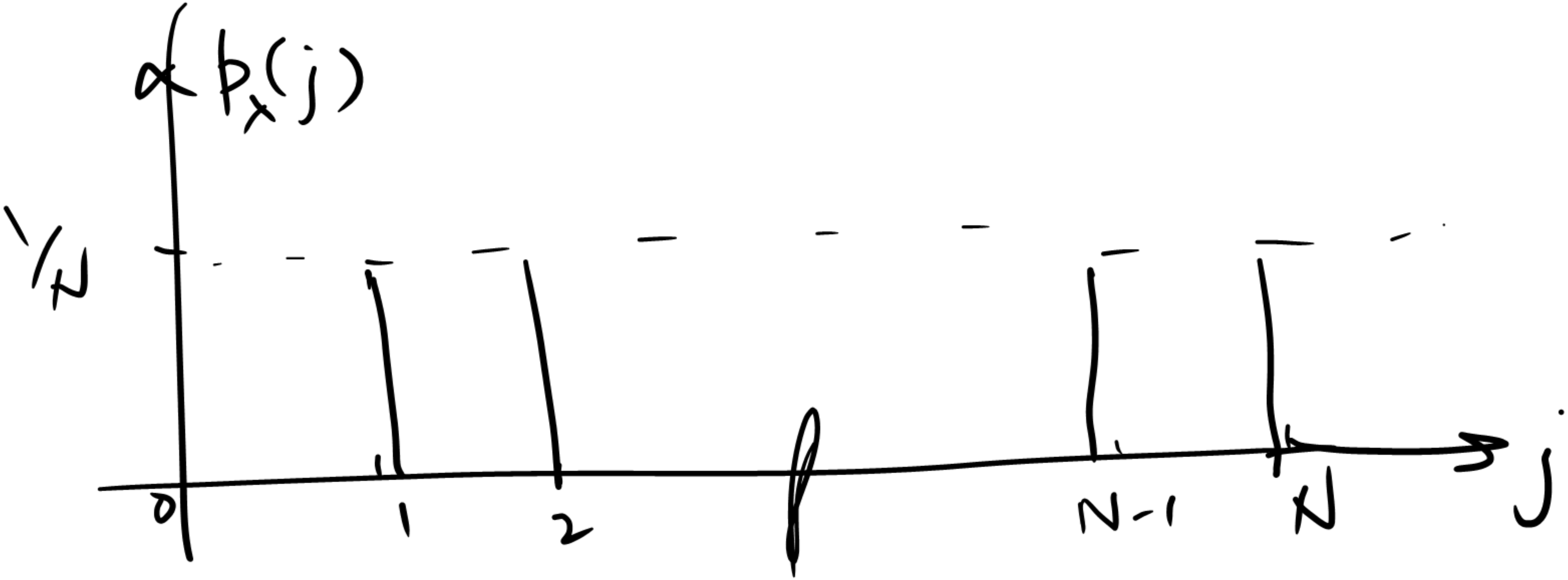
Special Discrete Distributions

1. Discrete Uniform Distribution

X takes values $1, 2, \ldots, N$

$$p_X(j) = P(X = j) = \frac{1}{N}, \quad j = 1, \ldots, N$$



$$E(X) = \sum_{j=1}^{N} \frac{j}{N} = \frac{(N+1)}{2}$$

$$E(X^2) = \sum_{j=1}^{N} \frac{j^2}{N} = \frac{(N+1)(2N+1)}{6}$$

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \frac{N^2-1}{12}$$

$$\underline{\underline{N=12}}, \quad \sigma^2 = \frac{143}{12}, \quad \sigma \simeq 3.45$$

moments of all orders can be evaluated

⊛ → Find $\mu_3, \mu_4, \beta_1, \beta_2$

$$M_X(t) = E\left(e^{tX}\right) = \sum_{j=1}^{N} \frac{e^{tj}}{N}$$

$$= \begin{cases} \dfrac{e^t\left(e^{Nt}-1\right)}{N\left(e^t-1\right)}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

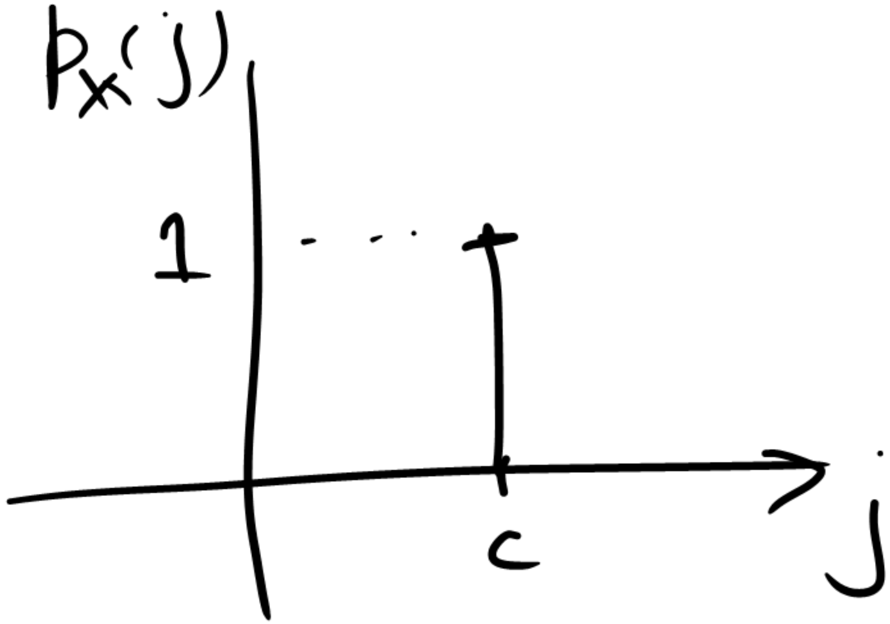
2. Degenerate Distribution

$$P(X = c) = 1, \quad c \in R$$

$$E(X^k) = c^k$$

$k = 1, 2, \ldots$

$$V(X) = 0$$



3. Bernoullian Trial

→ A random expt which results in two possible outcomes

→ success → (s)

→ failure → (f)

$X(s) = 1, \qquad X(f) = 0$

Bernoulli r.v.

$P(X=1) = p, \qquad P(X=0) = (1-p)$

$E(X) = p, \qquad \mu'_k = E(X^k) = p$

$\mu'_2 = p, \qquad V(X) = p - p^2 = p(1-p) = pq$

$0 < p < 1$

$M_X(t) = E(e^{tX}) = (1-p) + pe^t$

$= (q + pe^t)$

4. <u>Binomial Distribution</u> : Suppose n independent Bernoullian trials are conducted with identical prob. of success 'p' in each trial.

$X \to$ no. of successes in n trials

$X \to 0, 1, 2, \cdots, n$

$$p_X(k) = P(X=k) = \binom{n}{k} p^k q^{n-k},$$

$$k = 0, 1, \ldots, n$$

Binomial Distⁿ

$$\sum_{k=0}^{n} p_X(k) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}$$

$$= (q+p)^n = 1.$$

$$E(X) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=1}^{n} \frac{n! \; p^{k} \; q^{n-k}}{(k-1)! \, (n-k)!}$$

$$= np \sum_{k=1}^{n} \frac{(n-1)! \; p^{k-1} \; q^{n-1-\overline{k-1}}}{(k-1)(n-1-\overline{k-1})!}$$

$k-1 = j$

$$= np \sum_{j=0}^{n-1} \binom{n-1}{j} p^{j} q^{n-1-j}$$

$$= np\,(q+p)^{n-1} = np$$

We calculate factorial moments in order to get higher moments

$$E(X^2) = E\{X(X-1)\} + E(X)$$

$$= n(n-1)p^2 + np$$

$$Var(X) = E(X^2) - (EX)^2$$

$$= n(n-1)p^2 + np - n^2p^2$$

$$= np(1-p) = npq < np$$

So in a binomial distⁿ

variance < mean

$$E\,X(X-1) = \sum_{k=0}^{n} k(k-1)\binom{n}{k} p^{k} q^{n-k}$$

$$= \sum_{k=2}^{n} \frac{n!\, p^{k}\, q^{n-k}}{(k-2)!\,(n-k)!}$$

$$= n(n-1)\,p^{2} \sum_{k=2}^{n} \frac{(n-2)!\, p^{k-2}\, q^{n-2-\overline{k-2}}}{(k-2)!\,(n-2-\overline{k-2})!}$$

$k-2=j$

$$= n(n-1)p^2 \sum_{j=0}^{n-2} \binom{n-2}{j} p^j q^{n-2-j}$$

$$= n(n-1)p^2 (q+p)^{n-2}$$

$$= n(n-1)p^2$$

$$E\,X(X-1)(X-2) = n(n-1)(n-2)\,p^2$$

$$E(X^3) = E\,X(X-1)(X-2) + 3\cdot E X^2 - 2\,E X$$

$$= \cdots\cdots \quad *$$

$$\mu_3 = E(X-np)^3 = np(1-p)(1-2p)$$

$$\beta_1 = \frac{\mu_3}{\sigma^3} = \frac{np(1-p)(1-2p)}{[np(1-p)]^{3/2}} = \frac{(1-2p)}{(np(1-p))^{1/2}}$$

$\beta_1 = 0 \quad p = 1/2$

< 0 for $p > 1/2$

> 0 for $p < \frac{1}{2}$

For $p = \frac{1}{2}$

$$P(X=j) = \binom{n}{j}\left(\frac{1}{2}\right)^n = P(X=n-j)$$

$j = 0, 1, \ldots, n$

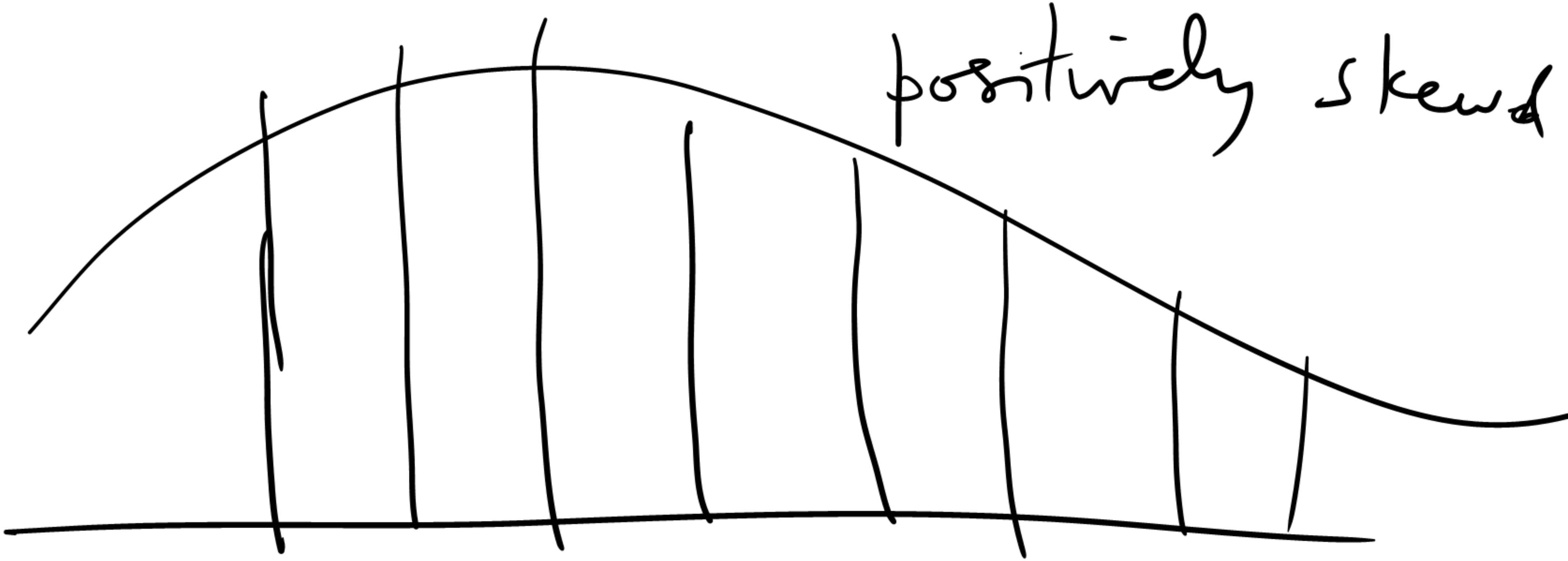
So it is perfectly symmetric

$p = 1/2$

For $\underline{\underline{p > 1/2}}$ $\binom{n}{j} p^j q^{n-j}$

negatively skewed $p > 1/2$

$p < 1/2$



$$\mu_4 = 3(npq)^2 + npq(1-6pq)$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} - 3$$

$$= \frac{1-6pq}{npq} \begin{cases} = 0 & \text{if } pq = 1/6 \\ > 0 & \text{if } pq < 1/6 \\ < 0 & \text{if } pq > 1/6 \end{cases}$$

$$p^2 - p + \frac{1}{6} = 0$$

$$M_X(t) = E\left(e^{tX}\right)$$

$$= \sum_{k=0}^{n} \left(e^{tk}\right) \binom{n}{k} p^{k} q^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left(pe^{t}\right)^{k} q^{n-k}$$

$$= \left(q + pe^{t}\right)^{n}$$

Ex: It is observed that 5% of people do not report to board flights after booking of tickets. So for a 50 seat flight airline sells 52 tickets. For a given flight what is the prob that every passenger who reports gets the seat?

$X \to$ no of passengers reporting for the flight

$X \sim \text{Bin}(52, 0.95)$

Reqd. $P(X \leq 50)$

$= 1 - \left(P(X=51) + P(X=52)\right)$

$= 1 - \binom{52}{51}(.95)^{51}(0.05) - (0.95)^{52}$

$\cong 0.74$

So prob that a passenger reporting may not get a seat is $\simeq 0.26$??

5. Geometric Distribution

Suppose independent Bernoullian trials are performed under identical conditions with prob.

of success p in each trial.

$X \to$ no of trials needed for the first success

$X \to 1, 2, 3, \ldots$

$$p_X(k) = P(X=k) = q^{k-1} p, \quad k = 1, 2, \ldots$$

$$\sum_{k=1}^{\infty} p_X(k) = \sum_{k=1}^{\infty} q^{k-1} p$$

$$= p\left(1+q+q^2+\cdots\right) = \frac{p}{1-q} = 1$$

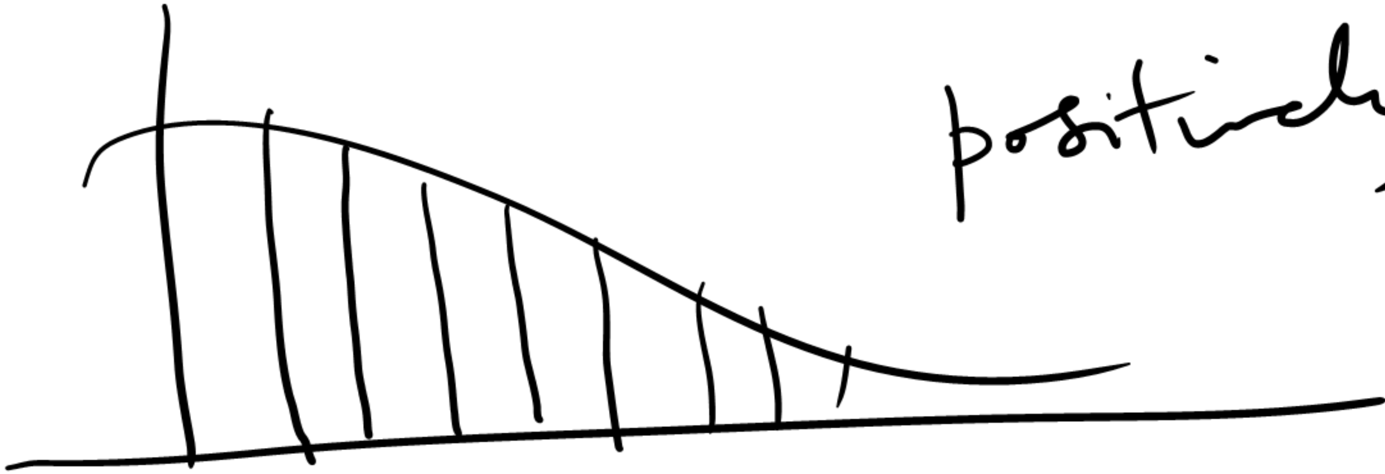
$$\mu_1' = E(X) = \sum_{j=1}^{\infty} j\, q^{j-1} p = \frac{p}{(1-q)^2} = \frac{1}{p}$$

$$(1-q)^{-(k+1)} = \sum_{j=k}^{\infty} \binom{j}{k} q^{j-k}$$

$$\boxed{0<q<1} \quad = \sum_{i=0}^{\infty} \binom{k+i}{k} q^i$$

$$E(X^2) = EX(X-1) + EX = \left(\frac{q+1}{p^2}\right)$$

$$V(X) = \frac{q+1}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}$$



positively skewed

$$M_X(t) = \sum_{k=1}^{\infty} e^{tk} q^{k-1} p$$

$$= pe^t \sum_{k=1}^{\infty} (qe^t)^{k-1}$$

$$= \frac{pe^t}{1-qe^t}, \quad 0 < qe^t < 1$$

$$t < (-\log q)$$