Random variable $\quad E \quad (\Omega, \mathcal{F})$

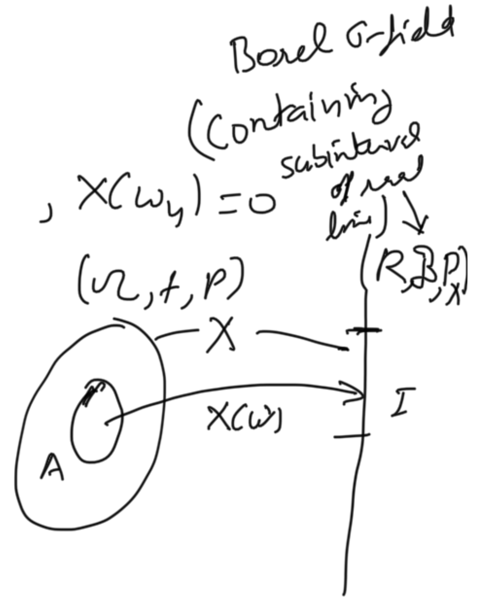
A rule X which assigns a real value $X(\omega)$ to each $\omega \in \Omega$ (sample space) is called a r.v., i.e., X is a fn whose domain is sample space Ω of outcomes ω and whose range is (some subset of) the real numbers.

Example $\quad E$: Toss the coin two times

$$\Omega = \{\underbrace{HH}_{\omega_1}, \underbrace{HT}_{\omega_2}, \underbrace{TH}_{\omega_3}, \underbrace{TT}_{\omega_4}\}$$

$$\mathcal{F} = \mathbb{P}(\Omega)$$

X : counting # of heads



$$X(\omega_1) = 2, \quad X(\omega_2) = 1 = X(\omega_3), \quad X(\omega_4) = 0$$

$$A_1 = \{\omega_2, \omega_3\} \equiv I = \{1\}$$

$$A_2 = \{\omega_2, \omega_3, \omega_4\} \equiv I = [0, 1]$$

Event A

$$P(A) = \underline{P(\omega : X(\omega) \in I)} = P_X(I)$$

$$A \equiv I$$

equivalent event

X is r.v. if

$$\{\omega : X(\omega) \in I\} \in \mathcal{F} \text{ (is an event)}$$

$$\equiv \{\omega : X(\omega) \le x\} \in \mathcal{F}$$

$\equiv \quad X^{-1}(-\infty, x] \in \mathcal{F}$ is an event

(CDF) Cummulative distribution function or distribution function : $(\Omega, \mathcal{F}, P)$

For event $[X \le x] \equiv \{\omega : X(\omega) \le x\}$, we define

$$F_X(x) = P(X \le x) = P(\underbrace{\{\omega : X(\omega) \le x\}}) \quad , x \in \mathbb{R}$$
$$= P_X((-\infty, x])$$

Example: $\Omega = \{HH, HT, TH, TT\}$

X : # of heads $\qquad \mathcal{F} = \mathbb{P}(\Omega)$

$R_X = \{0, 1, 2\}$

prob. mass function

pmf

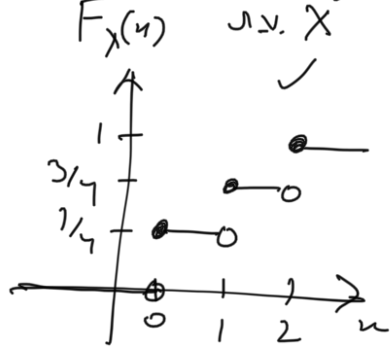
$\hookrightarrow p_X(x) = P(X = x)$

X	0	1	2
$p_X(x) = P(X=x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

CDF

$$F_X(x) = P(X \le x) = \sum_{x_i \le x} p(x_i) \qquad \text{discrete type r.v. } X$$

$$= \begin{cases} 0 & , x < 0 \\ \frac{1}{4} & , 0 \le x < 1 \\ 3/4 & , 1 \le x < 2 \\ 1 & ; x \ge 2 \end{cases}$$



Example A cathode ray tube is aged to failure

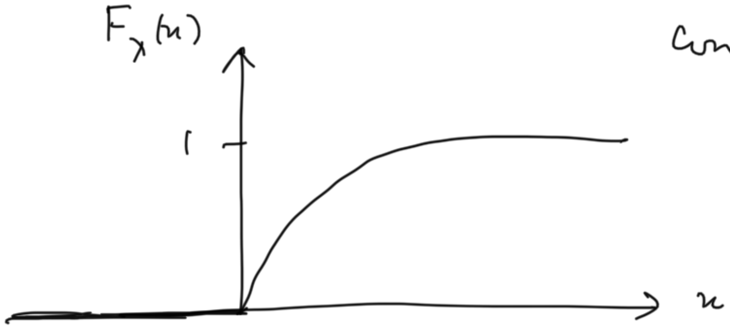
$\Omega = \{t \mid t \ge 0\} = [0, \infty)$

X : life of cathode ray tube till failure

CDF

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & x \geq 0 \end{cases}, \quad \lambda > 0$$

continuous case ✓



pdf (prob. density fn)

$\rightarrow f_X(x) = \frac{d}{dx} F_X(x)$

<u>Properties of CDF</u> (i) $0 \leq F_X(x) \leq 1, \ \forall x$

(ii) $F_X(x)$ is non-decreasing in x, i.e.,

$x_1 < x_2 \Rightarrow F_X(x_1) \leq F_X(x_2)$

(iii) $F_X(x)$ is right continuous, i.e., $F_X(x+) = F_X(x)$

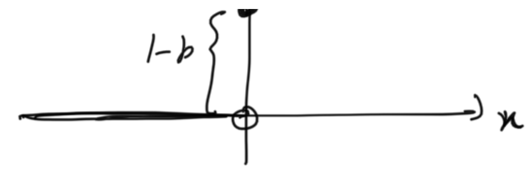
$$F_X(x+) = \lim_{\delta \to 0} F_X(x+\delta) = F_X(x)$$

(iv) $\lim_{x \to +\infty} F_X(x) = 1, \quad \lim_{x \to -\infty} F_X(x) = 0$

$$P(X = a) = F_X(a) - F_X(a-)$$

— x —

<u>mixed type r.v.</u>



Example Let X be the lifetime of an instrument which may fail immediately on installation with prob. $(1-p)$ or it may live upto age x with prob. $p(1-e^{-\lambda x})$, $x>0$, $\lambda>0$

$$F_X(x) = P(X \leq x)$$

$$= \begin{cases} 0 & , x<0 \\ (1-p) + p(1-e^{-\lambda x}) & , x \geq 0 \end{cases}$$

X mixed type.

—x—

$(1,2,3,4)$ $\quad \dfrac{1+2+3+4}{4} = 1\times\frac{1}{4} + 2\times\frac{1}{4} + 3\times\frac{1}{4} + 4\times\frac{1}{4}$

$1\times p_1 + 2\times p_2 + 3\times p_3 + 4\times p_4$

$\sum_{i=1}^{4} p_i = 1$

Discrete dist

Benoulli trail $Bern(p)$

$1 \text{ trail} \to S \quad p$
$\phantom{1 \text{ trail}} \to F \quad q$ $\quad p+q=1$

X counts # of successes $\in [0,1]$

$P(X=0) = q \quad ; P(X=1) = p$

✓ $E(X) = \sum_{x=0}^{1} x p(x) = 0 \times q + 1 \times p = p$

$E(X^2) = \sum x^2 p(x) = 0 \times q + 1^2 \times p = p$

$V(X) = E(X^2) - (E(X))^2 = p - p^2 = p(1-p) = pq$

Binomial distn : $X \sim Bin(n,p)$

n indep Bernoulli trail

X : # of successes in n trail $\in \{0,1,2,\ldots,n\}$

$$P(X=x) = \begin{cases} \binom{n}{x} p^x q^{n-x} & , x = 0,1,\ldots,n \\ 0 & \text{o.w.} \end{cases}$$

$E(X) = np$

$V(X) = npq$

Geometric distn : Let indep. Bernoulli trails are conducted till we observe a success.

X = # of trials to get a success

$$p(x) = P(X=x) = \begin{cases} q^{x-1} p & , x = 1,2,\ldots \\ 0 & \text{o.w.} \end{cases}$$

q q .. q p

F F ..F S

$X \sim Geo(p)$

memoryless property

$$P(X>m) = P(X > m+n \mid X > n)$$

Sol

$$P(X>m) = \sum^{\infty} P(X=x) = \sum^{\infty} p q^{x-1}$$

$x = m+1$ $\qquad\qquad$ $x = m+1$

$$= p\left[q^m + q^{m+1} + q^{m+2} + \cdots\right]$$

$$= pq^m\left[1 + q + q^2 + \cdots\right] = pq^m \times \frac{1}{1-q}$$

$$= \frac{\cancel{p}q^m}{\cancel{p}} = q^m \checkmark$$

$$P(X > m+n \mid X > n) = \frac{P(X > m+n,\, X > n)}{P(X > n)} = \frac{P(X > m+n)}{P(X > n)}$$

$$= \frac{q^{m+n}}{q^n} = q^m .$$

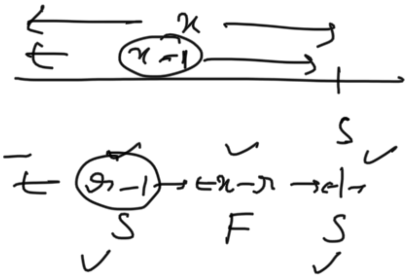
<u>Negative Binomial distn:</u> $X \sim NB(r, p)$

Let <u>indep Bernoulli trails are conducted till</u>

<u>we have r successes</u>.

Let r.v. X : # of trails required to get <u>r successes</u>

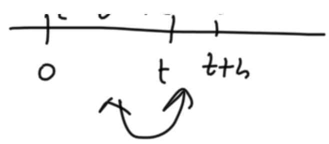
$$P(X = x) = \begin{cases} \binom{x-1}{r-1} p^r q^{x-r}, & x = r, r+1, \ldots \\ 0 & \text{o.w.} \end{cases}$$



— x —

P.P. $\quad$ ($*$) $\quad$ <u>$h \to$ small</u>

$$P_0(t+h) = P(N(0, t+h] = 0)$$



$$= P(\{N(0,t]=0\} \cap \{N(t,t+h]=0\})$$

$$= P(N(0,t]=0) \cdot P(N(t,t+h)=0) \quad | \text{ indp increment}$$

$$= P_0(t) \cdot P(N(h)=0) \quad | \text{ Stationary increments}$$

$$= P_0(t) \cdot (1 - \lambda h + o(h))$$

$$\Rightarrow \frac{P_0(t+h) - P_0(t)}{h} = -\lambda P_0(t) + \frac{o(h)\, P_0(t)}{h}$$

$h \to 0$

$$\frac{d P_0(t)}{dt} = -\lambda P_0(t)$$

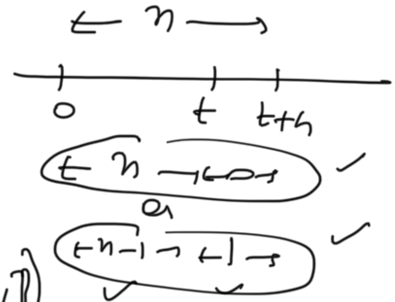
$$P_0(t) = C e^{-\lambda t}$$

$$P_0(0) = 1 \quad \Rightarrow C = 1$$

$$P_0(t) = e^{-\lambda t}$$

$h \to$ small

$$P_n(t+h) = P(N(0,t+h) = n)$$



$$= P(\underbrace{\{\{N(0,t)=n\} \cap \{N(t,t+h)=0\}\}}$$

$$\cup \{\underbrace{\{N(0,t)=n-1\} \cap \{N(t,t+h)=1\}}\})$$

$$= P_n(t)\, P_0(h) + P_{n-1}(t)\, P_1(h)$$

$$= P_n(t)\, (1 - \lambda h + o(h)) + P_{n-1}(t) \cdot (\lambda h + o(h))$$

$$\Rightarrow \frac{P_n(t+h) - P_n(t)}{} = -\lambda (P_n(t) - P_{n-1}(t)) + o(h)$$

h $\qquad$ $\overline{h}$

$h \to 0$ $\quad$ $$\frac{dP_n(t)}{dt} = -\lambda\left(P_n(t) - P_{n-1}(t)\right)$$

Know $\quad P_0(t) = e^{-\lambda t}$ ✓

<u>Assume</u> $\quad P_{n-1}(t) = \dfrac{e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}$ ✓ $\qquad$ Use mathematical induction ✓

$\to \quad P_n(t)$

$$\frac{dP_n(t)}{dt} = -\lambda P_n(t) + \lambda \frac{e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}$$

$$e^{\lambda t}\frac{dP_n(t)}{dt} + \lambda e^{\lambda t} P_n(t) = \frac{\lambda^n t^{n-1}}{(n-1)!}$$

$$\frac{d}{dt}\left(e^{\lambda t} P_n(t)\right) = \frac{\lambda^n}{(n-1)!} t^{n-1}$$

$$e^{\lambda t} P_n(t) = \frac{\lambda^n}{n!} t^n$$

$$\Rightarrow \quad P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \ldots$$

— x —