Random Variable and Distribution Functions

Lesson 4: Joint Distribution: Discrete Random Variable

Two random variables X and Y are said to be jointly distributed if they are defined on the same probability space.

Let us assume that two random variables X and Y take the values x_1, x_2, x_3 and $y_1, y_2, y_{3,} y_4$ such that there are total 12 pairs of values $(x_i, y_j), i = 1, 2, 3$ and $j = 1, 2, 3, 4$. We make the product set $\{x_1, x_2, x_3\} \times \{y_1, y_2, y_{3,} y_4\}$ into a probability space by defining the probability of the ordered pair (x_i, y_j) to be $P(X = x_i, Y = y_j)$ which we write as $p(x_i, y_j)$. The function p defined on the product set by

$$p_{ij} = P(X = x_i \cap Y = y_j) = p(x_i, y_j)$$

is called the joint probability function of X and Y and is usually represented in the form of the following table:

$X \backslash Y$	y_1	y_2	y_3	y_4	Total
x_1	p_{11}	p_{12}	p_{13}	p_{14}	p_1
x_2	p_{21}	p_{22}	p_{23}	p_{24}	p_2
x_3	p_{31}	p_{32}	p_{33}	p_{34}	p_3
Total	p^1	p^2	p^3	p^4	1

where

$$\sum_i \sum_j p_{ij} = p_{11} + p_{12} + .. + p_{14} + p_{21} + \cdots + p_{24} + p_{31} + \cdots + p_{34} = 1$$

The row totals (p_i) are called marginal probabilities of X and the column totals (p^j) are known as marginal probabilities of Y. From the table, we have

Sum of the marginal probabilities (of X or of Y) = $\sum_i p_i = \sum_j p^j = 1$

Marginal Probabilities:

(i) For X:

X	x_1	x_2	x_3	Total
$p(x)$ (marginal)	p_1	p_2	p_3	1

(i) For Y:

Y	y_1	y_2	y_3	y_4	Total
$p(y)$ (marginal)	p^1	p^2	p^3	p^4	1

Joint Probability Distribution Function:

Let (X, Y) be a two dimensional random variable. Then their joint distribution function is denoted by $F(x, y)$ and is defined by

$$F(x, y) = P(X \leq x, Y \leq y)$$

Problems:

Ex.1. The following table gives the joint distribution of X and Y:

X \ Y	2	3	**7**
1	0.1	0.25	0.05
3	0.3	0.15	0.15

(i) Write the marginal distribution of X and Y

(ii) Find the probabilities $P(X < Y)$, $P(2X + Y \geq 9)$

Solution:

(i) Marginal distribution of X:

X	1	3
$p(x)$	0.1 + 0.25 + 0.05 = 0.4	0.3 + 0.15 + 0.15 = 0.6

Marginal distribution of Y:

Y	2	3	7
$p(y)$	0.4	0.4	0.2

(ii) $P(X < Y) = P(X = 1, Y = 2) + P(X = 1, Y = 3) + P(X = 1, Y = 7)$

$+ P(X = 3, Y = 7)$

= 0.1 + 0.25 + 0.05 + 0.15 = 0.55.

Also, $P(2X + Y \geq 9) = P(X = 1, Y = 7) + P(X = 3, Y = 3) + P(X = 3, Y = 7)$

= 0.05 + 0.15 + 0.15 = 0.35.

Ans.

Ex.2. The following table gives the joint distribution of X and Y:

X \ Y	0	1	2
1	0.3	0.2	0.1
2	0.1	0	0.3

(i) Are X and Y independent?

(ii) Determine the correlation coefficient between X and Y.

Solution: The joint probability distribution is given by

X \ Y	0	1	2	Total
1	0.3	0.2	0.1	0.6
2	0.1	0	0.3	0.4
Total	0.4	0.2	0.4	1

(i) The two random variables X and Y will be independent when

$$E(XY) = E(X)E(Y)$$

From the table we calculate the following expected values

$$E(X) = \sum_x xp(x) = 1 \times 0.6 + 2 \times 0.4 = 1.4 --- (1)$$

$$E(Y) = 0 \times 0.4 + 1 \times 0.2 + 2 \times 0.4 = 1 --- (2)$$

Also

$$E(XY) = \sum_i \sum_j p_{ij}(x_i \times y_j)$$

$$= 0.3(1 \times 0) + 0.2(1 \times 1) + 0.1(1 \times 2) + 0.1(2 \times 0) + 0(2 \times 1) + 0.3(2 \times 2)$$

= 1.6 ------- (3)

From equations (1), (2) and (3), it is clear that

$$E(XY) = 1.6 \neq 1.4 = E(X)E(Y)$$

Hence the two random variables X and Y are not independent.

(ii) The correlation coefficient between X and Y is given by

$$r = \frac{\mathrm{Cov}(X, Y)}{\sigma_x \times \sigma_y} --- (4)$$

where

$$\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y) = 1.6 - 1.4 = 0.2$$

We also have

$$E(X^2) = 1^2 \times 0.6 + 2^2 \times 0.4 = 2.2$$

$$E(Y^2) = 0^2 \times 0.4 + 1^2 \times 0.2 + 2^2 \times 0.4 = 1.8$$

Then

$$\mathrm{Var}(X) = E(X^2) - \{E(X)\}^2 = 2.2 - 1.4^2 = 0.24 \rightarrow \sigma_x = 0.489$$

$$\mathrm{Var}(Y) = E(Y^2) - \{E(Y)\}^2 = 1.8 - 1^2 = 0.8 \rightarrow \sigma_y = 0.894$$

Therefore from equation (4), we get

$$r = \frac{0.2}{0.489 \times 0.894} = 0.457$$

Ans.

Ex.3. The marginal probabilities of X and Y are given in the following table:

Y \ X	5	7	Total
3	-	-	1/3
6	-	-	2/3
Total	1/2	1/2	1

If $\mathrm{Cov}(X, Y) = -\frac{1}{2}$, obtain the cell probabilities.

Solution: Let us re-write the table by taking the probability $P(X = 5, Y = 3) = p$

Y \ X	5	7	Total
3	p	$\frac{1}{3} - p$	$\frac{1}{3}$
6	$\frac{1}{2} - p$	$\frac{1}{6} + p$	$\frac{2}{3}$
Total	$\frac{1}{2}$	$\frac{1}{2}$	1

We have $E(X) = \frac{1}{2} \times 5 + \frac{1}{2} \times 7 = 6$ and $E(Y) = \frac{1}{3} \times 3 + \frac{2}{3} \times 6 = 5$

and $E(XY) = p(3 \times 5) + \left(\frac{1}{3} - p\right)(3 \times 7) + \left(\frac{1}{2} - p\right)(6 \times 5) + \left(\frac{1}{6} + p\right)(6 \times 7)$

$= 6p + 29$

Now $\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = 6p + 29 - 30 = 6p - 1$ which is given as $-\frac{1}{2}$, i.e,

$$6p - 1 = -\frac{1}{2} \to p = \frac{1}{12}$$

Hence the other cell probabilities are

$$\frac{1}{2} - p = \frac{1}{2} - \frac{1}{12} = \frac{5}{12}$$

$$\frac{1}{3} - p = \frac{1}{3} - \frac{1}{12} = \frac{1}{4}$$

$$\frac{1}{6} + p = \frac{1}{6} + \frac{1}{12} = \frac{1}{4}$$

With the above cell probabilities, the completed joint distribution becomes

$Y \backslash X$	5	7	Total
3	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{3}$
6	$\frac{5}{12}$	$\frac{1}{4}$	$\frac{2}{3}$
Total	$\frac{1}{2}$	$\frac{1}{2}$	1

Ans.