Assignment-Joint Distribution

1. Let the continuous random variables X and Y have the joint density

   $f(x,y) = \begin{cases} 3\,x & 0 \le y \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$

   Then find

   (a) $P(X \le 1/2, 1/4 < Y < 3/4)$.

   (b) $f_X(x)$ and $f_Y(y)$.

   (c) $f(x|y)$ $(0 < y < 1)$, $f(x|y = 1/2)$.

   (d) Are X and Y independent ?

   (e) Find $E(4\,X - 3\,Y)$.

   (f) Find $E(X\,Y)$.

2. Let the continuous random variables X and Y have the joint density

   $f(x,y) = \begin{cases} x^2 + x\,y/3 & 0 \le x \le 1 \text{ and } 0 \le y \le 2 \\ 0 & \text{otherwise.} \end{cases}$

   Find conditional expectation $E(X|Y = 1/2)$.

3. Let the continuous random variables X and Y have the joint density

   $f(x,y) = \begin{cases} \frac{1}{64}\,e^{-y/8} & 0 \le x \le y \le \infty \\ 0 & \text{otherwise.} \end{cases}$

   Find $Cov(X,Y)$.

4. Let $f(x,y) = \begin{cases} \frac{1}{28}(4\,x + 2\,y + 1) & 0 \le x < 2, 0 \le y < 2 \\ 0 & \text{otherwise.} \end{cases}$

   (a) Find $E(X\,Y)$.

   (b) $Cov(X\,Y)$.

   (c) $\rho_{X\,Y}$.

5. Find the value of c to make $f_{X\,Y}(x,y)$ a valid joint pdf.

   $f_{X\,Y}(x,y) = c\,x \quad x > 0, \quad y > 0, \quad 2 < x + y < 3.$

6. The random variables X and Y are independent and have the pdf's as follow

   $$f_X(x) = \begin{cases} x\,\mathrm{e}^{-\frac{1}{2}x^2} & x \ge 0 \\ 0 & \text{elsewhere.} \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} y\,\mathrm{e}^{-\frac{1}{2}x^2} & y \ge 0 \\ 0 & \text{elsewhere.} \end{cases}$$

   Find the probability that X is less than or equal to KY, where K is constant.

7. Let the continuous random variables X and Y have the joint density

   $f(x,y) = \begin{cases} 8\,x\,y & 0 < y < x < 1 \\ 0 & \text{otherwise.} \end{cases}$

   Then find

(a) $P(Y \leq 1/2, X < Y + 1/4)$.

(b) $P(X < 3/4 | Y = 1/6)$.

(c) $E(X|Y = 1/6)$.

8. Let the continuous random variables X and Y have the joint density

$$f(x, y) = \begin{cases} x + y & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Compute the correlation coefficient ρ_{XY}

9. The joint pdf of the random variables X and Y is given by

$$f(x, y) = \begin{cases} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) & 0 < x < 1, 0 < y < 2 \\ 0 & \text{otherwise.} \end{cases}$$

Evaluate $P(Y > 1/2 | X < 1/2)$.

10. An engineer is studying early morning traffic patterns at a particular intersection. The observation period begins at 5:30 AM. Let X denote the time of arrival of first vehicle from north-south direction. Let Y denote the first arrival time from east-west direction. Time is measured in fraction of an hour after 5:30 AM. Assume the density for (X, Y) is given by

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{x} & 0 < y < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find $P(Y \leq 0.25)$ and $P(X < 0.5 | Y > 0.25)$.

11. Suppose that the joint pdf of two random variables x and Y is

$$f(x, y) = \frac{1}{4\pi} \mathrm{e}^{-\frac{1}{2}\left(x^2+y^2\right)} \qquad -\infty < x, y < \infty$$

Find $P(-\sqrt{2} < (X + Y) < 2\sqrt{2})$.

12. Let $X_1 \sim \mathrm{N}(3, 9)$, $X_2 \sim \mathrm{N}(0, 1)$ and $X_3 \sim \mathrm{N}(1, 1)$ are independent random variables. Further let $Z_1 = 2X_1 + 3X_2 - 4X_3$ and $Z_2 = 3X_1 - 2X_2 + X_3$.
Find the correlation coefficient between Z_1 and Z_2.

13. Given that

$$f(x, y) = xe^{-x((y+1)}; \quad x \geq 0, \ y \geq 0.$$

Find $E[Y|X = x]$.

14.

$X \backslash Y$	1	2	3
1	1/3	1/6	1/6
2	1/6	1/12	1/6
3	1/12	1/12	0

(a) Find $E(XY)$.

(b) $Cov(XY)$.

(c) ρ_{XY}.

15. Given the random variables X and Y with their joint probability distribution as

$X \setminus Y$	-1	0	1
-1	1/8	1/8	1/8
0	1/8	0	1/8
1	1/8	1/8	1/8

(i) Find Cov(X,Y)(ii) Are X and Y independent ?

16. Let $X \sim N(0,1)$. Find pdf of $Y = \mathrm{e}^X$.

17. The random variable X is uniformly distributed over the internet (-1,3). Find the pdf of the random variable $Y = X^2$.

18. Let the random variable X is uniformly distributed over the unit interval. Find the pdf of the random variable $Y = -\lambda \ln X$.

19. Let $Z \sim N(0,1)$. Obtain the distribution of Z^2.

20. If e^{3t+8t^2} is m.g.f. of the random variable X, find $P(-1 \leq X \leq 9)$.

21. Let $X_1, X_2, \ldots, X_n$ be cont. $i.i.d.$ r.v. with pdf $f(x)$ and cdf $F(x)$. Find the pdf's of $Y_1 = min\{X_1, X_2, \ldots, X_n\}$ and $Y_2 = max\{X_1, X_2, \ldots, X_n\}$. Further if $X \sim exp(2)$, then find pdf of Y_1 and Y_2.

22. Let $X \sim N(0, \sigma^2), Y \sim N(0, \sigma^2)$. X and Y are independent. Further let us define $U = X^2 + Y^2$ and $V = X/Y$. Find the joint pdf of U and V, i.e., $f_{UV}(u, v)$. Are U and V independent ?

23. X_1 and X_2 are independent exponential random variables each having parameters λ. Find the joint density of $Y_1 = X_1 + X_2$ and $Y_2 = \mathrm{e}^{X_1}$.

24. Suppose (X, Y) is a bivariate normal random variable with parameters $\mu_X, \sigma_X, \mu_Y, \sigma_Y$ and ρ_{XY}. Show that the conditional density function of Y given $X = x$ is

$$N\left(\mu_Y + \rho_{XY}\frac{\sigma_Y}{\sigma_X}(X - \mu_X),\ \sigma_Y^2(1 - \rho_{XY}^2)\right)$$

25. Let X and Y denote the heart rate (in beats per minute) and average power output (in watts) for a 10 min. cycling time trial performed by a professional cyclist. Assume that X and Y have a bivariate normal distribution with parameters $\mu_x = 180, \mu_y = 400, \sigma_x = 10, \sigma_y = 50, \rho = 0.9$. Find

    (a) $E(Y|X = 170)$.
    (b) $E(Y|X = 200)$.
    (c) $V(Y|X = 170)$.
    (d) $V(Y|X = 200)$.
    (e) $P(Y \leq 380|X = 170)$.
    (f) $P(Y \geq 450|X = 200)$.

26. The joint pdf of the temperature in degrees Centigrade at two localities is at a particular time of day on a particular day, assumed to be a joint normal pdf

$$f(x,y) = \frac{1}{4\pi\sqrt{3}} e^{-\frac{2}{3}\left\{\left(\frac{x-20}{2}\right)^2 - \frac{(x-20)(y-20)}{4} + \left(\frac{y-20}{2}\right)^2\right\}} \qquad -\infty < x, y < \infty$$

Find the conditional pdf $f_{Y|X}(y|x)$. if the temperature at location 1, X, is 21^0C, what is the probability that the temperature at location 2, Y, is between 20 and 22^0C.

27. The amount of rainfall recorded (in inches) at Alipore weather station in April is a random variable X_1 and the amount of rainfall recorded in the same station in May is a random variable X_2. Let (X_1, X_2) have a bivariate normal distribution with parameters $\mu_1 = 6, \mu_2 = 4, \sigma_1 = 1, \sigma_2 = 0.5, \rho = 0.1$. Find

    (a) $P(X_1 \leq 5)$.

    (b) $P(X_2 \leq 5 | X_1 \leq 5)$.

    (c) $E(X_1 | X_2 = 6)$.

28. Suppose that the random variables X and Y have bivariate normal pdf with $\mu_x = 6, \mu_y = 4, \sigma_x^2 = 4, \sigma_y^2 = 10$ and $\rho_{xy} = 1/2$. Find

    (a) $P(5 < Y < 6.5)$.

    (b) $P(5 < Y < 6.5 | x = 2)$.

29. Let us assume that the distribution of grades for a particular group of students, where X and Y represent the grade point average in high school and the first year college, respectively, follow a bivariate normal distribution with parameters $\mu_x = 3.2, \mu_y = 2.4, \sigma_x = 0.4, \sigma_y = 0.6$ and $\rho = 0.6$. Evaluate the following probabilities:
    (i) $P(Y < 1.8)$
    $(ii) P(Y < 1.8 | X = 2.5)$.

30. Let X_i $(i = 1, 2, ..., 10)$ be independent random variables, each distributed over $U(0,1)$. Calculate approximately $P\left(\sum_{i=1}^{10} X_i > 6\right)$.

31. Let $Y_1, Y_2, ..., Y_{15}$ be a random sample of size 15 from the p.d.f.

    $$f_Y(y) = \begin{cases} 3(1-y)^2 & 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

    Using the central limit theorem find the approximate value of $P(1/8 < \overline{Y} < 3/8)$. ($\overline{Y}$ denotes the sample mean)