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- Introduction to Reinforcement Learning
- Bandits
- Markov Decision Processes

Machine learning – Block 8

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This week's lectures

- Introduction to Reinforcement learning

Introduction to Reinforcement Learning

- Another type of Machine Learning:
  - Supervised Learning
  - Unsupervised Learning
  - Reinforcement learning

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Introduction to Reinforcement Learning

- Another type of Machine Learning:
  - Supervised Learning
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  - Reinforcement learning
- Computational approach of learning from interaction
- Closest to human and animal learning: trial, error, and planning.
- The learner is *not* told which actions to take

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  - Reinforcement learning
- Computational approach of learning from interaction
- Closest to human and animal learning: trial, error, and planning.
- The learner is *not* told which actions to take
- Connections to:
  - Game Theory
  - Control Theory
  - Multi-agent systems
  - Swarm intelligence
  - Information theory
  - Statistics

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Introduction to Reinforcement Learning

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  1. Closed-loop: early actions affects later actions
  2. No direct instructions
  3. Reward signals over a long period of time

Recent Achievements

- RL agent won over Lee Sedol in 2016: AlphaGo

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- Elevator scheduling
- A/B testing and personalized recommendations
- Board games such as backgammon, chess and checkers

The different parts in RL

1. The Agent: The learning agent.

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5. Return: The aggregated reward over a long period.

The different parts in RL

1. Agents:
   1.1 Have a goal (maximize return)
   1.2 Sense aspect of their environment
   1.3 Choose actions
   1.4 Possibility to improve performance over time

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1. Agents:
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   1.4 Possibility to improve performance over time
2. Usually an uncertainty about the environment
3. Represent uncertainty of environment: Probability

Sub-elements of agents

1. Policy: How the agent choose actions. Determines behaviour.
2. Model: The agent's model of the environment. Used for planning

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- Outside agent: Reward signal: The instant value of an action
- Problem: Balance the trade-off between long-term and short-term rewards

Supervised, Unsupervised and Reinforcement Learning

1. Static vs. Dynamic

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Supervised, Unsupervised and Reinforcement Learning

1. Static vs. Dynamic
2. No Gold Standard
3. Multiple-Decision Process: Return vs. reward
4. Need for exploration
5. Evaluates actions - not only instruct actions

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Exploration vs Exploitation

- Goal: Maximize the return (the total reward), i.e.

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- Goal: Maximize the return (the total reward), i.e.
- Exploit the best actions
- Explore to know the best actions

Evolution vs Learning

- Set a policy without learning: Evolutionary Methods
- Good when agent cannot sense the environment

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- Good when agent cannot sense the environment
- Example: Bacteria don't learn, they evolve

Setting the goal for the Agent

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- Example: If you want the agent to do something quick, give -1 per action.

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- Setting the goal: defining the reward signal (reward function)
- Example: If you want the agent to do something quick, give -1 per action.
- We should give rewards for correct behaviour
- Do not use reward to guide how to reach the goal
- Be careful what you wish for...

Section 2

Bandits

The k-armed bandit problem

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- This is a tabular method/problem:
  We can represent the actions in a table.
- Tabular methods works in small problems
  e.g. A/B testing and dynamic web pages.

Exploration vs. Exploitation

- Two types of actions:
  1. Exploitation: Choose the action with highest expected reward (short term)
  2. Exploration: Choose action to improve $Q_t(a)$, but reduces the reward (long term)
- The conflict between exploration and exploitation

ϵ-greedy

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  - Large $V(R_t)$

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  - Large $V(R_t)$
  - Large $\mathcal{A}$
  - Non-stationarity

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Bandit example

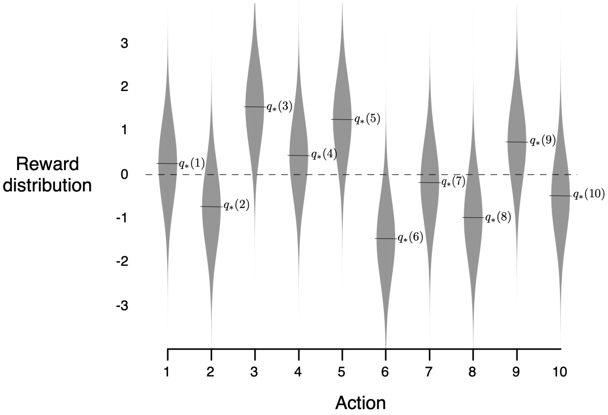

Figure: The 10-armed bandit environment (Sutton and Barto, 2017, Fig. 2.1)

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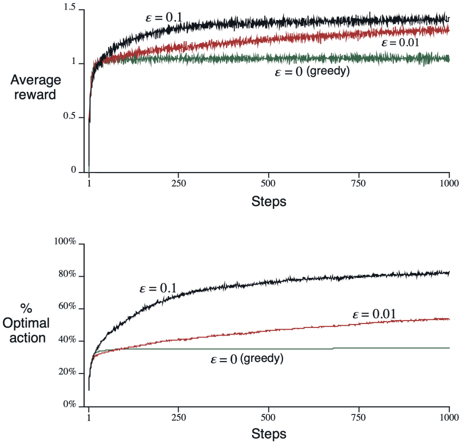

Figure: The ϵ-greedy algorithm result in the 10-armed bandit (Sutton and Barto, 2017, Fig. 2.2)

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Bandit example: Optimistic initialization

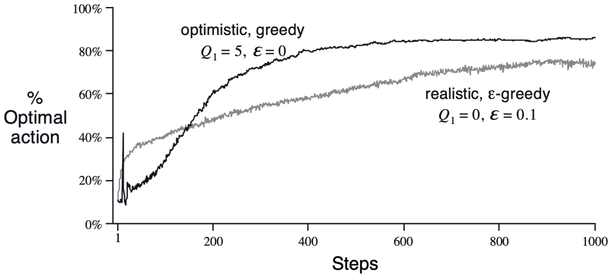

Figure: The ϵ-greedy algorithm and optimistic initialization (Sutton and Barto, 2017, Fig. 2.3)

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Efficient computation and non-stationarity

- Compute $Q_t(a)$ on the fly:

$$Q_T(a) = Q_{T-1} + \frac{1}{N_t(a)}(R_{t,A_t=a} - Q_{T-1}(a))$$

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  - $\alpha(t) = \frac{1}{N_t(a)}$: Average reward
- $Q_T(a) \to q^\star(a)$, if:
  1. $\sum_t^\infty \alpha_t = \infty$
  2. $\sum_t^\infty \alpha_t^2 < \infty$
- Where have we seen these criterias before?

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The ϵ-greedy algorithm

A simple bandit algorithm

Initialize, for $a = 1$ to k:

$Q(a) \leftarrow 0$
$N(a) \leftarrow 0$

Repeat forever:

$$A \leftarrow \begin{cases} \arg\max_a Q(a) & \text{with probability } 1-\varepsilon \quad \text{(breaking ties randomly)} \\ \text{a random action} & \text{with probability } \varepsilon \end{cases}$$

$R \leftarrow bandit(A)$
$N(A) \leftarrow N(A) + 1$
$Q(A) \leftarrow Q(A) + \frac{1}{N(A)}\left[R - Q(A)\right]$

Figure: The ϵ-greedy algorithm

The Upper-Confidence-Bound method

- Explore based on our uncertainty of $Q_t(a)$

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The Upper-Confidence-Bound method

- Explore based on our uncertainty of $Q_t(a)$
- The Upper-Confidence-Bound (UCB) method

$$A_t = \arg\max_a \left(Q_t + c\sqrt{\frac{\log t}{N_t(a)}} \right)$$

An analogy:

$$A_t = \arg\max_a \left(Q_t + c\sqrt{\frac{\hat{\sigma}^2(a)}{N_t(a)}} \right)$$

The UCB algorithm

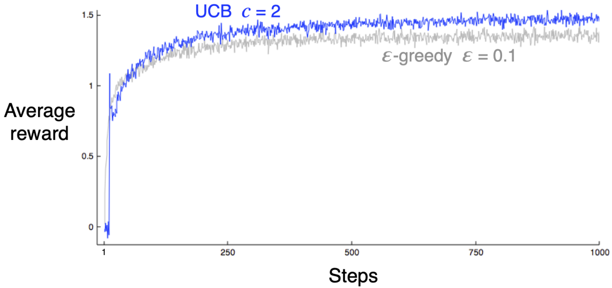

Figure: The UCB algorithm

The Bayesian Bandit: Thompson sampling

- A Bayesian Bandit
  1. Setup a likelihood for R, $p(R|a, \theta)$
  2. Setup a prior for θ, $p(\theta)$
  3. Compute posterior for θ, $p(\theta|R, a)$
  4. Choose action A_t proportional to

  $$\int I[\mathbb{E}(R|\theta, a^\star) = \arg\max_{a'} \mathbb{E}(R|\theta, a')]p(\theta|R, a)d\theta$$

  where I is the indicator function.
- Repeat step 3-4

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- Repeat step 3-4
- Monte Carlo approximation of step 4:
  1. Draw one sample from the posterior $\tilde{\theta}$

  $$\tilde{\theta} \sim p(\theta|R, a)$$

  2. Conditional on $\tilde{\theta}$, choose action A_t

  $$A_t = \arg\max_{a'} \mathbb{E}(R|\tilde{\theta}, a')]$$

Section 3

Markov Decision Processes

The Markov Decision process

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- An action might change the environment.
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- Bandit does not have a *state*.
- An action might change the environment.
- An action might be different in different states
- Example: In chess, we want to make a move based on the current position of all pieces
- To capture this we use a Markov Decision process
- One of the most important concepts in Reinforcement Learning

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The Markov Decision process

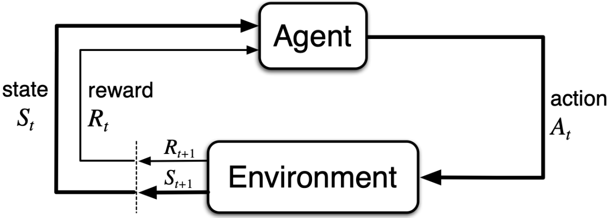

Figure: The (finite) Markov Decision Process (Sutton and Barto, 2017, Fig 3.1)

- States $S_t \in \mathcal{S}$: Basis for action
- Actions $A_t \in \mathcal{A}$
- Rewards $R_t \in \mathbb{R}$

The Markov Decision process

- Boundry between Agent and Environment:
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The Markov Decision process

- Boundry between Agent and Environment:
  - The total control of the action
  - Reward is external to agent: Pain and pleasure
  - The agent should not be able to change the reward function
- The policy ($\pi(A_t|S_t = s)$):
  - We make an action given the current state S_t
- The goal: (Again) maximize return $G_t = R_{t+1} + ... + R_T$

Return and discount

- Two type of interactions
  - Episodic: $T < \infty$, has terminal state
  - Continuing: $T = \infty$
- Discounting:

$$G_t = R_{t+1} + \gamma R_{t+3} + \gamma^2 R_{t+3} + \ldots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

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- Discount rate γ:
    - $0 \leq \gamma \leq 1$

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- Discount rate γ:
  - $0 \leq \gamma \leq 1$
  - $\gamma = 1$: No discount
  - $\gamma = 0$: Full discount: Only next reward counts
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Return and discount

- Two type of interactions
  - Episodic: $T < \infty$, has terminal state
  - Continuing: $T = \infty$
- Discounting:

$$G_t = R_{t+1} + \gamma R_{t+3} + \gamma^2 R_{t+3} + \ldots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

- Discount rate γ:
  - $0 \leq \gamma \leq 1$
  - $\gamma = 1$: No discount
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- For episodic problem we assume $R_{T+i} = 0$ for all $i \in \mathbb{N}^+$

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- Bellman equation is the basis for computing $v_\pi(s)$ (not part of this course)

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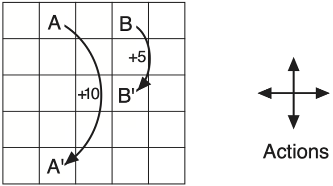
The value function



3.3	8.8	4.4	5.3	1.5
1.5	3.0	2.3	1.9	0.5
0.1	0.7	0.7	0.4	-0.4
-1.0	-0.4	-0.4	-0.6	-1.2
-1.9	-1.3	-1.2	-1.4	-2.0

Figure: The gridworld equiprobable policy value function

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- We might also estimate $v_\star(s)$ better for commonly encountered states

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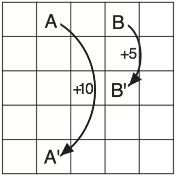
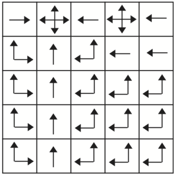
The value function



22.0	24.4	22.0	19.4	17.5
19.8	22.0	19.8	17.8	16.0
17.8	19.8	17.8	16.0	14.4
16.0	17.8	16.0	14.4	13.0
14.4	16.0	14.4	13.0	11.7

v_*



π_*

Figure: The gridworld optimal value function and policy