Let $(\Omega, \mathcal{B}, P)$ be a probability space.

Properties of the Probability Function

1. $P(\phi) = 0$

Pf. In Axiom P_3, take $E_1 = \Omega$

$E_i = \phi$, $i = 2, 3, \ldots\ldots$. Then P_3 gives

$$P(\Omega) = P(\Omega) + P(\phi) + P(\phi) + \cdots\cdots$$

$\Rightarrow \quad 1 = 1 + P(\phi) + P(\phi) + \cdots$

$\Rightarrow \quad P(\phi) = 0$

2. P is a finitely additive function.

<u>Pf</u>. In P_3, take $E_i = \phi$ for $i = n+1, n+2, \ldots$

Then $P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i) + P(\phi) + P(\phi) + \cdots$

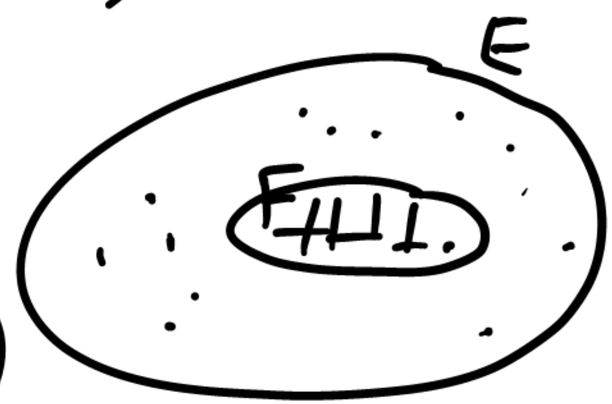
Since $P(\phi) = 0$, we get

$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$ where events E_i's are pairwise disjoint.

3. P is a monotonic increasing function

i.e. if $F \subset E$, then $P(F) \le P(E)$

& $P(E-F) = P(E) - P(F)$



Pf. $E = F \cup (E-F)$

So $P(E) = P(F) + P(E-F)$

$\Rightarrow 0 \le P(E-F) = P(E) - P(F)$

$\Rightarrow P(F) \le P(E)$

4. So for any $E \in \mathcal{B}$, $0 \leq P(E) \leq 1$.

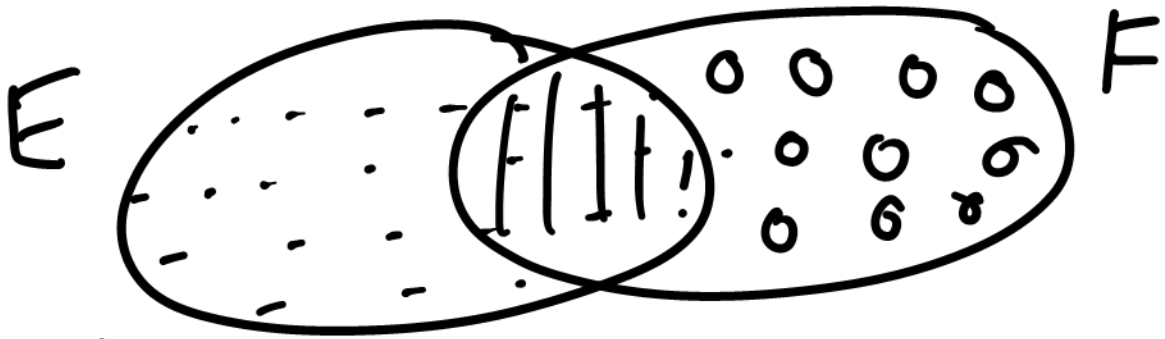
5. $P(E^c) = 1 - P(E)$

Pf. $E \cup E^c = \Omega \Rightarrow P(E) + P(E^c) = 1$

Addition Rule: Let $E, F \in \mathcal{B}$, then

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Pf.



We can write

$$E \cup F = E \cup (F - E \cap F)$$

So

$$P(E \cup F) = P(E) + P(F - (E \cap F))$$

$$= P(E) + P(F) - P(E \cap F)$$

For three events E, F & G,

$$P(E \cup F \cup G) = P(E) + P(F) + P(G)$$

$$- P(E \cap F) - P(F \cap G) - P(G \cap E)$$

$$+ P(E \cap F \cap G)$$

General Addition Rule: Let $E_1, \dots, E_n \in \mathcal{B}$

Then $P\left(\bigcup_{i=1}^{n} E_i\right) = S_1 - S_2 + S_3 + \cdots$

$$+ (-1)^{n+1} S_n$$

where $S_1 = \sum_{i=1}^{n} P(E_i)$,

$$S_2 = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i<j}}^{n} P(E_i \cap E_j)$$

$$S_3 = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{\substack{k=1 \\ i<j<k}}^{n} P(E_i \cap E_j \cap E_k)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$S_n = \quad P\left(\bigcap_{i=1}^{n} E_i\right)$$

<u>Pf.</u> We will use Principle of Mathematical Induction.

For $n=1$, the statement is always true.

For $n=2$, it is addition rule.

Assume the result to be true for $n=k$.

Now we want to prove for $n=k+1$.

$$P\left(\bigcup_{i=1}^{k+1} E_i\right) = P\left(\underbrace{\left(\bigcup_{i=1}^{k} E_i\right)} \cup \underbrace{E_{k+1}}\right)$$

$$= P\Big(\underbrace{\bigcup_{i=1}^{k} E_i}\Big) + P(E_{k+1}) - P\Big(\Big(\bigcup_{i=1}^{k} E_i\Big) \cap E_{k+1}\Big)$$

$$= \sum_{1}^{k} P(E_i) - \sum_{\substack{1 \\ i<j}}^{k} \sum_{1}^{k} P(E_i \cap E_j) + \cdots + (-1)^{k+1} P\Big(\bigcap_{1}^{k} E_i\Big)$$

$$+ P(E_{k+1}) - P\Big(\bigcup_{1}^{k} \big(\underbrace{E_i \cap E_{k+1}}\big)\Big)$$

$$= \sum_{1}^{k+1} P(E_i) - \sum_{\substack{1 \\ i<j}}^{k} \sum_{1}^{k} P(E_i \cap E_j) + \cdots + (-1)^{k+1} P\Big(\bigcap_{1}^{k} E_i\Big)$$

$$- \Big[\sum_{1}^{k} P(E_i \cap E_{k+1}) - \sum_{\substack{1 \\ i<j}}^{k} \sum_{1}^{k} P(E_i \cap E_j \cap E_{k+1}) + \cdots$$

$$+ (-1)^{k+1} P\left(\bigcap_{1}^{k+1} E_i\right) \Big]$$

$$= \sum_{1}^{k+1} P(E_i) - \sum_{1}^{k+1} \sum_{\substack{1 \\ i<j}}^{k+1} P(E_i \cap E_j)$$

$$+ \sum_{1}^{k+1} \sum_{1}^{k+1} \sum_{\substack{1 \\ i<j<m}}^{k+1} P(E_i \cap E_j \cap E_m) -$$

$$\cdots\cdots + (-1)^{k+2} P\left(\bigcap_{i=1}^{k+1} E_i\right)$$

So the statement is true for $n = k+1$.

Subadditivity of Probability Function

Let $E_1, \ldots, E_n \in \mathcal{B}$, then

$$P\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} P(E_i)$$

Let $E_1, E_2, \ldots \in \mathcal{B}$, then

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} P(E_i)$$

Bonferroni's Inequality: For any events $E_1, E_2, \ldots, E_n \in \mathcal{B}$,

$$\sum_{i=1}^{n} P(E_i) - \sum_{\substack{1 \\ i<j}}^{n} \sum_{1}^{n} P(E_i \cap E_j) \leq P\left(\bigcup_{i=1}^{n} E_i\right)$$

$$\leq \sum_{i=1}^{n} P(E_i)$$

Boole's Inequality: Let $E_1, E_2, \ldots \in \mathcal{B}$

then $$P\left(\bigcap_{i=1}^{\infty} E_i\right) \geq 1 - \sum_{i=1}^{\infty} P(E_i^c)$$

(*) Prove all the above results.

Some Problems:

1. Suppose we have 7 balls to be placed

randomly in 7 cells. Find the probability that exactly one cell remains empty (identifiable)

Sol^n: One empty cell can be selected in 7 ways. Now one cell will have 2 balls. This cell can be selected in 6 ways from remaining cells. Two balls can be selected in $\binom{7}{2}$ ways. Remaining 5 balls can be placed in 5 cells (one in each cell) in $5!$ ways.

So the reqd prob. $= \dfrac{7 \times 6 \times \binom{7}{2} \times 5!}{7^7}$

$$= \frac{2160}{16807} = 0.1285$$

2. In a certain univ. 50% of faculty own a desktop computer, 25% own a laptop and 10% own both a desktop & a laptop. What is the prob that a randomly selected faculty will have a desktop or laptop but

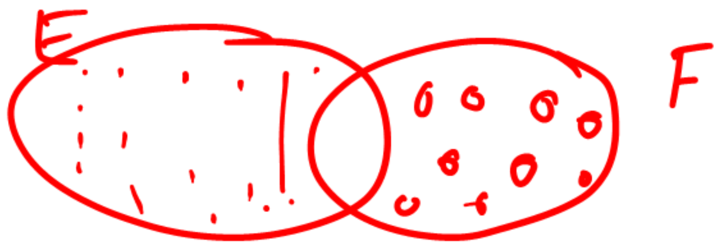
not both?

Sol^n E → faculty has a desktop

F → . . . . . . a laptop

$P(E) = 0.5$, $P(F) = 0.25$, $P(E \cap F) = 0.1$

Reqd Prob.



$= P(E-F) + P(F-E)$

$= P(E) - P(E \cap F) + P(F) - P(E \cap F)$

$= 0.5 + 0.25 - 0.2 = 0.55$

3. A box contains n balls marked 1 to n.

Two balls are drawn in succession with replacement. Find the prob that numbers on balls are consecutive (ignore the order).

Soln. If first number is 1 or n, then there is only one option for the second number. For numbers between 2 to $n-1$ there are 2 options each. So the total number of possibilities is $(n-2)\times 2 + 2\times 1$

$$= 2(n-1)$$

So the reqd prob = $\dfrac{2(n-1)}{n^2}$

<u>Conditional Probability</u>: Let $(\Omega, \mathcal{B}, P)$ be a probability space and $F \in \mathcal{B}$, with $P(F) > 0$. We define the conditional probability of an event E given F as

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

Multiplication Rule : Let $P(F) > 0$

$$P(E \cap F) = P(E \mid F)\, P(F)$$

General Multiplication Rule : Let $E_1, \ldots, E_n \in \mathcal{B}$, with $P\left(\bigcap_{i=1}^{n} E_i\right) > 0$. Then

$$P\left(\bigcap_{i=1}^{n} E_i\right) = P(E_1)\, P(E_2 \mid E_1)\, P(E_3 \mid E_1 \cap E_2) \cdots \cdots P\left(E_n \,\middle|\, \bigcap_{i=1}^{n-1} E_i\right)$$

* Prove the above result

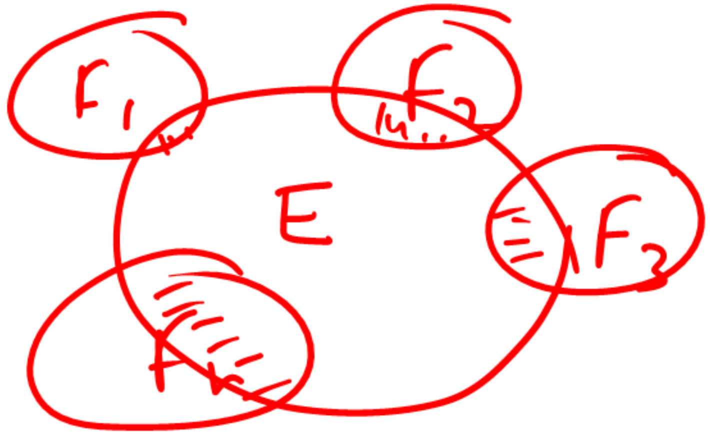
Theorem of Total Probability: Let $F_1, F_2, \ldots, F_n$ be pairwise disjoint events and $F = \bigcup_{i=1}^{n} F_i$. Let $P(F_i) > 0$, $i = 1 \ldots n$. Then for any event E,

$$P(E \cap F) = \sum_{i=1}^{n} P(E \mid F_i) P(F_i)$$

Pf. $P(E \cap F) = P\left(E \cap \left(\bigcup_{i=1}^{n} F_i\right)\right)$

$$= P\left(\bigcup_{i=1}^{n} (E \cap F_i)\right) = \sum_{i=1}^{n} P(E \cap F_i)$$

$$= \sum_{i=1}^{n} P(E|F_i)\,P(F_i)$$



<u>Special Case</u>: If $F = \Omega$, then the above theorem can be written as:

Let $F_1, \ldots, F_n$ be exhaustive events and pairwise disjoint, then

$$P(E) = \sum_{i=1}^{n} P(E|F_i)\,P(F_i)$$

Example : Chips are produced by 3 manufacturing industries F_1, F_2, F_3. It is known that 1% of product of F_1, 5% from F_2 & 10% from F_3 is defective.

A factory assembling systems using these chips procures 40% from F_1 & 30% from F_2 & 30% from F_3. What is the prob that a randomly selected chip in the factory assembled product is defective?

Let $D \rightarrow$ chip is defective

$$P(D) = P(D|F_1)P(F_1) + P(D|F_2)P(F_2) + P(D|F_3)P(F_3)$$

$$= 0.01 \times 0.4 + 0.05 \times 0.3 + 0.1 \times 0.3$$

$$= 0.049$$

Bayes Theorem: [Thomas Bayes (1763)]

Let $F_1, F_2, \ldots, F_n$ be pairwise disjoint and exhaustive events with $P(F_i) > 0$, $i = 1 \ldots n$.

Let $E \in \mathcal{B}$, with $P(E) > 0$. Then

$$P(F_i \mid E) = \frac{P(E \mid F_i)\, P(F_i)}{\sum_{j=1}^{k} P(E \mid F_j)\, P(F_j)}$$

prior prob.

posterior prob.

<u>Pf.</u> $$P(F_i \mid E) = \frac{P(E \cap F_i)}{P(E)}$$

$$= \frac{P(E|F_i)\,P(F_i)}{\sum_{j=1}^{n} P(E|F_j)\,P(F_j)} .$$

Ex. Suppose a randomly selected chip is found to be defective, what is the prob that it was supplied by F_i, $i = 1, 2, 3$.

Solⁿ: $$P(F_1|D) = \frac{P(D|F_1)\,P(F_1)}{P(D)} = \frac{0.01 \times 0.4}{0.049}$$

$$= \frac{4}{49} = 0.08$$

$$P(F_2 | D) = \frac{0.05 \times 0.3}{0.049} = \frac{15}{49} \simeq 0.31$$

$$P(F_3 | D) = \frac{0.1 \times 0.3}{0.049} = \frac{30}{49} \simeq 0.61$$

Independence of Events :

Suppose occurrence of event F does not have any effect on occurrence of another event E. That is,

$$P(E \mid F) = P(E)$$

ie $\dfrac{P(E \cap F)}{P(F)} = P(E)$ or

$$P(E \cap F) = P(E)\,P(F) \quad (*)$$

So we call events E and F to be independent if $P(E \cap F) = P(E)P(F)$.

Example: Tossing of Two fair dice

$E \to$ even no on first dice

$F \to$ odd no on second dice

$$P(E) = \frac{18}{36} = \frac{1}{2}, \quad P(F) = \frac{18}{36} = \frac{1}{2}$$

$$P(E \cap F) = \frac{9}{36} = \frac{1}{4}.$$

So E & F are independent.

For independence of 3 events $E, F, G,$

These are four conditions :

$P(E \cap F) = P(E)P(F)$, $P(F \cap G) = P(F)P(G)$

$P(G \cap E) = P(G)P(E)$, $P(E \cap F \cap G) = P(E)P(F)P(G)$.

Generalizing this concept, if events $E_1, E_2, \ldots, E_n$ are independent, we have

$(2^n - n - 1)$ conditions :

$P(E_i \cap E_j) = P(E_i)P(E_j) \quad \forall\, i<j,$ $\binom{n}{2}$

$P(E_i \cap E_j \cap E_k) = P(E_i)P(E_j)P(E_k) \quad \forall\, i<j<k$ $\binom{n}{3}$

$$P\left(\bigcap_{i=1}^{n} E_i\right) = \prod_{i=1}^{n} P(E_i) \quad , \quad \binom{n}{n}$$

$$\binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n} = 2^n - n - 1$$

<u>Ex.</u> Let two fair dice are tossed

$E \to$ odd on first, $F \to$ odd on second

$G \to$ odd sum.

Check if E, F, G are independent.

1. Six cards are drawn with replacement from an ordinary deck. What is the prob that each of the four suits will be represented at least once among the six cards?

<u>Soln</u> $E \to$ all suits appear at least once.

$E^c \to$ at least one suit will not appear.

$F_i \to$ i^{th} suit does not appear

$i=1 \to$ spade, $i=2 \to$ heart, $i=3 \to$ diamond

$i=4 \to$ club.

$$E^c = \bigcup_{i=1}^{4} F_i$$

$$P(F_i) = \left(\frac{3}{4}\right)^6 , \quad i = 1, 2, 3, 4$$

$$P(F_i \cap F_j) = \left(\frac{1}{2}\right)^6 , \quad i \neq j$$

$$P(F_i \cap F_j \cap F_k) = \left(\frac{1}{4}\right)^6 , \quad P\left(\bigcap_{i=1}^{4} F_i\right) = 0$$

By general addition rule

$$P(E^c) = 4 \times \left(\frac{3}{4}\right)^6 - 6 \times \left(\frac{1}{2}\right)^6 + 4 \times \left(\frac{1}{4}\right)^6$$

$$= \frac{317}{512} \cong 0.62$$

$$P(E) \cong 0.38$$