Introduction to Probability
Chapter 2: Conditional Probability

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Outline

1. Conditional Probability
2. Independence
3. Theorem of total probability
4. Bayes' theorem

References

1. Probability and statistics in engineering by Hines et al (2003) Wiley.
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Conditional Probability

Consider a family having two children, then $\Omega = \{BB, GB, BG, GG\}$, $n(\Omega) = 4$. Consider the event A: both the children are girls. Then $P(A) = 1/4$.
If some information in the form of "B: at least one of the children is a girl" is known. Then reduced sample space is $B = \{GB, BG, GG\}$. Then the probability of A given the condition B is $P(A|B) = 1/3$.
Note that $P(A|B) \geq P(A)$.
$P(A|B) = \frac{n(A \cap B)}{n(B)} = \frac{n(A \cap B)/n(\Omega)}{n(B)/n(\Omega)} = \frac{P(A \cap B)}{P(B)}$, provided $P(B) > 0$.

Definition

Let probability model be (Ω, f, P). Then the conditional probability of $A \in f$ given B is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) > 0.$$

Multiplication rule

The probability that n events $A_1, A_2, \ldots, A_n \in f$ occur in a sequence is

$$P(\cap_{i=1}^{n} A_i) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap A_2 \cdots \cap A_{n-1}),$$

provided $P(A_1) > 0,\ P(A_1 \cap A_2) > 0, \ldots, P(A_1 \cap A_2 \cdots \cap A_{n-1}) > 0.$

Example

A bag contains 5 red, 5 white and 4 blue balls. If someone draws 3 balls one by one without replacement, then the probability that three balls will be drawn in the sequence red-white-blue is

$$\begin{aligned} P(R_1 \cap W_2 \cap B_3) &= P(B_3|W_2 \cap R_1)P(W_2|R_1)P(R_1) \\ &= P(R_1)P(W_2|R_1)P(B_3|W_2 \cap R_1) \\ &= \frac{5}{14} \times \frac{5}{13} \times \frac{4}{12}. \end{aligned}$$

Here note that $R_1,\ W_2,\ B_3$ are dependent events.

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Here note that $R_1,\ W_2,\ B_3$ are independent events.

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In a war game, submarine S_1 targets S_2, and both S_2 and S_3 target S_1. The probabilities of S_1, S_2 and S_3 hitting their targets are $1/2$, $2/3$ and $1/3$ respectively. They shoot simultaneously. We want to determine the conditional probability that S_2 hits the target and S_3 does not given that S_1 is hit. Here the required probability is

$$\begin{aligned} P(S_2 \cap \bar{S_3} | S_1 \text{ is hit}) &= \frac{P(S_2)P(\bar{S_3})}{P(S_2 \cup S_3)} \\ &= \frac{\frac{2}{3} \times \frac{2}{3}}{\frac{2}{3} + \frac{1}{3} - \frac{2}{3} \times \frac{1}{3}} \\ &= \frac{4}{7} \end{aligned}$$

Independence

Two events are independent if the occurrence of one does not effect the occurrence or nonoccurrence of the other.

Definition

Events A and B are independent if $P(A|B) = P(A)$. Hence $P(A \cap B) = P(A)P(B)$ and also $P(B|A) = P(B)$.

- If A and B are independent, then A and $\bar{B}$ are independent.
- If A and B are independent, then $\bar{A}$ and B are independent.
- If A and B are independent, then $\bar{A}$ and $\bar{B}$ are independent.

Example

Example

Suppose that $P(A) = 0.4$, $P(B) = 0.5$, and A and B are independent events. Determine $P(A^c \cup B^c)$. Note that

$$\begin{aligned} P(A^c \cup B^c) &= P(A^c) + P(B^c) + P(A^c \cap B^c) \\ &= 1 - P(A) + 1 - P(B) - P(A^c)P(B^c) \\ &= 1 - 0.4 + 1 - 0.5 - (1 - 0.4)(1 - 0.5) \\ &= 0.8, \end{aligned}$$

it follows since A and B are independent, then $\bar{A}$ and $\bar{B}$ are also independent.

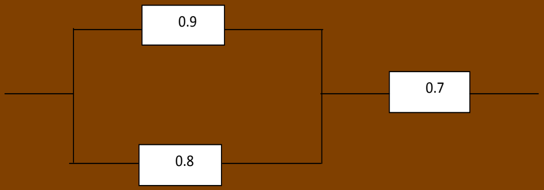
Definition

The n events $A_1, \ldots, A_n$ are mutually independent if and only if the probability of intersection of any $2, 3, \ldots, n$ of these sets is product of their respective probabilities, i.e., for $r = 2, 3, \ldots, n$,

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_r}).$$

Example

Consider the following electronic system (see diagram), which shows the probabilities of the system components operating properly (i.e, the reliability of the components). Assume that each component operates independently. Find the system reliability, i.e., the probability that the entire system operates?



Solution: Since components are mutually independent. Hence

$$\text{System reliability} = (1 - (1 - 0.9)(1 - 0.8)) \times 0.7 = 0.686$$

Example

Consider the experiment of rolling two fair dice repeatedly and independently until a total of 5 or a total of 7 appears. We want to determine the probability that a total of 5 is rolled before a total of 7 is rolled.

Solution: Let event A_i denote that 5 is rolled before a 7 in the ith trail and experiment terminates.

$$P(A_i) = P\left(\left\{\bigcap_{j=1}^{i-1}(5\cup 7)^c\right\}\cap 5\right) =^{\text{indep}} \left(\frac{26}{36}\right)^{i-1} \times \frac{4}{36}$$

$$\text{Now } P(5 \text{ before } 7) = P\left(\bigcup_{i=1}^{\infty} A_i\right) =^{\text{disjoint}} \sum_{i=1}^{\infty} P(A_i)$$

$$= \sum_{i=1}^{\infty}\left(\frac{26}{36}\right)^{i-1} \times \frac{4}{36} = \frac{4/36}{1-\frac{26}{36}} = \frac{2}{5}.$$

Theorem of total probability

Events $E_1, \ldots, E_n$ are mutually exclusive and exhustive, and event A is caused by happening of $E_1, \ldots, E_n$, then

$$P(A) = \sum_{i=1}^{n} P(A|E_i)P(E_i),$$

here $P(E_i) > 0, \; i = 1, 2, \ldots, n.$

Bayes' theorem

Events $E_1, \ldots, E_n$ are mutually exclusive and exhustive, and event A is caused by happening of $E_1, \ldots, E_n$, then for $i = 1, 2, \ldots, n$

$$P(E_i|A) = \frac{P(A|E_i)P(E_i)}{\sum_{j=1}^{n} P(A|E_j)P(E_j)},$$

here $P(A) > 0$ and $P(E_i) > 0, \ i = 1, 2, \ldots, n.$

Example

In a town there are 200 car drivers, 500 two-wheeler drivers and 20 bus drivers. There is a probability 0.01, 0.03 and 0.15 respectively for an accident involving car, two-wheeler and bus. One of the drivers meets with an accident, what is the probability that he/she was driving a car?

Solution: Let event A,B,C denote the events that the chosen driver drives a car, a two-wheeler, a bus, respectively. Let event E denote an accident. Here $P(A) = \frac{200}{200+500+20} = \frac{20}{72}$, $P(B) = \frac{50}{72}$, $P(C) = \frac{2}{72}$. Also $P(E|A) = 0.01$, $P(E|B) = 0.03$ and $P(E|C) = 0.15$. Now the required probability, using Bayes' theorem, is

$$\begin{aligned} P(A|E) &= \frac{P(A)P(E|A)}{P(A)P(E|A) + P(B)P(E|B) + P(C)P(E|C)} \\ &= \frac{\frac{20}{72} \times 0.01}{\frac{20}{72} \times 0.01 + \frac{50}{72} \times 0.03 + \frac{2}{72} \times 0.15} \\ &= 0.1 \end{aligned}$$

Example

Consider the experiment of rolling two fair dice repeatedly and independently until a total of 5 or a total of 7 appears. We want to determine the probability that a total of 5 is rolled before a total of 7 is rolled.

Solution: Let event A denote that 5 is rolled before a 7. Then $A = \bigcup_{i=1}^{n}(A \cap B_i)$, where B_i is the event that the game terminates in ith roll. Now the required probability, using theorem of total probability, is

$$
\begin{aligned}
P(A) &= \sum_{i=1}^{\infty} P(A \cap B_i) \\
&= \sum_{i=1}^{\infty} \left(\frac{26}{36}\right)^{i-1} \times \frac{4}{36} \\
&= \frac{4/36}{1-\frac{26}{36}} = \frac{2}{5}.
\end{aligned}
$$

Summary

Since there may be some information available about the outcome of the trail in a given experiment, hence we introduced the concept of the conditional probability. Also if this information is irrelevant to the event under considration from there comes the definiton of the independence of the events. These definitions can be used to find the reliabilities of the series-parallel or parallel-series structures. Hence examples are provided for the same. In the last theorem of total probability and Bayes' theorem were presented.