Stochastic Process : (S.P.)

→ A S.P. is a family of r.v.'s $\{X(t), t \in T\}$, defined on a given probability space, indexed by the parameter t, where $t \in T$.

→ The values assumed by the r.v. $X(t)$ are called states and set of all possible values form the state space (S) of the process

types (1) discrete state, discrete parameter/time SP

(2) " " , continuous " / " SP

(3) continuous state, " " / " SP

(4) " " , discrete " / " SP.

Example: Consider a queueing system with jobs arriving at random point in time, queueing for service and departing from the system after service completion.

(a) $X(t)$ # of jobs in the system at time t.

$$S = \{0, 1, 2, 3, \ldots\}, \quad T = \{t ; t \geq 0\}$$

$\{X(t)\}$ discrete state, continuous parameter SP.

(b) W_k time that the k^{th} customer has to wait in the system before receiving service.

$$S = \{x,\ x \geq 0\},\quad T = \{1, 2, 3, --\}$$

$\{W_k, k \in T\}$ continuous state, discrete parameter SP.

(c) $Y(t)$ cummulative service requirement (experience) of all jobs in the system at time t.

$$S = [0, \infty),\quad T = [0, \infty)$$

$\{Y(t)\}$ cont. state, cont. parameter SP.

(d) Let N_k # of jobs in the system at the time of the departure of the k^{th} customer (after service completion).

$$S = \{0, 1, 2, ..\},\quad T = \{1, 2, 3, --\}$$

$\{N_k, k \in T\}$ discrete state, discrete parameter S.P.

—x—

Discrete Time Markov Chain (DTMC):

discrete state, discrete parameter/time SP

SP $\{X_n, n = 0, 1, 2, --\}$ that takes on a finite or countable number of possible values.

$X_n = i \equiv$ process is in state i at time/step/transition n

$\{X_n\}$ DTMC if $\qquad \{i, j, i_0, i_1, .. -\} \in S$ State space

$$P(X_{n+1}=j \mid X_0=i_0, X_1=i_1, \dots, X_n=i)$$

$$= P(X_{n+1}=j \mid X_n=i)$$

$$= p_{ij}^{(1)}(n) \rightarrow \text{transition probability}$$

$$= p_{ij}^{(1)} \rightarrow \text{stationary transition probability}$$

(Homogeneous M.C.)

$$= p_{ij}$$

$$\left(= P(X_{m+1}=j \mid X_m=i) = P(X_1=j \mid X_0=i)\right)$$

$$S = \{0, 1, 2, \dots\}$$

$$p_{ij}^{(1)} = p_{ij} = P(X_{n+1}=j \mid X_n=i) \quad , i,j \in S$$

final state   initial state

$$P = P^{(1)} = \begin{array}{c|c} & j \longrightarrow \\ i \downarrow & \begin{array}{cccc} 0 & 1 & 2 & \dots \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \end{array} & \begin{bmatrix} p_{00} & p_{01} & p_{02} & \dots \\ p_{10} & p_{11} & p_{12} & \dots \\ p_{20} & p_{21} & p_{22} & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} \end{array}$$

Transition Probability Matrix (TPM)

$$0 \le p_{ij} \le 1, \ \forall i, \forall j$$

for fixed i

$$\sum_j p_{ij} = 1$$

$$p_{ij} = p_{ij}^{(1)}$$

Example (1) Consider a game of ladder climbing. There are 5 levels in the game, level 1 is lowest (bottom) and level 5 is the highest (top). A player start at the bottom. Each time, a fair coin is tossed.

If it turns up heads, the player moves up one rung. If tails, the player moves down to the very bottom. Once at the top level, the player moves to the very bottom if tails turns up and stays at the top if head turns up.

Let X_n be the <u>level</u> of the game in the n^{th} step/transition.

Find S, TPM

Sol statespace $S = \{1, 2, 3, 4, 5\}$

$\{X_n\}$ DTMC

$$p_{ij} = P(X_{n+1} = j \mid X_n = i)$$

tpm

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \end{array}$$

(2) Let $\{X_n\}_{n=0,1,2,\dots}$ be a sequence of i.i.d. discrete r.v. with $P(X_1 = j) = \left(\frac{1}{2}\right)^{j+1}$ $\forall j = 0, 1, 2, \dots$

Determine whether each of the following chain is

Markovian or not. If so find its corresponding state space (S) and tpm.

(i) $\{S_n\}_{n=0,1,2,\ldots}$ where $S_n = \sum_{i=1}^{n} X_i$

(ii) $\{M_n\}_{n=0,1,2,\ldots}$, where $M_n = \max\{X_1, X_2, \ldots, X_n\}$

Sol (i) $S_n = \sum_{i=1}^{n} X_i$

$\{S_n\}$ DTMC with $S = \{0,1,2,3,\ldots\}$

$p_{ij} = P(S_{n+1} = j \mid S_n = i)$

tpm

$$P = \begin{array}{c|ccccc} i \downarrow \; j \rightarrow & 0 & 1 & 2 & 3 & \cdots \\ \hline 0 & 1/2 & 1/4 & 1/8 & 1/16 & \cdots \\ 1 & 0 & 1/2 & 1/4 & 1/8 & \cdots \\ 2 & 0 & 0 & 1/2 & 1/4 & \cdots \\ 3 & 0 & 0 & 0 & 1/2 & \cdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

(ii)

Ans $S = \{0,1,2,\ldots\}$

$$P = \begin{bmatrix} 1/2 & 1/4 & 1/8 & 1/16 & \cdots \\ 0 & 3/4 & 1/8 & 1/16 & \cdots \\ 0 & 0 & 7/8 & 1/16 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \end{bmatrix}$$

Example (transformation of a process into M.C.)

Suppose that whether or not it rains today depends on previous weather conditions through the last two days. Suppose that if it has rained for the past two days, then it will rain tommorow with prob. 0.7; if it has rained today but not yesterday, then it will rain tommorow with prob. 0.5; if it has rained yesterday but not today, then it will rain tommorow with prob. 0.4; if it has not rained in the past two days, then it will rain tommorow with prob. 0.2.

Let Y_n : weather condition on n^{th} day.

$Y_n \mid Y_{n-1}, Y_{n-2}$, ~~X~~ not M.C

Let X_n : State at any time is determined by the weather condition during both that day and the previous day

$$\underbrace{X_{n+1}}_{\text{tom, today}} \mid \underbrace{X_n}_{\text{today, yest}}, \cancel{X_{n-1}}$$

X_n DTMC

State X_n	Rained yesterday	Rained today
0	✓	✓
1	×	✓
2		

$p_{ij} = P(X_{n+1}=j \mid X_n=i)$

2	✓	×
3	×	×

$i, j \in S = \{0, 1, 2, 3\}$

$X_{n+1} = j$

tpanghen (X_n)

$$
P = \begin{array}{c c} & \begin{array}{cccc} \text{today ✓ tomorrow ✓} & \text{today × tomorrow ✓} & ✓\times & \times\times \\ 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} \text{yesterday today ✓✓ } 0 \\ \times ✓ \ 1 \\ ✓ \times \ 2 \\ \times\times \ 3 \end{array} & \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix} \end{array}
$$

$X_n = i$

—x—

n- step transition probability

$$p_{ij}^{(n)} = P(X_{m+n} = j \mid X_m = i) = P(X_n = j \mid X_0 = i)$$

$$0 \le p_{ij}^{(n)} \le 1 \quad , \forall i, \forall j \qquad (X_n) \text{ DTMC}$$

for fixed i $\qquad S = \{0, 1, 2, \ldots\}$ State space

$$\sum_j p_{ij}^{(n)} = 1$$

$$
P^{(n)} = \left(\left(p_{ij}^{(n)}\right)\right) = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & \cdots \end{array} \ j \rightarrow \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \end{array} \ i \downarrow & \begin{bmatrix} p_{00}^{(n)} & p_{01}^{(n)} & p_{02}^{(n)} & \cdots \\ p_{10}^{(n)} & p_{11}^{(n)} & p_{12}^{(n)} & \cdots \\ p_{20}^{(n)} & p_{21}^{(n)} & p_{22}^{(n)} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \end{array}
$$

Chapman Kolomgrov equations :

$$p_{ij}^{(m+n)} = \sum p_{ik}^{(m)} p_{kj}^{(n)} = \sum p_{ik}^{(n)} p_{kj}^{(m)}$$

$$p_{ij} = \sum_k p_{ik}\, p_{kj} = \sum_k p_{ik}\, p_{kj} \qquad i,j,k \in S$$

Sol: $(i,j)^{th}$ element of $P^{(m+n)}$

$$p_{ij}^{(m+n)} = P(X_{m+n}=j \mid X_0=i)$$

$$= \sum_k P(X_{m+n}=j, X_n=k \mid X_0=i)$$

thm of total prob.

$$= \sum_k P(X_{m+n}=j \mid X_n=k, X_0=i) \cdot P(X_n=k \mid X_0=i)$$

$$\boxed{P(AB|C) = \frac{P(ABC)}{P(C)} \times \frac{P(BC)}{P(BC)} = P(A|BC)\,P(B|C)}$$

(X_n) DTMC

$$= \sum_k P(X_{m+n}=j \mid X_n=k)\, P(X_n=k \mid X_0=i)$$

$$= \sum_k p_{kj}^{(m)}\, p_{ik}^{(n)} = \sum_k p_{ik}^{(n)}\, p_{kj}^{(m)}$$

$$p_{ij}^{(m+n)} = \sum_k p_{ik}^{(n)}\, p_{kj}^{(m)}$$

$$P^{(m+n)} = \begin{array}{c} \\ \\ i \end{array}\overset{P^{(n)}}{\begin{pmatrix} - & - \\ p_{i0}^{(n)} & p_{i1}^{(n)} \cdots \end{pmatrix}} \overset{P^{(m)}}{\begin{pmatrix} \vdots & \overset{j}{p_{0j}^{(m)}} \\ & p_{1j}^{(m)} \\ \vdots & \vdots \end{pmatrix}}$$

$$P^{(m+n)} = P^{(n)} P^{(m)}$$

$$P^{(1)} = P$$

$$P^{(2)} = P^{(1)} P^{(1)} = P.P = P^2$$

$$P^{(n)} = P^n$$

Example Suppose that the chance of rain tomorrow depends on previous weather conditions only through whether or not it is raining today and not on past weather condition. Suppose also that if it rains today, then it will rain tomorrow with probability α; and if it does not rain today, then it will rain tomorrow with prob. β.

X_n : weather condition on n^{th} day DTMC

$X_n \in \{0,1\}$ 0: rain, 1: not raining

$S = \{0,1\}$ $p_{ij} = P(X_{n+1}=j \mid X_n = i)$

$i,j \in S$

tpm

$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{pmatrix} \end{array}$$

Let $\alpha = 0.7$ and $\beta = 0.3$. Calculate the prob. that it will rain 2 days from today given that it is raining today.

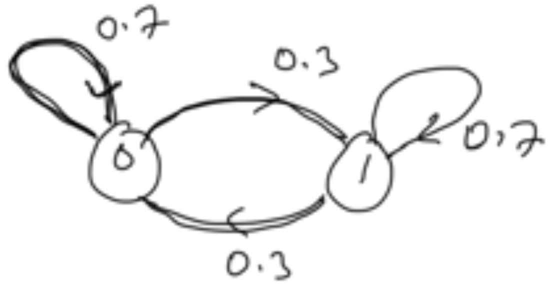
$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix} \end{array} = \left(\left(p_{ij}^{(1)}\right)\right)$$

$$p_{00}^{(2)} = P(X_{n+2}=0 \mid X_n=0) = ?$$

$$P^{(2)} = P^2 = P.P = \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix}$$

$$= \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} \boxed{0.58} & 0.42 \\ 0.42 & 0.58 \end{pmatrix} \end{array}$$

$$p_{00}^{(2)} = 0.58$$



$$p_{00}^{(2)} = 0.7 \times 0.7 + 0.3 \times 0.3 = 0.58$$

—x—

PMF of step X_n: $\{X_n\}$ DTMC

state space $S = \{0, 1, 2, --\}$

$$p_i^{(n)} = P(X_n = i) \qquad , i \in S$$

pmf of X_n is

$$\underset{\sim}{p}^{(n)} = \left(p_0^{(n)}, p_1^{(n)}, --, p_i^{(n)}, ---\right) ✓$$

$$\text{s.t. } \sum_i p_i^{(n)} = 1$$

initial state pmf of X_0 is

$$\underset{\sim}{p}^{(0)} = \left(p_0^{(0)}, p_1^{(0)}, ---\right) \qquad \text{s.t. } \sum_i p_i^{(0)} = 1$$

T.S. $\qquad \boxed{\underset{\sim}{p}^{(n)} = p^{(n-1)} P} \qquad P \leftarrow$ tpm

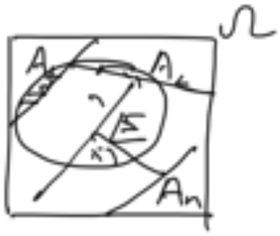
$$\left(p_0^{(n)}, p_1^{(n)}, --, \boxed{p_i^{(n)}}, --\right) = \left(\underline{p_0^{(n-1)}, p_1^{(n-1)}, -- p_i^{(n-1)}}, --\right) \begin{pmatrix} p_{00} & p_{01} & -- & p_{0i} & -- \\ p_{10} & p_{11} & -- & p_{1i} & -- \end{pmatrix}$$

T.S. $\boxed{p_i^{(n)} = \sum_k p_k^{(n-1)} p_{ki}}$

Sol $p_i^{(n)} = P(X_n = i) = \sum_k P(X_n = i, X_{n-1} = k)$

$= \sum_k P(X_n = i \mid X_{n-1} = k) \cdot P(X_{n-1} = k)$

Using thm of total prob.



$\cup A_i = \Omega$

$A_i \cap A_j = \phi$

$P(E) = \sum_i P(A_i \cap E)$

$= \sum_k p_{ki} \, p_k^{(n-1)}$

$\Rightarrow \boxed{\underset{\sim}{p}^{(n)} = \underset{\sim}{p}^{(n-1)} P}$

$\underset{\sim}{p}^{(n)} = \underset{\sim}{p}^{(n-2)} P^2$

$\underset{\sim}{p}^{(n)} = \underset{\sim}{p}^{(0)} P^n$

$P(A|B) = \frac{P(AB)}{P(B)}$

$\Leftrightarrow P(AB) = P(A|B) P(B)$

$P \to$ tpm

Example: Let (X_n) DTMC, with state space $S = \{1, 2, 3\}$, and

$$P = \begin{array}{c c} \text{tpm} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & \underline{0.2} \\ 0.3 & \underline{0.4} & \underline{0.3} \end{pmatrix} \end{array} \quad \checkmark$$

pmf of X_0 is $(0.7, \boxed{0.2}, 0.1) = \underset{\sim}{p}^{(0)}$

$P(X_0 = 2)$

(i) $P(X_0 = 2, X_2 = 3, X_1 = 3, X_3 = 2)$

$= P(X$

$= P(X_3=2, X_2=3, X_1=3, X_0=2)$

$= P(X_3=2 \mid X_2=3, \cancel{X_1=3, X_0=2}) \cdot P(X_2=3 \mid X_1=3, \cancel{X_0=2})$

$\cdot P(X_1=3 \mid X_0=2) \cdot P(X_0=2)$ | $P(ABCD) = P(A \mid BCD)\, P(B \mid CD)\, P(C \mid D)\, P(D)$

(X_n) DTMC

$= P(X_3=2 \mid X_2=3) \cdot P(X_2=3 \mid X_1=3) \cdot P(X_1=3 \mid X_0=2) \times 0.2$

$= p_{32}\; p_{33}\; p_{23} \times 0.2$

$= 0.4 \times 0.3 \times 0.2 \times 0.2 = 0.0048$

(ii) $P(X_2=3, X_1=3 \mid X_0=2)$ | $P(AB \mid C) = \frac{P(ABC)}{P(C)} \times \frac{P(BC)}{P(BC)}$

$= P(A \mid BC) \cdot P(B \mid C)$

$= P(X_2=3 \mid X_1=3, \cancel{X_0=2})$

$\cdot P(X_1=3 \mid X_0=2)$ (X_n) DTMC

$= P(X_2=3 \mid X_1=3) \cdot P(X_1=3 \mid X_0=2)$

$= p_{33}\; p_{23} = 0.3 \times 0.2 = 0.06$

(iii) $P(X_2=3)$ X_2 $\underset{\sim}{p}^{(2)} = \left(p_1^{(2)}, p_2^{(2)}, p_3^{(2)}\right)$

$p_3^{(2)} = P(X_2=3)$

$\underset{\sim}{p}^{(n)} = \underset{\sim}{p}^{(n-1)} P$ $P \leftarrow$ tpm

X_1 $\underset{\sim}{p}^{(1)} = \underset{\sim}{p}^{(0)} P = (0.7, 0.2, 0.1) \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$

$$= (0.22, 0.43, 0.35)$$

X_2

$$\underset{\sim}{p}^{(2)} = \underset{\sim}{p}^{(1)} P = (0.22, 0.43, 0.35) \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

$$= (0.385, 0.336, \boxed{0.279})$$

$$P(X_2 = 3) = p_3^{(2)} = 0.279$$

—x—

Classification of states: $\{X_n\}$ DTMC

$p_{ij}^{(n)} = P(X_n = j \mid X_0 = i)$ $\quad S = \{0, 1, 2, --\}$ $\quad i, j \in S$

Def^n $i \to j$ state j is accessible from state i if $p_{ij}^{(n)} > 0$ for some n.

Def^n $i \leftrightarrow j$ state i and j communicate with each other if $i \to j$ and $j \to i$

Result $\quad i \leftrightarrow j,\ j \leftrightarrow k \Rightarrow i \leftrightarrow k \quad ,i, j, k \in S$

Sol

Given $\left[\ p_{ij}^{(n)} > 0 \quad p_{jk}^{(m)} > 0\right.$ $\quad \exists\, n, m$ s.t. $i \to j,\ j \to k$

Now

$$p_{ik}^{(m+n)} = \sum_{l} p_{il}^{(n)} p_{lk}^{(m)} \geq p_{ij}^{(n)} p_{jk}^{(m)} > 0$$

$$\therefore i \to k$$

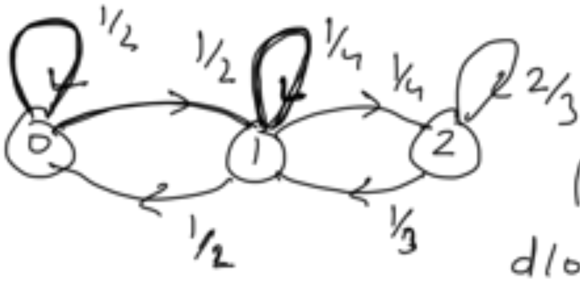
III^{ly} $k \to i$ $\quad \therefore i \leftrightarrow k$

Def^n $\{X_n\}$ DTMC is irreducible or indecomposable if every state communicates with every other state, Otherwise it is reducible.

eg (X_n) DTMC tpm

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 2 & 0 & \frac{1}{3} & \frac{2}{3} \end{array}$$

$S = \{0, 1, 2\}$



(X_n) irreducible/countable

$d(0) = \gcd\{1, 2, 3, \ldots\} = 1$

$d(0) = d(1) = d(2) = 1$

All state has period 1 or aperiodic

$0 \leftrightarrow 1 \leftrightarrow 2$

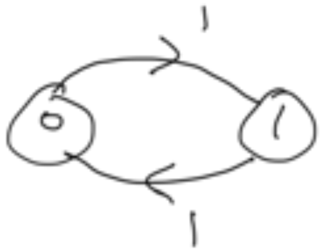
<u>Def^n</u> period of state i; $d(i)$;

$d(i)$ is $\underline{\gcd}$ of $I^+ = \{1, 2, 3, \ldots\}$ n s.t. $p_{ii}^{(n)} > 0$

(If $p_{ii}^{(n)} = 0 \; \forall n \geq 1$, define $d(i) = 0$)

eg (X_n) DTMC tpm

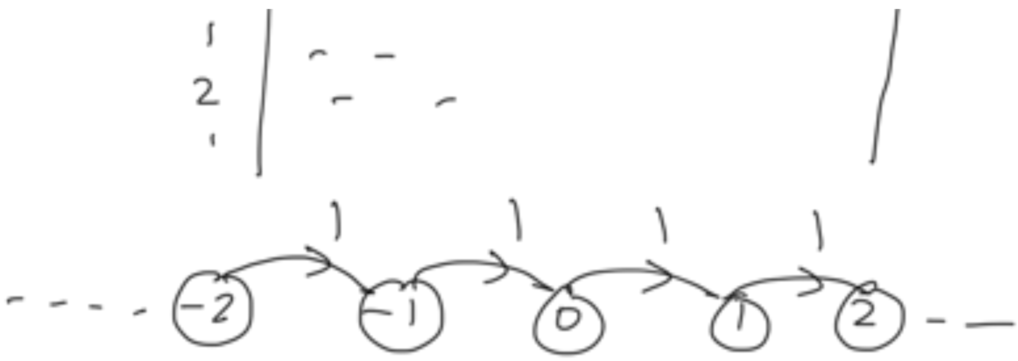
$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}$$



$d(0) = \gcd\{2, 4, 6, \ldots\} = 2 = d(1)$

eg

$$\begin{array}{c|cccccccc} & \cdots & -2 & -1 & 0 & 1 & 2 & \cdots \\ \hline \vdots & & & & & & & \\ -2 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots \\ -1 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots \end{array}$$



$d(i) = 0$

—x—

(X_n) DTMC, $\quad i \in S$

$f_i^{(n)}$ or $f_{ii}^{(n)} = P(X_n = i,\ X_k \neq i, k = 1, 2, \dots, n-1 \mid X_0 = i)$: prob of first visit to state i in n transitions / steps starting from state i

recurrence time prob

$$f_{ii}^{(0)} = 1$$

$$f_{ii} \text{ or } f_i = f_i^{(1)} + f_i^{(2)} + f_i^{(3)} + \dots = \sum_{n=1}^{\infty} f_i^{(n)}$$

prob. of ever visiting state i

Def^n $\quad f_i = 1$, return to state i is certain, starting from state i

i recurrent state

Def^n $\quad f_i < 1$, return to state i is uncertain

i transient state.

—x—

Let
$$I_n = \begin{cases} 1 & \text{if } X_n = i \\ 0 & \text{if } X_n \neq i \end{cases}$$

$\sum^{\infty} I_n$: # of time periods ...

$\sum_{n=1} I_n$: # of time periods, The process is in state i

$$E\left(\sum_{n=1}^{\infty} I_n \,\middle|\, X_0 = i\right) = \sum_{n=1}^{\infty} E(I_n \mid X_0 = i)$$

$$= \sum_{n=1}^{\infty} \left[1 . P(X_n = i \mid X_0 = i) + 0\, P(X_n \neq i \mid X_0 = i)\right]$$

$$= \sum_{n=1}^{\infty} P(X_n = i \mid X_0 = i)$$

$$= \sum_{n=1}^{\infty} p_{ii}^{(n)}$$

i recurrent $\Leftrightarrow \sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty \Leftrightarrow f_i = 1$

i transient $\Leftrightarrow \sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty \Leftrightarrow f_i < 1$

Defn Let i recurrent

m_{ii} or $m_i = \sum_{n=1}^{\infty} n f_i^{(n)}$: mean recurrence time (expected time the process return to state i)

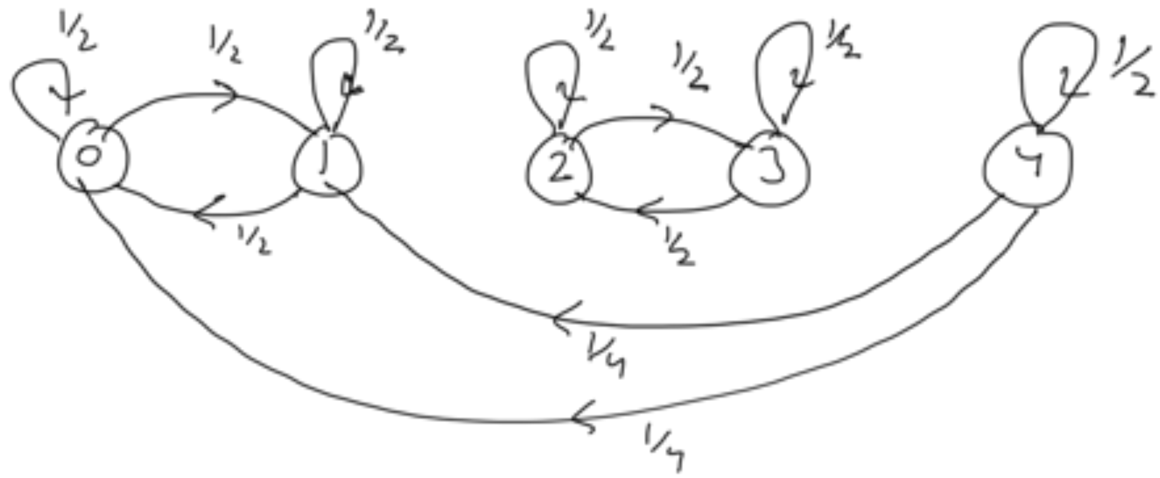
→ If $m_i = \infty$ $\quad$ i null recurrent

→ If $m_i < \infty$ $\quad$ i non-null recurrent/positive recurrent.

Example : $\{X_n\}$ DTMC $\quad S = \{0,1,2,3,4\}$ tpm

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccccc} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{array}\right. \end{array}$$

$$\begin{array}{c} 3 \\ 4 \end{array}\begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$



Classes $\{0,1\}$ $\{2,3\}$ $\{4\}$ $\quad 0 \leftrightarrow 1 \quad 2 \leftrightarrow 3 \quad 4$

$\{0,1\}$: recurrent aperiodic; $\{2,3\}$: recurrent aperiodic; $\{4\}$: transient. Reducible M.C

$$f_0 = f_0^{(1)} + f_0^{(2)} + f_0^{(3)} + \cdots$$

$$= \frac{1}{2} + \frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} + \cdots$$

$$= \frac{1}{2}\left[1 + \frac{1}{2} + \frac{1}{4} + \cdots\right] = \frac{1}{2}\times\frac{1}{1-\frac{1}{2}} = 1$$

State 0 recurrent

—x—

P1 $\quad \underbrace{i \leftrightarrow j, \ i \text{ recurrent}} \Rightarrow j$ recurrent

Given $\exists\, n, m \quad p_{ij}^{(n)} > 0,\ p_{ji}^{(m)} > 0;\ \sum_{\nu=1}^{\infty} p_{ii}^{(\nu)} = \infty$

$$p_{jj}^{(m+n+v)} \geq p_{ji}^{(n)} p_{ii}^{(v)} p_{ij}^{(m)} \quad \Big] \text{using C.K. equation}$$

$$\sum_{v} p_{jj}^{(m+n+v)} \geq p_{ji}^{(n)} p_{ij}^{(m)} \underbrace{\sum_{v} p_{ii}^{(v)}}_{\infty}$$

$$\Rightarrow \sum_{n} p_{jj}^{(n)} = \infty$$

, i.e. j recurrent state

P2 $\underline{i \text{ transient}}$, $i \leftrightarrow j \Rightarrow j$ transient

Sol On contrary suppose j recurrent, since $i \leftrightarrow j$

$\Rightarrow i$ recurrent (Using P1)

a contradiction

P3 In a <u>finite state M.C.</u> all states can not be transient.

P4 In a finite state, <u>irreducible</u> M.C. all states are recurrent.

Sol using P3, P1

P5 In irreducible M.C, all states are recurrent or transient.

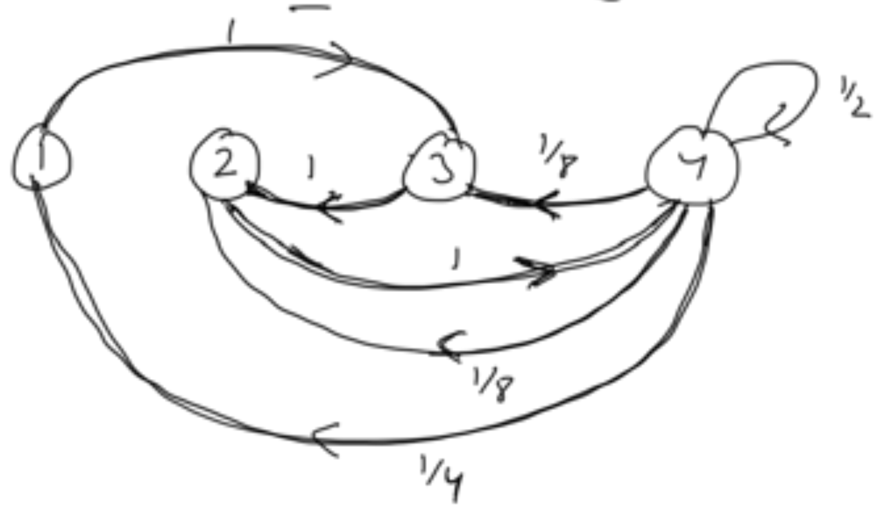
<u>Example</u>: $\{X_n\}$ DTMC $S = \{1,2,3,4\}$

tpm

$$P = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

classify the states

$$4\left[\ \tfrac{1}{4}\quad \tfrac{1}{8}\quad \tfrac{1}{8}\quad \tfrac{1}{2}\ \right]$$



$1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4$ Irreducible M.C.

Class $\{1,2,3,4\}$

Finite state, irreducible M.C all states are positive recurrent. (using P4)

mean recurrence time for state 4

$$m_4 = \sum_{n=1}^{\infty} n f_4^{(n)}$$

$$= 1\times\frac{1}{2} + 2\times\frac{1}{8} + 3\times\frac{1}{8} + 4\times\frac{1}{4} + 0 + 0 \cdots$$

$$= \frac{17}{8} < \infty$$

State 4 positive recurrent / non-null recurrent.

$f_4^{(1)} = \frac{1}{2}$, $f_4^{(2)} = \frac{1}{8}\times 1$,

$f_4^{(3)} = \frac{1}{8}\times 1\times 1$

$f_4^{(4)} = \frac{1}{4}\times 1\times 1\times 1$

$f_4^{(5)} = 0$

— x —

Gambler's Ruin problem

$i = 0, 1, 2, \ldots N$

initial capital Rs i Aim Rs N

Z_i : i^{th} bet / step / transition / time

$Z_1, Z_2, \ldots$ are independent

$P(Z_i = 1) = p$, $P(Z_i = -1) = 1-p = q$

$X_n = Z_1 + Z_2 + \dots + Z_n + i$

: fortune of the gambler after n steps

$p_{ij} = P(X_{n+1} = j \mid X_n = i)$; $i, j \in S = \{1, 2, \dots, N\}$

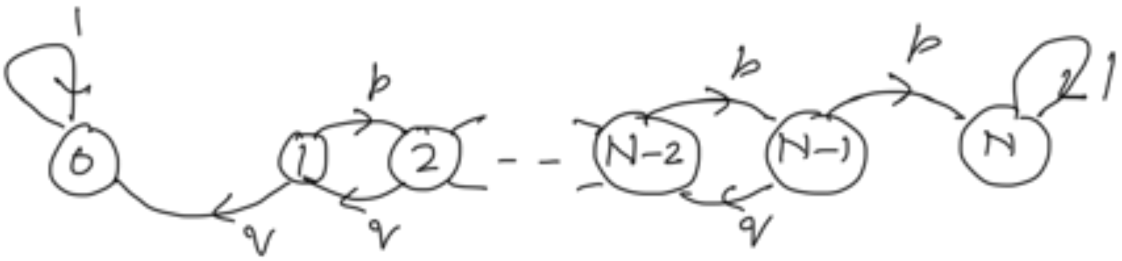
$\{X_n\}_{n=1,2,\dots}$ DTMC

$p_{0,0} = 1 = p_{N,N}$

$p_{i,i+1} = p$, $p_{i,i-1} = q$, $i = 1, \dots, N-1$

tpm

$$P = \begin{array}{c c} & \begin{array}{cccccc} 0 & 1 & 2 & - & - & N \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ \\ N \end{array} & \left[\begin{array}{cccccc} 1 & 0 & 0 & - & - & 0 \\ q & 0 & p & - & - & 0 \\ 0 & q & 0 & - & - & 0 \\ - & - & & - & & \\ - & & - & - & & \\ 0 & 0 & 0 & - & - & 1 \end{array}\right] \end{array}$$



Classes $\{0\}$ $\{1, 2, \dots, N-1\}$ $\{N\}$

recurrent /absorbing — transient — recurrent/ absorbing

T_0 = time he broke

$= \inf\{n : X_n = 0\}$

T_N = time he has Rs N

$= \inf \{ n : X_n = N \}$

$P_i = P(T_N < T_0)$: prob. that starting with Rs i, the gamblers fortune will reach N before reaching 0 ?

$$P_i = P(T_N < T_0 | Z_1 = 1) P(Z_1 = 1) + P(T_N < T_0 | Z_1 = -1) P(Z_1 = -1)$$

$$P_i = P_{i+1} p + P_{i-1} q$$

$$p P_i + q P_i = P_{i+1} p + P_{i-1} q$$

$$\Rightarrow (P_{i+1} - P_i) p = q (P_i - P_{i-1})$$

$$\Rightarrow P_{i+1} - P_i = \frac{q}{p} (P_i - P_{i-1})$$

$P_0 = 0, P_N = 1$

$i=1$ $\quad P_2 - P_1 = \frac{q}{p} (P_1 - P_0) = \frac{q}{p} P_1$

$i=2$ $\quad P_3 - P_2 = \frac{q}{p} (P_2 - P_1) = \left(\frac{q}{p}\right)^2 P_1$

$\vdots$

$$P_i - P_{i-1} = \left(\frac{q}{p}\right)^{i-1} P_1$$

$$P_i - P_1 = P_1 \left[\frac{q}{p} + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{i-1} \right]$$

$$P_i = P_1 \left[1 + \left(\frac{q}{p}\right) + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{i-1} \right]$$

$$= \left[1 - \left(\frac{q}{p}\right)^i \right.$$

$$P_i = \begin{cases} \dfrac{1-\left(\frac{q}{p}\right)^i}{1-\frac{q}{p}} P_1 & , \quad \frac{q}{p} \neq 1 \\ i P_1 & , \quad \frac{q}{p} = 1 \end{cases} \qquad \text{— (1)}$$

$$P_N = 1$$

If $i = N$ in (1), we have

$$P_1 = \begin{cases} \dfrac{1-\frac{q}{p}}{1-\left(\frac{q}{p}\right)^N} & , \quad \frac{q}{p} \neq 1 \\ \dfrac{1}{N} & , \quad \frac{q}{p} = 1 \end{cases} \qquad \text{— (2)}$$

using (2) in (1)

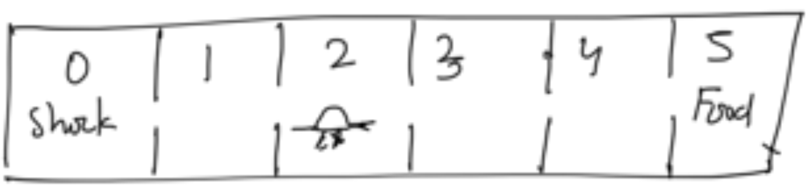
$$P_i = \begin{cases} \dfrac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N} & , \quad \frac{q}{p} \neq 1 \Leftrightarrow p \neq \frac{1}{2} \\ \dfrac{i}{N} & , \quad \frac{q}{p} = 1 \Leftrightarrow p = \frac{1}{2} \end{cases}$$

If $N \to \infty$, then

$$P_i = \begin{cases} 1-\left(\frac{q}{p}\right)^i & , \quad \frac{q}{p} < 1 \Leftrightarrow p > \frac{1}{2} \\ 0 & , \quad \frac{q}{p} \geq 1 \Leftrightarrow p \leq \frac{1}{2} \end{cases}$$

—x—

Example (assgn 1) A rat is put into the linear maze as shown below



each step $1/4 \leftarrow \quad \rightarrow 3/4$

Sol P(rat finds the food before getting shocked)

Gamblers ruin problem

$$i=2\,,\ N=5\,,\ p=\tfrac{3}{4} \qquad \frac{q}{p}=\frac{1}{4}\times\frac{4}{3}=\frac{1}{3}\neq 1$$

$$P_5=\frac{1-\left(\frac{q}{p}\right)^{i}}{1-\left(\frac{q}{p}\right)^{N}}=\frac{1-\left(\frac{1}{3}\right)^{2}}{1-\left(\frac{1}{3}\right)^{5}}=0.892$$

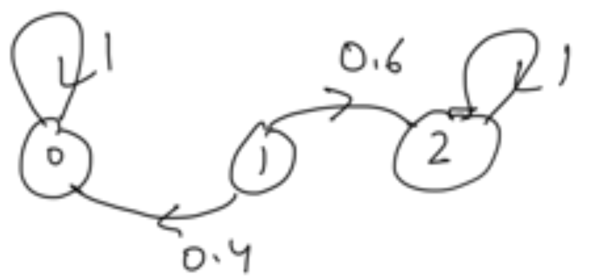
$1-P_5 = P$(rat get shocked before getting food)

<u>Example</u> (Ozamp) $\{X_n\}$ DTMC tpm

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0.4 & 0 & 0.6 \\ 2 & 0 & 0 & 1 \end{array}$$

Starting with 1, determine the prob. that the M.C ends in state 0.

Sol.



absorbing/recurrent

{0} {1} {2}

transient

$p=0.6\,,\ q=0.4$

$$\frac{q}{p}=\frac{4}{6}=\frac{2}{3}\neq 1$$

$$\text{Reqd. prob} = 1-\frac{1-\left(\frac{2}{3}\right)^{1}}{1-\left(\frac{2}{3}\right)^{2}}=1-\frac{\frac{1}{3}}{5/9}$$

$$=0.4$$

—x—

Q. $x,y\in S$ $\quad x\leftrightarrow y$, then $d(x)=d(y)$

Sol. $d(x) = \gcd\{n \geq 1 : p_{xx}^{(n)} > 0\}$

since $x \leftrightarrow y$

$p_{xy}^{(m)} > 0$, $p_{y,x}^{(n)} > 0$ for some m, n

$$p_{y,y}^{(n+m)} \geq p_{y,x}^{(n)} \, p_{x,y}^{(m)} > 0$$

$$p_{yy}^{(n+s+m)} \geq p_{y,x}^{(n)} \, p_{xx}^{(s)} \, p_{xy}^{(m)} > 0$$

$d(y)$ divides both $n+m$ and $n+s+m$

$\therefore$ $d(y)$ divides every s with $p_{xx}^{(s)} > 0$

$\Rightarrow$ $d(y)$ divides gcd of such s

$\Rightarrow$ $d(y)$ divides $d(x)$

Repeat by changing the roles

$x \leftrightarrow y \Rightarrow d(y)$ divides $d(x)$

period $d(x)$ and $d(y)$ divides each other $\Rightarrow$ they must be equal.

— x —

Limiting Probabilities

long run behaviour of M.C.'s:

finite state space $\rightarrow$ $S = \{0, 1, \ldots, N\}$

Regular TPM Given tpm P is regular if P^k has all its elt > 0.

Such tpm or corresponding MC is called regular

The most important fact concerning a regular M.C is the existance of a limiting prob. distn $\underset{\sim}{\Pi} = (\Pi_0, \ldots, \Pi_N)$ when $\Pi_j > 0 \ \forall j = 0, 1, \ldots, N$, $\sum_j \Pi_j = 1$ and this distn is indep. of initial state.

regular tpm $P = ((p_{ij}))$, we have the convergence

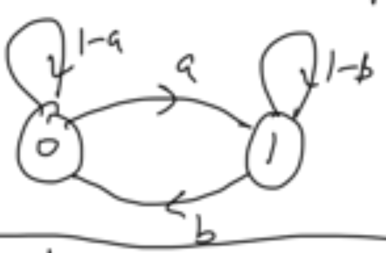
$$\lim_{n \to \infty} p_{ij}^{(n)} = \Pi_j > 0 \quad \forall j = 0, 1, \ldots, N$$

$$\text{i.e., } \lim_{n \to \infty} P(X_n = j \mid X_0 = i) = \Pi_j > 0 \ \forall j = 0, 1, \ldots, N$$

This convergence means that, in long run $(n \to \infty)$, the prob. of finding the M.C in state j is $\simeq \Pi_j$; no matter in which state the chain begins at time 0.

(A) Example tpm

$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \end{array}, \quad 0 \le a, b \le 1$$



here $|1-a-b| < 1$

irreducible, aperiodic, finite state M.C. $\quad$ ~~$a = b = 0$ or $a = b = 1$~~ $\to$ we will see separately

P regular tpm

$$P^n \, ?$$

$$p_{00} = 1-a, \quad p_{01} = a, \quad p_{10} = b, \quad p_{11} = 1-b$$

$$p_{01}^{(n-1)} = 1 - p_{00}^{(n-1)}$$

$$p_{00}^{(n)} = p_{00}^{(n-1)} p_{00} + p_{01}^{(n-1)} p_{10} = (1-a) p_{00}^{(n-1)} + b(1 - p_{00}^{(n-1)})$$

$$= b + (1-a-b)\, p_{00}^{(n-1)}$$

$$= b + (1-a-b)\, p_{00}^{(n-1)} \quad , n > 1$$

$$= b + (1-a-b)\left[b + (1-a-b)\, p_{00}^{(n-2)} \right]$$

$$= b + b(1-a-b) + (1-a-b)^2\, p_{00}^{(n-2)}$$

$$\vdots$$

$$= b + b(1-a-b) + b(1-a-b)^2 + \cdots + b(1-a-b)^{n-2}$$

$$+ p_{00}(1-a-b)^{n-1}$$

$$p_{00} \to (1-a)$$

$$= b \sum_{k=0}^{n-2} (1-a-b)^k + (1-a)(1-a-b)^{n-1}$$

$$\sum_{k=0}^{n-2} (1-a-b)^k \to \frac{1-(1-a-b)^{n-1}}{1-(1-a-b)}$$

$$= \frac{b}{a+b} + \frac{a(1-a-b)^n}{a+b}$$

$\therefore$ Similarly

$$P^n = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} \frac{b + a(1-a-b)^n}{a+b} & \frac{a - a(1-a-b)^n}{a+b} \\ \frac{b - b(1-a-b)^n}{a+b} & \frac{a + b(1-a-b)^n}{a+b} \end{bmatrix} \end{array} \quad (2)$$

Note that $|1-a-b| < 1$, $0 < a, b < 1$

thus $|1-a-b|^n \to 0$ as $n \to \infty$

$$\lim_{n\to\infty} P^n = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ \\ \end{array} & \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ b & \end{bmatrix} \end{array}$$

$$1\left[\frac{b}{a+b} \quad \frac{a}{a+b}\right]$$

Let

$$P(X_0=0)=\frac{1}{3}, \quad P(X_0=1)=\frac{2}{3}$$

initial state pmf $\quad \underset{\sim}{p}^{(0)}=\left(\frac{1}{3}, \frac{2}{3}\right)$

$$\underset{\sim}{p}^{(n)}=\underset{\sim}{p}^{(n-1)}P=\underset{\sim}{p}^{(0)}P^n$$

$$=\left(\frac{1}{3}, \frac{2}{3}\right) P^n$$

Take P^n from (*)

$$\lim_{n\to\infty} \underset{\sim}{p}^{(n)}=\left(\frac{b}{a+b}, \frac{a}{a+b}\right)$$

Rk: ① Finite state aperiodic irreducible M.C. is regular and recurrent.

② If a tpm P of N states is regular, the P^{N^2} will have no zero elts. Equivalently if P^{N^2} is not strictly +ve, then M.C is not regular.

eg

$$P=\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \quad \text{regular tpm}$$

$$\underset{P}{\begin{pmatrix} + & + & 0 \\ + & 0 & + \\ + & 0 & + \end{pmatrix}} \underset{P}{\begin{pmatrix} + & + & 0 \\ + & 0 & + \\ + & 0 & + \end{pmatrix}} = \underset{P^2}{\begin{pmatrix} + & + & + \\ + & + & + \\ + & + & + \end{pmatrix}}$$

Thm: Let P be a regular tpm on states $0,1,\dots,N$. ($S = \{0,1,\dots,N\}$)
Then the limiting dist $\underset{\sim}{\Pi} = (\Pi_0, \Pi_1, \dots, \Pi_N)$ is the unique non-negative sol^n of equations

$$\Pi_j = \sum_{k=0}^{N} \Pi_k \, p_{kj} \quad , j = 0,1,\dots,N$$

$$\sum_{k=0}^{N} \Pi_k = 1$$

$$\longleftrightarrow \quad \underset{\sim}{\Pi} = \underset{\sim}{\Pi} P, \quad \sum_{k=0}^{N} \Pi_k = 1$$

Sol: $\because$ MC regular, we have limiting prob.

$$\lim_{n\to\infty} p_{ij}^{(n)} = \Pi_j \quad \text{for which } \sum_{k=0}^{N} \Pi_k = 1$$

$$P^n = P^{n-1} P$$

$$p_{ij}^{(n)} = \sum_{k=0}^{N} p_{ik}^{(n-1)} p_{kj} \quad , j = 0,1,\dots,N$$

$n \to \infty$

as $n \to \infty$, $p_{ij}^{(n)} \to \Pi_j$, $p_{ik}^{(n-1)} \to \Pi_k$

$$\Pi_j = \sum_{k=0}^{N} \Pi_k \, p_{kj} \quad \text{as claimed}$$

T.S. sol^n is unique. Suppose that $x_0, x_1, \dots, x_N$ solves

$$x_j = \sum_{k=0}^{N} x_k \, p_{kj} \quad \text{for } j = 0,1,\dots,N \quad (1), \quad \sum_{k=0}^{N} x_k = 1$$

We wish to show $x_j = \Pi_j$

Now $$\underbrace{\sum_{j=0}^{N} x_j \, p_{jl}}_{1} = \sum_{j=0}^{N} \sum_{k=0}^{N} x_k \, p_{kj} \, p_{jl} = \sum_{k=0}^{N} x_k \sum_{j=0}^{N} p_{kj} \, p_{jl}$$

$y_l^{(2)}$ p_{kl}

using ① ↓

$$x_l = \sum_{k=0}^{N} x_k \, p_{kl}^{(2)}$$

repeating the argument

$$x_l = \sum_{k=0}^{N} x_k \, p_{kl}^{(n)} \quad , l = 0, 1, \ldots, N$$

$$n \to \infty \quad , \quad p_{kl}^{(n)} \to \pi_l$$

$$\therefore \quad x_l = \sum_{k=0}^{N} x_k \pi_l = \pi_l \quad , l = 0, 1, \ldots, N$$

$\therefore$ $x_l = \pi_l$ as claimed.

—x—

Example An No claim discount (NCD) system has the discount classes E_0 (no discount), E_1 (20% discount) and E_2 (40% discount). Movement in the system is determined by the rule whereby one step back one discount level (or stays in E_0) with one claims in a year, and returns to a level of no discount if more than one claim is made. A claim-free year results in a step up to a higher discount level (or one remains in class E_2 if already there). NCD system

NCD class	E_0	E_1	E_2
% discount	0	20	40

% discount	0	20	40
annual premium	100	80	60

If we suppose that for anyone in this scheme the probabilities of one claim in a year is 0.2 while the prob. of two or more claims is 0.1.

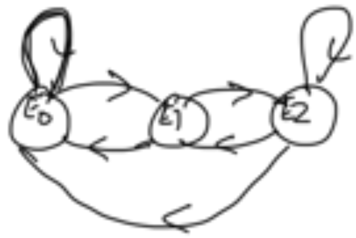
(i) Find tpm

$$P = \begin{array}{c} \\ E_0 \\ E_1 \\ E_2 \end{array} \begin{array}{c} \begin{array}{ccc} E_0 & E_1 & E_2 \end{array} \\ \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.3 & 0 & 0.7 \\ 0.1 & 0.2 & 0.7 \end{bmatrix} \end{array}$$ regular

(ii) In long run, what proportion of time is the process in each of three discount classes.

$$\overset{P}{\begin{pmatrix} + & + & 0 \\ + & 0 & + \\ + & + & + \end{pmatrix}} \overset{P}{\begin{pmatrix} + & + & 0 \\ + & 0 & + \\ + & + & + \end{pmatrix}} = \begin{pmatrix} + & + & + \\ + & + & + \\ + & + & + \end{pmatrix} P^2$$

regular



irreducible, aperiodic, finite State M.C.

$\Rightarrow$ regular

$$\underset{\sim}{\Pi} = (\Pi_0, \Pi_1, \Pi_2)$$

$$\underset{\sim}{\Pi} = \underset{\sim}{\Pi} P$$

$$\Pi_0 + \Pi_1 + \Pi_2 = 1$$

$$(\Pi_0, \Pi_1, \Pi_2) = (\Pi_0, \Pi_1, \Pi_2) \begin{pmatrix} 0.3 & 0.7 & 0 \\ \dots \end{pmatrix}$$

$$\begin{pmatrix} 0.3 & 0 & 0.7 \\ 0.1 & 0.2 & 0.7 \end{pmatrix}$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

$$\left.\begin{aligned} \pi_0 &= 0.3\pi_0 + 0.3\pi_1 + 0.1\pi_2 \\ \pi_1 &= 0.7\pi_0 \qquad\quad + 0.2\pi_2 \\ \pi_0 &+ \pi_1 + \pi_2 = 1 \end{aligned}\right\}$$

$\pi_0 = 0.1860$, $\pi_1 = 0.2442$, $\pi_2 = 0.5698$

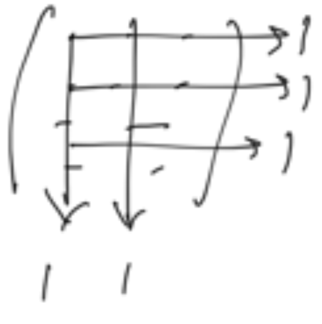
(iii) Find the average annual premium paid

$= 100 \times 0.1860 + 80 \times 0.2442 + 60 \times 0.5698$

$= 72.324$

—x—

Doubly Stochastic Matrices : tpm doubly stochastic $P = ((p_{ij}))$



columns sum to one as well as rows

$$\sum_k p_{ik} = \sum_k p_{kj} = 1 \;\forall i, j$$

Prop. 1) $S = \{0, 1, \ldots, N-1\}$

P regular tpm and doubly stochastic, then the unique limiting dist is the uniform dist $\underset{\sim}{\pi} = \left(\frac{1}{N}, \frac{1}{N}, \ldots \frac{1}{N}\right)$

Sol (solution from last theorem)

$$\pi_j = \sum_k \pi_k p_{kj}$$

$$\sum \pi_k = 1$$

only need to check that   is a sol.

$$\frac{1}{N} = \sum_k \frac{1}{N} p_{kj} = \frac{1}{N}\underbrace{\left(\sum_k p_{kj}\right)}_{1} = \frac{1}{N}$$

$\because$ doubly stochastic tpm

— x —

<u>Example</u>: Let Y_n be the sum of n indep rolls of a fair die and consider the problem of determining with what prob. Y_n is a multiple of 7 in the long run.

Let X_n be the remainder when Y_n is divided by 7

$X_n \in \{0, 1, \ldots, 6\}$ $\quad$ X_n is a M.C. $S = \{0, 1, \ldots, 6\}$

tpm

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 0 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 0 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 0 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 \end{pmatrix} \end{array}$$

regular and doubly stochastic

$$\Pi = \left(\frac{1}{}, \frac{1}{}, \ldots, \frac{1}{}\right)$$

$(7, 7, \ldots, 7)$

Y_n is a multiple of 7 iff $X_n = 0$

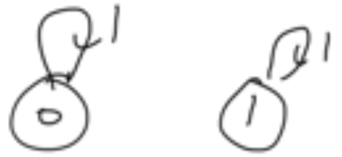
$\therefore$ limiting prob. that Y_n is a multiple of 7 is $\frac{1}{7}$.

— x —

Ⓐ Example (i) $a = b = 0$

$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 0 & 1 \end{array} = I$$

$P^n = I$

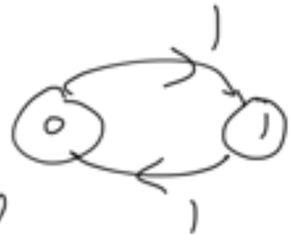


(ii) $a = b = 1$

periodic M.C. (with period 2)

$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}, \quad P^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad P^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \ldots$$

$$P^n = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if } n \text{ even} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \text{if } n \text{ odd} \end{cases}$$



$d(0) = \gcd(2, 4, 6, \ldots) = 2 = d(1)$

Here limiting prob. DNE.

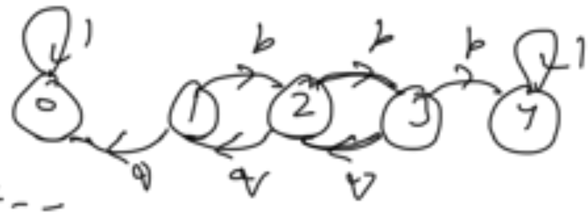
stationary prob. distn $\underline{\pi} = (\pi_0, \pi_1)$

$$\left.\begin{array}{l} \underline{\pi} = \underline{\pi} P \\ \pi_0 + \pi_1 = 1 \end{array}\right] \Rightarrow \left.\begin{array}{l} \pi_0 = \pi_1 \\ \pi_0 + \pi_1 = 1 \end{array}\right] \Rightarrow \pi_0 = \pi_1 = \frac{1}{2}$$

$$\underset{\sim}{\pi} = \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

—x—

Example



$$f_{2,1} = q + pq^2 + p^2q^3 + p^3q^4 + \cdots$$

$$= q + pq(q + pq^2 + \cdots) = q + pq\, f_{2,1}$$

$$\Rightarrow f_{2,1} = \frac{q}{1-pq}$$

If $p = 0.4$

$$f_{2,1} = \frac{0.6}{1-0.6\times 0.4} = \frac{0.6}{1-0.24} = \frac{0.6}{0.76} = 0.78$$

Gambler's ruin problem

$$f_{2,1} = P\{\text{Start } 1 \text{ goes broke before reaching } 3\}$$

$$= 1 - \frac{1-\left(\frac{0.6}{0.4}\right)^1}{1-\left(\frac{0.6}{0.4}\right)^3} = 0.78$$

—x—

Mean time spent in transient states:

finite state M.C

$T = \{1, 2, \ldots, t\}$ set of transient states

$$P_T = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1t} \\ \cdots & \cdots & & \cdots \\ \cdots & \cdots & & \\ p_{t1} & p_{t2} & \cdots & p_{tt} \end{bmatrix}$$ sum of rows < 1

$i, j \in T$

s_{ij} : expected number of time periods that the MC is in state j, given that it starts in state i.

Let $\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

$I_{n,j} = \begin{cases} 1 & \text{if } X_n = j \\ 0 & \text{o.w.} \end{cases}$

$$s_{ij} = \delta_{ij} + E\left(\sum_{n=1}^{\infty} I_{n,j} \,\Big|\, X_0 = i\right)$$

$$= \delta_{ij} + \sum_{n=1}^{\infty} E(I_{n,j} \mid X_0 = i)$$

$$= \delta_{ij} + \sum_{n=1}^{\infty} P(X_n = j \mid X_0 = i)$$

$$= \delta_{ij} + \sum_{n=1}^{\infty} p_{ij}^{(n)} \qquad \text{———— } (*)$$

$$p_{ij}^{(n)} \to \sum_k p_{ik} p_{kj}^{(n-1)} \quad \Big| \text{ Using CK eq's}$$

$$= \delta_{ij} + \sum_k p_{ik} \sum_{n=1}^{\infty} p_{kj}^{(n-1)}$$

$$= \delta_{ij} + \sum_k p_{ik} \left[\delta_{kj} + \sum_{n=2}^{\infty} p_{kj}^{(n-1)} \right]$$

$$= \delta_{ij} + \sum_k p_{ik} \underbrace{\left(\delta_{kj} + \sum_{n=1}^{\infty} p_{kj}^{(n)} \right)}_{s_{kj} \text{ from } (*)}$$

$$= \delta_{ij} + \sum_{k} p_{ik} s_{kj}$$

$$= \delta_{ij} + \sum_{k=1}^{t} p_{ik} s_{kj} ,$$

since it is impossible to go from a recurrent to a transient state $\Rightarrow s_{kj} = 0$, where k is a recurrent state

$$\therefore \boxed{s_{ij} = \delta_{ij} + \sum_{k=1}^{t} p_{ik} s_{kj}} \quad ; i,j \in T$$

$$S = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1t} \\ - & - & & \text{---} \\ s_{t1} & s_{t2} & \cdots & s_{tt} \end{bmatrix} \quad ; \quad I = \begin{bmatrix} 1 & 0 & \text{---} & 0 \\ 0 & 1 & \cdots & 0 \\ & \text{---} & & \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$S = I + P_T S$$

$$(I - P_T) S = I$$

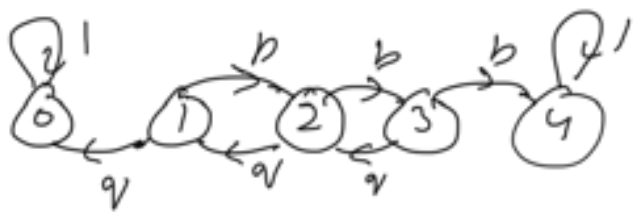
$$\Rightarrow \quad S = (I - P_T)^{-1}$$

<u>Example</u> : Consider the gamblers ruin problem with $p = 0.4$ and $N = 4$. Starting with 2 units, determine

a) the expected amt of time the gambler has 3 units.

b) " " " " " " " 1 unit.

Sol



Classes $\{0\}$ $\{1,2,3\}$ $\{4\}$

absorbing transient absorbing

$$T = \{1,2,3\}$$

$$P = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0.6 & 0 & 0.4 & 0 & 0 \\ 2 & 0 & 0.6 & 0 & 0.4 & 0 \\ 3 & 0 & 0 & 0.6 & 0 & 0.4 \\ 4 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$$P_T = \begin{bmatrix} 0 & 0.4 & 0 \\ 0.6 & 0 & 0.4 \\ 0 & 0.6 & 0 \end{bmatrix}$$

$$I - P_T = \begin{bmatrix} 1 & -0.4 & 0 \\ -0.6 & 1 & -0.4 \\ 0 & -0.6 & 1 \end{bmatrix}$$

$$S = (I - P_T)^{-1} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1.46 & 0.76 & 0.31 \\ 2 & 1.15 & 1.92 & 0.76 \\ 3 & 0.69 & 1.15 & 1.46 \end{array}$$

$$s_{2,3} = 0.76 \quad , \quad s_{2,1} = 1.15$$

(4) example (contd) Starting with state 2, what is the prob. that the gambler ever has a fortune of 1?

$$f_{2,1} = 0.78$$

—x—

$f_{i,j}$ = prob. that the M.C ever makes a transition into state j given that it starts in state i

$$s_{ij} = E(\text{time in } j \mid \text{start in } i)$$

$$= \underline{E(\text{time in } j \mid \text{start in } i, \text{ ever transit to } j)} \cdot f_{ij}$$

$+ E(\text{time } s_{ij} \mid \text{start } i, \text{ never transit to } j) \cdot (1 - f_{ij})$

$$= \underline{(\delta_{ij} + s_{jj}) f_{ij}} + \delta_{ij}(1 - f_{ij})$$

$$= \delta_{ij} + f_{ij}\, s_{jj}$$

$$\Rightarrow \boxed{f_{ij} = \frac{s_{ij} - \delta_{ij}}{s_{jj}}}$$

example (4)

$$f_{2,1} = \frac{s_{2,1} - \delta_{2,1}}{s_{11}} = \frac{1.15 - 0}{1.46} = 0.78$$

—x—

Branching Process:

each individual by the end of its lifetime produced j new offspring with prob $p_j < 1 \ \forall j \geq 0$

(let independently of number produced by other individuals)

X_0 size of zeroth generation

X_1 : all offsprings of zeroth generation or first generation

X_n : size of n^{th} generation.

$X_n \in \{0, 1, 2, \ldots\} = S \rightarrow$ State space

(X_n) DTMC

$f_{00} = 1$ 0 recurrent

the popln will either die out or it's size will

Converge to ∞.

mean # of offsprings of a single individual $\mu = \sum_{j=0}^{\infty} j p_j$

var. of " " " " " " $\sigma^2 = \sum_{j=0}^{\infty} (j-\mu)^2 p_j$

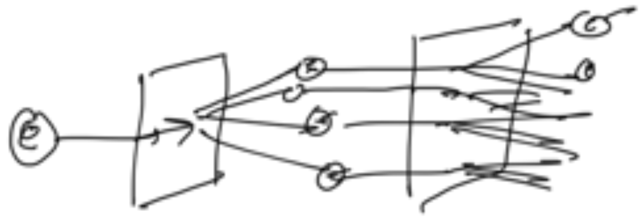
Let $X_0 = 1$

$$X_n = Z_1 + Z_2 + \cdots + Z_{X_{n-1}} = \sum_{i=1}^{X_{n-1}} Z_i$$

where Z_i # of offsprings of ith individual of the $(n-1)^{st}$ generation

$E(Z_i) = \mu$, $V(Z_i) = \sigma^2$

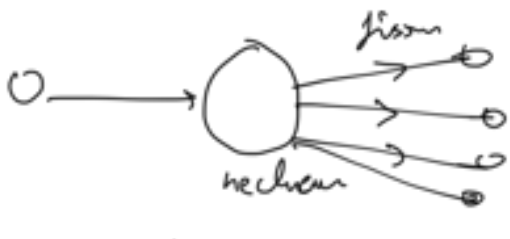
Application (i) electron multiplier



X_n # of electron emitted from n^{th} plate due to electron emanating from $(n-1)^{th}$ plate

(X_n) Branching process

(ii) Neutron chain reaction



X_n

(iii) Survival of family name

$$E(X_n) = E\left(\underline{E(X_n | X_{n-1})}\right)$$

$$= E\left(E\left(\sum_{i=1}^{X_{n-1}} Z_i \,\middle|\, X_{n-1}\right)\right)$$

$$\underbrace{\qquad\qquad}_{X_{n-1}\mu}$$

$$= E(X_{n-1}\mu)$$

$$= \mu\, E(X_{n-1})$$

$$E\left(\sum_{i=1}^{X_{n-1}} Z_i \,\middle|\, X_{n-1}=x\right) = E\left(\sum_{i=1}^{x} Z_i\right) = \sum_{i=1}^{x} \underbrace{E(Z_i)}_{\mu} = x\mu$$

$E(X_0)=1$, $E(X_1)=\mu$, $E(X_2)=\mu^2$, --- ---, $\boxed{E(X_n)=\mu^n}$ ✓

$$V(X_n) = E\big[\underbrace{V(X_n|X_{n-1})}_{\sigma^2 X_{n-1}}\big] + V\big[\underbrace{E(X_n|X_{n-1})}_{\mu X_{n-1}}\big]$$

$$= \sigma^2 E(X_{n-1}) + \mu^2 V(X_{n-1})$$

$$= \sigma^2\mu^{n-1} + \mu^2 V(X_{n-1}) \quad \text{—— } *$$

$$= \sigma^2\mu^{n-1} + \mu^2\left[\sigma^2\mu^{n-2} + \mu^2 V(X_{n-2})\right]$$

| using *

$$V\left(\sum_{i=1}^{X_{n-1}} Z_i \,\middle|\, X_{n-1}=x\right) = V\left(\sum_{i=1}^{x} Z_i\right)$$

Z_i indep

$$= \sum_{i=1}^{x} \underbrace{V(Z_i)}_{\sigma^2} = x\sigma^2$$

$$= \sigma^2\left[\mu^{n-1}+\mu^n\right] + \mu^4 V(X_{n-2})$$

$$= \sigma^2(\mu^{n-1}+\mu^n) + \mu^4\left(\sigma^2\mu^{n-3} + \mu^2 V(X_{n-3})\right)$$

$$= \sigma^2(\mu^{n-1}+\mu^n+\mu^{n+1}) + \mu^6 V(X_{n-3})$$

$$\sigma^2(\mu^{n-1} \quad \mu^n \quad \mu^{2n-2}) \quad \mu^{2n}$$

$$= \sigma^{2}(\mu^{n-1} + \mu^{n} + \cdots + \mu^{2n-2}) \quad + \mu^{2n}\underbrace{(V(X_0))}_{\to 0}$$

$$= \sigma^2 \mu^{n-1}(1 + \mu + \cdots + \mu^{n-1})$$

$$\therefore V(X_n) = \begin{cases} \sigma^2 \mu^{n-1}\left(\dfrac{1-\mu^n}{1-\mu}\right) & \text{if } \mu \neq 1 \\ n\sigma^2 & \text{if } \mu = 1 \end{cases}$$

$\rightarrow$ $$u_{n+1} = P(X_{n+1} = 0) = \sum_j P(X_{n+1} = 0 \mid X_1 = j)\, p_j$$

$$= \sum_j \left(P(X_n = 0)\right)^j p_j$$

$$= \sum_j u_n^j p_j$$

$$\therefore \boxed{u_{n+1} = \sum_j u_n^j p_j}$$

$\rightarrow$ <u>Prob of ultimate extinction</u> (Π_0)

Π_0 1 prob. that popl^n will eventually die out (under the assumption that $X_0 = 1$)

$$\Pi_0 = \lim_{n \to \infty} P(X_n = 0 \mid X_0 = 1)$$

$\rightarrow$ $\Pi_0 = 1$ if $\mu < 1$

$$\mu^n = E(X_n) = \sum_{j=\cancel{0}1}^{\infty} j\, P(X_n = j)$$

$$\geq \sum_{j=1}^{\infty} 1 \cdot P(X_n = j)$$

$$= P(X_n \geq 1)$$

Since $\mu^n \to 0$ if $\mu < 1$ as $n \to \infty$

$$P(X_n \geq 1) \to 0$$

, i.e., $P(X_n = 0) \to 1$, i.e., $\pi_0 = 1$

→ It can shown that $\pi_0 = 1$ even when $\mu = 1$

→ When $\mu > 1$ it turns out $\pi_0 < 1$

$\pi_0 = P($ popl^n dies out$)$

$$= \sum_{j=0}^{\infty} P(\text{popl}^n \text{ dies out} \mid X_1 = j)\, p_j$$

$$\leq \sum_{j=0}^{\infty} \pi_0^j\, p_j$$

$$\boxed{\pi_0 = \sum_{j=0}^{\infty} \pi_0^j\, p_j} \longrightarrow ✡$$

When $\mu > 1$, it can be shown that π_0 is the smallest +ve number satisfying equation (✡).

Example (i) $X_0 = 1$, $p_0 = \frac{1}{2}$, $p_1 = \frac{1}{4}$, $p_2 = \frac{1}{4}$

$$\pi_0 = ?$$

Sol $\mu = 0 \times \frac{1}{2} + 1 \times \frac{1}{4} + 2 \times \frac{1}{4} = \frac{3}{4} \leq 1$

$$\therefore \pi_0 = 1$$

(ii) $X_0 = 1$; $p_0 = \frac{1}{4}$, $p_1 = \frac{1}{4}$, $p_2 = \frac{1}{2}$

$$\pi_0 = ?$$

Sol

$$\mu = 0 \times \frac{1}{4} + 1 \times \frac{1}{4} + 2 \times \frac{1}{2} = \frac{5}{4} > 1$$

$$\pi_0 = \sum_{j=0}^{\infty} \pi_0^{\,j}\, p_j \qquad \Big| \text{ using ✡}$$

$$\Rightarrow \pi_0 = \frac{1}{4} + \frac{\pi_0}{4} + \frac{\pi_0^2}{2}$$

$$\Rightarrow 2\pi_0^2 - 3\pi_0 + 1 = 0$$

$$\Rightarrow \pi_0 = \overset{\times}{1},\ \overset{\checkmark}{\frac{1}{2}}$$

$$\therefore \pi_0 = \frac{1}{2}$$