Random variable

$E \quad (\Omega, \mathcal{F})$ SPA, PSP

30 CT-1	24.02.22	Thus	12–1:00
10 CT-2	10.03.22	Thu	12–12:30
~~50~~ CT→3	07.04.22	Thu	12–1:30
10 → assign, attend, project			

A rule X which assigns ~~a real~~ value $X(\omega)$ to each $\omega \in \Omega$ (sample space) is called a r.v., i.e., X is a fn whose domain is sample space Ω of outcomes ω and whose range is (some subset of) the real numbers.

Example $\quad E$: Toss the coin two times

$$\Omega = \{\underbrace{HH}_{\omega_1}, \underbrace{HT}_{\omega_2}, \underbrace{TH}_{\omega_3}, \underbrace{TT}_{\omega_4}\}$$

$$\mathcal{F} = \mathbb{P}(\Omega)$$

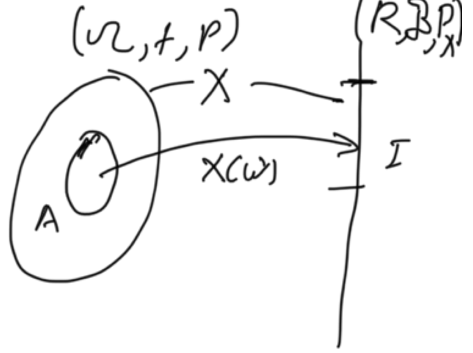
X : counting # of heads

$$X(\omega_1) = 2, \quad X(\omega_2) = 1 = X(\omega_3), \quad X(\omega_4) = 0$$

$$A_1 = \{\omega_2, \omega_3\} \equiv I = \{1\}$$

$$A_2 = \{\omega_2, \omega_3, \omega_4\} \equiv I = [0, 1]$$

Borel σ-field (containing subintervals of real line)



Event A

$$P(A) = \underline{P(\omega : X(\omega) \in I)} = P_X(I)$$

$$A \equiv I$$

equivalent event

X is r.v. if

$$\{\omega : X(\omega) \in I\} \in \mathcal{F} \text{ (is an event)}$$

$$\equiv \{\omega : X(\omega) \le x\} \in \mathcal{F}$$

$\equiv \quad X^{-1}(-\infty, x] \in \mathcal{F}$ is an event

(CDF) Cummulative distribution function or distribution function : $(\Omega, \mathcal{F}, P)$

For event $[X \le x] \equiv \{\omega : X(\omega) \le x\}$, we define

$$F_X(x) = P(X \le x) = P(\underbrace{\{\omega : X(\omega) \le x\}}) \quad , x \in \mathbb{R}$$
$$= P_X((-\infty, x])$$

Example: $\Omega = \{HH, HT, TH, TT\}$

X : # of heads $\qquad \mathcal{F} = \mathbb{P}(\Omega)$

$R_X = \{0, 1, 2\}$

prob. mass function

pmf

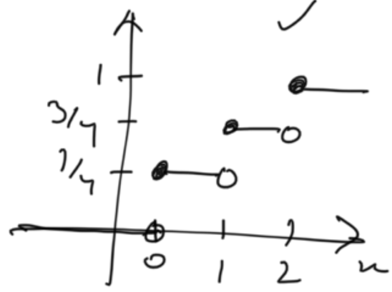
$\hookrightarrow p_X(x) = P(X = x)$

X	0	1	2
$p_X(x) = P(X=x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

CDF

$$F_X(x) = P(X \le x) = \sum_{x_i \le x} p(x_i) \qquad \text{discrete type r.v. } X$$

$$= \begin{cases} 0 & , x < 0 \\ \frac{1}{4} & , 0 \le x < 1 \\ 3/4 & , 1 \le x < 2 \\ 1 & ; x \ge 2 \end{cases}$$

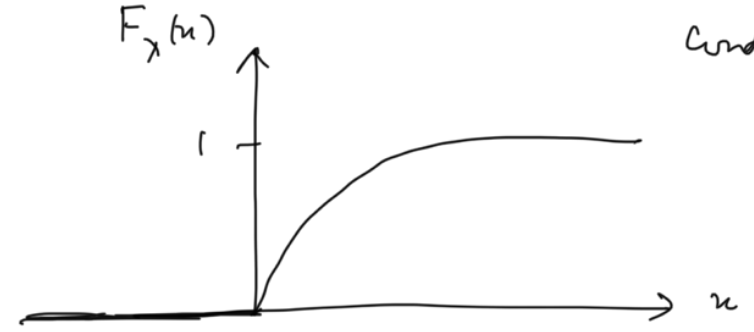


Example A cathode ray tube is aged to failure

$\Omega = \{ t \mid t \ge 0\} = [0, \infty)$

X : life of cathode ray tube till failure

CDF
$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & x \geq 0 \end{cases}, \quad \lambda > 0$$



continuous case ✓

pdf (prob. density fn)
$$f_X(x) = \frac{d}{dx} F_X(x)$$

Properties of CDF

(i) $0 \leq F_X(x) \leq 1$, $\forall x$

(ii) $F_X(x)$ is non-decreasing in x, i.e.,
$$x_1 < x_2 \Rightarrow F_X(x_1) \leq F_X(x_2)$$

(iii) $F_X(x)$ is right continuous, i.e., $F_X(x+) = F_X(x)$
$$F_X(x+) = \lim_{\delta \to 0} F_X(x+\delta) = F_X(x)$$

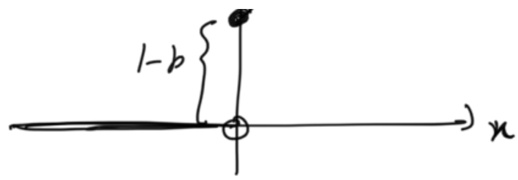
(iv) $\lim_{x \to +\infty} F_X(x) = 1$, $\lim_{x \to -\infty} F_X(x) = 0$

$$P(X = a) = F_X(a) - F_X(a-)$$

— x —

mixed type r.v.

Example Let X be the lifetime



of an instrument which may fail immediately on installation with prob. $(1-p)$ or it may live upto age x with prob. $p(1-e^{-\lambda x})$, $x>0$, $\lambda>0$

$$F_{X}(x) = P(X \leq x)$$

$$= \begin{cases} 0 & , x<0 \\ (1-p)+p(1-e^{-\lambda x}) & , x \geq 0 \end{cases}$$

X mixed type r.v.

—x—

$(1,2,3,4)$ $\qquad \dfrac{1+2+3+4}{4} = 1\times\frac{1}{4} + 2\times\frac{1}{4} + 3\times\frac{1}{4} + 4\times\frac{1}{4}$

$$1\times p_1 + 2\times p_2 + 3\times p_3 + 4\times p_4$$

$$\sum_{i=1}^{4} p_i = 1$$

Discrete dist

Benoulli trail $\quad Ber(p)$

$$1 \text{ trail} \begin{cases} \to S & p \\ \to F & q \end{cases} \qquad p+q=1$$

X counts # of successes $\in \{0,1\}$

$$P(X=0) = q \quad ; P(X=1) = p$$

$$E(X) = \sum_{x=0}^{1} x\,p(x) = 0 \times q + 1 \times p = p$$

$$E(X^2) = \sum x^2 p(x) = 0 \times q + 1^2 \times p = p$$

$$V(X) = E(X^2) - (E(X))^2 = p - p^2 = p(1-p) = pq$$

Binomial distn : $X \sim Bin(n, p)$

n indep Bernoulli trail

X : # of successes in n trail $\in \{0, 1, 2, \ldots, n\}$

$$P(X=x) = \begin{cases} \binom{n}{x} p^x q^{n-x} & , x = 0, 1, \ldots, n \\ 0 & \text{o.w.} \end{cases}$$

$E(X) = np$

$V(X) = npq$

Geometric distn : Let indep. Bernoulli trails are conducted till we observe a success.

X = # of trails to get a success

$$p(x) = P(X=x) = \begin{cases} q^{x-1} p & , x = 1, 2, \ldots \\ 0 & \text{o.w.} \end{cases}$$

$$\begin{array}{cccc} q & q & \cdots q & p \\ F & F & \cdots F & S \end{array}$$

$X \sim Geo(p)$

memoryless property

$$P(X > m) = P(X > m+n \mid X > n)$$

Sol

$$P(X > m) = \sum^{\infty} P(X = x) = \sum^{\infty} p q^{x-1}$$

$$\sum_{x=m+1} P(X=x) = \sum_{x=m+1} p q^{x-1}$$

$$= p\left[q^{m} + q^{m+1} + q^{m+2} + \cdots\right]$$

$$= pq^{m}\left[1 + q + q^{2} + \cdots\right] = pq^{m} \times \frac{1}{1-q}$$

$$= \frac{\cancel{p}q^{m}}{\cancel{p}} = q^{m} \checkmark$$

$$P(X > m+n \mid X > n) = \frac{P(X > m+n,\, X > n)}{P(X > n)} = \frac{P(X > m+n)}{P(X > n)}$$

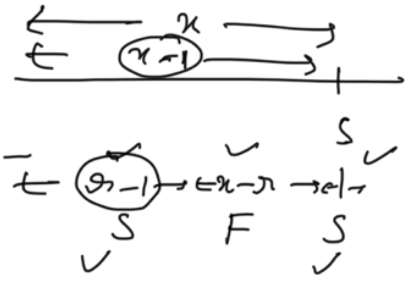
$$= \frac{q^{m+n}}{q^{n}} = q^{m}.$$

<u>Negative Binomial distn:</u> $X \sim NBL(r, p)$

Let <u>indep Bernoulli trails are conducted till we have r successes.</u>

Let r.v. X : # of trials required to get <u>r successes</u>

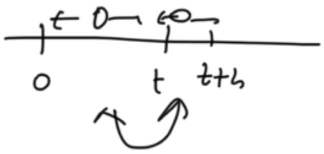
$$P(X=x) = \begin{cases} \binom{x-1}{r-1} p^{r} q^{x-r}, & x = r, r+1, \ldots \\ 0 & \text{o.w.} \end{cases}$$



— x —

P.P. ✱ <u>$h \to$ small</u>

$$P_0(t+h) = P(N(0, t+h] = 0) \quad \leftarrow 0 \rightarrow$$



$$= P(\{N(0,t]=0\} \cap \{N(t,t+h]=0\})$$

$$= P(N(0,t]=0) \cdot P(N(t,t+h)=0) \quad \Big|\ \text{indep increments}$$

$$= P_0(t) \cdot P(N(h)=0) \quad \Big|\ \text{stationary increments}$$

$$= P_0(t) \cdot (1-\lambda h+o(h))$$

$$\Rightarrow \frac{P_0(t+h)-P_0(t)}{h} = -\lambda P_0(t) + \frac{o(h)\,P_0(t)}{h}$$

$h \to 0$

$$\frac{d P_0(t)}{dt} = -\lambda P_0(t)$$

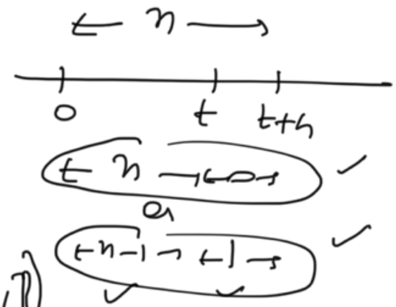
$$P_0(t) = C e^{-\lambda t}$$

$$P_0(0) = 1 \Rightarrow C = 1$$

$$P_0(t) = e^{-\lambda t}$$

$h \to$ small

$$P_n(t+h) = P(N(0,t+h)=n)$$



$$= P\Big(\underbrace{\{N(0,t)=n\} \cap \{N(t,t+h)=0\}}$$

$$\cup \underbrace{\{N(0,t)=n-1\} \cap \{N(t,t+h)=1\}}\Big)$$

$$= P_n(t)\,P_0(h) + P_{n-1}(t)\,P_1(h)$$

$$= P_n(t)\,(1-\lambda h+o(h)) + P_{n-1}(t)\cdot(\lambda h+o(h))$$

$$\Rightarrow \frac{P_n(t+h)-P_n(t)}{h} = -\lambda(P_n(t) - P_{n-1}(t)) + \frac{o(h)}{h}$$

$$\frac{\qquad}{h} = \lambda\,(P_n(t) - P_{n-1}(t)) + \frac{o(h)}{h}$$

$h \to 0$

$$\frac{dP_n(t)}{dt} = -\lambda\left(P_n(t) - P_{n-1}(t)\right)$$

know $\quad P_0(t) = e^{-\lambda t}$ ✓

<u>Assume</u> $\quad P_{n-1}(t) = \dfrac{e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}$ ✓ ✓

Use mathematical induction

$\longrightarrow \quad \underline{P_n(t)}$

$$\frac{dP_n(t)}{dt} = -\lambda P_n(t) + \lambda\,\frac{e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}$$

$$e^{\lambda t}\frac{dP_n(t)}{dt} + \lambda e^{\lambda t} P_n(t) = \frac{\lambda^n t^{n-1}}{(n-1)!}$$

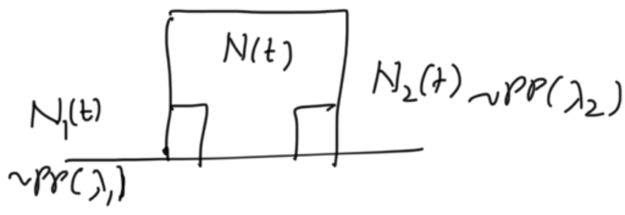
$$\frac{d}{dt}\left(e^{\lambda t} P_n(t)\right) = \frac{\lambda^n}{(n-1)!}\, t^{n-1}$$

$$e^{\lambda t} P_n(t) = \frac{\lambda^n}{n!}\, t^n$$

$$\Rightarrow \quad P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \ldots$$

—×—

(1)



$N_1(t)$ & $N_2(t)$ are indep

mgf of $N(t)$ $\quad N(t) = N_1(t) + N_2(t) \sim P.P.(\lambda_1 + \lambda_2)$

$$M_{N(t)}(u) = M_{\sum_{i=1}^{2} N_i(t)}(u) = E\left(e^{u\sum_{i=1}^{2} N_i(t)}\right)$$

$$= \prod_{i=1}^{2} M_{N_i(t)}(u)$$

$\because$ of indep $N_1(t)$ & $N_2(t)$

$$= e^{\lambda_1 t(e^u - 1)} \cdot e^{\lambda_2 t(e^u - 1)}$$

$$= e^{(\lambda_1+\lambda_2)t(e^u-1)}$$

$N(t) \sim P.P.(\lambda_1 + \lambda_2)$

— x —

$N(t) \sim P.P.(\lambda)$

$N_m(t)$ $\quad$ $m \sim p$

$N_F(t)$ $\quad$ $F \sim q$ $\quad p+q=1$

$N(t) = N_m(t) + N_F(t)$

Then $N_m(t) \sim P.P.(\lambda p)$

Indep. $N_F(t) \sim P.P.(\lambda q)$

— x —

$$\lim_{\substack{n\to\infty \\ np=\lambda}} \boxed{\binom{n}{x}} p^x (1-p)^{n-x} = \lim_{n\to\infty} \underbrace{\frac{n(n-1)\cdots(n-x+1)}{n \cdot n \cdots n}}_{x\text{ times}} \frac{\lambda^x}{x!}\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

$$\left(1-\frac{\lambda}{n}\right)^{n} \times \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= 1 \times \frac{\lambda^x}{x!} \times e^{-\lambda} \times 1$$

$$= \frac{e^{-\lambda}\lambda^x}{x!}$$

$n!$

—x—

Gamma & P.P

$$f_{S_r}(t) = -\frac{d}{dt} P(S_r > t)$$

$$= -\frac{d}{dt}\left[e^{-\lambda t} \sum_{i=0}^{r-1} \frac{(\lambda t)^i}{i!} \right]$$

$$= -\left[-\lambda e^{-\lambda t} \sum_{i=0}^{r-1} \frac{(\lambda t)^i}{i!} + e^{-\lambda t} \sum_{i=\cancel{0}\,1}^{r-1} \frac{\cancel{i}(\lambda t)^{i-1}\lambda}{\cancel{i!}\,(i-1)!} \right]$$

$$= \lambda e^{-\lambda t}\left[\sum_{i=0}^{r-1} \frac{(\lambda t)^i}{i!} - \sum_{j=0}^{r-2} \frac{(\lambda t)^j}{j!} \right] \qquad i-1=j$$

$$= \lambda e^{-\lambda t} \frac{(\lambda t)^{r-1}}{(r-1)!} = \frac{\lambda^r}{(r-1)!} e^{-\lambda t} t^{r-1}$$

$$\Gamma r$$

$S_r \sim \text{Gamma}(r, \lambda)$.

—x—

<u>Example</u>: A committee of 3 is to be selected from a pool of 10 teachers consisting of 3 Professors, 3 Associate Professors and 4 Assistant Professors. Let

X # of Prof. in committee

Y # of Asso. Prof committee

selected

$$R_X = \{0,1,2,3\}$$

$$R_Y = \{0,1,2,3\}$$

(X,Y)

$$R_{(X,Y)} = \{(i,j) : i = 0,1,2,3;\ j = 0,1,2,3,\ i+j \le 3\}$$

(X,Y) joint pmf

$$p(i,j) = P(X=i, Y=j) = \frac{\binom{3}{i}\binom{3}{j}\binom{4}{3-i-j}}{\binom{10}{3}},$$

$i+j \le 3$, $i,j = 0,1,2,3$

$Y=j \downarrow$ \ $X=i \rightarrow$	0	1	2	3	$p_Y(j) = P(Y=j)$
0	$4/120$	$18/120$	$12/120$	$1/120$	$\frac{35}{120} = P(Y=0)$
1	$18/120$	$36/120$	$9/120$	0	$\frac{63}{120}$
2	$12/120$	$9/120$	0	0	$\frac{21}{120}$
3	$1/120$	0	0	0	$\frac{1}{120}$
$p_X(x) = P(X=i)$	$\frac{35}{120}$	$\frac{63}{120}$	$\frac{21}{120}$	$\frac{1}{120}$	

$P(X=0)$ $P(X=1)$

$$P(X=0) = \sum_j P(X=0, Y=j)$$

marginal pmf of r.v. X

$X=i$	0	1	2	3
$p_X(i)$	$\frac{35}{120}$	$\frac{63}{120}$	$\frac{21}{120}$	$\frac{1}{120}$

$E(X)$

$$V(X) = E(X^2) - (E(X))^2$$

Conditional pmf of X given Y=1

$$p_{X|Y=1}(x) = \frac{p(x,1)}{p_Y(1)} \quad , x = 0,1,2,3$$

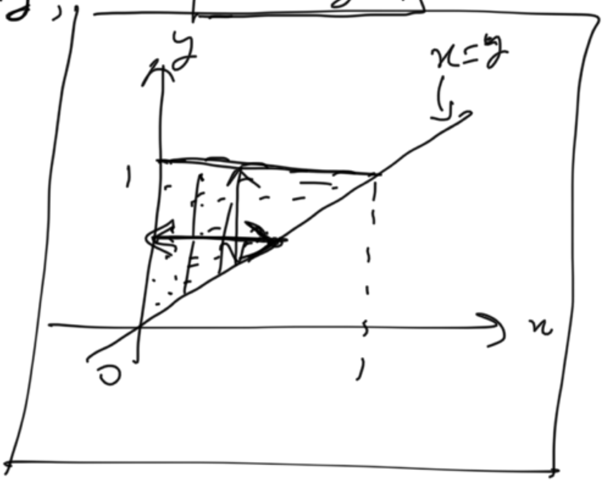
$$p_{X|Y=1}(0) = \frac{p(0,1)}{p_Y(1)} = \frac{18/120}{63/120} = \frac{18}{63}$$

x	0	1	2	3	Total
$p_{X\|Y=1}(x)$	$\frac{18}{63}$	$\frac{36}{63}$	$\frac{9}{63}$	0	1

$$E(X|Y=1) = \sum x\, p_{X|Y=1}(x) = 1 \times \frac{36}{63} + 2 \times \frac{9}{63}$$

Example

$$f(x,y) = ky, \quad \boxed{0 < x < y < 1}$$



$$\iint f(x,y)\, dx\, dy = 1$$

$$\Leftrightarrow \int_0^1 \int_0^y ky\, dx\, dy = 1$$

$$\Leftrightarrow k \int_0^1 y^2\, dy = 1$$

$$\Leftrightarrow \frac{k}{3} = 1 \Leftrightarrow k = 3$$

$$f_X(x) = \int f(x,y)\, dy = \int_x^1 3y\, dy = \frac{3}{2}(1 - x^2), \quad 0 < x < 1$$

$$f_Y(y) = \int f(x,y)\, dx = \int_0^y 3y\, dx = 3y^2, \quad 0 < y < 1$$

—x—

Example

— $f(x,y) = 6x^2y \quad , \quad 0 \le x \le 1, \ 0 \le y \le 1$

$$f_X(x) = \int_0^1 6x^2y \, dy = 3x^2, \quad 0 \le x \le 1$$

$$f_Y(y) = \int_0^1 6x^2y \, dx = 2y \quad , \quad 0 \le y \le 1$$

$$f(x,y) = f_X(x) \, f_Y(y)$$

X and Y are independent $\Rightarrow Cov(X,Y) = 0$

— x —

$$Cov(X,Y) = E(X-\mu_X)(Y-\mu_Y)$$

$$= E(XY - \mu_X Y - X\mu_Y + \mu_X \mu_Y)$$

$$= \sum_x \sum_y (xy - \mu_X y - x\mu_Y + \mu_X\mu_Y) \, p(x,y)$$

$$= \underline{\sum_x \sum_y xy \, p(x,y)} - \mu_X \sum_y y \underbrace{\boxed{\sum_x p(x,y)}}_{p_Y(y)}$$

$$- \mu_Y \sum_x x \underbrace{\boxed{\sum_y p(x,y)}}_{p_X(x)} + \mu_X\mu_Y \underbrace{\left(\sum_x \sum_y p(x,y)\right)}_{1}$$

$$= E(XY) - \mu_X\mu_Y - \cancel{\mu_Y\mu_X} + \cancel{\mu_X\mu_Y}$$

$$= \underline{E(XY) - \mu_X\mu_Y}$$

— x —

If $\underline{X \text{ and } Y \text{ are indep}}$, then $Cov(X,Y) = 0$

Given $\underline{p(x,y) = p_X(x) p_Y(y)}$ T.S. $E(XY) = \ldots$

$\ldots = \mu_X \mu_Y$

Sol $$E(XY) = \sum_x \sum_y xy\, p(x,y)$$

$$= \sum_x \sum_y xy\, p_X(x)\, p_Y(y) \quad \Big| \because X \,\&\, Y \text{ indep (given)}$$

$$= \underbrace{\sum_x x\, p_X(x)}\Big(\underbrace{\sum_y y\, p_Y(y)}_{\mu_Y}\Big)$$

$$= \mu_Y \mu_X$$

—x—

$$E(E(X|Y)) = E(X)$$

$$E(\underbrace{E(X|Y)}_{\psi(Y)}) = \sum_y \underbrace{E(X|y)}\, p_Y(y)$$

$$= \sum_y \Big(\sum_x x\, \underbrace{p_{X|y}(x)}\Big)\, p_Y(y)$$

$$= \sum_x x \sum_y \frac{p(x,y)}{\cancel{p_Y(y)}}\, \cancel{p_Y(y)}$$

$$= \sum_x x \underbrace{\Big(\sum_y p(x,y)\Big)}_{p_X(x)}$$

$$= \sum_x x\, p_X(x)$$

$$= E(X)$$

—x—

$$Y = a_0 + \sum_{i=1}^{n} a_i X_i \qquad , a_i \in \mathbb{R}$$

$$E(X_i) = \mu_i \quad , V(X_i) = \sigma_i^2$$

$$E(Y) = E\left(a_0 + \sum_{i=1}^{n} a_i X_i\right)$$

$$= a_0 + \sum_{i=1}^{n} a_i \underbrace{E(X_i)}_{\mu_i} = a_0 + \sum_{i=1}^{n} a_i \mu_i$$

$$Y = a_0 + a_1 X_1 + a_2 X_2$$

$$E(Y) = E(a_0 + a_1 X_1 + a_2 X_2)$$

$$= \sum_{x_1} \sum_{x_2} (a_0 + a_1 x_1 + a_2 x_2)\, p(x_1, x_2)$$

$$= a_0 \times 1 + a_1 \underbrace{\sum_{x_1} x_1\, p_{X_1}(x_1)}_{\mu_1} + a_2 \underbrace{\sum_{x_2} x_2\, p_{X_2}(x_2)}_{\mu_2}$$

$X_1, \ldots, X_n$ indep.

$$V\left(a_0 + \sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 V(X_i) = \sum_{i=1}^{n} a_i^2 \sigma_i^2$$

$$V(Y) = V(a_0 + a_1 X_1 + a_2 X_2)$$

$$V(X) = E\left((X - E(X))^2\right)$$

$$= E\left(a_0 + a_1 X_1 + a_2 X_2 - E(a_0 + a_1 X_1 + a_2 X_2)\right)^2$$

$$= E\left(\cancel{a_0} + a_1 X_1 + a_2 X_2 - \cancel{a_0} - a_1 E(X_1) - a_2 E(X_2)\right)^2$$

$$= E\left(a_1 (X_1 - E(X_1)) + a_2 (X_2 - E(X_2))\right)^2$$

$$= E(a_1(X_1 - E(X_1)) + a_2(X_2 - E(X_2)))$$

$$= E\left(a_1^2 \underbrace{(X_1 - E(X_1))^2} + a_2^2 (X_2 - E(X_2))^2 + 2a_1 a_2 \underbrace{(X_1 - E(X_1))(X_2 - E(X_2))} \right)$$

$$= a_1^2 V(X_1) + a_2^2 V(X_2) + 2 a_1 a_2 Cov(X_1, X_2)$$

$$V\left(a_0 + \sum_{i=1}^{n} a_i X_i \right) = \sum_{i=1}^{n} a_i^2 V(X_i) + 2 \sum_{i}\sum_{j}_{i<j} a_i a_j Cov(X_i, X_j)$$

—x—

$S_n = X_1 + X_2$ $X_1, X_2 \to$ indep

$$m_{\sum_{i=1}^{2} X_i}(t) = m_{S_n}(t) = E(e^{tS_n}) = E(e^{t(X_1+X_2)})$$

$$= E(e^{tX_1} e^{tX_2})$$

$$= \int\int e^{tx_1} e^{tx_2} \underbrace{f_{X_1,X_2}(x_1, x_2)}_{f_{X_1}(x_1)\, f_{X_2}(x_2) \ \because \text{indep}} dx_1 dx_2$$

$$= \int e^{tx_2} f_{X_2}(x_2) \Big(\underbrace{\int e^{tx_1} f_{X_1}(x_1) dx_1}_{E(e^{tX_1})} \Big) dx_2$$

$$= E(e^{tX_1}) E(e^{tX_2})$$

$$= m_{X_1}(t)\, m_{X_2}(t)$$

—x—

(1) $X_i \overset{indep}{\sim} B(n_i, p)$, $i = 1, \ldots, m$

n_e

$$M_{X_i}(t) = (q+pe^t)^{n_i}$$

$$S_m = \sum_{i=1}^{m} X_i$$

$$M_{S_m}(t) = \prod_{i=1}^{m} M_{X_i}(t) = (q+pe^t)^{n_1} \cdots (q+pe^t)^{n_m}$$

$$= (q+pe^t)^{n_1+n_2+\cdots+n_m}$$

$$S_m \sim Bin\left(\sum_{i=1}^{m} n_i, p\right)$$

— x —

✓ $X_i \sim N(\mu_i, \sigma_i^2)$ $\quad X_1, \ldots, X_n$ indep

$$M_{X_i}(t) = e^{\mu_i t + \frac{1}{2}\sigma_i^2 t^2}$$ ✓

$$S_m = \sum_{i=1}^{m} X_i$$

$$M_{S_m}(t) = \prod_{i=1}^{m} M_{X_i}(t)$$

$$= e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2} \cdots e^{\mu_m t + \frac{1}{2}\sigma_m^2 t^2}$$

$$= e^{\left(\sum_{i=1}^{m}\mu_i\right)t + \frac{1}{2}\left(\sum_{i=1}^{m}\sigma_i^2\right)t^2}$$

$$S_m \sim N\left(\sum_{i=1}^{m}\mu_i, \sum_{i=1}^{m}\sigma_i^2\right)$$

— x —

LLN

X_i indep μ, σ^2

$$S_n = \sum_{i=1}^{n} X_i, \quad \bar{X}_n = \frac{S_n}{n}$$

for any $\epsilon > 0$

$$P(|\bar{X}_n - \mu| < \epsilon) \to 1 \text{ as } n \to \infty$$

Sol

$$V(S_n) = V\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} V(X_i) = n\sigma^2$$

$$V(\bar{X}_n) = V\left(\frac{S_n}{n}\right) = \frac{1}{n^2} \times n\sigma^2 = \frac{\sigma^2}{n}$$

Chebyshev's inequality

$$E(\bar{X}_n) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) \xrightarrow{} \mu$$

$$= \frac{\cancel{n}\mu}{\cancel{n}} = \mu$$

$$P(|X-\mu| \le k) \ge 1 - \frac{\sigma^2}{k^2}$$

small ϵ

$$\underline{P(|\bar{X}_n - \mu| \le \epsilon)} \ge 1 - \frac{\sigma^2}{n\epsilon^2}$$

$\to 1$ as $n \to \infty$

— x —

eg $X = \#$ on top face of dice

i w.p. $\frac{1}{6}$ $i = 1, \ldots, 6$

$$\underline{\mu} = E(X) = \frac{1+2+3+4+5+6}{6} = 3.5$$

experiment

$(3, 5, 4) \ldots \to 4$

$\bar{X}_n$

— x —

<u>BVN</u> (X, Y) Joint density

$$\underline{f(x,y)} = \underbrace{\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-Q/2}}, \text{ where}$$

$$Q = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]$$

$$\frac{1}{1-\rho^2}\left[\left(\frac{\quad}{\sigma_1}\right)^2 - 2\rho\left(\frac{\quad}{\sigma_1}\right)\left(\frac{\quad}{\sigma_2}\right) + \left(\frac{\quad}{\sigma_2}\right)^2\right]$$

$$= \frac{1}{1-\rho^2}\left[\left(\frac{y-\mu_2}{\sigma_2} - \rho\,\frac{x-\mu_1}{\sigma_1}\right)^2 + (1-\rho^2)\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right]$$

$$f(x,y) = \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}_{\to\, Q_1(x)} \times$$

$$\underbrace{\frac{1}{\sqrt{2\pi}\,\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)\sigma_2^2}\left[(y-\mu_2) - \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)\right]^2\right\}}_{\to\, Q_2(x,y)}$$

$$= Q_1(x) \times Q_2(x,y)$$

$$f_X(x) = \int f(x,y)\,dy = Q_1(x)\underbrace{\int Q_2(x,y)\,dy}_{\to\, 1}$$

$$= Q_1(x)$$

$$X \sim N(\mu_1, \sigma_1^2)$$

$$Y \sim N(\mu_2, \sigma_2^2)$$

$$[Y|X=x] \sim f_{[Y|X=x]}(y) = \frac{f(x,y)}{f_X(x)} = \frac{\cancel{Q_1(x)}\,Q_2(x,y)}{\cancel{Q_1(x)}}$$

$$= Q_2(x,y)$$

$$[Y|X=x] \sim N\Big(\underbrace{\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)}_{\to\, E(Y|X=x)},\ \underbrace{\sigma_2^2(1-\rho^2)}_{V(Y|X=x)}\Big)$$

$$[X|Y=y] \sim N\left(\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y-\mu_2),\ \sigma_1^2(1-\rho^2)\right)$$

$$E(Y|x) = E(Y|X=x) = \mu_2 + \rho \frac{\sigma_2}{\sigma_1}(x-\mu_1)$$

X depth

$$\underbrace{[Y|X_i = x_i]}_{} \equiv \tilde{Y}_i \qquad \check{X} = x_i$$

$$E(XY|x) = x\, E(Y|x) = \mu_2 x + \rho \frac{\sigma_2}{\sigma_1}(x^2 - \mu_1 x)$$

$$E(XY|X) = \mu_2 X + \rho \frac{\sigma_2}{\sigma_1}(X^2 - \mu_1 X)$$

$$E(X) = E(E(X|Y))$$

$$E(XY) = E(E(XY|X))$$

$$= \mu_2 E(X) + \rho \frac{\sigma_2}{\sigma_1}(E(X^2) - \mu_1 \underline{E(X)})$$

$$\sigma_1^2 = E(X^2) - \mu_1^2$$

$$= \mu_1\mu_2 + \rho \frac{\sigma_2}{\sigma_1}(\sigma_1^2 + \cancel{\mu_1^2} - \cancel{\mu_1^2})$$

$$= \mu_1\mu_2 + \rho\,\sigma_1\sigma_2$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = \cancel{\mu_1\mu_2} + \rho\sigma_1\sigma_2 - \cancel{\mu_1\mu_2}$$

$$= \rho\,\sigma_1\sigma_2$$

$$Corr(X,Y) = \frac{Cov(X,Y)}{\quad} = \rho$$