Basic Probability

Lesson 2: Axiomatic Approach to Probability

Under this approach, the probability can be deduced from mathematical concepts. To start with, some concepts are laid down. Then some statements are made in respect to the properties possessed by these concepts. These are called 'axioms' of the theory.

Random Experiment and Sample Space:

Any probabilistic situation that can be 'mathematically described' or 'modelled' is known as random experiment. Each performance in a random experiment E is called a trial. The result of a trial in a random experiment is called an outcome / an elementary event / a sample point. The totality of all sample points constitutes the sample space S. It serves as the universal set for all queries related to the random experiment E. The events are therefore, non-empty subsets of S. Hence all the laws of set theory hold for algebra of events.

Set Theoretic Approach to Probabilistic Events

Statement	Meaning in Terms of Set Theory
Complementary event of A	$\bar{A}$
At least one of the events A or B occurs	$A \cup B$
Both the events A and B occurs	$A \cap B$
Neither A nor B occurs	$\bar{A} \cap \bar{B}$
Event A occurs and B does not occur	$A \cap \bar{B}$
Exactly one of the events A or B occurs	$(A - B) \cup (B - A) = A \Delta B$
Events A and B are mutually exclusive	$A \cap B = \emptyset$

With the notion of set theory, we now define probability in the following way:

Probability Function:

$P(A)$ is called the probability function defined on $\mathscr{B}$, which is the collection of all subsets of the sample space S of events, if the following axioms hold:

(i) For each $A \in \mathscr{B}$, $P(A)$ is defined, real and ≥ 0 ----- axiom of positivity

(ii) $P(S) = 1$ ------- axiom of certainty

(iii) If $\{A_n\}$ is any finite or infinite sequence of disjoint events in $\mathscr{B}$, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

--------- axiom of additivity

Note:

1. The set function P defined on $\mathscr{B}$, taking its values on the real line and satisfying the above three conditions is called the probability measure

2. The triplet (S, $\mathscr{B}$, P) is often known as the probability space

Theorems/Results on Probabilities of Events:

Th.1: Probability of the impossible event is zero, i.e, $P(\emptyset) = 0$.

[* Though $P(A) = 0$ does not necessarily mean that $A = \emptyset$]

Th.2: Probability of the complementary event $\bar{A}$ of A is given by

$$P(\bar{A}) = 1 - P(A)$$

Th.3: For any two events A and B,

(i) $P(\bar{A} \cap B) = P(B) - P(A \cap B)$

(ii) $P(A \cap \bar{B}) = P(A) - P(A \cap B)$

Th.4: Addition Theorem/Total Probability Theorem:

Probability of the union of any two events A and B is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Note: For any three events A, B and C, we can write

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

Th.5: If $B \subset A$, then $P(B) \leq P(A)$

Problems:

Ex.1. An urn contains 4 tickets numbered 1, 2, 3, 4 and another contains 6 tickets numbered 2, 4, 6, 7, 8, 9. If one of the two urns is chosen at random and a ticket is drawn at random from the chosen urn, find the probabilities that the ticket drawn bears the number (i) 2 or 4 (ii) 3 (iii) 1 or 9.

Solution: The ticket can be randomly drawn in either of the following two ways:

I. 1st urn is chosen and then a ticket is drawn
II. 2nd urn is chosen and then a ticket is drawn

Since the probability of choosing any urn is ½, hence, the required probability will be = $\frac{1}{2}[P(I) + P(II)]$

(i) In this case, $P(I) = \frac{1+1}{4} = \frac{2}{4} = \frac{1}{2}$ and $P(II) = \frac{1+1}{6} = \frac{2}{6} = \frac{1}{3}$

Hence, the reqd. prob. = $\frac{1}{2}\left[\frac{1}{2} + \frac{1}{3}\right] = \frac{5}{12}$

(ii) In this case, $P(I) = \frac{1}{4}$ and $P(II) = 0$

Hence, the reqd. prob. = $\frac{1}{2}\left[\frac{1}{4} + 0\right] = \frac{1}{8}$

(iii) In this case, $P(I) = \frac{1+0}{4} = \frac{1}{4}$ and $P(II) = \frac{0+1}{6} = \frac{1}{6}$

Hence, the reqd. prob. = $\frac{1}{2}\left[\frac{1}{4} + \frac{1}{6}\right] = \frac{5}{24}$

Ans.

Ex.2. A card is drawn from a well-shuffled pack of playing cards. What is the probability that it is either a spade or an ace?

Solution: Let the two events be defined as : A: a spade is drawn

B: an ace is drawn

Then $P(A) = \frac{^{13}C_1}{^{52}C_1} = \frac{13}{52}$ and $P(B) = \frac{^{4}C_1}{^{52}C_1} = \frac{4}{52}$. We need to find $P(A \cup B)$. By addition theorem, we know that, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

Ans.

Ex.3. The probability that a student passes a Physics test is 2/3 and the probability that he passes both a Physics test and an English test is 14/45. The probability that he passes at least one test is 4/5. What is the probability that he passes the English test?

Solution. Let us consider that A: the event that he passes a Physics test

B: the event that he passes an English test

We have, $P(A) = \frac{2}{3}$, $P(A \cap B) = \frac{14}{45}$ and $P(A \cup B) = \frac{4}{5}$. Then from addition theorem, we can write $P(B) = P(A \cup B) - P(A) + P(A \cap B)$

$$= \frac{4}{9}$$

Ans.

Ex.4. Three newspapers A, B and C are published in a city. It is estimated from a survey that 20% people read A, 16% read B, 14% read C, 8% read A and B, 5% read A and C, 4% read B and C and 2% people read all the three newspapers. What percentage of people reads at least one of the papers?

Solution: Let us define the events as, A: percentage of people reading newspaper A

B: percentage of people reading newspaper B

C: percentage of people reading newspaper C

Then we have,

$$P(A) = 0.2, \quad P(B) = 0.16, \quad P(C) = 0.14, \quad P(A \cap B) = 0.08$$

$$P(A \cap C) = 0.05, \quad P(B \cap C) = 0.04, \quad P(A \cap B \cap C) = 0.02$$

Using the addition theorem for three arbitrary events, we can write,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C)$$

$+P(A \cap B \cap C)$

$= 0.2 + 0.16 + 0.14 – 0.08 – 0.04 – 0.05 + 0.02$

$= 0.35.$

That means, 35% people of the city reads at least one newspaper.

Ans.