Control & Instrumentation Lab, Autumn 2021-22

Session 5: Root Locus Technique

Note to the students: Please solve all the problems by hand and then verify using MATLAB. Show to the TAs how you have solved them by hand.

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CSE LAB SESSION-5

1. Consider the characteristic equation

$$s(s+2)(s+3) + K(s+1) = 0$$

Determine $K = 0$ and $K = \infty$ points and the number of branches of the root loci.

ANSWER:

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1. $S(S+2)(S+3) + K(S+1) = 0$

$$1 + K\frac{(S+1)}{S(S+2)(S+3)} = 0.$$

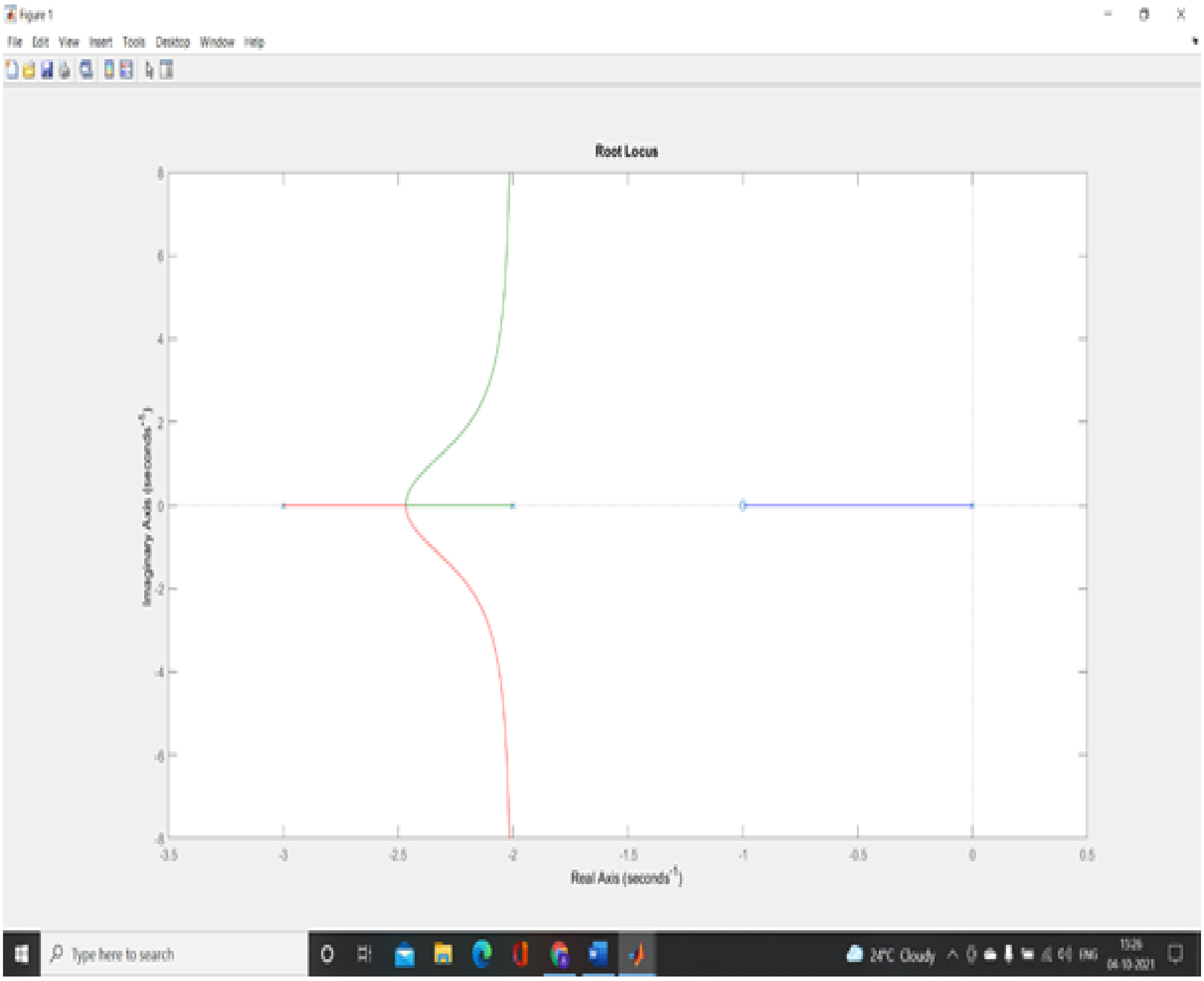
$\underline{K=0}$

$\Rightarrow S=0, S=-2, S=-3$ are roots of ch eq.

$\underline{K\to\infty}$

$S=\infty$, $S=-1$ are roots of ch eq.

No. of branches = No. of closed loop poles = 3.



2. Consider the characteristic equation

$$s(s+2)(s+1+j)(s+1-j)+K=0.$$

Show the symmetry of the root loci.

ANSWER:

2. $s(s+2)(s+1+j)(s+1-j)+K=0$.

$$1+\frac{K}{s(s+2)(s+1+j)(s+1-j)}=0$$

open loop poles $= 0, -2, -1-j, -1+j$

open loop zeros $= -$

$$\text{Centroid of asymptotes} = \frac{\Sigma \text{ finite poles} - \Sigma \text{ finite zeros}}{\text{no of finite poles} - \text{no of finite zeros}}$$

$$= \frac{(0-2-1-1)-0}{4-0}$$

$$= -1.$$

$$\text{Angle of asymptotes} = \frac{(2n+1)\pi}{4-0} = 45^\circ, 135^\circ, 225^\circ, 315^\circ$$

Breakaway point

$$K = -s(s+2)(s+1+j)(s+1-j)$$

$$= -(s^2+2s)(s^2+2s+2)$$

$$= -(s^4+4s^3+6s^2+4s)$$

For breakaway, $\frac{dK}{ds}=0$.

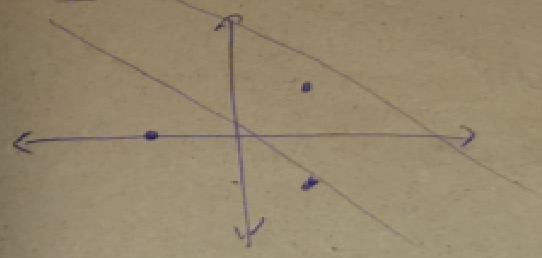
$$-[4s^3+12s^2+12s+4]=0$$

$$s^3+3s^2+3s+1=0.$$

$$(s+1)^3=0$$

$s = -1, -1, -1$.

Root locus:-



Imaginary axis crossover:-

$$s^4 + 4s^3 + 6s^2 + 4s + K = 0$$

Routh Table:-

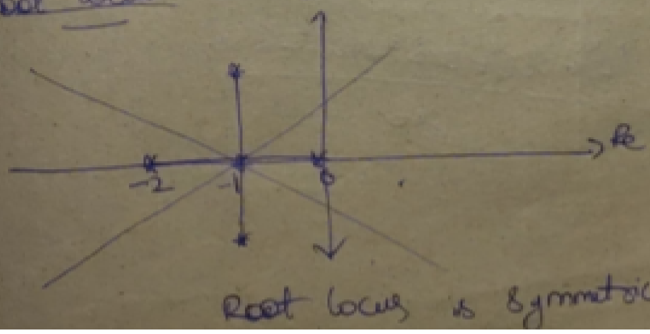
s^4	1	6	K
s^3	4	4	
s^2	5	K	→ A(s) - Auxillary eqn.
s^1	$\frac{5-K}{5}$		→ shld be zero ⇒ K = 5.
s^0	K		

$$A(s) = 5s^2 + K = 0$$

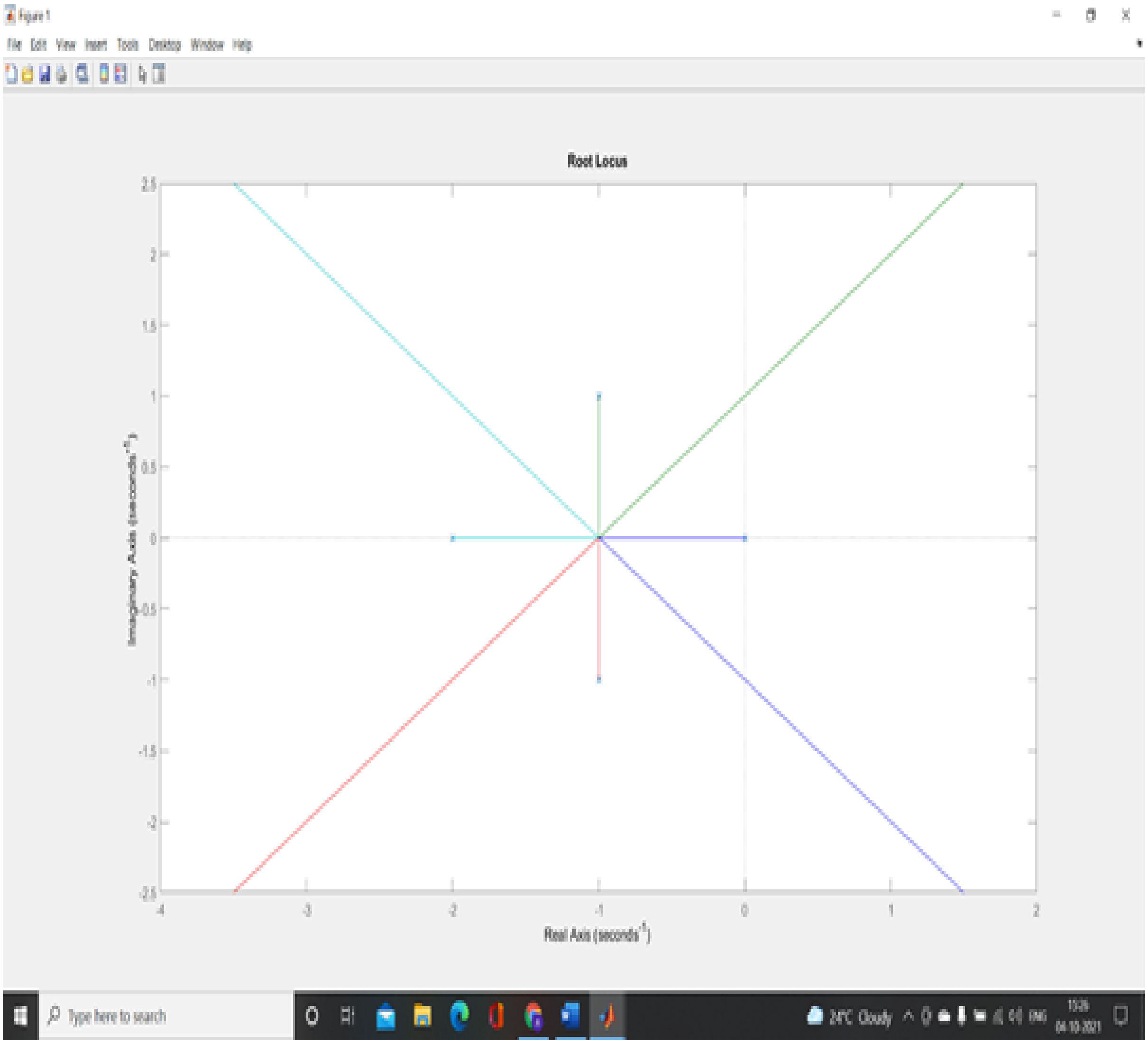
$$5s^2 + 5 = 0$$

$s = \pm j$ → jω crossover points.

Root locus:-



Root locus is symmetric.



3. Consider the characteristic equation

$$s(s+4)(s^2+2s+2)+K(s+1)=0$$

Compute the angles of asymptotes and intersect of asymptotes.

ANSWER:

3) $s(s+4)(s^2+2s+2)+k(s+1)=0$

$\Rightarrow 1+k\cdot\frac{s+1}{s(s+4)(s^2+2s+2)}$

open loop poles : $0, -4, -1\pm j$

open loop zeroes : -1

Angle of asymptotes $= \frac{(2n+1)\pi}{\text{no of poles} - \text{no of zeros}}$

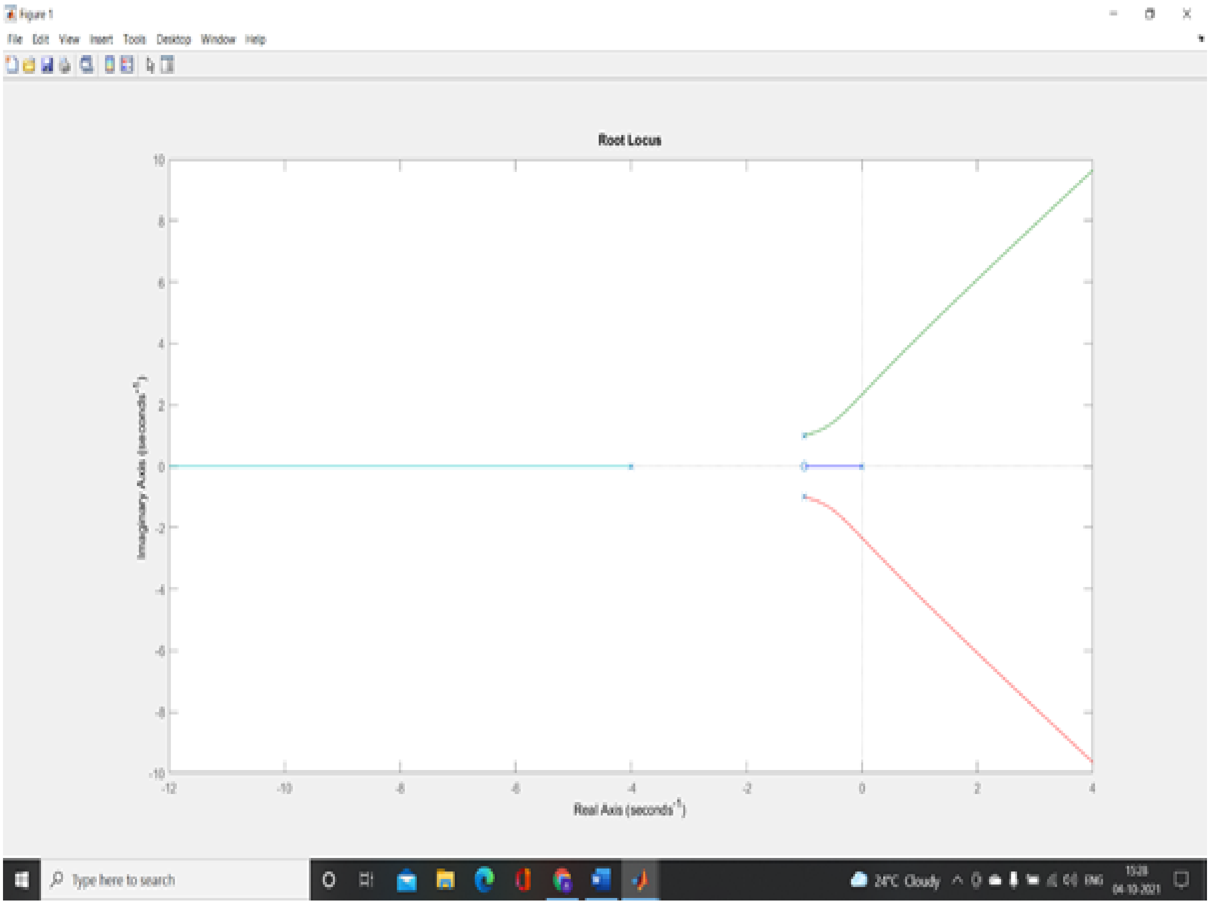
$= \frac{(2n+1)\pi}{4-1}$

$= 60°, 180°, 300°, \ldots$

Intersection of asymptotes $= \frac{\Sigma(\text{finite poles}) - \Sigma(\text{finite zeroes})}{\text{no of poles} - \text{no. of zeroes}}$

$= \frac{(0-4-1+j-1-j)-(-1)}{4-1}$

$= -5/3 = -1.67$



4. Consider the characteristic equation

$$s(s+3)(s^2+2s+2)+K=0$$

Show root loci on the real axis and compute the angle of departure at $s_1 = -1 + j1$.

ANSWER:

4) $s(s+3)(s^2+2s+2)+k=0$

$$1+\frac{k}{s(s+3)(s^2+2s+2)}=0.$$

Open loop poles $\rightarrow$ $0, -3, -1\pm j$

open loop zeros $\rightarrow$ -

$$\text{Centroid} = \frac{(0-3-1+j-1-j)-0}{4-0} = -5/4$$

$$\text{Angle of asymptotes} = (2n+1)\frac{180°}{4-0}$$

$$= 45°, 135°, 225°, 315°$$

Breakaway point:

$$k = -s(s+3)(s^2+2s+2)$$

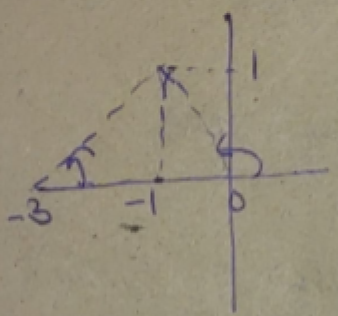
$$= -(s^4+5s^3+8s^2+6s)$$

$$\frac{dk}{ds} = -[4s^3+15s^2+16s+6]=0$$

$$\Rightarrow s = -2.3$$

other two are not on real axis

Angle of departure for $s = -1+j$

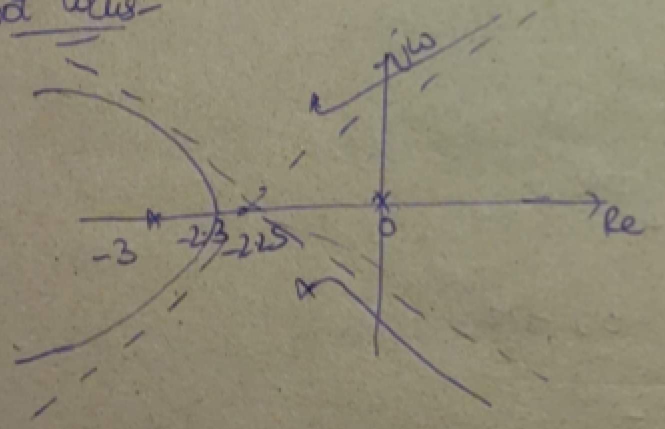


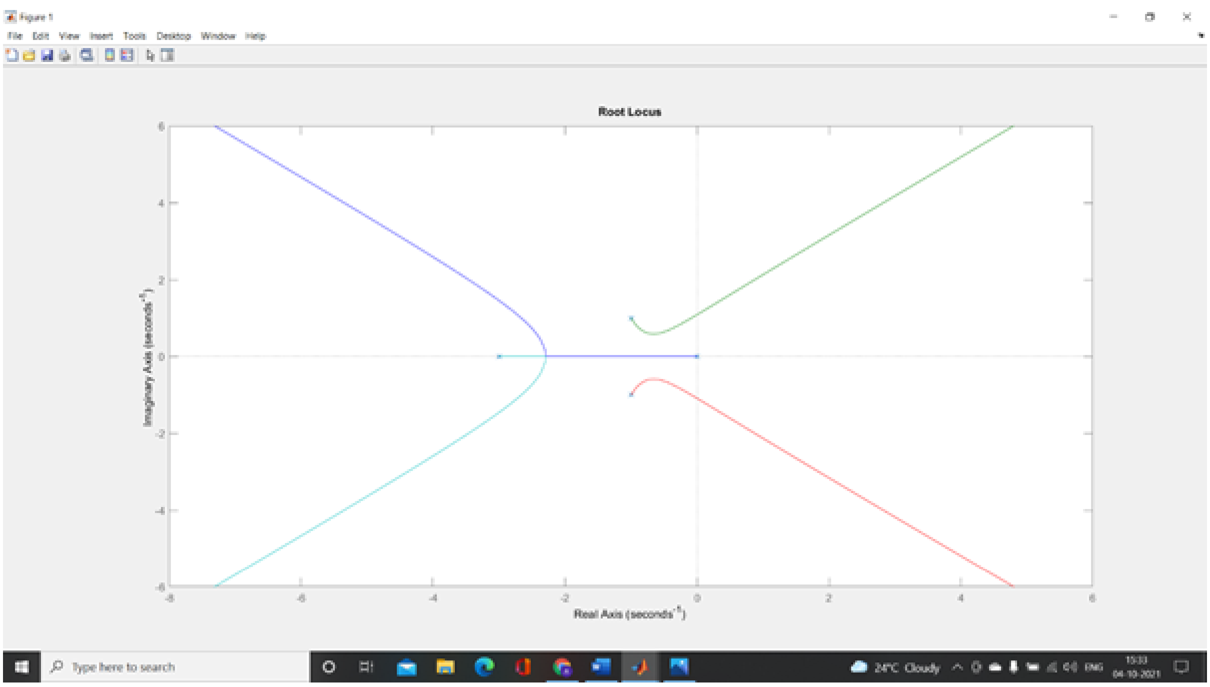
$$-135^\circ - \tan^{-1}(1/2) - 90^\circ - \theta = 180^\circ$$

$$\theta = -431.56^\circ$$

$$= -71.56^\circ$$

Root Locus:





5. Consider the characteristic equation

$$s(s+4)(s^2+4s+20)+K=0.$$

Determine the intersections of the root loci with the imaginary axis and breakaway points on the root loci.

ANSWER:

5. $S(s+4)(s^2+4s+20)+K=0$.

$$1+\frac{K}{(s)(s+4)(s^2+4s+20)}=0.$$

Open loop pole $\rightarrow 0, -4, -2 \pm 4j$

Open loop zeroes $\rightarrow \phi$ -

Breakaway points:-

$$K=-s(s+4)(s^2+4s+20)$$

$$K=(-s^2-4s)(s^2+4s+20)$$

$$=-(s^4+8s^3+36s^2+80s)$$

$\frac{dk}{ds} = 0 \Rightarrow -(4s^3 + 24s^2 + 72s + 80) = 0$

$\Rightarrow s^3 + 6s^2 + 18s + 20 = 0$

$-8 + 24 - 36 + 10$

$s = -2$ ~~-5.3~~ (only root on real axis)

jω axis crossover:-

Eqn is $s^4 + 8s^3 + 36s^2 + 80s + k = 0$

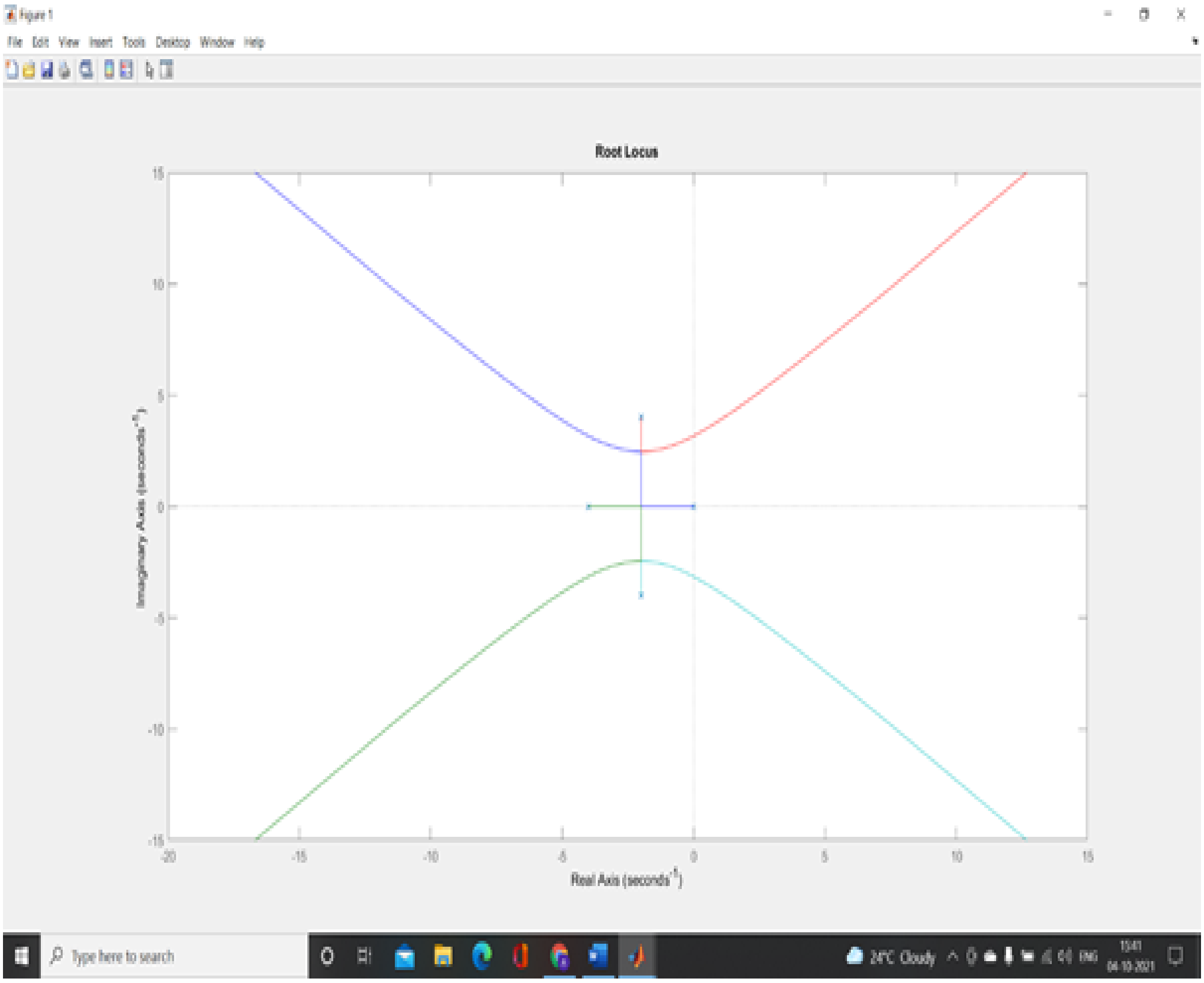
Routh Table:-

s^4	1	36	k
s^3	8/1	80/10	
s^2	26	k	} A(s)
s^1	$\frac{260-k}{26}$	} should be zero k=260	
s^0	k		

$A(s) = 26s^2 + k = 0 \Rightarrow 26s^2 + 260 = 0$

$\Rightarrow s = \pm\sqrt{10}\,j$

∴ jω axis crossover points are $\pm\sqrt{10}j$,



6. Construct the root locus, given the characteristic equation

$$s(s+5)(s+6)(s^2+2s+2)+K(s+3)=0.$$

(a) manually, and (b) using MATLAB

ANSWER:

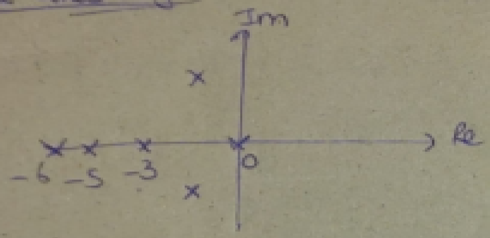
6. $s(s+5)(s+6)(s^2+2s+2)+k(s+3)=0$.

$$1+\frac{k(s+3)}{s(s+5)(s+6)(s^2+2s+2)}=0$$

Open loop poles $\Rightarrow 0, -5, -6, -1\pm j$

Open loop zeroes $\rightarrow -3$.

Real axis segment:-



$$\text{Centroid} = \frac{(0-5-6-1+j-1-j)-(-3)}{5-1}$$

$$= -\frac{10}{4} = -2.5$$

Angle of asymptotes $= \frac{(2n+1)\pi}{4} = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4 \ldots$

Breakaway points:-

$$k = -\frac{s(s+5)(s+6)(s^2+2s+2)}{s+3}$$

$$= -\frac{(s^5+13s^4+54s^3+82s^2+60s)}{s+3}$$

$$\frac{dk}{ds}=0.$$

$$\Rightarrow \frac{-(s+3)(5s^4+52s^3+162s^2+164s+60)+(s^5+13s^4+54s^3+82s^2+60s)}{(s+3)^2}=0$$

$$4s^5+54s^4+264s^3+568s^2+492s+180=0$$

$$\Rightarrow s=-5.526,\ -0.66\pm 0.47j,\ -3.33\pm 1.2j$$

$j\omega$ axis crossover:

Eqn $s^5+13s^4+54s^3+82s^2+(60+k)s+3k=0$

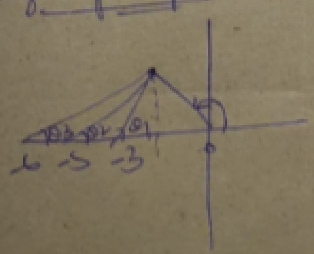
s^5	1	54	$60+k$
s^4	13	82	$3k$
s^3	$\frac{620}{13}$	$\frac{780+10k}{13}$	
s^2	$\frac{40700-130k}{620}$	$3k$	$\rightarrow A(s)$
s^1	$\frac{-130k^2-84760k+3174400}{52910-169k}$		$\rightarrow$ shld be zero $\Rightarrow k=35.52, -687.82$
s^0	$3k$		

A(s) doesn't have imaginary roots

$$A(s)=\frac{40700-130k}{620}s^2+3k=0$$

$$\Rightarrow s=\pm 1.353j$$

Angle of departure:

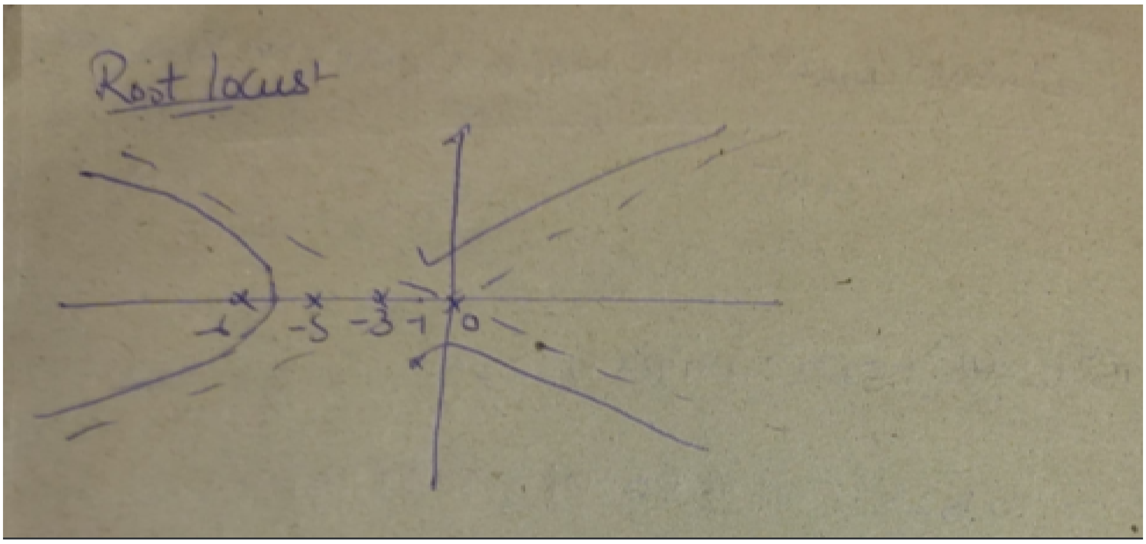


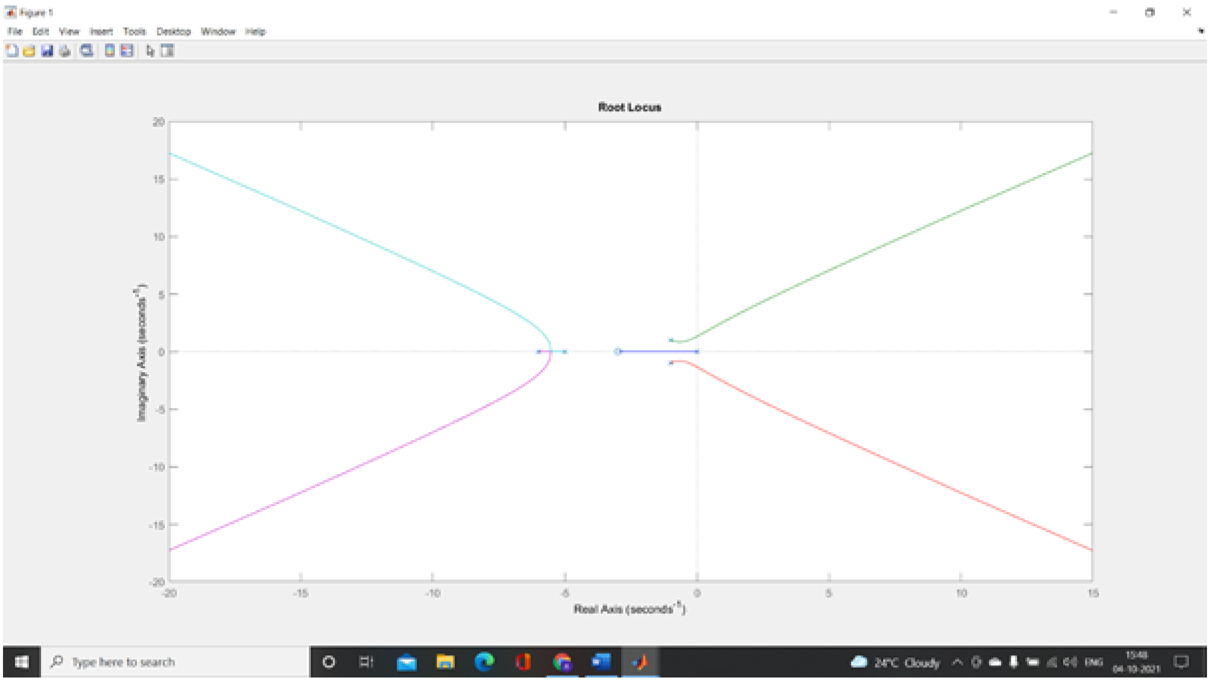
$$-135+\theta_1-\theta_2-\theta_3-90-\theta=180^\circ$$

$$-135^\circ+\tan^{-1}(1/2)-\tan^{-1}(1/4)$$

$$-\tan^{-1}(1/5)-90-\theta=180^\circ$$

$$\boxed{\theta=-403.78^\circ=-43.78^\circ}$$





ADDITIONAL:

Question (1): A simplified form of the open-loop transfer function of an airplane with an autopilot in the longitudinal mode is

$$G(s)H(s) = \frac{K(s+a)}{s(s-b)(s^2+2\xi\omega_n s+\omega_n{}^2)}, \qquad a>0, \quad b>0$$

Such a system involving an open-loop pole in the right-half s plane may be conditionally stable. Sketch the root loci when $a = b = 1. \xi = 0.5$ and $\omega_n = 4$. Find the range of gain K for stability.

Additional:-

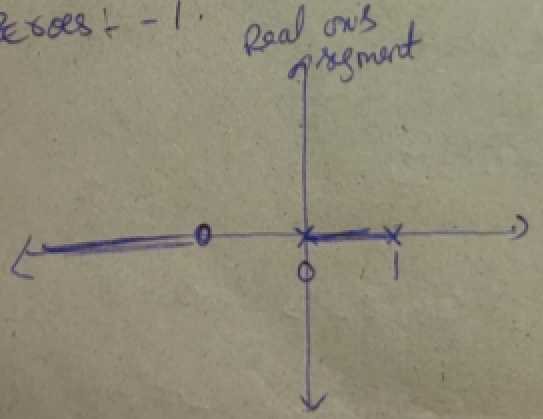
① $G(s)H(s) = \dfrac{k(s+a)}{s(s-b)(s^2+2\xi\omega_n s+\omega_n^2)}$ $\quad a>0,\ b>0$

ch eq $\quad s(s-1)(s^2+4s+16) + k(s+1) = 0$

$$1 + \frac{k(s+1)}{s(s-1)(s^2+4s+16)}$$

Poles:- $0,\ 1,\ \cancel{\pm}\ -2 \pm 2\sqrt{3}\,j$

Zeroes:- -1.



Centroid = $\frac{(0+1-4)-(-1)}{3}$

$= -2/3$

Angle of asymptotes $= (2k+1)\pi/3$

$= 60°, 180°, 270°$,

Break away points :-

$$\frac{dk}{ds} = -\frac{(s^4+3s^3+12s^2-16s)}{s^2-s}$$

$= 2(s+2)$ (upon simplification)

$$\frac{dk}{ds} = 0$$

$\Rightarrow \boxed{s=-2}$

Stability : RH Criteria

$s^4+3s^3+12s^2+s(k-16)+k=0$

R Table

s^4	1	12	k
s^3	3	k-16	
s^2	$52-k/3$	k	
s^1	$\frac{k^2-59k+832}{k-52}$		
s^0	k		

} → A(s)

} → > 0

$\Rightarrow k = \frac{59}{2} \pm \frac{3}{2}\sqrt{A}$

$= 35.687$ or 23.319

$A(s) = \frac{52-k}{3}s^2 + k = 0$

$\Rightarrow s = \sqrt{\frac{3k}{k-52}}$

Here, s should be on imaginary axis

$\Rightarrow \quad K = \dfrac{59 \pm 3\sqrt{17}}{2}$

$\Rightarrow s = \pm 2.56j \;, \pm 1.2j$

Range of K for stability:-

From the Routh table & characteristic eqn for stability,

$K - 16 > 0, \; K > 0, \; \dfrac{52-K}{3} > 0, \; \dfrac{K^2 - 59K + 832}{K - 32} > 0$

$\Rightarrow 16 < K < 52 \;, \; K^2 - 59K + 832 < 0$

$\Rightarrow K > 23.32 \;\&\; K < 35.68$

$$\boxed{23.32 < K < 35.68}$$

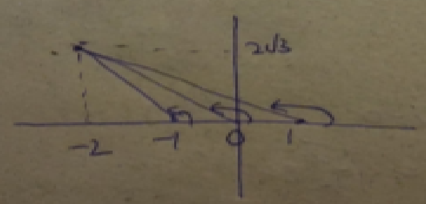
Angle of departure:-

$\left[180^\circ - \tan^{-1}(2\sqrt{3})\right] - \left(180^\circ - \tan^{-1}(\sqrt{3})\right) - \left[180^\circ - \tan^{-1}\left(\tfrac{2}{\sqrt{3}}\right)\right] - 90 - \theta) = 180^\circ$

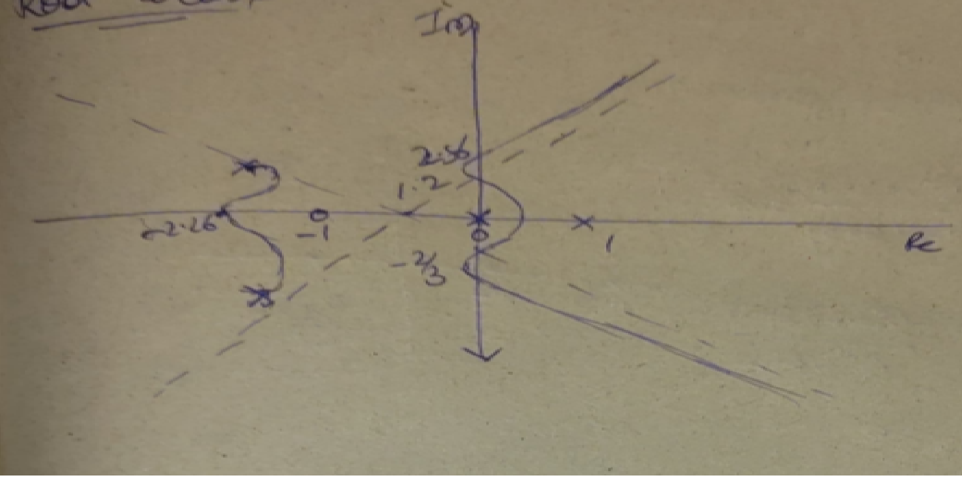
$\Rightarrow \tan^{-1}(\sqrt{3}) - \tan^{-1}\left(\tfrac{2}{\sqrt{3}}\right) - \tan^{-1}(2\sqrt{3}) - 270 - \theta = 180^\circ$

$60^\circ + 49.12^\circ - 180^\circ - 270^\circ - 73.8^\circ = \theta$

$\Rightarrow \theta = -54.7^\circ$

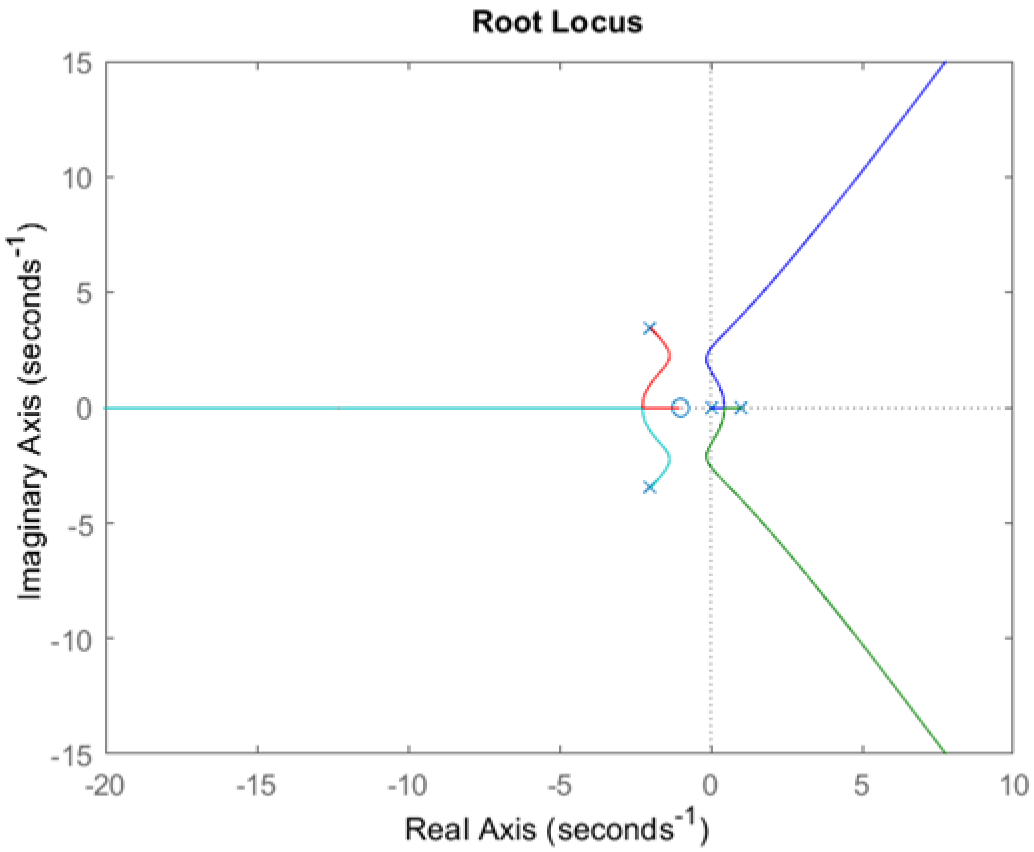


Root locus:-



MATLAB CODE:

```
%   Part A Root Locus
sys = zpk ([-1],[0, -2+3.456j, -2-3.456j, 1], 1);
rlocus(sys)
```



Question (3): A unity feedback system with open-loop transfer function is given below

$$G(s) = \frac{K(s^2 - 2s + 2)}{(s + 2)(s + 3)}$$

plot root locus and find valid break away point, value of K for stability?

3. $G(s) = \dfrac{k(s^2-2s+2)}{(s+2)(s+3)}$.

Sol. Stability :-

Ch :- $(s+2)(s+3) + k(s^2-2s+2) = 0$.

$s^2+5s+6+ks^2-2ks+2k=0$.

$(1+k)s^2+(5-2k)s+6+2k=0$.

All Coeff shld be of same sign.

$1+k>0$, $5-2k>0$ $6+2k>0$

$k>-1$ $k<2.5$ $k>-3$

$\Rightarrow -1<k<2.5$

(or)

$1+k<0$ $5-2k<0$ $6+2k<0$

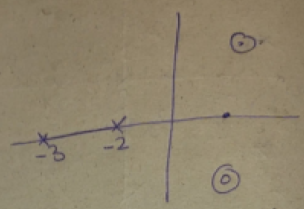
$k<-1$ $k>2.5$ $k<-3$

No mutual soln.

∴ Coeff shld be +ve

For break away point:

$$1 + k\frac{(s^2-2s+2)}{(s+2)(s+3)} = 0$$



$$k = \frac{-(s+2)(s+3)}{s^2-2s+2}$$

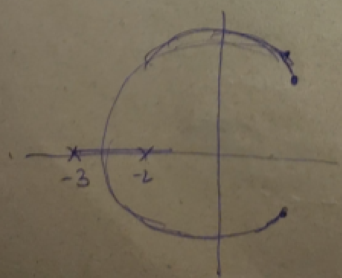
$$dk/ds = 0$$

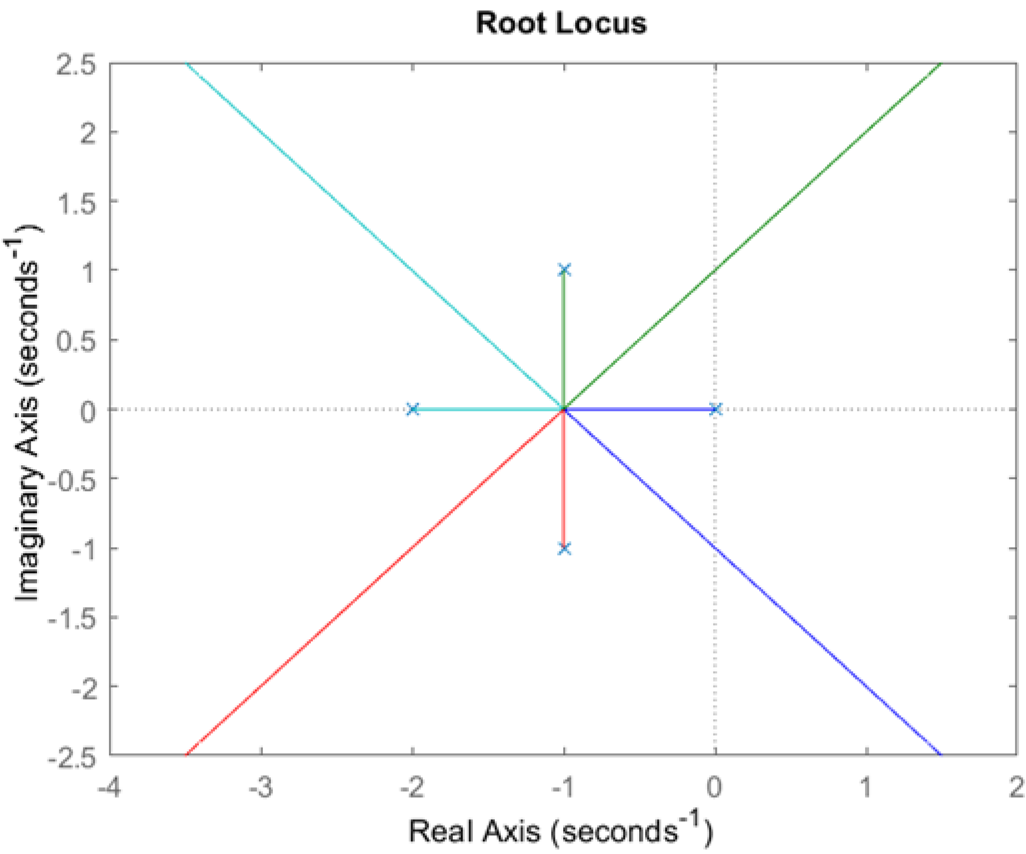
$$-\frac{(7s^2+8s-22)}{(s^2-2s+2)^2} = 0$$

Roots: $-\frac{4}{7} \pm \frac{\sqrt{170}}{7}$

Break away point: $-4/7 - \frac{\sqrt{170}}{7}$

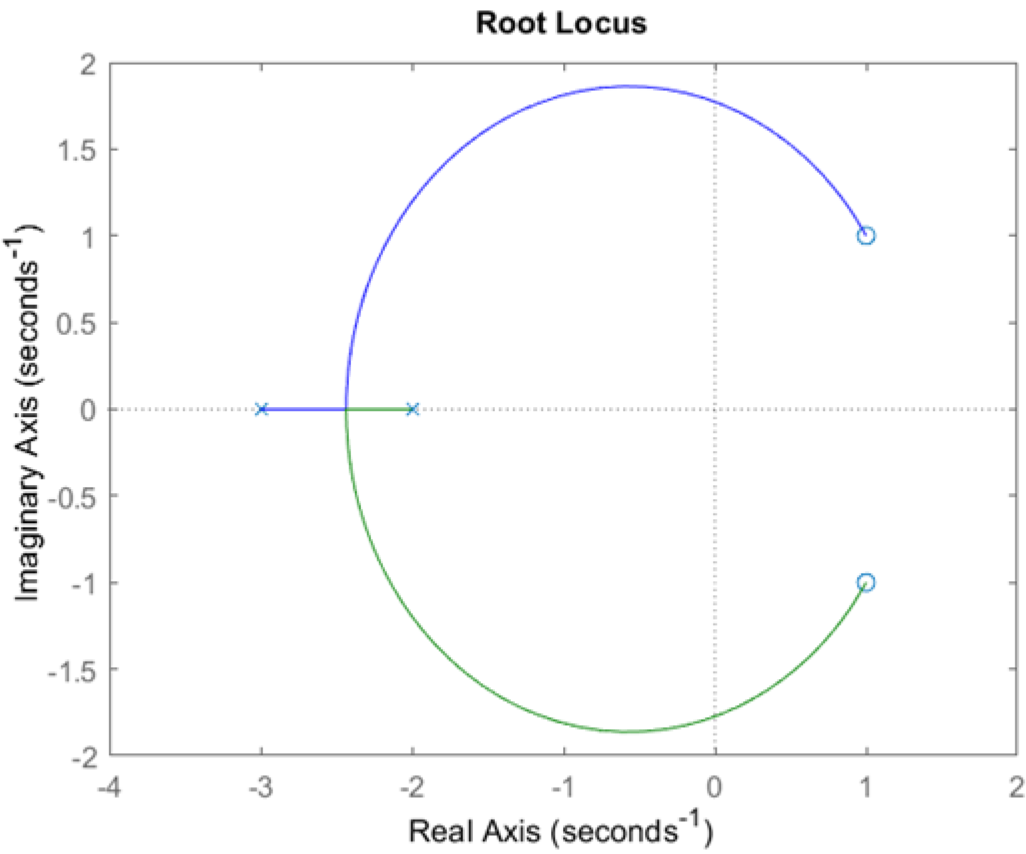
Root-locus:





MATLAB COMMAND:

```
%   Part C Root Locus
sys = zpk ([1+j, 1-j],[-2, -3], 1);
rlocus(sys)
```



Question (4): Draw the RL of the system, without finding all the break points. It is already given that one of the BP = centroid. No need to find the values of BPs.

a) $G(s)H(s)$ = K $s(s$ 2+2s+1.25)(s+2) ;

b) $G(s)H(s)$ = K $s(s$ 2+2s+2)(s+2) ;

c) $G(s)H(s)$ = K $s(s$ 2+2s+10)(s+2) ;

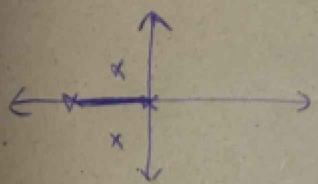
4.

(a) $G(s)H(s) = \dfrac{k}{s(s^2+2s+1.25)(s+2)}$

Ch :- $1 + \dfrac{k}{s(s^2+2s+1.25)(s+2)} = 0$.

Open loop poles → $0, -2, -1 \pm 0.5j$

Open loop zeros → none



Centroid $= \dfrac{(0-2-1+0.5j-1-0.5j)-0}{4-0}$

$= -1$.

Given Centroid is one of the breakpoints

Angle of asymptotes $= \dfrac{(2n+1)\pi}{4} = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$

<u>jω cross over</u>

Ch eq is $s^4+4s^3+5.25s^2+2.5s+k=0$.

<u>Routh Table</u>:

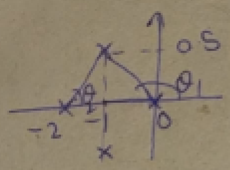
s^4	1	5.25	k
s^3	4	2.5	
s^2	4.625	k	→ Auxillary Eqn A(s)
s^1	$2.5-1.6k$		→ should be zero → $k = 1.563$
s^0	k		

→ $A(s) = 4.625s^2+k=0$

→ $s^2 = -\dfrac{1.563}{4.625}$

$s = \pm 0.58j$

Angle of departure:-



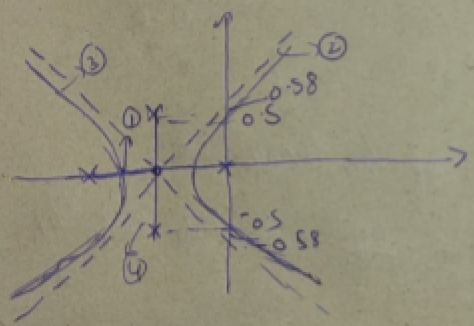
$$-\theta_1-\theta_2-90^\circ-\theta=180^\circ$$

$$-[180^\circ-\tan^{-1}(0.5)]$$

$$-\tan^{-1}(0.5)-90^\circ-\theta=180^\circ$$

$$\Rightarrow \theta=-450^\circ=-90^\circ$$

Root locus:-



(b) $G(s)H(s)=\dfrac{K}{s(s^2+2s+2)(s+2)}$

Ch eq:- $1+\dfrac{K}{s(s^2+2s+2)(s+2)}=0$

open loop poles :- $0, -1\pm j, -2$

" " zeroes :- None.

$$\text{Centroid}=\frac{(0-2-1+j-1-j)-0}{4-0}$$

$$=-1$$

Angle of asymptotes $=\dfrac{(2n+1)\pi}{4}=\pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4$

jω axis crossover:

Ch eq: $s(s^2+2s+2)(s+2)+K=0$

$\Rightarrow s^4+4s^3+6s^2+4s+K=0$

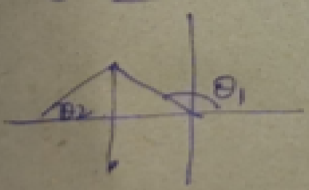
Routh Table:

s^4	1	6 K
s^3	4 1	4 1
s^2	5	K → (ROS)
s^1	$1-4/5$	→ K = 5
s^0	K	

$A(s)=5s^2+K=0$

$\Rightarrow s=\pm j$

Angle of Departure:



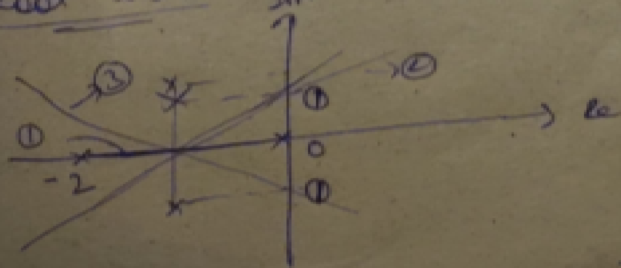
$-\theta_1-\theta_2-90^\circ-\theta=180^\circ$

$-[180^\circ-\tan^{-1}(1)]-\tan^{-1}(1)-90^\circ-\theta=180^\circ$

$\Rightarrow -270^\circ-\theta=180^\circ$

$\Rightarrow \theta=-90^\circ$

Root locus:



(c) $G(s)H(s) = \frac{k}{s(s+2)(s^2+2s+10)}$

Ch. eq :- $1 + \frac{k}{s(s+2)(s^2+2s+10)} = 0$

Open loop poles → $0, -2, -1 \pm 3j$

" " zeroes → None

Centroid $= \frac{0 - 2 - 1 + 3j - 1 - 3j}{4} = -1$

Angle of asymptotes $= \frac{(2n+1)\pi}{4} = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$

$j\omega$ axis Cross over :-

$s^4 + 4s^3 + 14s^2 + 20s + k = 0$.

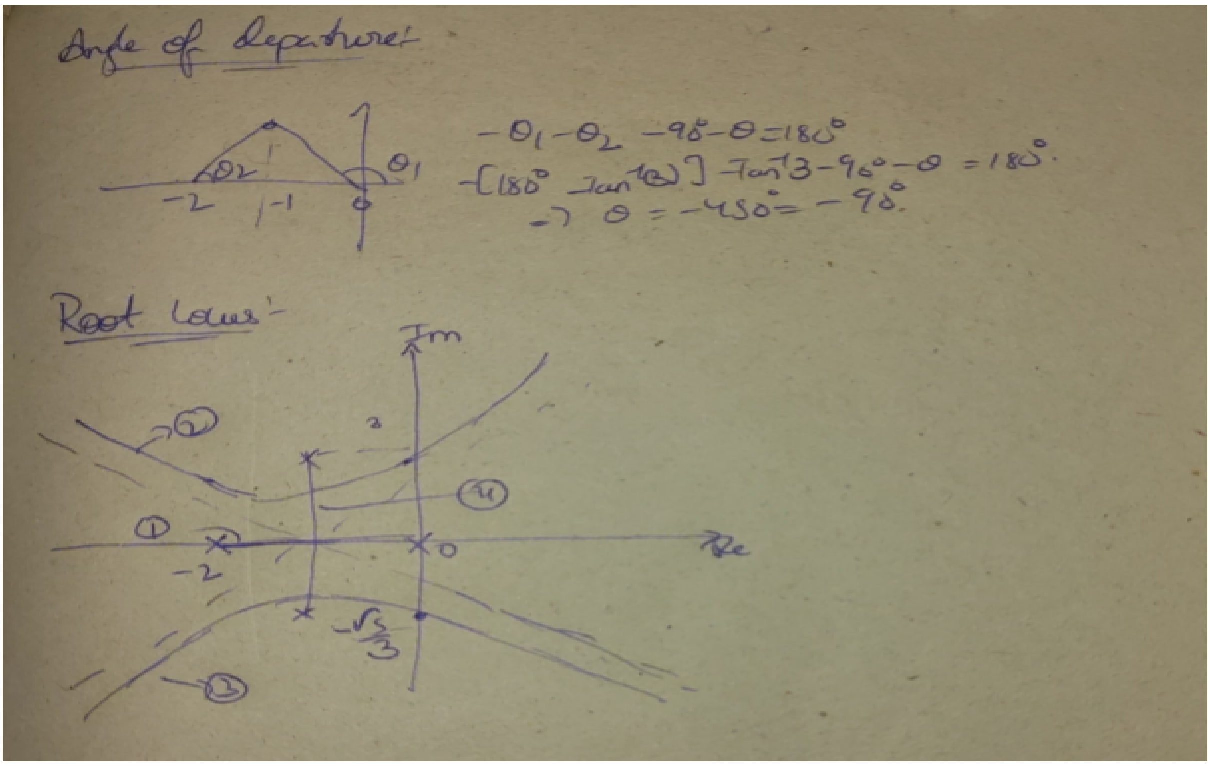
Routh Table :-

s^4	1	14	k
s^3	4	20	
s^2	9	k	
s^1	$5 - k/9$		
s^0	k		

$\rightarrow k = 45$ (45)

$9s^2 + k = 0$

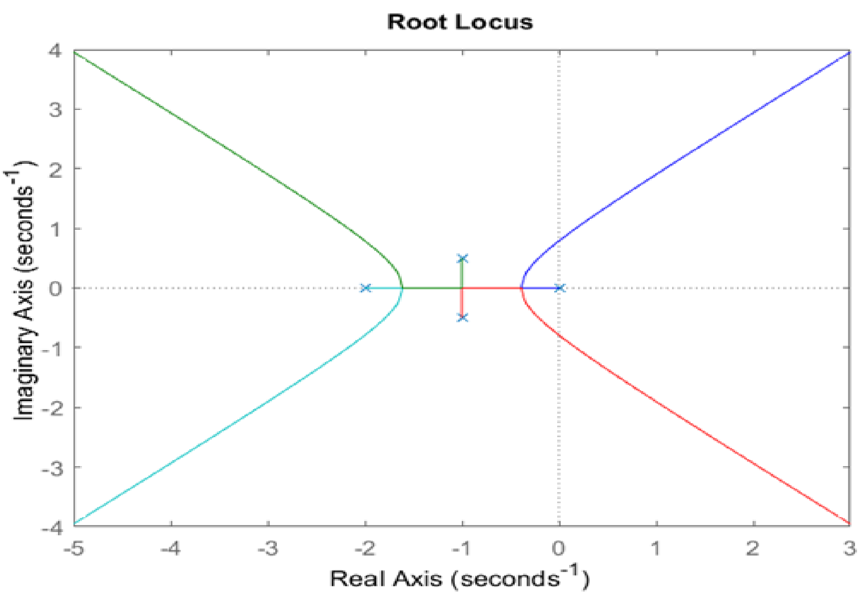
$\rightarrow s = \pm\sqrt{5}\, j$



PART A-

MATLAB COMMAND:

```
%   Part D_1 Root Locus
sys = zpk ([],[0, -2, -1+0.5j, -1-0.5j], 1);
rlocus(sys)
```



PART B-

MATLAB COMMAND:

```
%   Part D_2 Root Locus
sys = zpk ([],[0, -2, -1+j, -1-j], 1);
rlocus(sys)
```

PART C-

MATLAB COMMAND:

```
%   Part D_3 Root Locus
sys = zpk ([],[0, -2, -1+3j, -1-3j], 1);
rlocus(sys)
```