Poisson Process (contd)

$N_i \equiv N_i(t)$ # of type i event in $(0,t)$, $i=1,2$

$p \to$ probability of event is type 1
$q \to$,, ,, ,, -- type 2

$N(t) = N_1(t) + N_2(t) \sim PP(\lambda)$

$p+q=1$

Show $N_1(t) \sim PP(\lambda p)$

$N_2(t) \sim PP(\lambda q)$

Sol

$$P(N_1=n, N_2=m) = P(N_1=n, N_2=m \mid N=n+m)\,\frac{e^{-\lambda}\lambda^{n+m}}{(n+m)!}$$

λt

$$= {}^{n+m}C_n\, p^n q^m \frac{e^{-\lambda}\lambda^{n+m}}{(n+m)!}$$

$$= \frac{\cancel{(n+m)!}}{n!\,m!}\, p^n q^m e^{-\lambda p} e^{-\lambda q} \frac{\lambda^n \lambda^m}{\cancel{(n+m)!}}$$

$$= \frac{e^{-\lambda p}(\lambda p)^n}{n!}\;\frac{e^{-\lambda q}(\lambda q)^m}{m!}$$

$$P(N_1=n) = \sum_{m=0}^{\infty} P(N_1=n, N_2=m)$$

$$= \frac{e^{-\lambda p}(\lambda p)^n}{n!}, \quad n=0,1,2,\ldots$$

indep $\begin{cases} N_1(t) \sim PP(\lambda p) \\ N_2(t) \sim PP(\lambda q) \end{cases}$

— x —

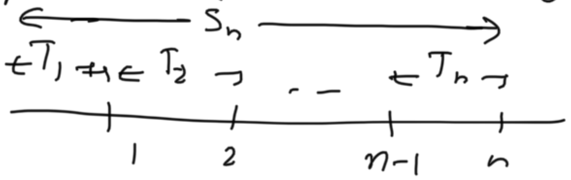
Interarrival and Waiting time distn:

$N(t) = \#$ of event $(0,t] \sim P.P.(\lambda)$ — λ: rate

$$P(N(t)=n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n = 0,1,2,\ldots$$

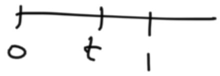
T_1 : time for first event

T_n : time elapsed btw $(n-1)^{st}$ and n^{th} event



$T_i \sim \exp(\lambda)$

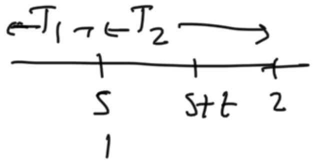
$T_1 > t \equiv N(0,t] = 0$



$$P(T_1 > t) = P(N(t)=0) = e^{-\lambda t}$$

$$T_1 \sim \exp(\lambda)$$

$$P(T_2 > t \mid T_1 = s) = P(N(s, s+t] = 0)$$



$$= P(N(t)=0)$$

$$= e^{-\lambda t}$$

$$P(T_2 > t) = E\left(P(T_2 > t \mid T_1)\right) = e^{-\lambda t}$$

$\therefore \quad T_2 \sim \exp(\lambda)$

$\therefore$ interarrival times $T_1, T_2, \ldots$ are i.i.d. expo. r.v's. with mean $\frac{1}{\lambda}$.

$$E\left(P(T_2 \geq t \mid T_1)\right) = \int P(\underbrace{T_2 > t \mid T_1 = t_1}) f_{T_1}(t_1)\, dt_1$$

$$= \int \left(\int_t^{\infty} f_{T_2 \mid T_1 = t_1}(t_2)\, dt_2 \right) f_{T_1}(t_1)\, dt_1$$

$$= \int_t \int^{\infty} f_{T_2 | T_1 = t_1}(t_2)\, f_{T_1}(t_1)\, dt_2\, dt_1$$

$$f_{T_1, T_2}(t_1, t_2)$$

$$= \int_t^{\infty} \left(\int f_{T_1, T_2}(t_1, t_2)\, dt_1 \right) dt_2$$

$$f_{T_2}(t_2)$$

$$= \int_t^{\infty} f_{T_2}(t_2)\, dt_2 = P(T_2 > t)$$

$S_n = \sum_{i=1}^{n} T_i$, $n \geq 1$, arrival time of the n^{th} event,

called waiting time until the nth event.

$\because T_i \overset{iid}{\sim} \exp(\lambda)$ $\qquad E(S_n) = \frac{n}{\lambda}$, $V(S_n) = \frac{n}{\lambda^2}$.

$S_n \sim \text{Gamma}(n, \lambda)$

Sol mgf $\qquad M_{T_i}(t) = \left(1 - \frac{t}{\lambda}\right)^{-1}$

$$M_{S_n}(t) = \prod_{i=1}^{n} M_{T_i}(t) = \left(1 - \frac{t}{\lambda}\right)^{-n}$$

Also $\qquad S_n > t \equiv N(t) \leq n-1$ | see previous notes on Gamma & PP.

—x—

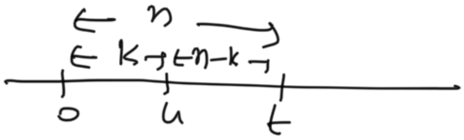
Binomial dist also arises in the context of P.P.

→ If $N(t) \sim PP(\lambda)$, Then for $0 < u < t$

$$[N(u) \mid N(t)=n] \sim \text{Bin}\left(n, \frac{u}{t}\right)$$

Sol. For $0 < u < t$, $0 \le k \le n$

$$P(N(u)=k \mid N(t)=n) = \frac{P(N(u)=k,\ N(t)=n)}{P(N(t)=n)}$$



$$= \frac{P(N(0,u]=k,\ N(u,t]=n-k)}{P(N(t)=n)}$$

$$= \frac{P(N(u)=k)\ P(N(t-u)=n-k)}{P(N(t)=n)}$$

using stationary & independent increments of P.P.

$$= \frac{\dfrac{\cancel{e^{-\lambda u}}(\cancel{\lambda} u)^k}{k!} \times \dfrac{\cancel{e^{-\lambda(t-u)}}(\cancel{\lambda}(t-u))^{n-k}}{(n-k)!}}{\dfrac{\cancel{e^{-\lambda t}}(\cancel{\lambda} t)^n}{n!}}$$

$$= \frac{n!}{k!\,(n-k)!}\left(\frac{u}{t}\right)^k\left(1-\frac{u}{t}\right)^{n-k}$$

—x—

Q. A

indep $\begin{cases} X \sim \text{Gamma}(\alpha, \lambda) \\ Y \sim \text{Gamma}(\beta, \lambda) \end{cases}$ let $U = \dfrac{X}{X+Y}$, $V = X+Y$

Then $U \sim \text{Beta}(\alpha, \beta)$, $V \sim \text{Gamma}(\alpha+\beta, \lambda)$

indep.

Sol $u = \dfrac{x}{x+y}$, $v = x+y$

$0 < u < 1, \quad v > 0$

$\Rightarrow x = uv$, $y = v - uv$

Joint pdf of X, Y

$$f_{X,Y}(x,y) = \frac{\lambda^{\alpha}}{\Gamma\alpha} x^{\alpha-1} e^{-\lambda x} \frac{\lambda^{\beta}}{\Gamma\beta} y^{\beta-1} e^{-\lambda y} \quad ; x>0, y>0$$

Joint pdf of U, V

$$f_{U,V}(u,v) = f_{X,Y}(x,y) \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix} = v(1-u) + uv = v$$

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = |v| = v$$

$$f_{U,V}(u,v) = \frac{\lambda^{\alpha+\beta}}{\Gamma\alpha\,\Gamma\beta} (uv)^{\alpha-1} (v(1-u))^{\beta-1} e^{-\lambda v} v$$

$$= \boxed{\frac{\Gamma\alpha+\beta}{\Gamma\alpha\,\Gamma\beta} u^{\alpha-1} (1-u)^{\beta-1}} \times \boxed{\frac{\lambda^{\alpha+\beta}}{\Gamma\alpha+\beta} e^{-\lambda v} v^{\alpha+\beta-1}}$$

$$= f_U(u) \quad f_V(v) \quad | \text{ Clearly}$$

$$\therefore \underbrace{U \sim \text{Beta}(\alpha,\beta) \quad , V \sim \text{Gamma}(\alpha+\beta)}_{\text{indep.}}$$

— x —

P.P. & Beta dist

$T_1, T_2, \ldots$ are i.i.d. $\exp(\lambda)$

m

$$S_m = \sum_{i=1} T_i$$

indep. $\begin{cases} S_m \sim \text{Gamma}(m, \lambda) \\ S_n - S_m \sim \text{Gamma}(n-m, \lambda) \end{cases}$

$$\frac{S_m}{S_n} = \frac{S_m}{S_m + (S_n - S_m)}$$

indep $\begin{cases} U = \dfrac{S_m}{S_n} \sim \text{Beta}(m, n-m) \\ V = S_n \sim \text{Gamma}(n, \lambda) \end{cases}$ | by the result (A)

—x—

<u>Example</u> Suppose customers stream into a drug store at the constant Poisson av. rate of 15 per hr. The pharmacy opens its door at 8:00 AM and closes at 8:00 PM. Given that the 100^{th} customer on a particular day walked in at 2:00 PM, we want to know what is the prob. that the 50^{th} customer came before noon.

Sol S_j : arrival time of the jth customer on that day

N(t) ~ P.P.

$n = 100, m = 50$

$$P(S_m < 4 \mid S_n = 6) = P\left(\frac{S_m}{S_n} < \frac{4}{6} \middle| S_n = 6\right)$$

$$= P\left(\frac{S_{50}}{S_{100}} < \frac{4}{6}\right)$$

$\Rightarrow U = \frac{S_m}{S_n}$ & S_n are indep.

$U = \frac{S_m}{S_n} \sim \text{Beta}(50, 50)$

$$\overset{CLT}{=} \Phi\left(\frac{4/6 - 1/2}{\sqrt{0.0025}}\right)$$

$= 0.9997$

$U = \frac{S_m}{S_n} \sim Beta(\underset{\alpha}{m}, \underset{\beta}{n-m})$

$E(U) = \frac{\alpha}{\alpha+\beta} = \frac{1}{2}$

$V(U) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{50 \times 50}{(100)^2 \times 101}$

$= 0.0025$

— x —

→ Let $N(t) \sim$ P.P. and one event take place in $(0,t]$. Then Y the r.v. describing the time of occurance of this Poisson event, has a continuous uniform dist $(0,t]$

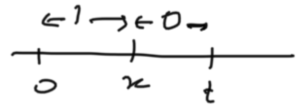
Sol

$$Y = [T_1 \mid N(t)=1]$$

$$N(t) \sim P.P.(\lambda)$$

$$P(N(t)=n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n = 0,1,2,\ldots$$

For $0 < x < t$, cdf of Y



$$P(Y \le x) = P(T_1 \le x \mid N(t)=1)$$

$$= \frac{P(N(0,x]=1,\ N(x,t]=0)}{P(N(t)=1)}$$

$$= \frac{P(N(x)=1) \cdot P(N(t-x)=0)}{P(N(t)=1)}$$

$$= \frac{\cancel{e^{-\lambda x}}\ \cancel{\lambda} x\ \cancel{e^{-\lambda(t-x)}}}{\cancel{e^{-\lambda t}}\ \cancel{\lambda} t}$$

$$= \frac{x}{t}$$

pdf of Y is $$f_Y(x) = \begin{cases} \frac{1}{t}, & 0<x<t \\ 0 & \text{o.w.} \end{cases}$$

$$Y = [T_1 \mid N(t)=1] \sim U(0,t)$$

$\delta/t \leftarrow P$(contains the event)

—x—

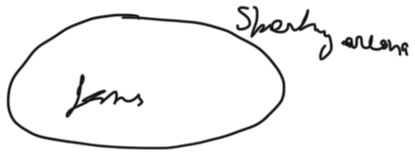
Compound P.P.

S.P. $\{X(t), t\geq 0\}$ is compound P.P. if

$$X(t) = \sum_{i=1}^{N(t)} Y_i, \quad t\geq 0, \text{ where } \{N(t), t\geq 0\} \text{ is a P.P.,}$$

and $\{Y_i, i\geq 1\}$ be a family of i.i.d. r.v's that is also indep. of $\{N(t), t\geq 0\}$.

Example ①

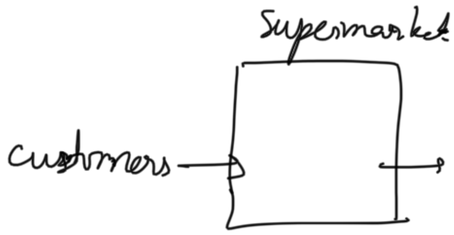


Modeling
- $N(t)$ # of buses in $(0,t)$ $\sim$ P.P.(λ)
- Y_i # of fans in i^{th} bus, Y_i iid

$X(t) = \sum_{i=1}^{N(t)} Y_i$ # of fans in sporting arena in $(0,t)$

→ Compound P.P.

②



. $N(t)$ # of customers leaving the supermarket indep $<$ Y_i amt of money spent by ith customer

Y_i iid.

$$X(t) = \sum_{i=1}^{N(t)} Y_i$$ total amt of money earned by supermarket $(0,t)$.

Compound P.P.

(3) If $Y_i \equiv 1$, then $X(t) = N(t)$ usual P.P.

$$E(X(t)) = E\left(\sum_{i=1}^{N(t)} Y_i\right) = E\left(\underbrace{E\left(\sum_{i=1}^{N(t)} Y_i \,\middle|\, N(t)\right)}_{N E(Y_1)}\right)$$

$\because E(X) = E(E(X|Y))$ $\because *$

$$* \quad E\left(\sum_{i=1}^{N(t)} Y_i \,\middle|\, N(t)=n\right) = E\left(\sum_{i=1}^{n} Y_i \,\middle|\, N(t)=n\right)$$

$$= E\left(\sum_{i=1}^{n} Y_i\right) \quad \Big|\because Y_i \text{ \& } N(t) \text{ indep}$$

$$= n E(Y_1) \quad \Big| Y_i \text{ iid}$$

$$E(X(t)) = E(N(t)E(Y_1)) = E(Y_1)\, E(N(t))$$

$$= \lambda t \; E(Y_1)$$

$N \equiv N(t)$

$$V(X(t)) = V\left(\sum_{i=1}^{N} Y_i\right) = E\left(\underbrace{V\left(\sum_{i=1}^{N} Y_i \,\middle|\, N\right)}\right) + V\left(\underbrace{E\left(\sum_{i=1}^{N} Y_i \,\middle|\, N\right)}\right)$$

$\because V(X) = E(V(X|Y)) + V(E(X|Y))$

$$V\left(\sum_{i=1}^{N} Y_i \,\middle|\, N=n\right) = V\left(\sum_{i=1}^{n} Y_i \,\middle|\, N=n\right)$$

$$= V\left(\sum_{i=1}^{n} Y_i\right) = n V(Y_1)$$

$$\therefore \; V(X(t)) = E(N\,V(Y_1)) + V(N\,E(Y_1))$$

$$= \underbrace{E(N)}_{\lambda t} V(Y_1) + [E(Y_1)]^2 \underbrace{V(N)}_{\lambda t}$$

$$= \lambda t \left[E(Y_1^2) - [E(Y_1)]^2 \right] + [E(Y_1)]^2 \lambda t$$

$$= \lambda t \; E(Y_1^2)$$

$$\boxed{\begin{aligned} E(X(t)) &= \lambda t \; E(Y_1) \\ V(X(t)) &= \lambda t \; E(Y_1^2) \end{aligned}}$$

✓ ✓

<u>Example</u> (1) Customers arrive at the ATM in accordance with P.P. with rate 12 per hr. The amt of money withdrawn on each transaction is a r.v. with mean \$30 and s.d. \$50 (A negative withdrawal means that money was deposited). The machine is in use for 15 hr daily. Approximate the prob. that the total daily withdrawal is less than \$6000.

$\lambda = 12$ per h

$E(Y_i) = 30$

$V(Y_i) = (50)^2$

$= E(Y_i^2) - (E(Y_i))^2$

Sol

$$X(15) = \sum_{i=1}^{N(15)} Y_i$$

↳ daily withdrawal

Y_i → amt withdrawn by ith person.

$N(t)$ # of person $(0,t)$

$E(X(15)) = 12 \times 15 \times 30$, $V(X(t)) = 12 \times 15 \times ((50)^2 + (30)^2)$

$= 5400$ $\qquad = 612000$

$P(X(15) < 6000) \overset{CLT}{=} \Phi\left(\frac{6000 - 5400}{\ldots}\right)$

$$= \Phi\left(\frac{\;\;\;\;\;\;\;}{\sqrt{612000}}\right)$$

$$= \Phi(0.767)$$

$$= 0.78.$$

—×—

(2) (a) Suppose that families migrate to an area at a Poisson rate $\lambda = 2$ per week. If the # of people in each family is indep. and takes on the values 1, 2, 3, 4 with resp. prob. $\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}$ then what is the expected value and var. of the individuals migrating to this area during a fixed five-week period?

Sol

$X(t)$: # of individual migrating to the area $(0, t]$

$N(t)$: # of families migrating to the area $(0,t]$ $\sim P.P.(\lambda)$ $\lambda = 2$ per week

Y_i : # of people in ith family Y_i iid

$$X(t) = \sum_{i=1}^{N(t)} Y_i \sim \text{compound P.P.}$$

$$E(X(t)) = \lambda t\; E(Y_1)$$

$$V(X(t)) = \lambda t\; E(Y_1^2)$$

$$E(Y_1) = 1 \times \frac{1}{6} + 2 \times \frac{1}{3} + 3 \times \frac{1}{3} + 4 \times \frac{1}{6} = \frac{5}{2}$$

$$E(Y_1^2) = 1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{3} + 3^2 \times \frac{1}{3} + 4^2 \times \frac{1}{6} = \frac{43}{6}$$

$$E(X(5)) = 2 \times 5 \times \frac{5}{2} = 25$$

$$V(X(5)) = 2 \times 5 \times \frac{43}{6} = \frac{215}{3}$$

(3) contd (g) Find the approximate prob. that at least 240 people migrate to the area within the next 50 weeks

Sol
$$E(X(50)) = 2 \times 50 \times \frac{5}{2} = 250$$

$$V(X(50)) = 2 \times 50 \times \frac{43}{6} = \frac{4300}{6}$$

$$P(X(50) \geq 240) \overset{CLT}{\approx} P\left(Z \geq \frac{239.5 - 250}{\sqrt{\frac{4300}{6}}}\right) \quad \Big| \because Z \sim N(0,1)$$

$$= 1 - \Phi(-0.3922)$$

$$= \Phi(0.3922)$$

$$= 0.6525$$

— x —