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- Introduction to Reinforcement Learning
- Bandits
- Markov Decision Processes

Machine learning – Block 8

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Autumn 2025

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- Introduction to Reinforcement Learning
- Bandits
- Markov Decision Processes

This week's lectures

- Introduction to reinforcement learning

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Introduction to Reinforcement Learning

- Another type of machine learning:
  - Supervised Learning
  - Unsupervised Learning
  - Reinforcement learning

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- Computational approach of learning from interaction

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- Another type of machine learning:
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- Closest to human and animal learning: trial, error, and planning.

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- The learner is *not* told which actions to take

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- Computational approach of learning from interaction
- Closest to human and animal learning: trial, error, and planning.
- The learner is *not* told which actions to take
- Connections to:
  - Game Theory
  - Control Theory
  - Multi-agent systems
  - Swarm intelligence
  - Information theory
  - Statistics

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Introduction to Reinforcement Learning

- Goal: maximize return over a sequence of actions

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- Three characteristics:
  1. Closed-loop: early actions affect later actions

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Introduction to Reinforcement Learning

- Goal: maximize return over a sequence of actions
- Three characteristics:
  1. Closed-loop: early actions affect later actions
  2. No instructions
  3. Reward signals over a long period of time

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Recent Achievements

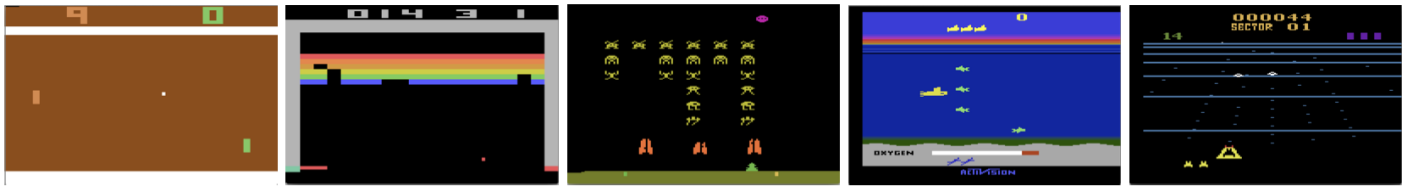

Figure: Mnih et al (2013) "Playing Atari with Deep Reinforcement Learning"

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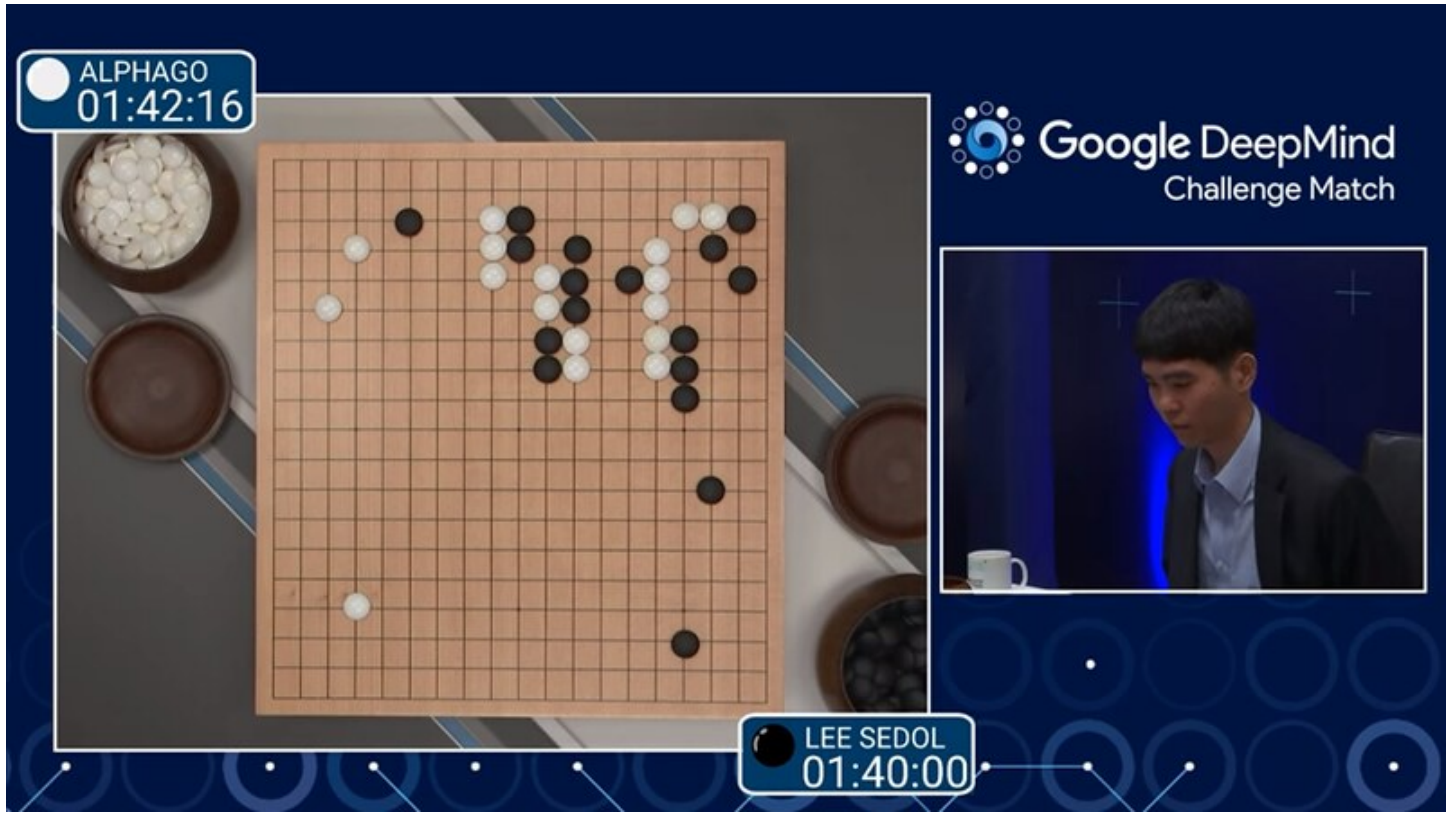

Figure: Lee Sedol vs. Alpha Go in 2016

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Recent Achievements, also

- Industry automation: RL is used to reduce the energy cost of datacenter cooling

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- Automated trading

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- Elevator scheduling

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Recent Achievements, also

- Industry automation: RL is used to reduce the energy cost of datacenter cooling
- Automated trading
- Elevator scheduling
- A/B testing and personalized recommendations

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The different parts in RL

1. The Agent: The learning agent.

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4. Reward: The evaluation of an action. A scalar value. "Pleasure and pain" of actions.

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2. The Environment: Where the agent performs actions.
3. Actions: Made by the agent and affect the environment.
4. Reward: The evaluation of an action. A scalar value. "Pleasure and pain" of actions.
5. Return: The aggregated reward over a long period.

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The different parts in RL

1. Agents:
   - 1.1 Have a goal (maximize return)
   - 1.2 Sense aspect of their environment
   - 1.3 Choose actions
   - 1.4 Possibility to improve performance over time

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The different parts in RL

1. Agents:
   - 1.1 Have a goal (maximize return)
   - 1.2 Sense aspect of their environment
   - 1.3 Choose actions
   - 1.4 Possibility to improve performance over time
2. Usually an uncertainty about the environment
3. Represent uncertainty of environment: Probability

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Sub-elements of agents

1. Policy: How the agent chooses actions. Determines behaviour.
2. Model: The agent's model of the environment. Used for planning

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3. Value function: The long-term value (the expected long-term return following a policy)

- Outside the agent: Reward signal: The instant value of an action
- Problem: Balance the trade-off between long-term and short-term rewards

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Key Distinctions in Reinforcement Learning

Reward vs. Return

- *Reward* R_{t+1}:
  - Immediate, scalar feedback from the environment.
  - Measures short-term desirability of an action.
- *Return* G_t:
  - Cumulative (possibly discounted) future reward.
  - Objective the agent seeks to maximize.

$$G_t = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

Policy vs. Value Function

- *Policy* $\pi(a \mid s)$:
  - Defines how the agent chooses actions.
  - A probability distribution over actions given a state.
- *Value function* $v_\pi(s)$:
  - Evaluates how good a state is under a policy.
  - Expected return starting from state s and following π.

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Supervised, Unsupervised and Reinforcement Learning

1. Static vs. Dynamic

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Supervised, Unsupervised and Reinforcement Learning

1. Static vs. Dynamic
2. No Gold Standard
3. Multiple-Decision Process: Return vs. reward
4. Need for exploration
5. Evaluates actions rather than being explicitly instructed which actions to take

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Exploration vs Exploitation

- Goal: Maximize the return (the total reward), i.e.

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- Goal: Maximize the return (the total reward), i.e.
- Exploit the best actions

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Exploration vs Exploitation

- Goal: Maximize the return (the total reward), i.e.
- Exploit the best actions
- Explore to know the best actions

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Evolution vs Learning

- Set a policy without learning: Evolutionary Methods
- Good when the agent cannot sense the environment

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Evolution vs Learning

- Set a policy without learning: Evolutionary Methods
- Good when the agent cannot sense the environment
- Example: Bacteria don't learn, they evolve

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Setting the goal for the Agent

- Setting the goal: defining the reward signal (reward function)
- Example: If you want the agent to do something quick, give -1 per action.

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Setting the goal for the Agent

- Setting the goal: defining the reward signal (reward function)
- Example: If you want the agent to do something quick, give -1 per action.
- We should give rewards for correct behaviour
- Do not use reward to guide how to reach the goal
- Be careful what you wish for...

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Bandits vs. Markov Decision Processes

Two levels of sequential decision-making

Bandits

- No state: each action is independent of past actions.
- No dynamics: actions do not change the environment.
- Feedback is myopic: reward depends only on the chosen action.
- Objective: identify and exploit the best action.

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Markov Decision Processes (MDPs)

- Explicit state describing the environment.
- Actions influence future states and rewards.
- Feedback is delayed: consequences unfold over time.
- Objective: plan a sequence of actions to maximize long-term return.

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Section 2

Bandits

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The k-armed bandit problem

- Goal: Maximize the total or average reward after N actions

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- This is a tabular method/problem:
  We can represent the actions in a table.
- Tabular methods works in small problems
  e.g. A/B testing and dynamic web pages.

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Exploration vs. Exploitation

- Two types of actions:
  1. Exploitation: Choose the action with highest expected reward (short term)
  2. Exploration: Choose action to improve $Q_t(a)$, but reduces the reward (long term)
- The conflict between exploration and exploitation

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ϵ-greedy

- ϵ-greedy: $P(\text{exploration}) = \epsilon$

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$$A_t \sim U(1, \ldots k)$$
- $Q_1(a) = 0$ (or used to encourage initial exploration)
- For any fixed $\epsilon > 0$, under stationarity and with sample-average updates,
$$Q_t(a) \xrightarrow{\text{a.s.}} q^\star(a) \quad \text{as } t \to \infty.$$

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ϵ-greedy

- We estimate $q^\star(a)$ using $Q_t(a)$ as

$$Q_T(a) = \frac{1}{N_T(a)} \sum_{t=1}^{T} R_t \, \mathbf{1}\{A_t = a\}.$$

where $N_T(a)$ is the total number of times action a has been taken at time point T.

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- When should we explore?
  - Large $V(R_t)$
  - Large $\mathcal{A}$
  - Non-stationarity

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The ϵ-greedy algorithm

A simple bandit algorithm

Initialize, for $a = 1$ to k:
$Q(a) \leftarrow 0$
$N(a) \leftarrow 0$

Repeat forever:
$$A \leftarrow \begin{cases} \arg\max_a Q(a) & \text{with probability } 1-\varepsilon \quad \text{(breaking ties randomly)} \\ \text{a random action} & \text{with probability } \varepsilon \end{cases}$$
$R \leftarrow bandit(A)$
$N(A) \leftarrow N(A) + 1$
$Q(A) \leftarrow Q(A) + \frac{1}{N(A)}\left[R - Q(A)\right]$

Figure: The ϵ-greedy algorithm

Bandit example

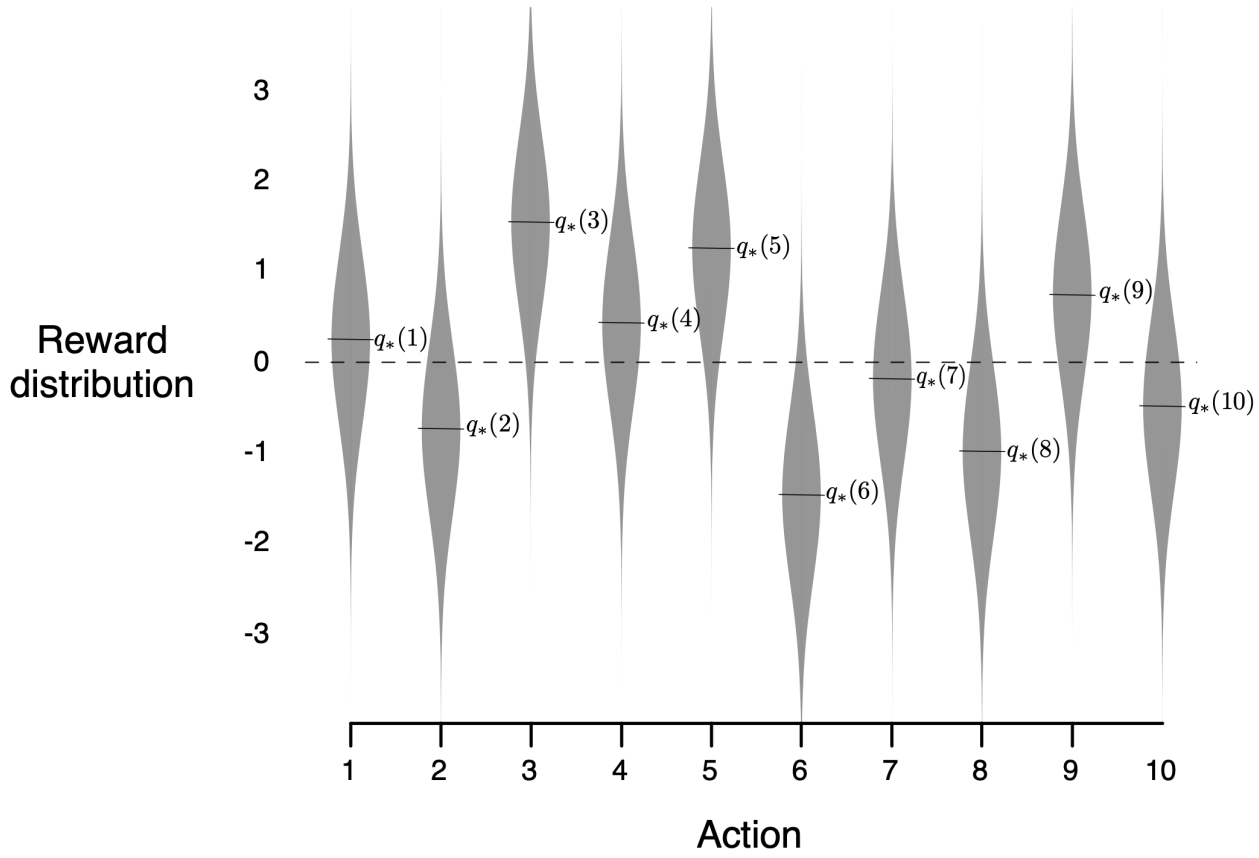

Figure: The 10-armed bandit environment (Sutton and Barto, 2017, Fig. 2.1)

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Bandit example

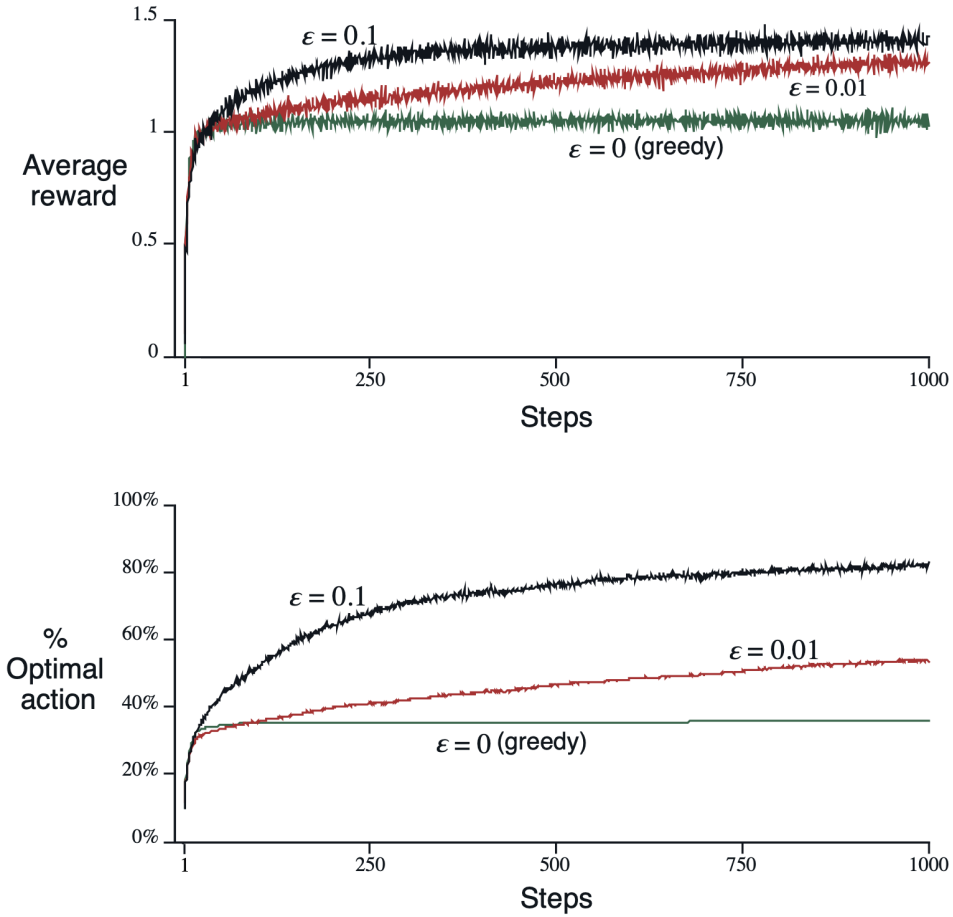

Figure: The ϵ-greedy algorithm result in the 10-armed bandit (Sutton and Barto, 2017, Fig. 2.2)

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Bandit example: Optimistic initialization

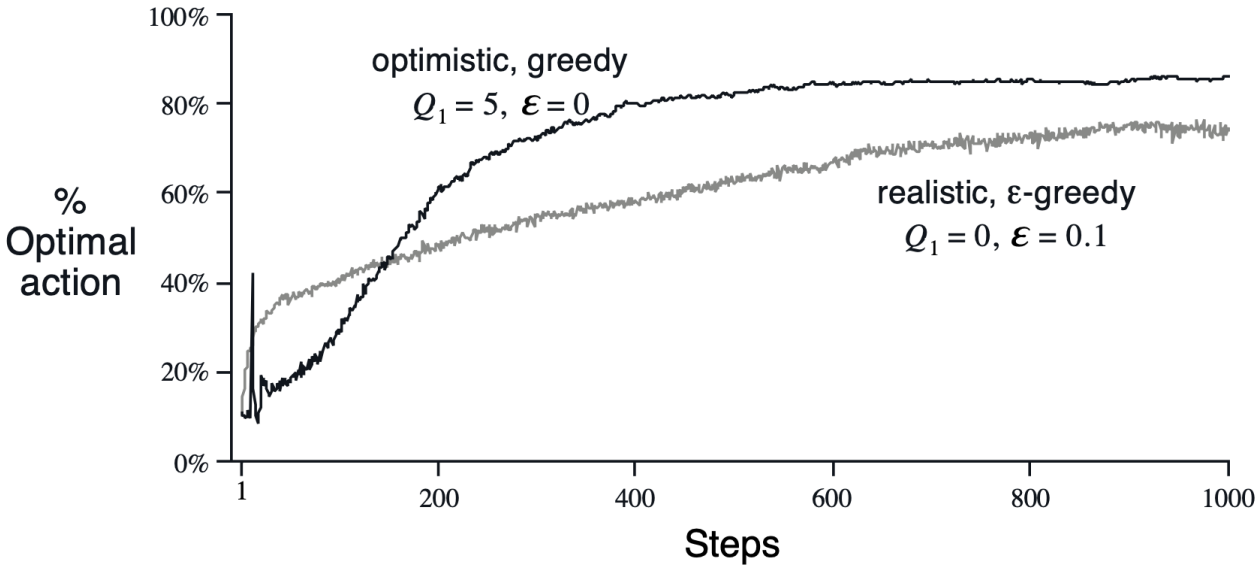

Figure: The ϵ-greedy algorithm and optimistic initialization (Sutton and Barto, 2017, Fig. 2.3)

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Efficient computation and non-stationarity

- Compute $Q_t(a)$ on the fly:

$$Q_T(a) = Q_{T-1} + \frac{1}{N_t(a)}(R_{t,A_t=a} - Q_{T-1}(a))$$

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$$Q_T(a) = Q_{T-1} + \frac{1}{N_t(a)}(R_{t,A_t=a} - Q_{T-1}(a))$$

- Handling non-stationarity:

$$Q_T(a) = Q_{T-1} + \alpha(t)(R_{t,A_t=a} - Q_{T-1}(a))$$

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Efficient computation and non-stationarity

$$Q_{t+1}(a) = Q_t(a) + \alpha_t \big(R_{t+1} - Q_t(a)\big).$$

- Examples:
  - $\alpha(t) = 1$: $Q_T(a) = R_{t,A_t=a}$

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  - $\alpha(t) = 0$: $Q_T(a) = Q_1(a)$
  - $\alpha(t) = \frac{1}{N_t(a)}$: Average reward

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Stochastic Approximation Conditions

$$Q_{t+1}(a) = Q_t(a) + \alpha_t\big(R_{t+1} - Q_t(a)\big).$$

Robbins–Monro conditions

For convergence of stochastic approximation algorithms, the learning rates must satisfy

$$\sum_{t=1}^{\infty} \alpha_t = \infty, \qquad \sum_{t=1}^{\infty} \alpha_t^2 < \infty.$$

Interpretation

- $\sum_t \alpha_t = \infty$: ensures continued learning and prevents premature convergence.
- $\sum_t \alpha_t^2 < \infty$: controls the variance of the stochastic noise and prevents divergence.

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Stochastic Approximation Conditions

$$Q_{t+1}(a) = Q_t(a) + \alpha_t\big(R_{t+1} - Q_t(a)\big).$$

$$\sum_{t=1}^{\infty} \alpha_t = \infty, \qquad \sum_{t=1}^{\infty} \alpha_t^2 < \infty.$$

- These are *not heuristics*, but results from stochastic approximation theory (Robbins & Monro, 1951).
- They guarantee convergence of value estimates under stationarity, sufficient exploration, and bounded rewards.
- Many RL algorithms (e.g. bandits, TD learning, Q-learning) rely on these conditions.

Examples

- Sample-average updates: $\alpha_t = \frac{1}{t}$ (satisfies both conditions).
- Constant step-size: $\alpha_t = \alpha$ (violates $\sum_t \alpha_t^2 < \infty$; useful in non-stationary settings).

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The Upper-Confidence-Bound method

- Explore based on our uncertainty of $Q_t(a)$

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The Upper-Confidence-Bound method

- Explore based on our uncertainty of $Q_t(a)$
- The Upper-Confidence-Bound (UCB) method

$$A_t = \arg\max_a \left(Q_t(a) + c\sqrt{\frac{\log t}{N_t(a)}} \right)$$

where $c > 0$ is a hyperparameter controlling exploration.

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The Upper-Confidence-Bound method

- Explore based on our uncertainty of $Q_t(a)$
- The Upper-Confidence-Bound (UCB) method

$$A_t = \arg\max_a \left(Q_t(a) + c\sqrt{\frac{\log t}{N_t(a)}} \right)$$

where $c > 0$ is a hyperparameter controlling exploration. An analogy:

$$A_t = \arg\max_a \left(Q_t + c\sqrt{\frac{\hat{\sigma}^2(a)}{N_t(a)}} \right)$$

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The UCB algorithm

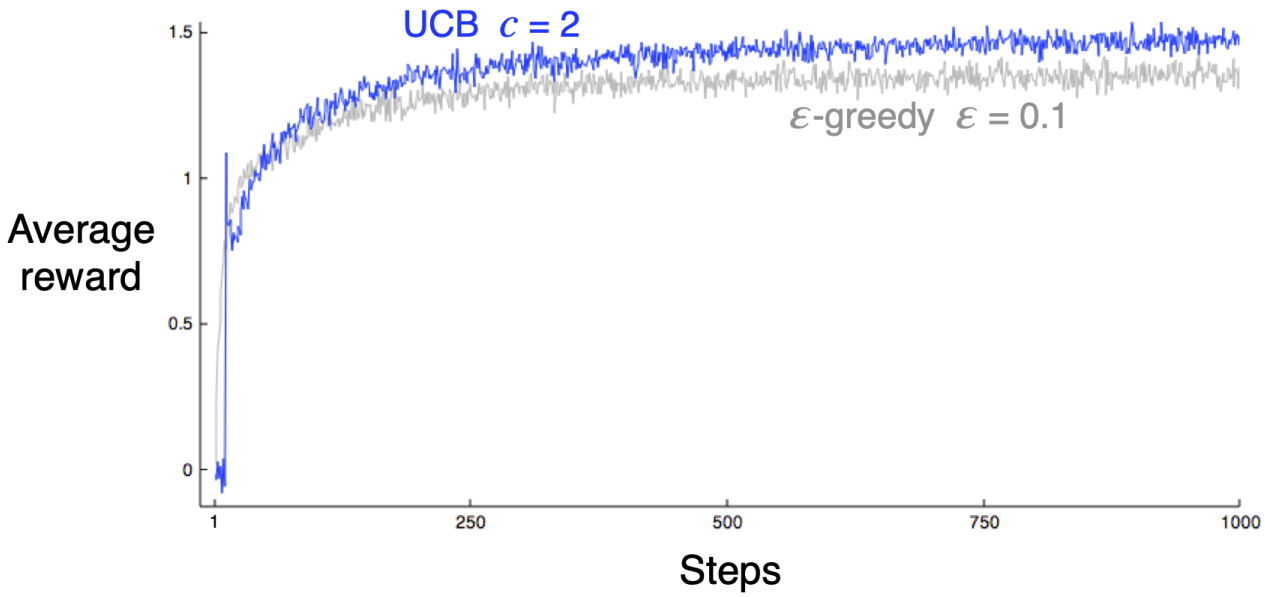

Figure: The UCB algorithm

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The Bayesian Bandit: Thompson sampling

- A Bayesian Bandit
  1. Setup a likelihood for R, $p(R|a, \theta)$
  2. Setup a prior for θ, $p(\theta)$
  3. Compute posterior for θ, $p(\theta|D, a)$
  4. Choose action A_t by

$$\theta \sim p(\theta \mid D), \qquad A_t = \arg\max_a \mathbb{E}[R \mid a, \theta]\,,$$

     where $D = \{(A_1, R_1), \ldots, (A_{t-1}, R_{t-1})\}$.
  5. Collect reward R_t.
- Repeat step 3-5. Step 4 can be approximated using Monte Carlo.

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Section 3

Markov Decision Processes

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The Markov Decision process

- Bandits don't have a *state*.

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- An action might change the environment.
- An action might be different in different states

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- Example: In chess, we want to make a move based on the current position of all pieces

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The Markov Decision process

- Bandits don't have a *state*.
- An action might change the environment.
- An action might be different in different states
- Example: In chess, we want to make a move based on the current position of all pieces
- To capture this we use a Markov Decision process
- One of the most important concepts in reinforcement learning

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The Markov Decision process

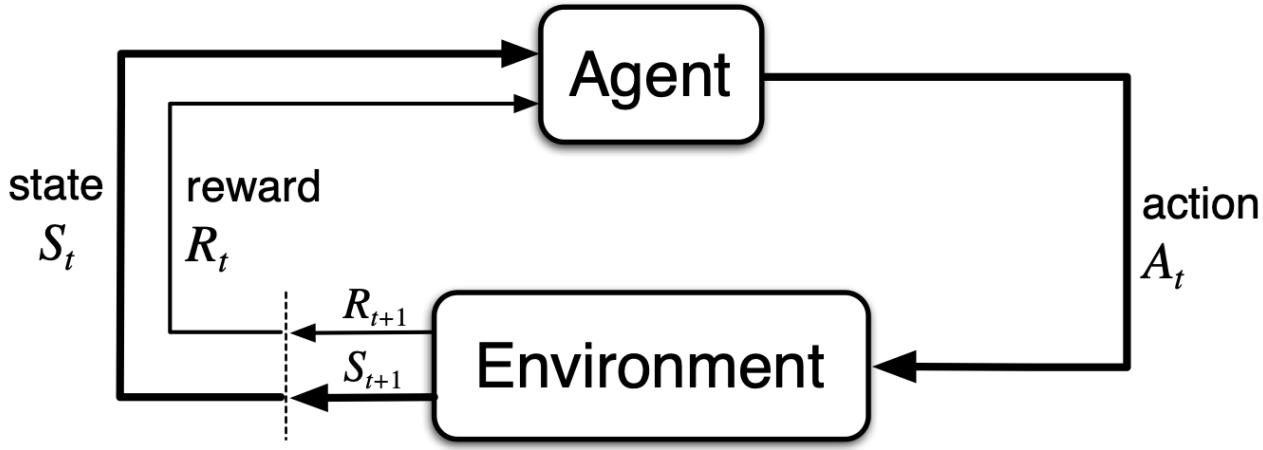

Figure: The (finite) Markov Decision Process (Sutton and Barto, 2017, Fig 3.1)

- States $S_t \in \mathcal{S}$: Basis for action
- Actions $A_t \in \mathcal{A}$
- Rewards $R_t \in \mathbb{R}$

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The Markov Decision process

- Boundary between Agent and Environment:
  - The total control of the action

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The Markov Decision process

- Boundary between Agent and Environment:
  - The total control of the action
  - Reward is external to agent: Pain and pleasure
  - The agent should not be able to change the reward function

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The Markov Decision process

- Boundary between Agent and Environment:
  - The total control of the action
  - Reward is external to agent: Pain and pleasure
  - The agent should not be able to change the reward function
- The policy ($\pi(A_t|S_t = s)$):
  - We make an action given the current state S_t
- The goal: (Again) maximize return $G_t = R_{t+1} + ... + R_T$

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Return and discount

- Two types of interactions
  - Episodic: $T < \infty$, has terminal state
  - Continuing: $T = \infty$
- Discounting:

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

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- For episodic problem we assume $R_{T+i} = 0$ for all $i \in \mathbb{N}^+$

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The Markov Decision Process

- The Markov Decision process (MDP):

$$P(S_{t+1} = s', R_{t+1} = r | S_t = s, A_t = a) \quad (1)$$

- An MDP is fully specified by the tuple $(\mathcal{S}, \mathcal{A}, P, R, \gamma)$.

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- Markov property:

$$P(S_{t+1} = s', R_{t+1} = r | S_1 = s, A_1 = a, ..., S_t = s, A_t = a) = P(S_{t+1} = s', R_{t+1} = r | S_t = s, A_t = a)$$

- The MDP is a good approximation or model:
  All models are wrong, but some are useful.

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The Markov Decision Process Marginals

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  - State-action rewards:

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  - State-transition probability:

$$p(s' | s, a) = P(S_{t+1} = s' | S_t = s, A_t = a)$$

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The Value function

- The value function $v_\pi(s)$:
  the long-term value of s given a policy $\pi(a|s)$, i.e.

$$v_\pi(s) = \mathbb{E}_\pi(G_t|S_t = s) = \mathbb{E}_\pi\left(\sum_{k=0}^{\infty} \gamma^k R_{t+k+1}|S_t = s\right)$$

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- Informally: How "good" is a state for the agent with the policy π.

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- Estimating $v_\pi(s)$ is one of the most important problem in RL

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- Value functions are recursive:

$$\begin{aligned} v_\pi(s) &= \mathbb{E}_\pi(G_t|S_t = s) \\ &= \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a)(r + \gamma v_\pi(s')) \end{aligned}$$

- This is the Bellman equation for $v_\pi(s)$:
  The relationship between the values of the state and its successor states.

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- This is the Bellman equation for $v_\pi(s)$:
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- Bellman equation is the basis for computing $v_\pi(s)$ (not part of this course)

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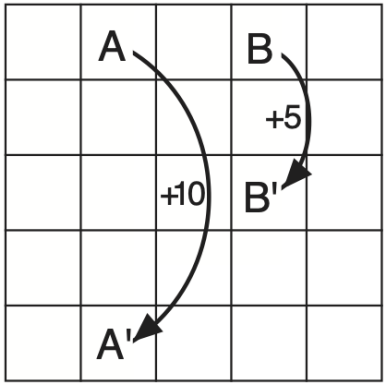
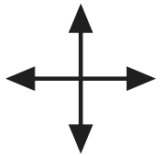
The value function





3.3	8.8	4.4	5.3	1.5
1.5	3.0	2.3	1.9	0.5
0.1	0.7	0.7	0.4	-0.4
-1.0	-0.4	-0.4	-0.6	-1.2
-1.9	-1.3	-1.2	-1.4	-2.0

Figure: The gridworld equiprobable policy value function

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The Optimal Policy

- A policy π is better than π' if $v_\pi(s) \geq v_{\pi'}(s)$ for all $s \in \mathcal{S}$.

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- The optimal value function: $v_{\pi_\star}(s) = v_\star(s)$
- The optimal policy $\pi_\star$ is greedy wrt $v_{\pi_\star}(s)$:
  The best long term strategy

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The Optimal Policy

- Computing optimal value function might be impossible: we need to estimate/re-estimate/approximate it
- Example: Chess, we cannot compute the optimal long-term moves, we need to approximate/estimate (based on computational budget)

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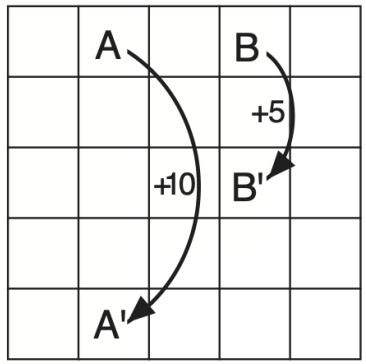
The Optimal Policy

- Computing optimal value function might be impossible: we need to estimate/re-estimate/approximate it
- Example: Chess, we cannot compute the optimal long-term moves, we need to approximate/estimate (based on computational budget)
- We might also estimate $v_\star(s)$ better for commonly encountered states

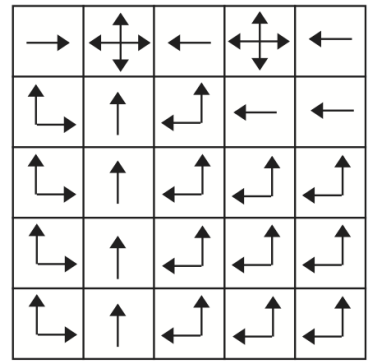
The value function



Gridworld

22.0	24.4	22.0	19.4	17.5
19.8	22.0	19.8	17.8	16.0
17.8	19.8	17.8	16.0	14.4
16.0	17.8	16.0	14.4	13.0
14.4	16.0	14.4	13.0	11.7

v_*



π_*

Figure: The gridworld optimal value function and policy