CS 590 Homework 2
Recurrences and Radix Sort

Part 1: Recurrences:

Q] Solve the following recurrences using the substitution method. Subtract off a lower-order term to make the substition proof work or adjust the guess in case the initial substitution fails

1) $T(n) = T(n-3) + 3\lg n$. Our guess: $T(n) = O(n \lg n)$
show $T(n) \le cn\lg n$ for some constant $c > 0$
(Note: $\lg n$ is monotonically increasing for $n > 0$)

Solution:

given $T(n) = T(n-3) + 3\lg n$ —— ①

Assumption: true for all $m < n$ all $m < n$.

To prove: $T(n) <= cn\lg n$ assuming $T(m) <= cm\lg m$ for n

Let $m = (n-3) < n$

$$= c(n-3)\lg(n-3) + 3\lg n$$
$$= cn\lg(n-3) - 3c\lg(n-3) + 3\lg n$$

$\therefore T(n) \le cn\lg(n-3) - 3c\lg(n-3) + 3\lg n$

$T(n) <= cn\lg n - d\lg n$

$$= c(n-3)\lg(n-3) - d\lg(n-3)$$
$$= cn\lg n - dn$$
$$<= cn\lg n - dn \quad \text{if } d >= 2$$

2) $T(n) = 4T(n/3) + n$ Our guess: $T(n) = O(n^{\log_3 4})$
show $T(n) <= cn^{\log_3 4}$ for some constant $c > 0$

Solution:

Given $T(n) = 4T(n/3) + n$ [assuming true for all $m < n$]

To prove: $T(n) <= cn^{\log_3 4}$ assuming $T(m) <= cm^{\log_3 4}$ for all $m < n$

$$T(n) = 4c(n/3)^{\log_3 4} + n$$
$$= (4/3^{\log_3 4}) \times (c\,n^{\log_3 4}) + n$$
$$= cn^{\log_3 4} + n \longrightarrow \text{fails}$$

Lets guess $T(n) = O(n^{\log_3 4} - n)$

$$T(n) = 4T(n/3) + n$$
$$<= 4c((n/3)^{\log_3 4} - n/3) + n$$
$$<= cn^{\log_3 4} - (4/3)cn + n$$
$$T(n) <= cn^{\log_3 4} \quad \text{for all} \quad c > 2$$

3) $T(n) = T(n/2) + T(n/4) + T(n/8) + n$. Our guess: $T(n) = O(n)$

show $T(n) <= cn$ for some constant $c > 0$

Solution:

given. $T(n) = T(n/2) + T(n/4) + T(n/8)$ (assuming for all $m < n$)

To prove $T(n) <= cn$ assuming $T(m) <= cm$ for all $m < n$

$$T(n) = c(n/2) + c(n/4) + c(n/8) + n$$
$$T(n) = ((4cn + 2cn + cn)/8) + 8n$$
$$T(n) = ((7/8)c + 1)n$$

$\therefore$ $\underline{T(n) <= cn}$ for all $c >= 8$

4) $T(n) = 4T(n/2) + n^2$ Our guess: $T(n) = O(n^2)$

Show $T(n) <= cn^2$ for some constant $c > 0$

Solution:

Given $T(n) = 4T(n/2) + n^2$ assuming true for all $m < n$

To prove $T(n) <= cn^2$ assuming $T(m) <= cm^2$ for all $m < n$

$$T(n) = 4c(n/2)^2 + n^2$$
$$T(n) = 4c(n^2/4) + n^2 = cn^2 + n^2$$
$$T(n) = n^2(c+1) \longrightarrow \text{Fails.}$$

Lets guess $T(n) = O(n^2 \log n)$

$$T(n) = 4T(n/2) + n^2$$
$$= 4c(n/2)^2 \log(n/2) + n^2$$
$$= cn^2 \log(n/2) + n^2$$
$$= cn^2 \log n - cn^2 \log 2 + n^2$$
$$<= cn^2 \log n \quad \text{for all } c > 1$$