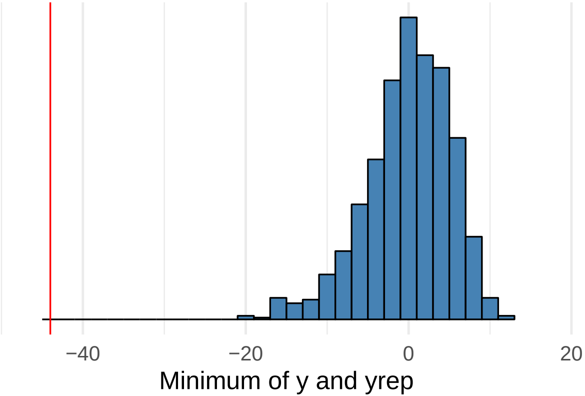

# Posterior predictive checking – example

- Good test statistic is ancillary (or almost)
  - a statistic $T(X)$ that does not depend on the parameters of the model are ancillary
    e.g. in a normal model with known $\sigma^2$,
    $$s^2 = \sum_i^n \frac{(x_i - \bar{x})^2}{n-1},$$
    is ancillary ($\mu$ cancel out).
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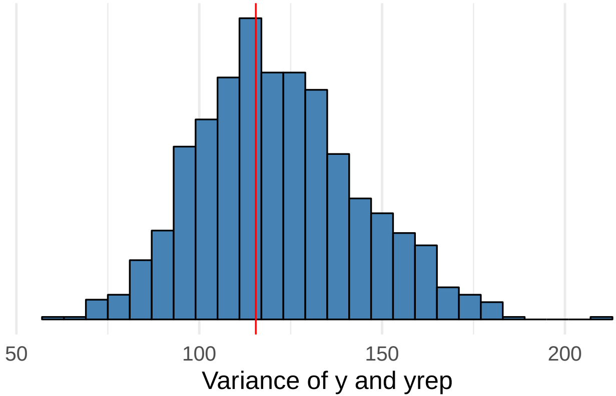
- We want to identify problems in data not captured by the model
- Connections:
  - Sufficient statistic: contains all the information about $\theta$
  - Ancillary statistic: contains no information about $\theta$

# Posterior predictive checking – bad example



# Posterior predictive $p$-value

- *Posterior predictive p-value*

$$
\begin{aligned}
p &= \Pr(T(y^{\mathrm{rep}}, \theta) \geq T(y, \theta) | y) \\
&= \int\int I_{T(y^{\mathrm{rep}}, \theta) \geq T(y, \theta)} p(y^{\mathrm{rep}} | \theta) p(\theta | y) dy^{\mathrm{rep}} d\theta
\end{aligned}
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where $I$ is an indicator function

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- having $(y^{\text{rep}\,(s)}, \theta^{(s)})$ from the posterior predictive distribution (Monte Carlo):

$$T(y^{\text{rep}(s)}, \theta^{(s)}) \geq T(y, \theta^{(s)}), \quad s = 1, \ldots, S$$

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- Posterior predictive $p$-value (ppp-value): could difference between the model and data arise by chance
- Not commonly used, since the distribution of test statistic $T(y, \theta)$ has more information

Subsection 1

## Marginal Predictive Checking

# Marginal and CV predictive checking

- Consider marginal predictive distributions $p(\tilde{y}_i|y)$ and each observation separately
  - marginal posterior p-values

$$p_i = \Pr(T(y_i^{\mathrm{rep}}) \leq T(y_i)|y)$$

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  - if $T(y_i) = y_i$

$$p_i = \Pr(y_i^{\text{rep}} \leq y_i|y)$$

- if $Pr(\tilde{y}_i|y)$ well calibrated, distribution of $p_i$ would be uniform between 0 and 1
  - holds better for cross-validation predictive tests: $Pr(\tilde{y}_i|y_{-i})$ (cross-validation)

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# Marginal predictive checking - Example

- Marginal tail area or Probability integral transform (PIT)

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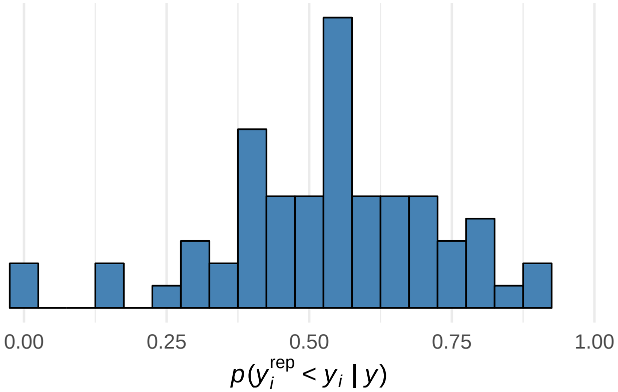

# Subsection 2

## Sensitivity analysis

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# Sensitivity analysis

- How much different choices in model structure and priors affect the results

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- How much different choices in model structure and priors affect the results
  - test different models and priors
  - alternatively combine different models to one model
    - e.g. hierarchical model instead of separate and pooled
    - e.g. $t$ distribution contains Gaussian as a special case
  - robust models are good for testing sensitivity to “outliers”
    - e.g. $t$ instead of Gaussian

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- Compare sensitivity of essential inference quantities
  - extreme quantiles are more sensitive than means and medians
  - extrapolation is more sensitive than interpolation

Subsection 3

## Example

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# Example: Exposure to air pollution

- Example from Jonah Gabry, Daniel Simpson, Aki Vehtari, Michael Betancourt, and Andrew Gelman (2019). Visualization in Bayesian workflow. https://doi.org/10.1111/rssa.12378
- Estimation of human exposure to air pollution from particulate matter measuring less than 2.5 microns in diameter ($\mathrm{PM}_{2.5}$)
  - Exposure to $\mathrm{PM}_{2.5}$ is linked to a number of poor health outcomes and a recent report estimated that $\mathrm{PM}_{2.5}$ is responsible for three million deaths worldwide each year (Shaddick et al., 2017)
  - In order to estimate the public health effect of ambient $\mathrm{PM}_{2.5}$, we need a good estimate of the $\mathrm{PM}_{2.5}$ concentration at the same spatial resolution as our population estimates.

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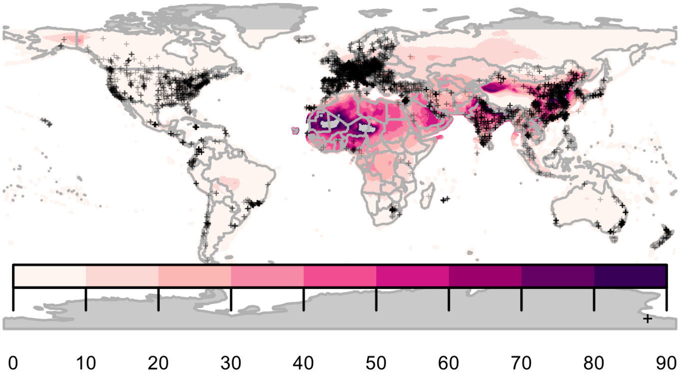
# Example: Exposure to air pollution

- Direct measurements of PM 2.5 from ground monitors at 2980 locations
- High-resolution satellite data of aerosol optical depth



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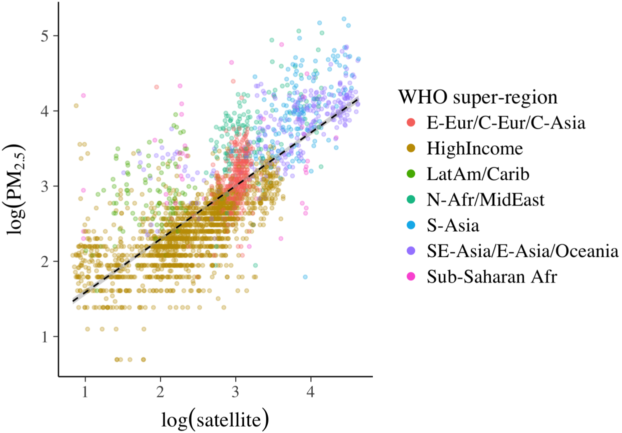

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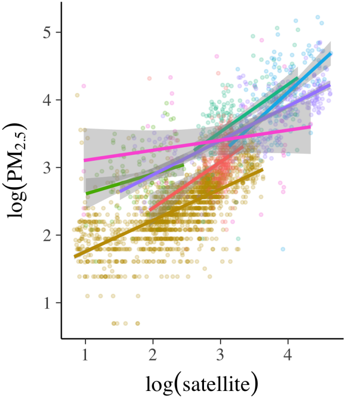


# Example: Exposure to air pollution

- Three models:
  1. Linear regression

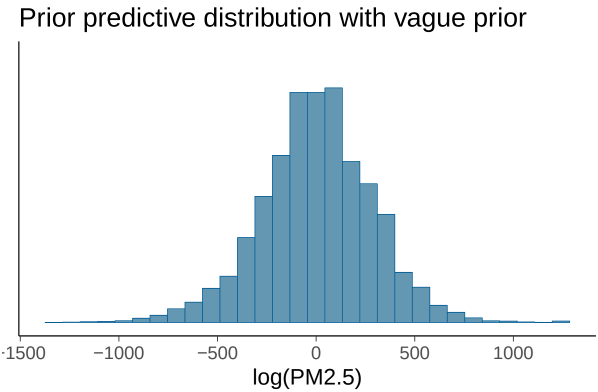
$$y_{ij} \sim N(\beta_0 + \beta_1 x_{ij}, \sigma^2)$$

  2. Hiearchical linear regression (WHO super regions)
  3. Hiearchical linear regression (clustered super regions)

$$y_{ij} \sim N(\beta_0 \beta_{0j} + (\beta_1 + \beta_{1j}) x_{ij}, \sigma^2)$$

where $y_{ij}$ is the log of $PM_{2.5}$ concentration and $x_{ij}$ is the satellite estimate, and $j \in \{1, ..., J\}$ is the super-region indicator.

# Example: Prior predictive checking



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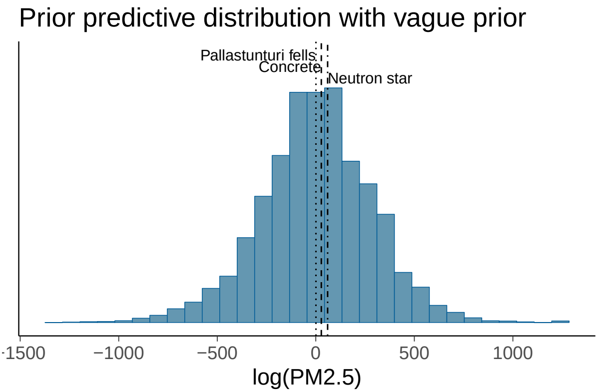

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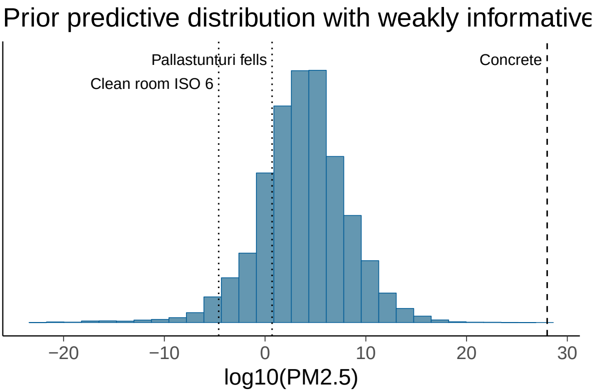

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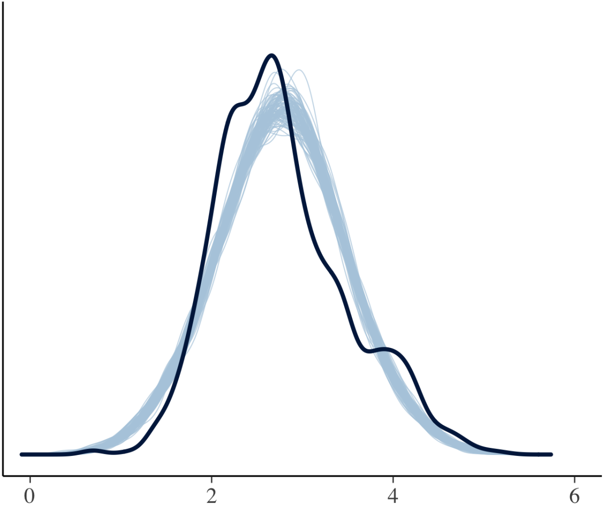


Figure: Model 1

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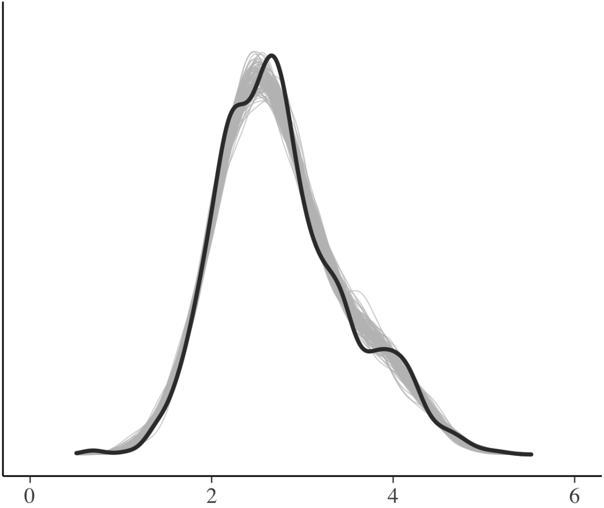


Figure: Model 2

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# Example: Marginal predictive distributions

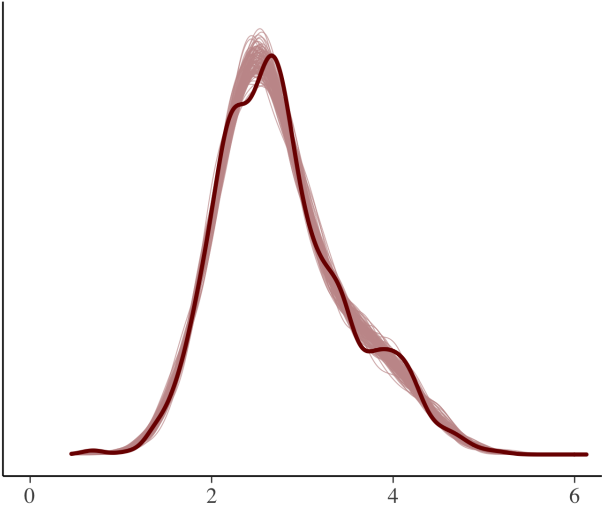
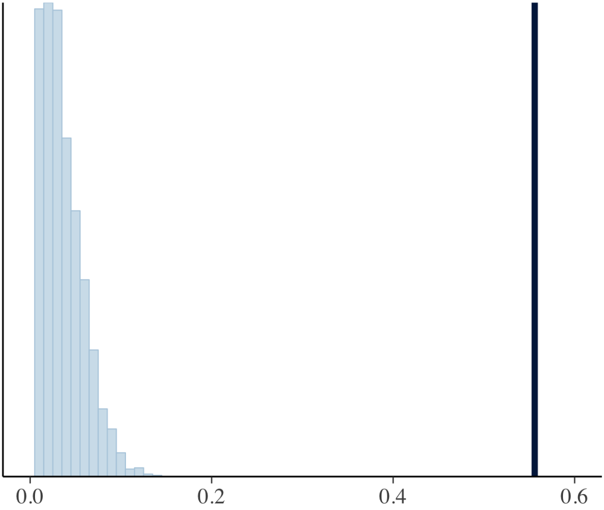



Figure: Model 3

# Example: Test statistic (skewness)

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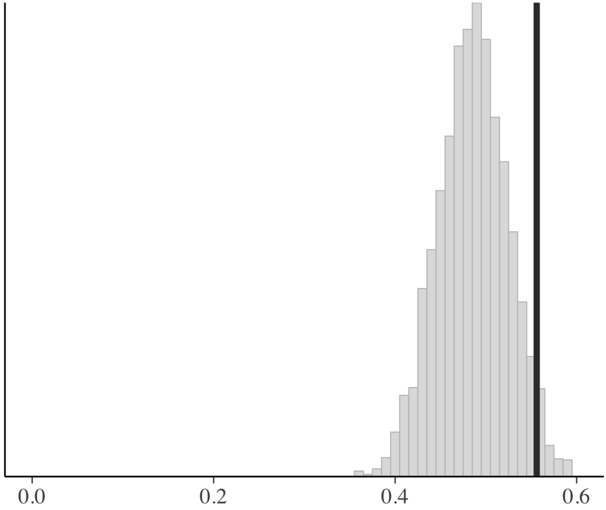


Figure: Model 2

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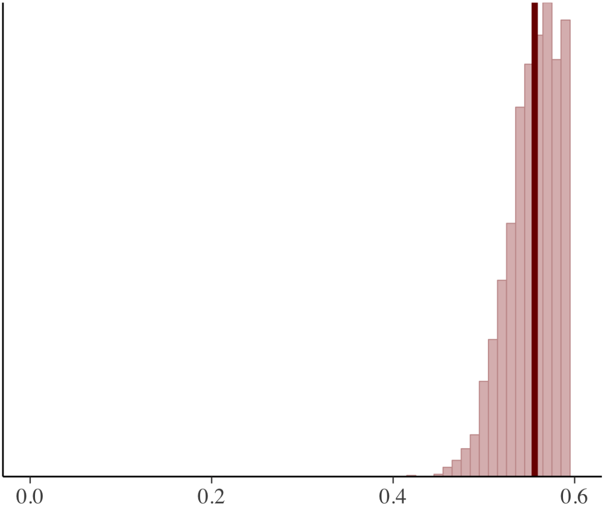


Figure: Model 3