Sample Questions

1 i) Coin toss results - HTT HHT HHHT

$n(H) = 6$, $n(T) = 4$.

So, MLE of bias $= p(H) = \dfrac{n(H)}{n(H)+n(T)}$

So, $p = \dfrac{6}{6+4} = 0.6$.

Now, for Bayesian estimate,

$$\text{posterior}(p \mid x) \propto \prod_{i=1}^{N} \text{prob}(X_i = x_i \mid p) \cdot \text{prior}(p)$$

Now $\text{prior}(p) = \text{Beta}(1,1)$ i.e. $a=1, b=1$

So,

$$\text{posterior}(p \mid x) \propto \prod_{i=1}^{N} p^{n(H)} \cdot (1-p)^{n(T)} \cdot \cancel{p^{a-1}} \cdot \cancel{(1-p)^{b-1}}$$

$$\propto p^{n(H)} (1-p)^{n(T)}$$

So, $\text{posterior}(p \mid x) \doteq \text{Beta}(n(H)+1, n(T)+1)$

and $p_{\text{Bayes}} = \dfrac{n(H)+1}{n(H)+n(T)+2}$

A) After toss 1 : $X = H$, $p = \dfrac{1+1}{1+0+2} = \dfrac{2}{3}$.

B) After toss 2 : $X = T$, $p = \dfrac{1+1}{1+1+2} = \dfrac{2}{4} = \dfrac{1}{2}$

C) After toss 3 : $X = T$, $p = \dfrac{1+1}{1+2+2} = \dfrac{2}{5}$

D) After toss 4, $X = H$, $p = \dfrac{2+1}{2+2+2} = \dfrac{3}{6} = \dfrac{1}{2}$.

E) After toss 5, $X = H$, $p = \dfrac{3+1}{3+2+2} = \dfrac{4}{7}$.

F) After toss 6, X = T, $p = \frac{3+1}{3+3+2} = \frac{4}{8} = \frac{1}{2}$

G) After toss 7, X = H, $p = \frac{4+1}{4+3+2} = \frac{5}{9}$

H) After toss 8, X = H, $p = \frac{5+1}{5+3+2} = \frac{6}{10} = \frac{3}{5}$.

I) After toss 9, X = H, $p = \frac{6+1}{6+3+2} = \frac{7}{11}$.

J) After toss 10, X = T, $p = \frac{6+1}{6+4+2} = \frac{7}{12}$.

ii) The number of buses passing a bus stop are

[3, 4, 2, 5, 3, 4, 4, 1, 2, 7, 4]

This ~~has~~ follows poisson distribution.

λ = average no. of buses each hour.

$$= \frac{3+4+2+5+3+4+4+1+2+7+4}{11}$$

$$= \frac{39}{11}$$

So,

λ_{new} = average no. of buses for each 2 hours.

$$= 2\lambda = \frac{78}{11}.$$

So,

$X \sim$ poisson (λ new) where $X \to$ No. of buses in each 2 hours.

So,

$$P(X=6) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x_0!}$$

$$= \frac{e^{-78/11} \cdot \left(\frac{78}{11}\right)^{6}}{6!}$$

$$= 0.147.$$

iii) The maximum and minimum temperatures follow Gaussian distribution.

Given observations:

(40, 22), (41, 24), (39, 20), (40, 20), (43, 21), (45, 25), (43, 22), (42, 22), (40, 20), (42, 23).

Let ~~the~~ $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim \mathcal{N}(\mu, \Sigma)$

where x_1 = max temp
x_2 = min temp

The mean vector μ will be $\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$

$$\mu_1 = \frac{40 + 41 + 39 + 40 + 43 + 45 + 43 + 42 + 40 + 42}{10}$$

$$= 41.5$$

$$\mu_2 = \frac{22 + 24 + 20 + 20 + 21 + 25 + 22 + 22 + 20 + 23}{10}$$

$$= 21.9.$$

So,
$$\mu = \begin{bmatrix} 41.5 \\ 21.9 \end{bmatrix}$$

Now, for covariance matrix:

~~[illegible]~~

$$\mathrm{Cov}(x_1, x_2) = E[(x_1 - E(x_1))(x_2 - E(x_2))^T]$$

$$\Sigma = \begin{bmatrix} \mathrm{Cov}(x_1, x_1) & \mathrm{Cov}(x_1, x_2) \\ \mathrm{Cov}(x_2, x_1) & \mathrm{Cov}(x_2, x_2) \end{bmatrix}$$

Now,

$$Cov(x_1, x_1) = \frac{1}{10}\left[\frac{40^2 + 41^2 + 39^2 + 40^2 + 43^2 + 45^2 + 43^2 + 42^2 + 40^2 + 42^2}{10} - 41.5^2\right]$$

$$= 0.305$$

$$Cov(x_2, x_2) = \frac{1}{10}\left[\frac{22^2 + 24^2 + 20^2 + 20^2 + 21^2 + 25^2 + 22^2 + 22^2 + 20^2 + 23^2}{10} - 21.9^2\right]$$

$$= 2.69.$$

$$Cov(x_1, x_2) = \frac{1}{10}\left[\frac{40\times22 + 41\times24 + 39\times20 + 40\times20 + 43\times21 + 45\times25 + 43\times22 + 42\times22 + 40\times20 + 42\times23}{10} - 41.5\times21.9\right]$$

$$= \cancel{2.993}\ 2.17.$$

So,

$$X \sim \mathcal{N}\left(\begin{bmatrix} 41.5 \\ 21.9 \end{bmatrix}, \begin{bmatrix} 0.305 & 2.17 \\ 2.17 & 2.69 \end{bmatrix}\right)$$

2) Coin bias : $p = 0.7$.

head — fair die rolled ($1/6$)

tail — loaded die rolled ($1/3$) (odd nos).

Sum on dice is 8.

possible combinations : ✓ 2+6 ← ~~die~~ fair die

= 3+5 ← loaded die & fair die

4+4 ← fair die

= 5+3 ← loaded & fair die

6+2 ← fair die.

$$P(\text{fair dice} \mid \text{sum} = 8)$$

$$= \frac{P(\text{sum} = 8 \mid \text{fair dice}) \cdot P(\text{fair dice})}{P(\text{sum} = 8 \mid \text{fair dice}) \cdot P(\text{fair dice}) + P(\text{sum} = 8 \mid \cancel{\text{not}}\ \overset{\text{one}}{}\ \text{fair dice}) \cdot P(\cancel{\text{not}}\ \overset{\text{one}}{}\ \text{fair dice}) + P(\text{sum} = 8 \mid \text{no fair dice})\, P(\text{no fair dice})}$$

$$P(\text{fair dice}) = 0.7^2 = 0.49.$$

~~P(not fair dice both time) = 1 - 0.49 = 0.51.~~

Now, $P(\text{one fair die}) = 0.42$. $P(\text{no fair dice}) = 0.09$

$$P(\text{sum} = 8 \mid \text{fair dice})$$

$$= \frac{5}{36}$$

$$P(\text{sum} = 8 \mid \cancel{\text{not fair dice}}\ \text{one fair dice})$$

$$= \cancel{\frac{3}{36} \cdot \frac{2}{9} \cdot + \left(\frac{1}{6} \times \frac{1}{3} \times 2\right) \times 2.}$$

$$= \cancel{\frac{2}{9} + \frac{2}{9} = \frac{4}{9}.}$$

$$= \left(\frac{1}{6} \times \frac{1}{3} \times 2\right) \times 2 = \frac{2}{9} = \frac{8}{36}$$

$$P(\text{sum} = 8 \mid \text{no fair die})$$

$$= \frac{1}{3} \times \frac{1}{3} \times 2 = \frac{2}{9} = \frac{8}{36}.$$

So, $P(\text{fair dice} \mid \text{sum} = 8)$

$$= \frac{0.49 \times \frac{5}{\cancel{36}}}{0.49 \times \frac{5}{\cancel{36}} + 0.42 \times \frac{8}{\cancel{36}} + 0.09 \times \frac{8}{\cancel{36}}} = 0.376.$$

3) $a_i \sim \mathcal{N}\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$

$b_i \sim \mathcal{N}\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \right)$

i) A coin has bias p is tossed.

head $\rightarrow$ a_i is drawn
tail $\rightarrow$ b_i is drawn. $\Big\} \rightarrow x_i$

$$E[x_i] = p \begin{bmatrix} 1 \\ 4 \end{bmatrix} + (1-p) \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 - p \\ 2 + 2p \end{bmatrix}.$$

ii) Let bias of the coin be p.

$x_1, x_2, \ldots, x_{10} \sim N_1$ or N_2.

$$p(x) = p(x, z=1) + p(x, z=2)$$

$$= p(z=1)\, p(x \mid z=1) + p(z=2)\, p(x \mid z=2)$$

$$= \cancel{p \cdot \frac{1}{2\pi|\Sigma|} e^{-\frac{1}{2}[x-4]}}$$

$$= p \cdot \frac{1}{2\pi|\Sigma_1|} e^{-\frac{1}{2}\left[(x-\mu_1)^T \Sigma^{-1} (x-\mu_1)\right]}$$

$$+ (1-p) \cdot \frac{1}{2\pi|\Sigma_2|} e^{-\frac{1}{2}\left[(x-\mu_2)^T \Sigma^{-1} (x-\mu_2)\right]}$$

So,

$$P(x_1, x_2, \ldots, x_{10}) = \prod_{i=1}^{10} \left(p \cdot \frac{1}{2\pi|\Sigma_1|} e^{-\frac{1}{2}\left[(x_i-\mu_1)^T \Sigma^{-1} (x_i-\mu_1)\right]} + (1-p) \cdot \frac{1}{2\pi|\Sigma_2|} e^{-\frac{1}{2}\left[(x_i-\mu_2)^T \Sigma^{-1} (x_i-\mu_2)\right]} \right)$$

Now, $\frac{dP(x_1, \ldots, x_{10})}{dp} = 0$ (obtain p)

ii) Checking mean stationarity,

$$\mu_1 = \frac{12+13+13+15+16+14+18}{7} = 14.428$$

$$\mu_2 = \frac{10+10+11+13+15+11+16}{7} = 12.286$$

$$\mu_3 = \frac{14+16+15+19+17+20+20}{7} = 17.143$$

$$\mu_4 = \frac{25+26+27+30+30+28+33}{7} = 28.429$$

Clearly, the mean for each time stamp is not nearly same, so data has not mean stationarity.

Checking covariance stationarity.

$$Cov(x_1, x_2) \approx Cov(x_2, x_3) \approx Cov(x_3, x_4)$$

for covariance stationarity

$Cov(x_1, x_2) = 4.86$

$Cov(x_2, x_3) = 3.29$

$Cov(x_3, x_4) = 4.93$

As the values are close the data has covariance stationarity.

Check for 4 hour differences, 8 hour, 12 hour & 16 hour differences.

5) Order 1 auto regressive process

a) $Y = \begin{bmatrix} 10 \\ 10 \\ 11 \\ 13 \\ 15 \\ 11 \\ 16 \end{bmatrix}$, $X = \begin{bmatrix} 12 & 1 \\ 13 & 1 \\ 13 & 1 \\ 15 & 1 \\ 16 & 1 \\ 14 & 1 \\ 18 & 1 \end{bmatrix}$, $\theta = \begin{bmatrix} a \\ b \end{bmatrix}$

So, $y = \theta^T x$ or $y_i = \theta^T x_i$

Using least squares.

$$\theta = (X^T X)^{-1} X^T y.$$

So, $a = 1.1333$, $a = -4.0667$

b) $Y = \begin{bmatrix} 14 \\ 18 \\ 15 \\ 18 \\ 17 \\ 20 \\ 20 \end{bmatrix}$, $X = \begin{bmatrix} 10 & 1 \\ 10 & 1 \\ 11 & 1 \\ 13 & 1 \\ 15 & 1 \\ 11 & 1 \\ 16 & 1 \end{bmatrix}$. $\theta = \begin{bmatrix} a \\ b \end{bmatrix}$

So, $\theta = (X^T X)^{-1} X^T y$

$a = -$ find similarly.

For order 2, take 2 features.

b) Build a spatial auto regressive process.

~~Repeat~~ ~~det~~ ~~ws~~

Coefficient of each influencing location $= \dfrac{w}{\text{distance}(s,s')}$ = a_i

So, $a_i = \begin{cases} \dfrac{w}{\text{dist}(s,s')} & \text{if } |d_{lat,lon} - d'_{lat-lon}| = 1 \\ 0 & \text{otherwise.} \end{cases}$

Let us construct the regression equations for first order

$$X_s = \mu_s + \sum_{i=1}^{s} a_i X_i \quad \text{(general form)}$$

~~ws distance ...~~

Calculating ~~mean~~ spatial mean for each location:

$$\mu_1 = \frac{4+5+5+6+5}{5} = 5.$$

$$\mu_2 = \frac{5+5+6+7+5}{5} = 5.6$$

$$\mu_3 = \frac{7+7+6+9+8}{5} = 7.4.$$

$$\mu_4 = \frac{3+3+3+5+5}{5} = 3.8.$$

$$\mu_5 = \frac{4+5+4+6+6}{5} = 5$$

$$\mu_6 = \frac{6+8+7+7+6}{5} = 6.8$$

$$\mu_7 = \frac{3+2+2+5+4}{5} = 3.2$$

$$\mu_8 = \frac{4+3+3+5+6}{5} = 4.2$$

$$\mu_9 = \frac{5+6+6+7+7}{5} = 6.2$$

So, for X_1, the auto regressive equations are

$$\underbrace{\begin{bmatrix}4\\5\\5\\6\\5\end{bmatrix}}_{X_1} = \underbrace{\begin{bmatrix}5\\5\\5\\5\\5\end{bmatrix}}_{\mu_1} + w \times \underbrace{\begin{bmatrix}5\\5\\6\\7\\5\end{bmatrix}}_{X_2} + w \times \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

$$\underbrace{4}_{X_{11}} = \underbrace{5}_{\mu_1} + \underbrace{5}_{X_{21}} w + \underbrace{3}_{X_{41}} w .$$

$$\underbrace{5}_{X_{12}} = \underbrace{5}_{\mu_1} + \underbrace{5w}_{X_{22}} + \underbrace{3w}_{X_{42}}$$

$$\underbrace{5}_{X_{13}} = \underbrace{5}_{\mu_1} + \underbrace{6w}_{X_{23}} + \underbrace{3w}_{X_{43}}$$

$$\underbrace{6}_{X_{14}} = \underbrace{5}_{\mu_1} + \underbrace{7w}_{X_{24}} + \underbrace{5w}_{X_{44}}$$

$$\underbrace{5}_{X_{15}} = \underbrace{5}_{\mu_1} + \underbrace{5w}_{X_{25}} + \underbrace{5w}_{X_{45}}$$

So,

$-1 = 8w$ ——— (1).

$0 = 8w$ ——— (2).

$0 = 9w$ ——— (3)

$1 = 12w$ ——— (4)

$0 = 10w$ ——— (5).

Least squares estimate is

$$w = (X^T X)^{-1} (X^T y)$$

$$= \left\{ \begin{bmatrix} 8 & 8 & 9 & 12 & 10 \end{bmatrix} \begin{bmatrix} 8\\8\\9\\10\\12 \end{bmatrix} \right\}^{-1} \begin{bmatrix} 8 & 8 & 9 & 12 & 10 \end{bmatrix} \begin{bmatrix} -1\\0\\0\\1\\0 \end{bmatrix}$$

$$= \frac{1}{449} \times (-8 + 12) = \frac{4}{449}.$$

similarly do for $x_2, x_3, \ldots$ and obtain w's.

final w = mean of all such w's.

=) Gaussian process:, $\mu = 0$,

$$c(s_1, s_2) = e^{-\frac{1}{2}\|s_1 - s_2\|} \; ; \; \|s_1 - s_2\| = \text{euclidean distance.}$$

Let value at other location be x, then,

$$\begin{bmatrix} 5 \\ x \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_{11}^2 & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \right)$$

where

$$\sigma_{11} = e^0 = 1$$

$$\sigma_{12} = e^{-\frac{1}{2}\cdot 2} = e^{-1}$$

$$\sigma_{21} = e^{-\frac{1}{2}\cdot 2} = e^{-1}$$

$$\sigma_{22} = e^0 = 1$$

So,

$$\begin{bmatrix} 5 \\ x \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & e^{-1} \\ e^{-1} & 1 \end{bmatrix} \right)$$

Now,

$$f\left(\begin{bmatrix} 5 \\ x \end{bmatrix}\right) = \frac{1}{2\pi|\Sigma|} e^{-\frac{1}{2}\left[\begin{bmatrix} 5 \\ x \end{bmatrix}^T \Sigma^{-1} \begin{bmatrix} 5 \\ x \end{bmatrix}\right]}$$

$$= \frac{1}{2\pi(1-e^{-2})} e^{-\frac{1}{2}\left(\begin{bmatrix} 5 \\ x \end{bmatrix}^T \frac{1}{1-e^{-2}} \begin{bmatrix} 1 & -e^{-1} \\ -e^{-1} & 1 \end{bmatrix} \begin{bmatrix} 5 \\ x \end{bmatrix}\right)}$$

$$= \frac{1}{2\pi(1-e^{-2})} e^{-\frac{1}{2(1-e^{-2})}\left(\begin{bmatrix} 5 \\ x \end{bmatrix}^T \cdot \begin{bmatrix} 5 - xe^{-1} \\ -5e^{-1} + x \end{bmatrix}\right)}$$

$$= \frac{1}{2\pi(1-e^{-2})} e^{-\frac{1}{2(1-e^{-2})}\left(25 - 5xe^{-1} - 5xe^{-1} + x^2\right)}$$

$$= \frac{1}{2\pi(1-e^{-2})} \cdot e^{-\frac{1}{2(1-e^{-2})}\left(x^2+25\right)}$$

Now taking maximum likelihood,

$$\frac{d}{dx} P\left(\begin{bmatrix} 5 \\ x \end{bmatrix}\right) = 0$$

$$\Rightarrow \frac{d}{dx} \frac{1}{2\pi(1-e^{-2})} e^{-\frac{1}{2(1-e^{-2})}\left(x^2+25\right)} = 0$$

$$= -\frac{1}{2(1-e^{-2})}(2x) \cdot e^{-\frac{1}{2(1-e^{-2})}\left(x^2+25\right)} = 0.$$

$$\Rightarrow \boxed{x = 0}.$$

So, most likely measurement for other location is 0.

8) Mean value at every location is 5.

Covariance function:

$$c(s_1, s_2) = e^{-\frac{1}{2}\|s_1 - s_2\|}$$

9) Spatial variable $X(s) = m(s) + Y(s)$.

$m(s)$ is fixed local component mean.

$Y(s)$ is global component following $GP(0, \Sigma)$

$$c(s, s') = \begin{cases} 1 & ; \ \text{dist}(s, s') < 2 \\ 0 & ; \ \text{otherwise.} \end{cases}$$

Let unknown spatial variable be x, then

condition vector

$$\begin{bmatrix} 5 \\ x \\ 6 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 5 \\ 6 \end{bmatrix} + Y(s).$$

So, $Y(s) = \begin{bmatrix} 0 \\ x-6 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

Now,

$P(Y(s)) = \mathcal{N}(0, \Sigma)$ where

$$\Sigma = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

So,

$$P(Y(s)) = \frac{1}{(2\pi)^{5/2} |\Sigma|} e^{-\frac{1}{2}\left[Y(s)^T \Sigma^{+} Y(s)\right]}$$

Σ^{+} (pseudo inverse) $= \begin{bmatrix} 0.5 & 0.25 & -0.5 & 0.25 & -0.5 \\ 0.25 & -0.125 & 0.25 & -0.125 & 0.25 \\ -0.5 & 0.25 & 0.5 & 0.25 & -0.5 \\ 0.25 & -0.125 & 0.25 & -0.125 & 0.25 \\ -0.5 & 0.25 & -0.5 & 0.25 & 0.5 \end{bmatrix}$

Now,

$$\Sigma^{+} Y(s) = \begin{bmatrix} -1.75 \\ 0.875 + 0.5x \\ -2.75 \\ 0.875 \\ -0.75 \end{bmatrix}$$

Now, $Y(s)^T \Sigma^{-1} Y(s)$

$$= \begin{bmatrix} 0 & x-6 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1.75 \\ 0.875 + 0.5x \\ -2.75 \\ 0.875 \\ -0.75 \end{bmatrix}$$

$$= (x-6)(0.875 + 0.5x) + 2.75 - 0.875 - 0.75$$

$$= 0.5x^2 - 2.125x - 5.25 + 2.75 - 0.875 - 0.75$$

$$= 0.5x^2 - 2.125x - 4.125$$

So,

$$P(Y(x)) = \frac{1}{2\pi|\Sigma|} e^{-\frac{1}{2}\left[0.5x^2 - 2.125x - 4.125\right]}$$

So, $\frac{dP}{dx} = 0$

$$\Rightarrow e^{-\frac{1}{2}\left[0.5x^2 - 2.125x - 4.125\right]} \cdot (x - 2.125) = 0$$

$$\Rightarrow \boxed{x = 2.125}$$

Now, for finding the missing mean, let it be m.

Considering 3 and 4 row,

$$Y_1 = \begin{bmatrix} 0 \\ 3-m \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0 \\ 3-m \end{bmatrix}, \quad Y_3 = \begin{bmatrix} -1 \\ 3-m \end{bmatrix}$$

$$Y_4 = \begin{bmatrix} 2 \\ 3-m \end{bmatrix}, \quad Y_5 = \begin{bmatrix} 1 \\ 5-m \end{bmatrix}$$

Considering temporal independence.

$$P(Y_1, Y_2, Y_3, Y_4, Y_5) = \prod_{i=1}^{5} P(y_i)$$

$$\propto \prod_{i=1}^{5} \frac{1}{2\pi|\Sigma|} e^{-\frac{1}{2}\left[Y_i^T \Sigma^{-1} Y_i \right]}$$

where $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \Sigma^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So, evaluating exponent term,

$$P \propto \prod_{i=1}^{5} \frac{1}{2\pi} e$$

$$Y_1^T \Sigma^{-1} Y_1 = \begin{bmatrix} 0 \\ 3-m \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3-m \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3-m \end{bmatrix} \begin{bmatrix} 0 \\ 3-m \end{bmatrix} = (3-m)^2$$

$$Y_2^T \Sigma^{-1} Y_2 = \begin{bmatrix} 0 & 3-m \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3-m \end{bmatrix} = (3-m)^2$$

$$Y_3^T \Sigma^{-1} Y_3 = \begin{bmatrix} -1 & 3-m \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3-m \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 3-m \end{bmatrix} \begin{bmatrix} -1 \\ 3-m \end{bmatrix} = 1 + (3-m)^2$$

$$Y_4^T \Sigma^{-1} Y_4 = \begin{bmatrix} 2 & 5-m \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5-m \end{bmatrix} = 4 + (5-m)^2$$

$$Y_5^T \Sigma^{-1} Y_5 = \begin{bmatrix} 1 & 5-m \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5-m \end{bmatrix} = 1 + (5-m)^2$$

So,

$$P \propto e^{-\frac{1}{2}\left[3(3-m)^2 + 2(5-m)^2 + 6 \right]}$$

$$\frac{d}{dm} P = 0 \Rightarrow 6(3-m) \times -1 + 4(5-m) \times -1 = 0$$

$$\Rightarrow -18 + 6m - 20 + 4m = 0$$

$$\Rightarrow 10m = 38 \Rightarrow \boxed{m = 3.8}$$

1b) $X(s,t) = a\,m(s) + (1-a)\,n(t) + e(s,t)$

s_1 : $X(s_1,t_1) = a\,m(s_1) + (1-a)\,n(t_1) + e(s_1,t_1)$

$X(s_1, t_2)$

$$X(s_1,t_1)^{(1)} = a\,m(s_1) + (1-a)\,n(t_1) + e_1$$

$$X(s_1,t_1)^{(2)} = a\,m(s_1) + (1-a)\,n(t_1) + e_2$$

$\vdots$

$$\bar{X}_{t_1} = \frac{N a\,m(s_1) + N(1-a)\,n(t_1)}{N} + \sum e_i$$

$$= a\,m(s_1) + (1-a)\,n(t_1) + \cancel{\frac{\sum e_i}{N}}^{\,0}$$

N is large enough

Here $\bar{X}_{t_1}$, a, $m(s_1)$ are known

So, $n(t_1) = \dfrac{\bar{X}_{t_1} - a\,m(s_1)}{(1-a)}$.

Similar approach to find $n(t_2)$, $n(t_3)$, ...

Now to find $m(s_2)$,

$$X(s_2,t_1)^{(1)} = a\,m(s_2) + (1-a)\,n(t_1) + e_1$$

$$X(s_2,t_2)^{(1)} = a\,m(s_2) + (1-a)\,n(t_1) + e_2$$

$\vdots$

$$X(s_2,t_1)^{(2)} = a\,m(s_2) + (1-a)\,n(t_1) + e_j$$

$\vdots$

$$\bar{X}_{s_2} = a\,m(s_2) + (1-a)\left[\alpha_1 n(t_1) + \alpha_2 n(t_2) + \dots\right] + \cancel{\frac{\sum e_i}{N}}^{\,0}$$

So,

$$m(s_2) = \frac{\bar{x}_{s_2} - (1-a)\left[\alpha_1 n(t_1) + \alpha_2 n(t_2) + \ldots \alpha_{12} n(t_{12})\right]}{a}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_{12}$ are no. of observations for location s_2 and time $t_1, t_2, \ldots, t_{12}$ respectively.

Similarly find $m(s_3)$, $m(s_4)$ ...

11) Given data,

	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
s_1	45	49	49	44	38	89	46	40	42	48
s_2	54	56	59	50	42	46	52	48	52	56
s_3	60	58	67	61	52	56	68	50	58	62
	t_{11}	t_{12}	t_{13}	t_{14}	t_{15}	t_{16}	t_{17}	t_{18}	t_{19}	t_{20}
s_1	46	48	42	39	32	38	58	52	47	48
s_2	55	57	51	48	40	48	68	59	51	55
s_3	62	61	57	53	52	58	66	64	58	66

Climatology for each location.

s_1 : $\dfrac{\sum_{i=1}^{20} x(s_1, t_i)}{20} = 44.45$

s_2 : $\dfrac{\sum_{i=1}^{20} x(s_2, t_i)}{20} = 52.35$

s_3 : $\dfrac{\sum_{i=1}^{20} x(s_3, t_i)}{20} = 59.45$.

Now calculating anomalies

	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
S_1	0.55	3.55	4.55	-0.45	-6.45	-5.45	1.55	-4.45	-2.45	3.55
S_2	1.65	3.65	6.65	-2.35	-10.35	-4.35	-0.35	-4.35	-0.35	3.65
S_3	0.55	-1.45	7.55	1.55	-7.45	-3.45	8.55	-9.45	-1.45	2.55
	t_{11}	t_{12}	t_{13}	t_{14}	t_{15}	t_{16}	t_{17}	t_{18}	t_{19}	t_{20}
S_1	1.55	3.55	-2.45	-5.45	-12.45	-6.45	13.55	7.55	2.55	3.55
S_2	2.65	4.65	-1.35	-4.35	-12.35	-4.35	15.65	6.65	-1.35	2.65
S_3	2.55	1.55	-2.45	-6.45	-7.45	-1.45	6.55	4.55	-1.45	6.55

Now, for ~~10~~ 90^{th} quantile for each location, 3^{rd} largest value is is found.

S_1 : 49

S_2 : 59

S_3 : 66.

For return period of climatology.

we need to find ~~how many~~ probability of observations less than or equal to climatology.

For S_1 : $P(X_{S_1} < 44.45) = \frac{9}{20} = 0.45$

So, return period $= \frac{1}{0.45} = 2.22$

For S_2 : $P(X_{S_2} < 52.35) = \frac{10}{20} = 0.5$

So, return period $= \frac{1}{0.5} = 2$

For S_3 : $P(X_{S_3} < 59.45) = \frac{10}{20} = 0.5$

So, return period $= 2$.

To check if positive anomalies and 90th quantile events are temporally coherent.

$Y(t) \rightarrow$ anomaly.

For S_1

$$P_{S_1}(Y(t) \geq 0) = \frac{4}{20} = 0.55$$

For S_2

$$P_{S_2}(Y(t) \geq 0) = \frac{9}{20} = 0.45$$

For S_3.

$$P_{S_3}(Y(t) > 0) = \frac{10}{20} = 0.5$$

Now, for S_1

$$P_{S_1}(Y(t) > 0 \mid Y(t-1) > 0) = \frac{\frac{7}{20}}{11/20} = 0.63$$

for S_2.

$$P_{S_2}(Y(t) > 0 \mid Y(t-1) > 0) = \frac{\frac{5}{20}}{9/20} = 0.55$$

for S_3,

$$P_{S_3}(Y(t) > 0 \mid Y(t-1) > 0) = \frac{\frac{4}{20}}{10/20} = 0.4.$$

~~As,~~ $\cancel{P_{S_i}(Y(t) > 0 \mid Y(t-1) > 0) < P_{S_i}(Y(t) > 0)}$

~~for all~~ $\cancel{i \in \{1, 2, 3\}}$

~~None of the locations are temporally coherent.~~

Now, to

For S_1, S_2, $P(Y(t) > 0 \mid Y(t-1) > 0) > P(Y(t) > 0)$

So, S_1, S_2 are temporally coherent.

To check for 90^{th} quantile event,

for s_1,

$$P_{s_1}(y(t) > 0) = \frac{3}{20}.$$

$$P_{s_1}(y(t) > 0 \mid y(t-1) > 0) = \frac{\frac{1}{20}}{\frac{3}{20}} = 1/3.$$

So, s_1 is temporally coherent for 90%ile event

Do similarly for others and so also for spatial coherence.

12) for (s_1, T_5) and (s_2, T_2)

we can use temporal average

So, $$x(s_1, T_5) = \frac{20+16+18+19+17}{5} = 18.$$

$$x(s_2, T_2) = \frac{25+23+24+21+21}{5} = 22.8.$$

Now, for (s_3, T_3) and (s_4, T_6)

temporal average cannot be used as variance across time is high

So, taking spatial average over (s_3, s_4)

$$x(s_3, T_3) = 26.$$

$$x(s_4, T_6) = 28.$$

Using low rank matrix factorisation,

Given matrix

$$\begin{bmatrix} 20 & 16 & 18 & 19 & x & 17 \\ 25 & x & 23 & 24 & 21 & 21 \\ 18 & 30 & x & 20 & 29 & 28 \\ 15 & 36 & 26 & 19 & 34 & x \end{bmatrix}_{4\times 6} = A \; B$$

where $A \in \mathbb{R}^{4\times 1}$

$B \in \mathbb{R}^{1\times 6}$.

So, $A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$, $B = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{bmatrix}$

Now,

$a_1 b_1 = 20$ —— ①

$a_1 b_2 = 16$ —— ②

$a_1 b_3 = 18$ —— ③

$a_1 b_4 = 19$ —— ④

$a_1 b_6 = 17$ —— ⑦

$a_2 b_1 = 25$	$a_3 b_1 = 18$	$a_4 b_1 = 15$
$a_2 b_3 = 23$	$a_3 b_2 = 30$	$a_4 b_2 = 36$
$a_2 b_4 = 24$	$a_3 b_4 = 20$	$a_4 b_3 = 26$
$a_2 b_5 = 21$	$a_3 b_5 = 29$	$a_4 b_4 = 19$
$a_2 b_6 = 21$	$a_3 b_6 = 28$	$a_4 b_5 = 34$

One solution can be assume $a_1 = 4$ and find other values. (may not be optimal)