Basic Statistics

Lesson 4: Moments: Mean and Variance

Moments:

The r-th moment about any arbitrary point x = A is given by

$$\mu_r' = \frac{1}{n}\sum_{i=1}^{n}(x_i - A)^r -- \text{ [ungrouped data]}$$

$$\mu_r' = \frac{1}{N}\sum_{i=1}^{n}f_i(x_i - A)^r -- \text{ [frequency distribution]}$$

Note:

(i) In particular, if A = 0, then we get the r-th moment about the origin as

$$\mu_r' = \frac{1}{n}\sum_{i=1}^{n}{x_i}^r -- \text{ [ungrouped data]}$$

$$\mu_r' = \frac{1}{N}\sum_{i=1}^{n}f_i{x_i}^r -- \text{ [frequency distribution]}$$

Substituting r = 1, we get the 1^{st} moment about the origin as

$$\mu_1' = \frac{1}{n}\sum_{i=1}^{n}x_i = \bar{x} -- \text{ [ungrouped data]}$$

$$\mu_1' = \frac{1}{N}\sum_{i=1}^{n}f_ix_i = \bar{x} -- \text{ [frequency distribution]}$$

These moments are known as the Raw moments. Thus, the first raw moment is the mean of the distribution.

(ii) In particular, if A = $\bar{x}$, then we get the r-th moment about the A.M $\bar{x}$ of the distribution as μ_r and defined by

$$\mu_r = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^r -- \text{ [ungrouped data]}$$

$$\mu_r = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^r -- \text{ [frequency distribution]}$$

- Substituting r = 1, we get the 1st moment about the mean as

$$\mu_1 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x}) = \bar{x} - \bar{x} = 0 -- \text{ [ungrouped data]}$$

$$\mu_1 = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x}) = \bar{x} - \bar{x}\frac{1}{N}\sum_{i=1}^{n} f_i = 0 -- \text{ [frequency distribution]}$$

- Substituting r = 2, we get the 2nd moment about the mean as

$$\mu_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sigma^2 = \text{variance} -- \text{ [ungrouped data]}$$

$$\mu_2 = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2 = \sigma^2 = \text{variance} -- \text{ [frequency distribution]}$$

These moments are known as the Central moments. Thus the 1st central moment is always zero and the 2nd central moment is the variance of the distribution. The 3rd and 4th central moments are used to measure the Skewness and the Kurtosis of the distribution respectively.

(iii) We have already seen that the alternative expression of σ^2 is

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} {x_i}^2 - \bar{x}^2$$

Hence the central and the raw moments are related by

$$\mu_2 = \frac{1}{n}\sum_{i=1}^{n} {x_i}^2 - \bar{x}^2 = {\mu_2}' - {\mu_1}'^2$$

Similarly, one can proceed to obtain the 3rd , 4th , …… central moments about the mean as

$$\mu_3 = {\mu_3}' - 3{\mu_2}'{\mu_1}' + 2{\mu_1}'^3$$
$$\mu_4 = {\mu_4}' - 4{\mu_3}'{\mu_1}' + 6{\mu_2}'{\mu_1}'^2 - 3{\mu_1}'^4$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Problems:

Ex.1. The first two moments of a distribution about the value 5 are 2 and 20. Find mean and variance of the distribution.

Solution: We have ${\mu_1}' = 2$ and ${\mu_2}' = 20$ around the point A = 5. Thus we have

$${\mu_1}' = \frac{1}{n}\sum_{i=1}^{n}(x_i - A)$$

$$\text{or,} \qquad 2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - 5)$$

$$\text{or,} \qquad 2n = \sum_{i=1}^{n} x_i - 5n \;\rightarrow\; \sum_{i=1}^{n} x_i = 7n - - - -(1)$$

Hence mean of the distribution will be

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{7n}{n} = 7$$

Similarly, we can write

$${\mu_2}' = \frac{1}{n}\sum_{i=1}^{n}(x_i - A)^2$$

$$\text{or,} \qquad 20 = \frac{1}{n}\sum_{i=1}^{n}(x_i - 5)^2$$

$$\text{or,} \qquad 20n = \sum_{i=1}^{n} {x_i}^2 - 10\sum_{i=1}^{n} x_i + 25n$$

$$\text{or,} -5n = \sum_{i=1}^{n} {x_i}^2 - 10 \times 7n \quad [\text{from eqn}.(1)]$$

$$\text{or,} \qquad \sum_{i=1}^{n} {x_i}^2 = 65n$$

Hence the variance of the distribution will be

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} {x_i}^2 - \bar{x}^2 = 65 - 49 = 16$$

Ans.

Ex.2. The first two moments about the value 3 are 2 and 10. Find the first two moments about the origin and about the mean.

Solution: Here we have $\mu_1{}' = 2$ and $\mu_2{}' = 10$ around the point A = 3. Thus we have

$$2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - 3) \rightarrow \sum_{i=1}^{n} x_i = 5n ---- (1)$$

and

$$10 = \frac{1}{n}\sum_{i=1}^{n}(x_i - 3)^2 \rightarrow \sum_{i=1}^{n} {x_i}^2 = 31n ---- (2)$$

(i) We need to find the first two moments about the origin (i.e, when A = 0) given by

$$\mu_1{}' = \frac{1}{n}\sum_{i=1}^{n} x_i \quad \text{and} \quad \mu_2{}' = \frac{1}{n}\sum_{i=1}^{n} {x_i}^2 ---- (3)$$

From equation (1) and the first expression of equation (3), we get the 1st moment about the origin as

$$\mu_1' = \bar{x} = 5 \quad ---(4)$$

Also, from equation (2) and the second expression of equation (3), we get the 2nd moment about the origin as $\mu_2' = 31$.

(ii) We need to find the first two moments about the mean (i.e, when A = $\bar{x}$) given by

$$\mu_1 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x}) \text{ and } \mu_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 ----(5)$$

We know that μ_1 is always zero. Now, the 2nd central moment from equation (5) can be written as

$$\mu_2 = \frac{1}{n}\left[\sum_{i=1}^{n} {x_i}^2 - 2\bar{x}\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \bar{x}^2\right]$$

$$= \frac{1}{n}[31n - 2 \times 5 \times 5n + 5^2 n] - [\text{using eqns.}(1), (2) \text{ and } (4)]$$

$$= \frac{6n}{n} = 6$$

Ans.