Poisson Process

$X(t) \to$ number of occurrences in the time interval of length t

$$P(X(t)=n) = P_n(t),$$

$$P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n=0,1,2,\ldots$$

<u>Example</u>: Suppose students join the class between 11:58 to 12:02 at the rate of 60 per minute.

What is the probability that

(i) no student joined in first 15 seconds?

(ii) Not more 30 students join in first minute (11:58 - 11:59)?

(iii) No. of students joining between 12:00 to 12:01 is between 60 to 90?

Sol^n: (i) $\lambda t = 60 \times \frac{1}{4} = 15$

$$P(X(1/4)=0) = e^{-15} \approx 3.06 \times 10^{-7}$$

$$P(X(1) \le 30) = \sum_{k=0}^{30} \frac{e^{-60}(60)^k}{k!}$$

(iii) $$P(60 < X(1) < 90) = \sum_{k=61}^{89} \frac{e^{-60}(60)^k}{k!}$$

Ex. Suppose customers arrive in a car showroom at the rate of 5 per minute. What is the prob that

(i) no customer came in a 1 minute period?

(ii) 2 customers in a 3 minutes period?

Sol^n (i) $\lambda = 5$, time $\to$ minutes.

$$P(X(1) = 0) = e^{-5} \approx 0.67 \times 10^{-2}$$

(ii) $P(X(3)=2) = \dfrac{e^{-15}(15)^2}{2!}$

$\approx 3.44 \times 10^{-5}$

Traditionally the distribution of the number of occurences in a Poisson process is referred to as a Poisson distribution and it was initially derived as a limiting distⁿ to a binomial distⁿ under certain conditions.

Theorem: Let X be a random variable with $Bin(n, p)$ distn.
As $n \to \infty$, $p \to 0$ such that $np \to \lambda$ (a real constant)
the pmf of X converges to

$$e^{-\lambda} \lambda^{x} / x!$$

Proof: The pmf of X is

$$p_X(x) = P(X=x) = \binom{n}{x} p^x (1-p)^{n-x},$$

$$x = 0, 1, 2 \cdots$$

$$= \frac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x}$$

$$= \frac{n!}{x!\,(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x} \quad \left\{ \begin{array}{l} np \approx \lambda \\ p \approx \frac{\lambda}{n} \end{array} \right.$$

$$= \frac{n(n-1)\cdots(n-x+1)}{n^x} \cdot \frac{\lambda^x}{x!}\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n}{n} \cdot \left(\frac{n-1}{n}\right) \cdots \left(\frac{n-x+1}{n}\right) \frac{\lambda^x}{x!}\left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$\rightarrow 1 \cdot \frac{\lambda^x}{x!} e^{-\lambda} \cdot 1 = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$x = 0, 1, 2, \cdots$$

This is called a Poisson Dist^n.

$$p_X(x) = P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2 \ldots\ldots$$

$$\sum_{x=0}^{\infty} p_X(x) = \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \boxed{\sum_{x=0}^{\infty} \frac{\lambda^x}{x!}}$$

$$= e^{-\lambda} e^{\lambda} = 1.$$

$$\mu_1' = E(X) = \sum_{x=\cancel{0}1}^{\infty} x \cdot \frac{e^{-\lambda} \lambda^{x}}{x!}$$

$$= e^{-\lambda} \lambda \boxed{\sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}} = e^{-\lambda} \lambda e^{\lambda}$$

$$= \lambda$$

$$\mu_2' = E(X^2) = E\, X(X-1) + E(X)$$

$$= \lambda^2 + \lambda$$

$$\mu_2 = \mathrm{Var}(X) = \mu_2' - \mu_1'^2 = \lambda$$

So in a Poisson dist^n mean and variance are the same.

$$s.d.(X) = \lambda^{1/2}.$$

$$\mu_3' = E(X^3) = E\,X(X-1)(X-2) + 3E\,X(X-1) + E(X)$$

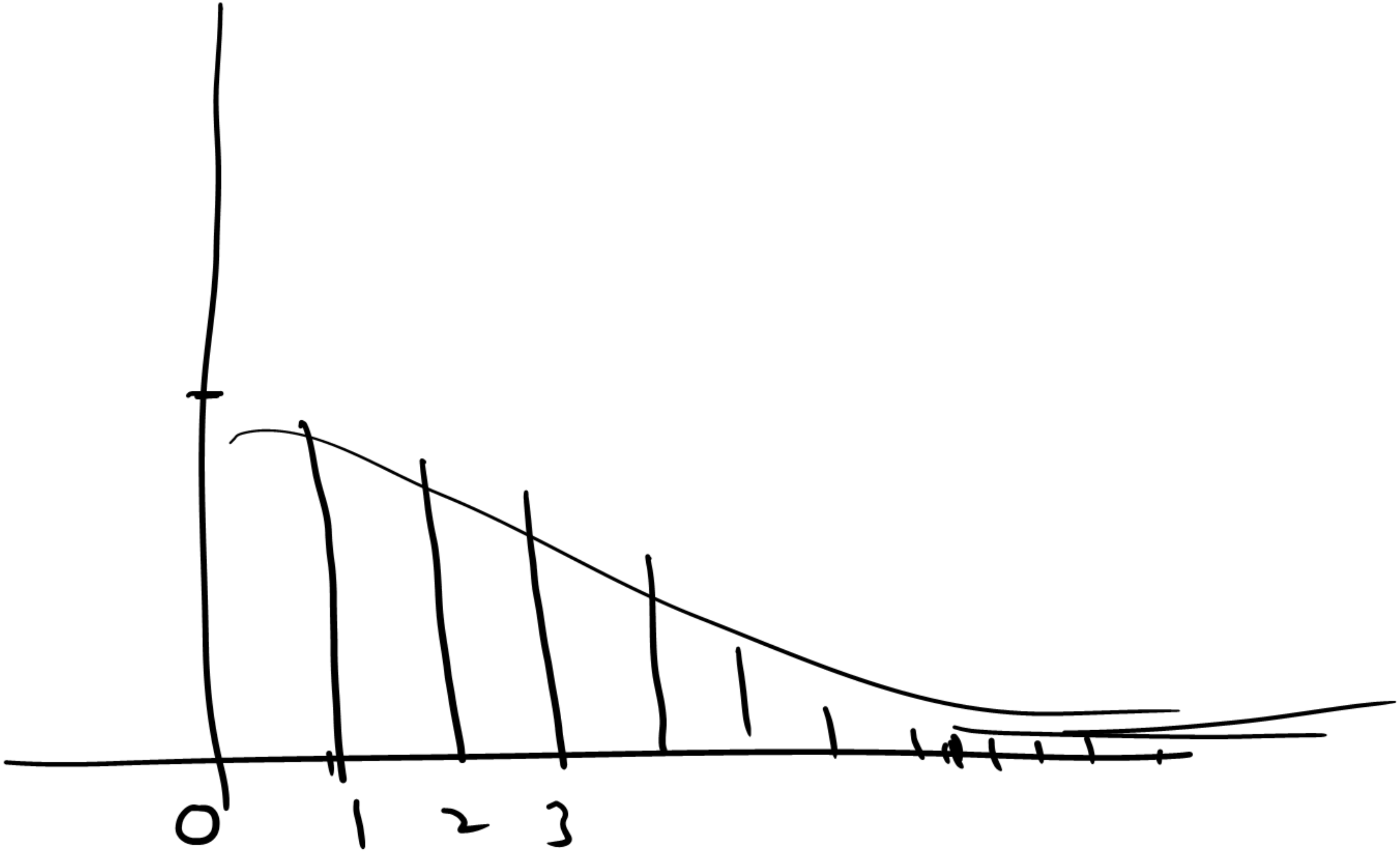
$$= \lambda^3 + 3\lambda^2 + \lambda$$

$$\mu_3 = \lambda \quad *$$

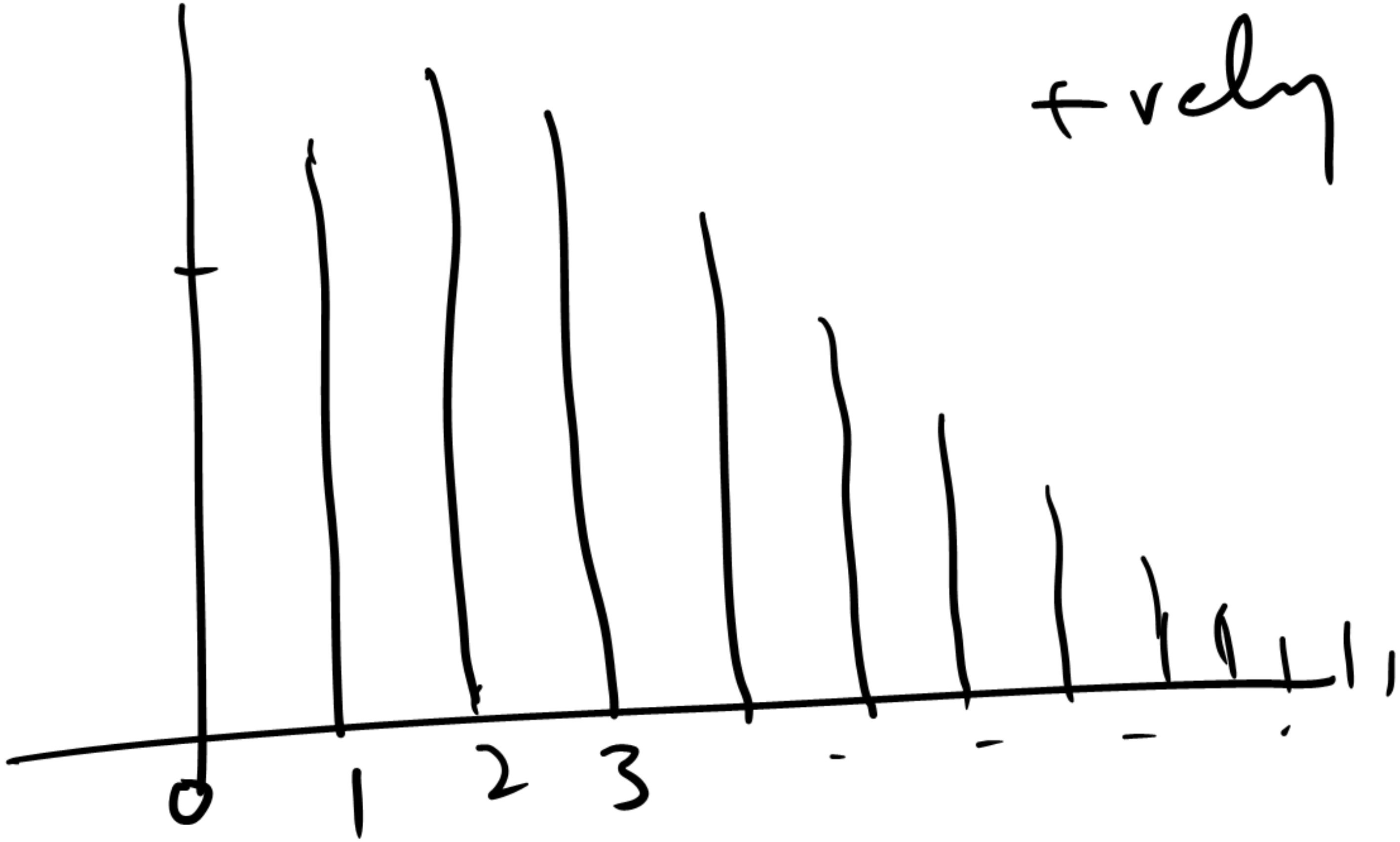
$$\mu_4' = E(X^4) = E\{X(X-1)(X-2)(X-3)\}$$
$$+ 6E\{X(X-1)(X-2)\}$$
$$+ 7E\{X(X-1)\} + E(X)$$
$$= \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$$
$$\mu_4 = \lambda + 3\lambda^2 \qquad \cancel{\alpha}$$

Measure of Skewness $\beta_1 = \dfrac{\mu_3}{\sigma^3} = \dfrac{\lambda}{\lambda^{3/2}}$

$= \dfrac{1}{\sqrt{\lambda}} > 0$ +vely skewed



$\lambda < 1$



+vely skewed

$\lambda > 1$

Measure of Kurtosis $\beta_2 = \dfrac{\mu_4}{\mu_2^2} - 3$

$$= \frac{\lambda + 3\lambda^2}{\lambda^2} - 3 = \frac{1}{\lambda} > 0 \text{ leptokurtic}$$

$$M_X(t) = E\left(e^{tX}\right)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\left(\lambda e^t\right)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}.$$

We can use mgf approach for proving the limiting result of binomial to Poisson.

MGF of a Bin (n, p) distn

$$M_X(t) = (q + pe^t)^n$$

$np \simeq \lambda$

$p = \lambda/n$

$$= (1-p+pe^t)^n$$

$$\cong \left[1+\frac{\lambda}{n}(e^t-1)\right]^n$$

$$\to e^{\lambda(e^t-1)}$$

which is mgf of $P(\lambda)$ distn.

Ex. Suppose the prob of surviving

a serious disease is 0.05.
What is the prob that out 100 patients of this disease less than 10 survive?

$X \to$ no of survivors

$X \sim \text{Bin}(100, 0.05)$

$n = 100, \quad p = 0.05, \quad np = 5 = \lambda$

$$P(X<10) = \sum_{x=0}^{9} \binom{100}{x} (0.05)^{x} (0.95)^{100-x}$$

$$\approx \boxed{\sum_{x=0}^{9} \frac{e^{-5} (5)^{x}}{x!}}$$

Special Continuous Distributions

1. Continuous Uniform Dist^n

$$f_X(x) = \begin{cases} k, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

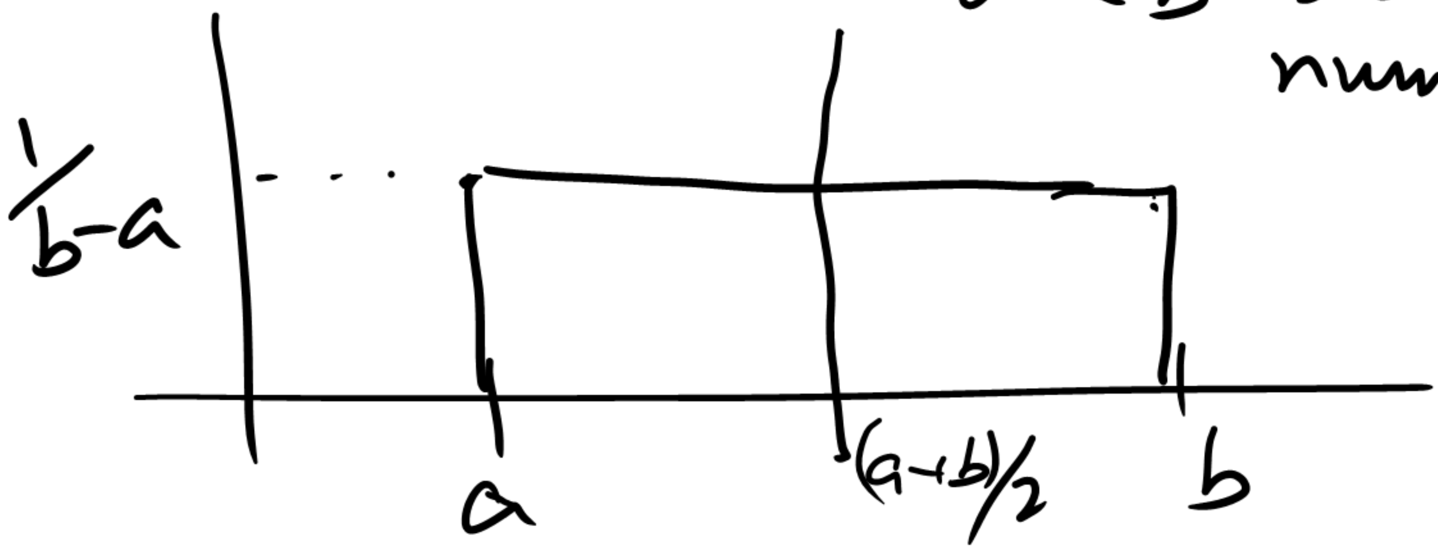
$$k \int_a^b dx = 1 \Rightarrow (b-a)k = 1$$

$$\Rightarrow k = \frac{1}{b-a}$$

So the pdf of a $U(a,b)$ is

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

$a < b$ real numbers



Also called a rectangular distn.

$$\mu_k' = E(X^k) = \int_a^b \frac{x^k}{b-a}\,dx$$

$$= \frac{b^{k+1} - a^{k+1}}{(k+1)(b-a)}, \quad k = 1, 2 \ldots$$

$$\mu_1' = E(X) = \frac{a+b}{2},$$

$$\mu_2' = E(X^2) = (a^2 + b^2 + ab)/3$$

$$\mu_2' = Var(X) = \frac{(b-a)^2}{12} \quad (= \sigma^2)$$

$$s.d.(X) = \sigma = \frac{b-a}{2\sqrt{3}}$$

$$M_X(t) = \int_a^b \frac{e^{tx}}{b-a}\,dx$$

$$= \begin{cases} \dfrac{e^{bt} - e^{at}}{t(b-a)}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

$$F_X(x) = \begin{cases} 0, & x \leq a \\ \dfrac{x-a}{b-a}, & a < x < b \\ 1, & x \geq b \end{cases}$$