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# Machine learning – Block 8


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Autumn 2022

# This week's lectures

- Introduction to Reinforcement learning

# Introduction to Reinforcement Learning

- Another type of Machine Learning:
  - Supervised Learning
  - Unsupervised Learning
  - Reinforcement learning

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- Another type of Machine Learning:
  - Supervised Learning
  - Unsupervised Learning
  - Reinforcement learning
- Computational approach of learning from interaction
- Closest to human and animal learning: trial, error, and planning.
- The learner is *not* told which actions to take

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  - Unsupervised Learning
  - Reinforcement learning
- Computational approach of learning from interaction
- Closest to human and animal learning: trial, error, and planning.
- The learner is *not* told which actions to take
- Connections to:
  - Game Theory
  - Control Theory
  - Multi-agent systems
  - Swarm intelligence
  - Information theory
  - Statistics

# Introduction to Reinforcement Learning

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- Three characteristics:
  1. Closed-loop: early actions affects later actions
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  3. Reward signals over a long period of time

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- Introduction to Reinforcement Learning
- Bandits
- Markov Decision Processes

# Recent Achievements

- RL agent won over Lee Sedol in 2016: AlphaGo

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- Automated trading
- Elevator scheduling
- A/B testing and personalized recommendations
- Board games such as backgammon, chess and checkers

# The different parts in RL

1. The Agent: The learning agent.

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5. Return: The aggregated reward over a long period.

# The different parts in RL

1. Agents:
   1.1 Have a goal (maximize return)
   1.2 Sense aspect of their environment
   1.3 Choose actions
   1.4 Possibility to improve performance over time

# The different parts in RL

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1. Agents:
   1.1 Have a goal (maximize return)
   1.2 Sense aspect of their environment
   1.3 Choose actions
   1.4 Possibility to improve performance over time
2. Usually an uncertainty about the environment
3. Represent uncertainty of environment: Probability

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# Sub-elements of agents

1. Policy: How the agent choose actions. Determines behaviour.
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3. Value function: The long-term value (the expected long-term return following a policy)

- Outside agent: Reward signal: The instant value of an action
- Problem: Balance the trade-off between long-term and short-term rewards

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# Supervised, Unsupervised and Reinforcement Learning

1. Static vs. Dynamic

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1. Static vs. Dynamic
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4. Need for exploration
5. Evaluates actions - not only instruct actions

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- Goal: Maximize the return (the total reward), i.e.

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# Exploration vs Exploitation

- Goal: Maximize the return (the total reward), i.e.
- Exploit the best actions
- Explore to know the best actions

# Evolution vs Learning

- Set a policy without learning: Evolutionary Methods
- Good when agent cannot sense the environment

# Evolution vs Learning

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- Set a policy without learning: Evolutionary Methods
- Good when agent cannot sense the environment
- Example: Bacteria don't learn, they evolve

# Setting the goal for the Agent

- Setting the goal: defining the reward signal (reward function)
- Example: If you want the agent to do something quick, give -1 per action.

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- Setting the goal: defining the reward signal (reward function)
- Example: If you want the agent to do something quick, give -1 per action.
- We should give rewards for correct behaviour
- Do not use reward to guide how to reach the goal
- Be careful what you wish for...

# The $k$-armed bandit problem

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- This is a tabular method/problem:
  We can represent the actions in a table.
- Tabular methods works in small problems
  e.g. A/B testing and dynamic web pages.

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# Exploration vs. Exploitation

- Two types of actions:
  1. Exploitation: Choose the action with highest expected reward (short term)
  2. Exploration: Choose action to improve $Q_t(a)$, but reduces the reward (long term)
- The conflict between exploration and exploitation

# $\epsilon$-greedy

- $\epsilon$-greedy: $P(\text{exploration}) = \epsilon$

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- We estimate $q^\star(a)$ using $Q_t(a)$ as

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  - Large $V(R_t)$
  - Large $\mathcal{A}$
  - Non-stationarity

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# Bandit example

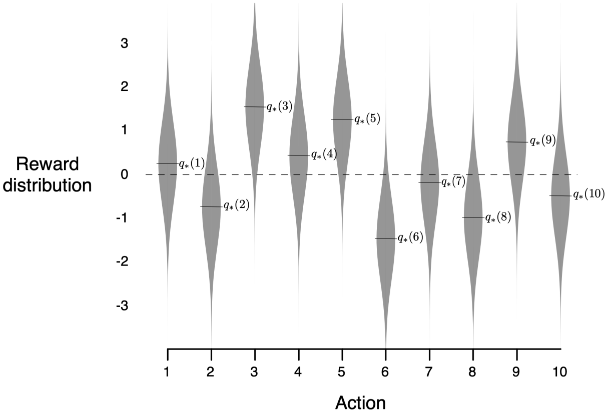


Figure: The 10-armed bandit environment (Sutton and Barto, 2017, Fig. 2.1)

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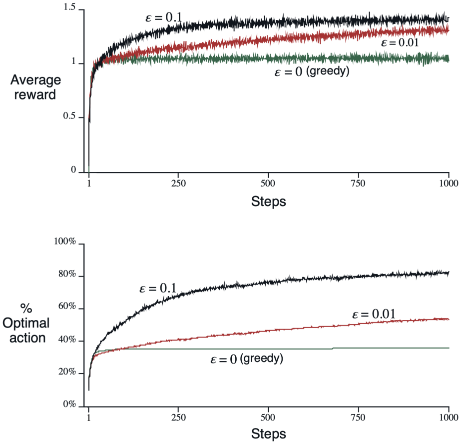


Figure: The $\epsilon$-greedy algorithm result in the 10-armed bandit (Sutton and Barto, 2017, Fig. 2.2)

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# Bandit example: Optimistic initialization

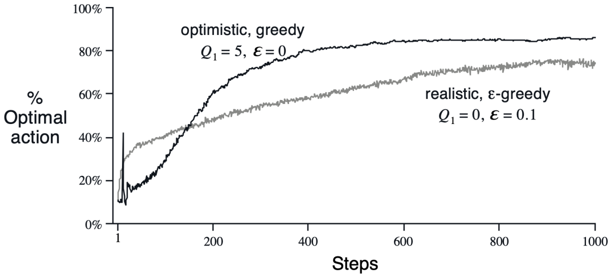


Figure: The $\epsilon$-greedy algorithm and optimistic initialization (Sutton and Barto, 2017, Fig. 2.3)

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# Efficient computation and non-stationarity

- Compute $Q_t(a)$ on the fly:

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  - $\alpha(t) = \frac{1}{N_t(a)}$: Average reward
- $Q_T(a) \to q^\star(a)$, if:
  1. $\sum_t^\infty \alpha_t = \infty$
  2. $\sum_t^\infty \alpha_t^2 < \infty$
- Where have we seen these criterias before?

# The $\epsilon$-greedy algorithm

**A simple bandit algorithm**

Initialize, for $a = 1$ to $k$:

$Q(a) \leftarrow 0$
$N(a) \leftarrow 0$

Repeat forever:

$$A \leftarrow \begin{cases} \arg\max_a Q(a) & \text{with probability } 1-\varepsilon \quad \text{(breaking ties randomly)} \\ \text{a random action} & \text{with probability } \varepsilon \end{cases}$$

$R \leftarrow bandit(A)$
$N(A) \leftarrow N(A) + 1$
$Q(A) \leftarrow Q(A) + \frac{1}{N(A)}\big[R - Q(A)\big]$

Figure: The $\epsilon$-greedy algorithm

# The Upper-Confidence-Bound method

- Explore based on our uncertainty of $Q_t(a)$

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- Explore based on our uncertainty of $Q_t(a)$
- The Upper-Confidence-Bound (UCB) method

$$A_t = \arg\max_a \left( Q_t + c\sqrt{\frac{\log t}{N_t(a)}} \right)$$

An analogy:

$$A_t = \arg\max_a \left( Q_t + c\sqrt{\frac{\hat{\sigma}^2(a)}{N_t(a)}} \right)$$

- Another (Bayesian) alternative is Thompson Sampling

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# The UCB algorithm

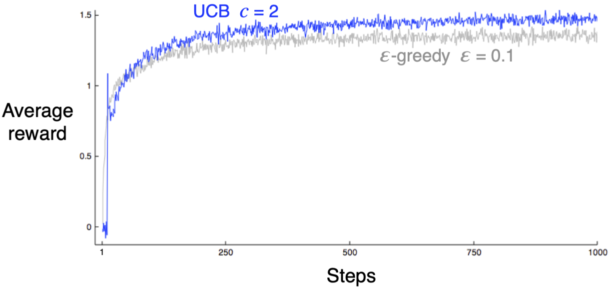


Figure: The UCB algorithm

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# The Markov Decision process

- Bandit does not have a *state*.
- An action might change the environment.
- An action might be different in different states
- Example: In chess, we want to make a move based on the current position of all pieces
- To capture this we use a Markov Decision process
- One of the most important concepts in Reinforcement Learning

# The Markov Decision process

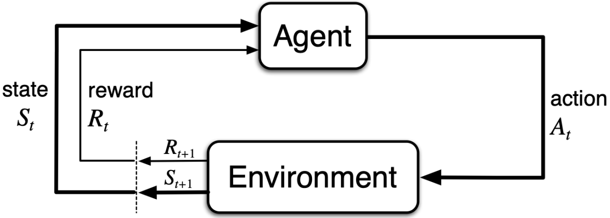


Figure: The (finite) Markov Decision Process (Sutton and Barto, 2017, Fig 3.1)

- States $S_t \in \mathcal{S}$: Basis for action
- Actions $A_t \in \mathcal{A}$
- Rewards $R_t \in \mathbb{R}$

# The Markov Decision process

- Boundry between Agent and Environment:
  - The total control of the action

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  - Reward is external to agent: Pain and pleasure
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# The Markov Decision process

- Boundry between Agent and Environment:
  - The total control of the action
  - Reward is external to agent: Pain and pleasure
  - The agent should not be able to change the reward function
- The policy ($\pi(A_t|S_t = s)$):
  - We make an action given the current state $S_t$
- The goal: (Again) maximize return $G_t = R_{t+1} + ... + R_T$

# Return and discount

- Two type of interactions
  - Episodic: $T < \infty$, has terminal state
  - Continuing: $T = \infty$
- Discounting:

$$G_t = R_{t+1} + \gamma R_{t+3} + \gamma^2 R_{t+3} + \ldots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

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  - $\gamma = 1$: No discount
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# Return and discount

- Two type of interactions
  - Episodic: $T < \infty$, has terminal state
  - Continuing: $T = \infty$
- Discounting:

$$G_t = R_{t+1} + \gamma R_{t+3} + \gamma^2 R_{t+3} + \dots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

- Discount rate $\gamma$:
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- For episodic problem we assume $R_{T+i} = 0$ for all $i \in \mathbb{N}^+$

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$$p(s' | s, a) = P(S_{t+1} = s' | S_t = s, A_t = a)$$

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- Bellman equation is the basis for computing $v_\pi(s)$ (not part of this course)

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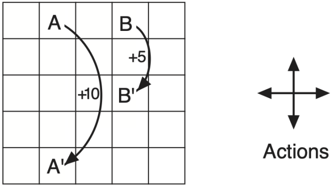


| | | | | |
|---|---|---|---|---|
| 3.3 | 8.8 | 4.4 | 5.3 | 1.5 |
| 1.5 | 3.0 | 2.3 | 1.9 | 0.5 |
| 0.1 | 0.7 | 0.7 | 0.4 | -0.4 |
| -1.0 | -0.4 | -0.4 | -0.6 | -1.2 |
| -1.9 | -1.3 | -1.2 | -1.4 | -2.0 |

Figure: The gridworld equiprobable policy value function

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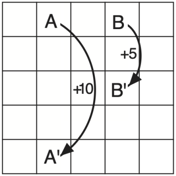
- We might also estimate $v_\star(s)$ better for commonly encountered states

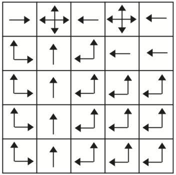
# The value function




Gridworld

| | | | | |
|---|---|---|---|---|
| 22.0 | 24.4 | 22.0 | 19.4 | 17.5 |
| 19.8 | 22.0 | 19.8 | 17.8 | 16.0 |
| 17.8 | 19.8 | 17.8 | 16.0 | 14.4 |
| 16.0 | 17.8 | 16.0 | 14.4 | 13.0 |
| 14.4 | 16.0 | 14.4 | 13.0 | 11.7 |

$v_*$



$\pi_*$

Figure: The gridworld optimal value function and policy