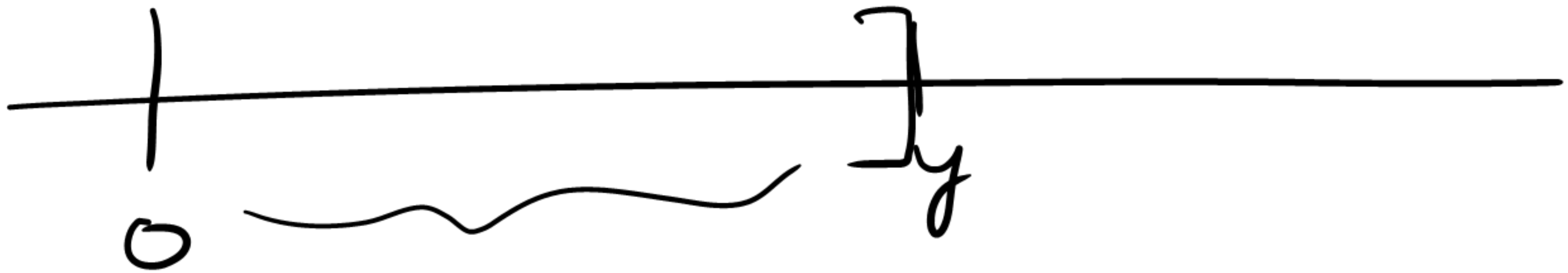
(Negative) Exponential Distribution

Let us consider $X(t)$ as a Poisson process with rate λ. We start observing the process from some initial time and let Y be the time till the first occurrence. We want the distribution of Y.

Clearly Y is a continuous r.v.



Consider

$$P(Y>y) = P(\text{there is no occurrence in } (0,y])$$

$$= P(X(y)=0) = \begin{cases} e^{-\lambda y}, & y>0 \\ 1, & y\le 0 \end{cases}$$

So cdf of Y is

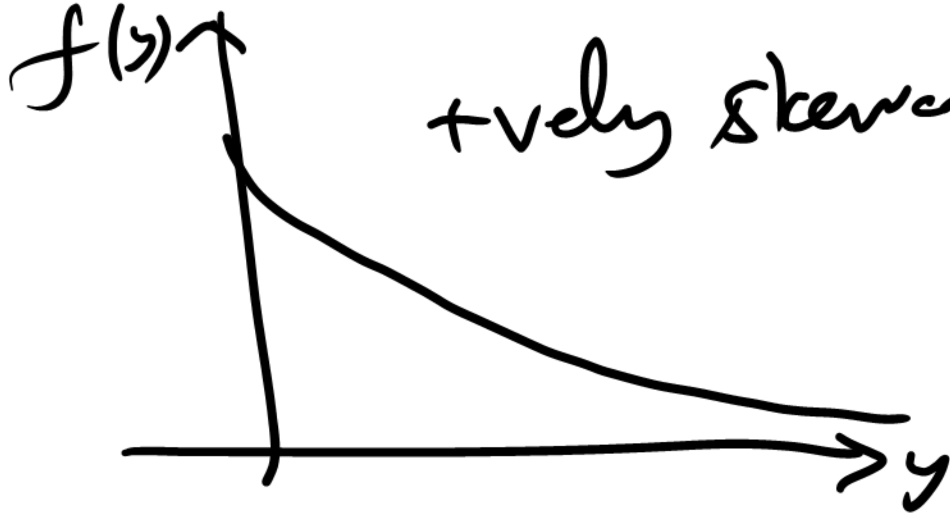
$$F_Y(y) = 1 - P(Y > y) = \begin{cases} 0, & y \le 0 \\ 1 - e^{-\lambda y}, & y > 0 \end{cases}$$

So the pdf of Y is

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda y}, & y > 0, \quad \lambda > 0 \\ 0, & \text{otherwise} \end{cases}$$

$\downarrow$

Exponential distn.



+vely skewed

$$\mu_k' = E(y^k)$$

$$= \int_0^\infty y^k \lambda e^{-\lambda y} dy$$

$$= \frac{\lambda \Gamma k+1}{\lambda^{k+1}} = \frac{\Gamma(k+1)}{\lambda^k}$$

$$\mu_1' = E(Y) = \frac{1}{\lambda}, \quad \mu_2' = \frac{2}{\lambda^2}$$

$V(Y) = \frac{1}{\lambda^2}$, $s.d.(Y) = \frac{1}{\lambda}$

$\mu_3' = \frac{6}{\lambda^3}$, $\mu_3 = \frac{2}{\lambda^3}$

$$\beta_1 = \frac{\mu_3}{\mu_2^{3/2}} = \frac{2/\lambda^3}{1/\lambda^3} = 2 > 0$$

The distn is always +vely skewed.

$\mu_4' = \frac{24}{\lambda^4}$, $\mu_4 = \frac{9}{\lambda^4}$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} - 3 = \frac{9/\lambda^4}{1/\lambda^4} - 3 = 6 > 0$$

The dist^n is always leptokurtic.

<u>Memoryless Property of Exponential Distribution</u> :

$$P(Y > a) = e^{-\lambda a}$$

$$P(\underbrace{Y > a+b}_{E} \mid \underbrace{Y > b}_{F}) = \frac{P(Y > a+b)}{P(Y > b)}$$

$$= \frac{e^{-\lambda(a+b)}}{e^{-\lambda b}} = e^{-\lambda a} = P(Y>a)$$

<u>Moment Generating Function</u>

$$M_Y(t) = E\left(e^{tY}\right) = \int_0^\infty e^{ty} \lambda e^{-\lambda y} dy$$

$$= \lambda \int_0^\infty e^{-y(\lambda-t)} dy = \frac{\lambda}{\lambda-t}, \; t<\lambda$$

Example: Suppose life of a electronic item (in months) follows an exponential distⁿ with mean 4 (months). What is prob that an item is working even after one year?

$$f_Y(y) = \frac{1}{4} e^{-y/4}, \quad y > 0$$

$$P(Y > 12) = e^{-12/4} = e^{-3} \approx 0.0498$$

$$\approx 0.05$$

What is the prob. that out of five randomly selected items at least one is working after one year?

$X \rightarrow$ no of items working after one year

$X \sim Bin(5, p)$; $p = 0.05$

$$P(X \geq 1) = 1 - P(X=0) = 1 - (0.95)^5$$

$$= 0.2262$$

2. Suppose light bulbs are produced at two plants A and B. The lives of bulbs (from A) has exponential $dist^n$ with mean 5 months and those from B has exponential $dist^n$ with mean 2 months. Plant B produces three times as many bulbs as plant A. What is the prob. that a randomly selected bulb from the market will have

life at least 5 months?

$X \rightarrow$ life of bulb

$$f_{X|A}(x) = \frac{1}{5} e^{-x/5}, \quad x > 0$$

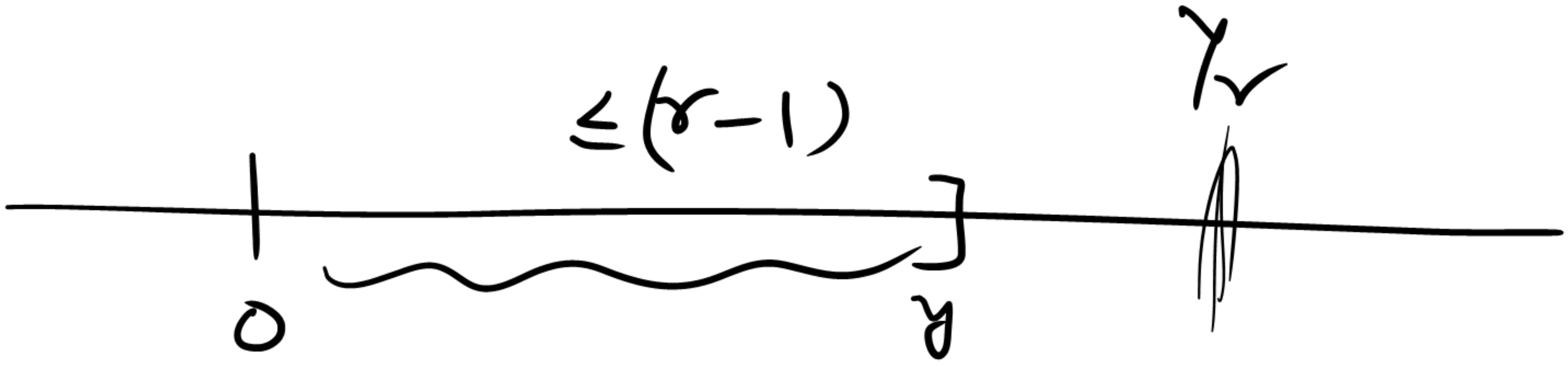
$$f_{X|B}(x) = \frac{1}{2} e^{-x/2}, \quad x > 0$$

$$P(X>5) = P(X>5|A)\,P(A) + P(X>5|B)\,P(B)$$

$$= e^{-5/5} \cdot \frac{1}{4} + e^{-5/2} \cdot \frac{3}{4} \simeq 0.1534$$

Gamma / Erlang Distribution

Consider Poisson process $X(t)$ with rate λ. Let Y_r to be the time till the r^{th} occurrence, $r = 1, 2, \ldots$ in the process.



To derive the distⁿ of Y_r :

$P(Y_r > y) =$ P(number of occurrences in $(0, y]$ is at most $(r-1)$)

$$= \begin{cases} P(X(y) \le r-1), & y > 0 \\ 1, & y \le 0 \end{cases}$$

So for $y > 0$

$$P(Y_r > y) = \sum_{j=0}^{r-1} \frac{e^{-\lambda y} (\lambda y)^j}{j!}$$

So cdf of Y_r

$$F_{Y_r}(y) = \begin{cases} 0, & y \le 0 \\ 1 - \sum_{j=0}^{r-1} \frac{e^{-\lambda y} (\lambda y)^j}{j!}, & y > 0 \end{cases}$$

The pdf of Y_r is then obtained by differentiating $F_{Y_r}(y)$ wrt y.

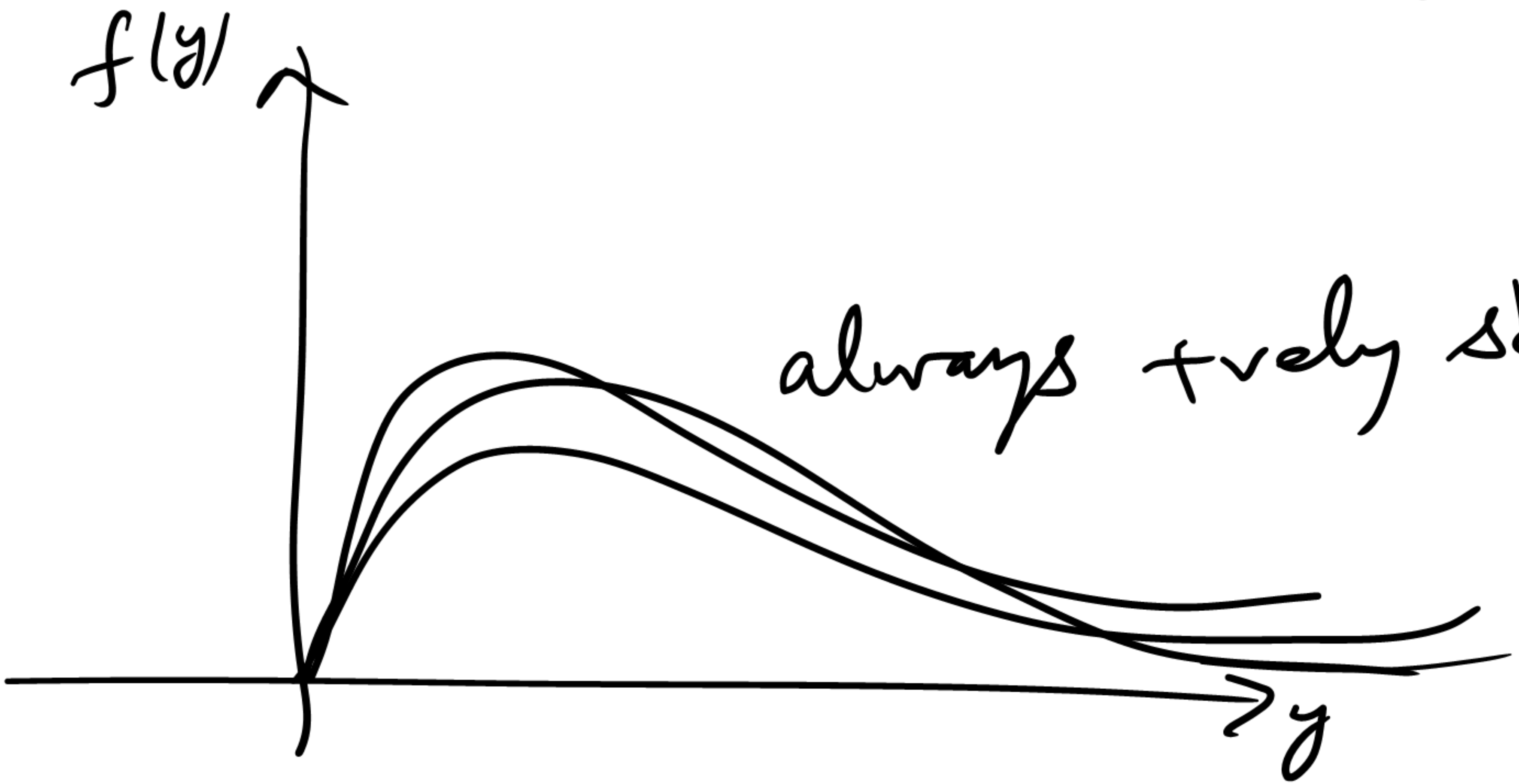
$$f_{Y_r}(y) = -\frac{d}{dy}\left[e^{-\lambda y} + (\lambda y)\, e^{-\lambda y} + \left(\frac{\lambda^2 y^2}{2!}\right) e^{-\lambda y} + \cdots + \frac{\lambda^{r-1} y^{r-1}}{(r-1)!}\, e^{-\lambda y} \right]$$

$$= -\left[-\lambda \cancel{e^{-\lambda y}} + \lambda \cancel{e^{-\lambda y}} - \lambda^2 \cancel{y e^{-\lambda y}} + \lambda^2 \cancel{y e^{-\lambda y}} - \cdots - \frac{\lambda^r y^{r-1}}{(r-1)!}\, e^{-\lambda y} \right]$$

$$= \frac{\lambda^r}{(r-1)!}\, e^{-\lambda y}\, y^{r-1}, \qquad y > 0$$

$$f_{Y_r}(y) = \frac{\lambda^r}{\Gamma(r)} e^{-\lambda y} y^{r-1}, \qquad y > 0$$

$\longrightarrow$ Erlang / Gamma distⁿ.



always +vely skewed

$$\mu_k' = E(Y_r^k) = \int_0^\infty y^k \cdot \frac{\lambda^r e^{-\lambda y} y^{r-1}}{\Gamma r} dy$$

$$= \frac{\lambda^r}{\Gamma r} \int_0^\infty e^{-\lambda y} y^{r+k-1} dy$$

$$= \frac{\lambda^r}{\Gamma r} \cdot \frac{\Gamma r+k}{\lambda^{r+k}} = \frac{\Gamma k+r}{\Gamma r} \cdot \frac{1}{\lambda^k}$$

$$\mu_1' = \frac{r}{\lambda}, \qquad \mu_2' = \frac{r(r+1)}{\lambda^2},$$

$$\mathrm{Var}(Y_r) = \frac{r}{\lambda^2}.$$

Example: CPU time required to execute some programs on a server has gamma distn with mean 40 (seconds) & s.d. 20 (seconds.). What is the prob that a random program will require less than 20 (seconds)?

$X \to$ time for execution of program

$X \sim \text{Gamma}(r, \lambda)$.

$$\frac{r}{\lambda} = 40, \qquad \frac{r}{\lambda^2} = 400$$

$$\Rightarrow r = 4, \quad \lambda = 1/10$$

$$f_X(x) = \left(\frac{1}{10}\right)^4 \cdot \frac{1}{\Gamma 4} e^{-\frac{x}{10}} x^3, \quad x > 0$$

$P(X < 20)$

$$= \int_0^{20} \frac{1}{6\,(10)^4} \cdot e^{-x/10}\, x^3\, dx$$

$$\frac{x}{10} = y, \quad \frac{dx}{10} = dy$$

$$= \int_0^2 \frac{1}{6} e^{-y} y^3\, dy \quad ,$$

$$= 1 - \int_2^\infty \frac{1}{6} e^{-y} y^3\, dy = 1 - \frac{19}{3} e^{-2}$$

$$\approx 0.1429$$

MGF $\quad M_{Y_r}(t) = E\left(e^{tY_r}\right)$

$$= \int_0^\infty e^{ty} \cdot \frac{\lambda^r}{\Gamma r} e^{-\lambda y} y^{r-1}\, dy$$

$$= \int_0^\infty \frac{\lambda^r}{\Gamma r} e^{-y(\lambda-t)} \cdot y^{r-1}\, dy$$

$$= \left(\frac{\lambda}{\lambda-t}\right)^r, \quad t<\lambda$$

Weibull Distribution :

$$f_X(x) = \begin{cases} \alpha\beta\, x^{\beta-1}\, e^{-\alpha x^{\beta}}, & x > 0 \\ & \alpha > 0, \beta > 0 \\ 0, & x \le 0 \end{cases}$$

$$P(X > a) = e^{-\alpha a^{\beta}},$$

$$F_X(x) = 1 - P(X > x) = \begin{cases} 1 - e^{-\alpha x^\beta}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

$$\mu_k' = E(X^k) = \int_0^\infty \alpha\beta\, x^{\beta+k-1}\, e^{-\alpha x^\beta}\, dx$$

$$y = x^\beta, \qquad \beta x^{\beta-1} dx = dy$$

$$= \int_0^\infty \alpha\, y^{k/\beta} e^{-\alpha y}\, dy = \frac{\alpha\, \Gamma\left(\frac{k}{\beta}+1\right)}{\alpha^{k/\beta+1}}$$

$$= \alpha^{-\frac{k}{\beta}}\, \Gamma\left(\frac{k}{\beta}+1\right)$$

$$\mu_1' = \alpha^{-1/\beta}\, \Gamma\left(\frac{1}{\beta}+1\right)$$

$$\mu_2' = \alpha^{-2/\beta}\, \Gamma\left(\frac{2}{\beta}+1\right)$$

$$\operatorname{Var}(X) = \alpha^{-2/\beta}\left[\Gamma\left(\frac{2}{\beta}+1\right) - \left\{\Gamma\left(\frac{1}{\beta}+1\right)\right\}^2\right]$$

$Y \rightarrow$ life of a system

$$R_Y(t) = P(Y > t) = 1 - F_Y(t)$$

= Reliability of the system at time t

We now define the failure rate at

time t of the system/process/equipment

$$\underset{h \to 0}{Lt} \; \frac{1}{h} P(\underbrace{t < Y \leq t+h}_{E} \mid \underbrace{Y > t}_{F})$$

$$= Z(t) \qquad (\text{or } H(t))$$

$$Z(t) = \underset{h \to 0}{Lt} \; \frac{1}{h} \frac{P(t < Y \leq t+h)}{P(Y > t)}$$

$$= \underset{h \to 0}{Lt} \left[\frac{F_Y(t+h) - F_Y(t)}{h} \right] \Big/ R_Y(t)$$

$$= \frac{f_y(t)}{R_y(t)} = \frac{f_y(t)}{1 - F_y(t)}.$$

$$Z(t) = -\frac{d}{dt}\log_e\left(1 - F_y(t)\right)$$

$$R_y(t) = 1 - F_y(t) = c\, e^{-\int Z(t)\,dt}$$

For exponential distn

$$Z(t) = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda,$$ which is independent of time t.

<u>Weibull distn</u>

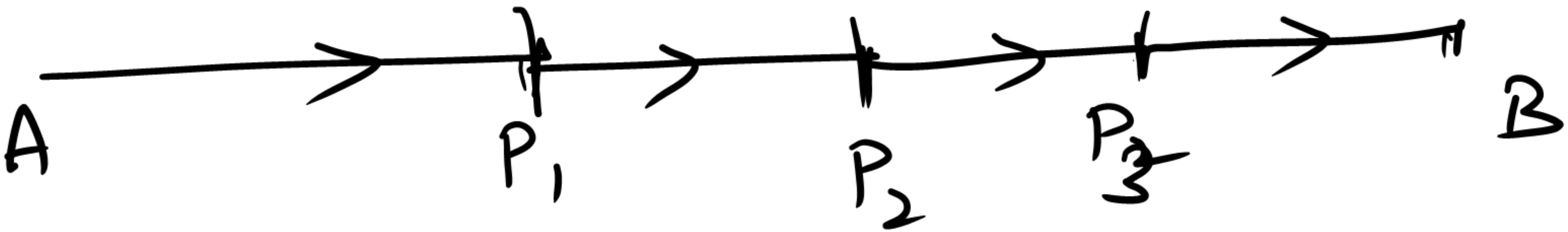
$$Z(t) = \frac{\alpha \beta t^{\beta-1} e^{-\alpha t^{\beta}}}{e^{-\alpha t^{\beta}}}$$

$$= \alpha\beta t^{\beta-1}$$

For $\beta > 1$, increasing failure rate

$\beta < 1$, decreasing failure rate

Reliability of Series Systems



Let a system consist of k independent components connected in a series.

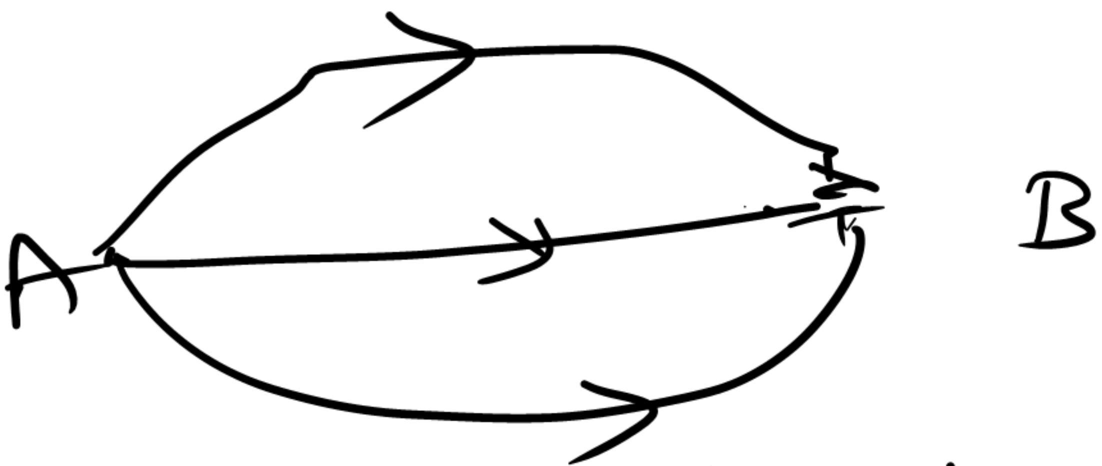
Let Y denote the life of the system and $X_1, X_2, \ldots, X_k$ denote the lives of k component. Suppose each component is functioning independently.

Reliability of system

$$R_X(t) = P(X > t) = P(X_1 > t, \dots, X_k > t)$$

$$= \prod_{i=1}^{k} P(X_i > t) = \prod_{i=1}^{k} R_i(t)$$

Reliability of a Parallel Systems



Suppose a system (with life X)

works if at least one of k independent components (with lives $X_1, \dots, X_k$) are working

$$R_X(t) = P(X > t) = 1 - P(X \le t)$$

$$= 1 - P(X_1 \le t, \dots, X_k \le t)$$

$$= 1 - \prod_{i=1}^{k} P(X_i \le t)$$

$$= 1 - \prod_{i=1}^{k} \left(1 - P(X_i > t)\right)$$

$$= 1 - \prod_{i=1}^{k} \left(1 - R_i(t)\right)$$