Random Variable and Distribution Functions

Lesson 3: Moments and Expectation

Discrete Random Variable:

Let X is a discrete random variable which takes the values $x_1, x_2, x_3, \ldots\ldots, x_n$ with probabilities $p_1, p_2, p_3, \ldots\ldots, p_n$ such that the probability mass function is given by $p(x)$, then

(i) Arithmetic mean:

$$\bar{x} = \sum_{i=1}^{n} x_i p_i \quad \text{or,} \quad \sum xp(x)$$

(ii) Moments:

(a) r-th moment about any point x = A: $\mu_r{}' = \sum_r (x - A)^r p(x)$

(b) r-th moment about the origin: $\mu_r{}' = \sum_r x^r p(x)$

(c) r-th moment about the mean: $\mu_r = \sum_r (x - \bar{x})^r p(x)$

Continuous Random Variable:

Let $f(x)$ be the probability density function of a random variable X defined between a and b, then

(i) Arithmetic mean:

$$\bar{x} = \int_a^b xf(x)dx$$

(ii) Moments:

(a) r-th moment about any point x = A: $\mu_r{}' = \int_a^b (x - A)^r f(x)dx$

(b) r-th moment about the origin: $\mu_r{}' = \int_a^b x^r f(x)dx$

(c) r-th moment about the mean: $\mu_r = \int_a^b (x-\bar{x})^r f(x)dx$

(iii) Median:

If M is the median, then $\int_a^M f(x)dx = \int_M^b f(x)dx = \frac{1}{2}$. Solution of these two equations gives the median.

(iv) Mode:

Mode is the value of x for which $f(x)$ is maximum. It is thus the solution of $f'(x) = 0$ with $f''(x) < 0$, provided it lies in $[a, b]$

Note: All other moment measures for skewness and kurtosis remain same

Problems:

Ex.1. A random variable X is defined as follows: $P(X=1) = p,\ P(X=0) = 1-p,\ 0 < p < 1$. Find mean and variance of the distribution.

Solution: We have the distribution given as

X	0	1
$p(x)$	1-p	p

Here X is a discrete random variable, so the mean and the variance of the distribution are given by

$$\bar{x} = \sum_{i=1}^{2} x_i p_i = 0 \times (1-p) + 1 \times p = p$$

$$\mu_2 = \mu_2' - {\mu_1'}^2 = \sum_{i=1}^{2} {x_i}^2 p_i - p^2 = p - p^2 = p(1-p)$$

Ans.

Ex.2. Find mean and standard deviation of the distribution with density function $f(x) = kx^2e^{-x},\ 0 < x < \infty$

Solution: For the density function $f(x)$, we always have

$$\int_{-\infty}^{\infty} f(x)dx = 1 \rightarrow k\int_{0}^{\infty} x^2e^{-x}dx = 1 \rightarrow k\Gamma(3) = 1 \rightarrow k = \frac{1}{\Gamma(3)} = \frac{1}{2\Gamma(1)} = \frac{1}{2}$$

Therefore mean of the distribution is given by

$$\bar{x} = \int_{0}^{\infty} x\left(\frac{x^2}{2}e^{-x}\right)dx = \frac{1}{2}\int_{0}^{\infty} x^3e^{-x}dx = \frac{1}{2}\Gamma(4) = 3$$

Also the variance of the distribution will be

$$\mu_2 = \mu_2{}' - \mu_1{}'^2 = \int_{0}^{\infty} x^2\left(\frac{x^2}{2}e^{-x}\right)dx - 3^2 = 12 - 9 = 3$$

Hence the standard deviation of the distribution will be $\sqrt{3}$.

Ans.

Ex.3. The probability density function of a random variable X is $f(x) = \frac{\pi}{10}\sin\left(\frac{\pi x}{5}\right), 0 \leq x \leq 5$. obtain the median of the distribution.

Solution: Let us consider that M is the median of the distribution. Then we have

$$\int_{0}^{M} \frac{\pi}{10}\sin\left(\frac{\pi x}{5}\right)dx = \frac{1}{2} \rightarrow \int_{0}^{M} \sin\left(\frac{\pi x}{5}\right)dx = \frac{5}{\pi} \rightarrow M = 2.5$$

Ans.

Mathematical Expectation:

The average value of a random phenomenon is termed as its mathematical expectation or expected values.

(i) For a discrete random variable X with mass function $p(x)$

$$E(X) = \sum_x xp(x)$$

provided the series is absolutely convergent, i.e, $\sum_x |x| p(x) < \infty$

(ii) For a continuous random variable X with density function $f(x)$

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

provided the integral is absolutely convergent, i.e, $\int_{-\infty}^{\infty} |x| f(x)dx < \infty$

Expected value of a function of a random variable:

Let X is a random variable with mass function $p(x)$ / density function $f(x)$. Let $g(.)$ is a function such that $g(X)$ is a random variable. Then expected value of $g(X)$ is defined as

$$E[g(X)] = \sum_x g(x)p(x) - - - - \text{(discrete)}$$

$$= \int_{-\infty}^{\infty} g(x)f(x)dx - - - \text{ (continuous)}$$

Particular case:

1. If $g(X) = X^r$, then $E[X^r] = \sum_x x^r p(x)$ ----- (discrete)

$= \int_{-\infty}^{\infty} x^r f(x)dx$ ----- (continuous)

$= \mu_r{}'$ = r-th moment about the origin

The mean and variance of the distribution can be found out from this expression.

2. If $g(X) = [X - E(X)]^r$, then

$E([X - E(X)]^r) = \sum_x (x - \bar{x})^r p(x)$ ----- (discrete)

$= \int_{-\infty}^{\infty} (x - \bar{x})^r f(x) dx$ ----- (continuous)

$= \mu_r$ = r-th moment about the mean

3. If $g(X) = c$, a constant, then $E(c) = \int_{-\infty}^{\infty} c f(x) dx = c$

Properties of mathematical expectation:

1. Addition Theorem:

   For two random variables X and Y and two constants a and b, we have $E(aX + bY) = aE(X) + bE(Y)$

2. Multiplication Theorem:

   If X and Y are independent random variables, then $E(XY) = E(X)E(Y)$

3. If X and $g(X)$ are both random variables and a is a constant, then

(i) $E[ag(X)] = aE[g(X)]$ (ii) $E[g(X) + a] = E[g(X)] + a$

4. If X is a random variable ≥ 0, then $E(X) \geq 0$

5. If X and Y are two random variables and $X < Y$, then $E(X) \leq E(Y)$

6. For a random variable X, $|E(X)| \leq E(|X|)$

7. If X and Y are two independent random variables, then

$$E[h(X)k(Y)] = E[h(X)]E[k(Y)]$$

8. If X is a random variable, then $\text{Var}(aX+b) = a^2\text{Var}(X)$, where a and b are constants

9. If X and Y are two random variables, then covariance between them is given by

$$\text{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

Problems:

Ex.1. The probability distribution of a commodity is given below:

Demand (X_i)	5	6	7	8	9	10
Probability (p_i)	0.05	0.1	0.3	0.4	0.1	0.05

Find the expected demand and its variance.

Solution: A discrete distribution is given, hence

$$E(X) = \sum_x xp(x) = 7.55$$

And

$$\text{Var}(X) = E(X^2) - \{E(X)\}^2 = \sum_x x^2p(x) - 7.55^2 = 1.25$$

Ans.