Special Probability Distribution

Lesson 1: Binomial Distribution

Bernoulli Trials:

Let us assume that a random experiment has two possible outcomes, which are complementary to each other. If the probability p $(0 < p < 1)$ of getting success at each trial of the experiment is constant, then the trials are called Bernoulli trials.

In a series of n independent trials of a random experiment, if the probability of 'success' in each trial is a constant p, the probability of 'failure' is $q = (1 - p)$, then the probability of x successes is given by the Binomial distribution with probability mass function

$$p(x) = P(X = x) = {}^{n}C_{x} p^{x} q^{n-x}, \quad x = 0, 1, 2, \dots, n$$

$$= 0, \quad \text{otherwise}$$

It can be easily checked that

(i) $p(x) \geq 0 \; \forall \; x = 0, 1, 2, \dots, n$

(ii) $\sum_{x=0}^{n} p(x) = 1$

Note:

(i) Binomial distribution is a discrete distribution

(ii) n and p are the two parameters of the distribution

(iii) Mean of the distribution = $\mu = E(X) = np$

(iv) Variance of the distribution = $\sigma^2 = V(X) = npq$

(v) Skewness of the distribution = $\frac{q-p}{\sqrt{npq}}$

(vi) Kurtosis of the distribution = $\frac{1-6pq}{npq}$

Problems:

Ex.1. If 20% of the bolts produced by a machine are defective, determine the probability that out of 4 bolts chosen, at most 2 bolts will be defective.

Solution: Let us consider that the event ‘a defective bolt is produced’ is a success.

Then the probability of getting a defective bolt from the machine = p = 0.2 and the probability of getting a non-defective bolt from the machine will be given by = q = 1 – p = 0.8.

Now, the chosen number of bolts is = n = 4.

Let X : the number of defective bolts obtained from the machine

Then the probability of getting x defective bolts out of these 4 chosen bolts is given by the probability mass function

$$P(X = x) = p(x) = {}^4C_x p^x q^{4-x}, \;\; x = 0, 1, 2, 3, 4$$

Therefore probability of getting at most 2 defective bolts is given by

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$= {}^4C_0 (0.2)^0 (0.8)^{4-0} + {}^4C_1 (0.2)^1 (0.8)^{4-1} + {}^4C_2 (0.2)^2 (0.8)^{4-2}$$

$$= 0.9728.$$

Ans.

Ex.2. A multiple choice test in Mathematics with 40 questions, each having 5 options, is given to a student. If the student guess all 40 questions, what are the mean and standard deviation of the number of correct answers?

Solution: Let us consider that the event ‘correct answer’ is a success. Then the probability of correct answer will be $p = \frac{1}{5} \rightarrow q = \frac{4}{5}$.

Since there are total 40 questions, therefore n = 40. Hence mean = $np = 8$ and variance = $npq = \frac{32}{5}$. Hence standard deviation = $\sqrt{\frac{32}{5}} = 2.53$.

Ans.

Ex.3. The probability that an entering college student will be a graduate is 0.4. Determine the probability that out of 5 entering students (i) none (ii) one (iii) at least one, will be graduate.

Solution: Let the event 'an entering college student will be a graduate' is to be called a success. Then $p = 0.4,\ q = 0.6,\ n = 5$.

Let X : the number of entering students being graduated

Then the probability of having x graduates out of the 5 entering students is given by the probability mass function

$$P(X = x) = p(x) = {}^5C_x p^x q^{5-x},\ \ x = 0, 1, 2, 3, 4, 5$$

(i) Probability that none of the 5 entering students will be graduate

$= P(X = 0) = {}^5C_0 (0.4)^0 (0.6)^{5-0} = 0.07776$

(ii) Probability that one of the 5 entering students will be graduate

$= P(X = 1) = {}^5C_1 (0.4)^1 (0.6)^{5-1} = 0.2592$

(iii) Probability that at least one of the 5 entering students will be graduate

$= P(X \geq 1) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)$

$= 1 - P(X = 0)$

$= 1 - 0.07776$

$= 0.92224$

Ans.

Ex. 4. The overall percentage of failures in a certain examination is 40. What is the probability that out of a group of 6 candidates, at least 4 passed the examination?

Solution: Let us consider that the event 'a candidate is passed in the examination' to be the success. Then we have $p = 0.6,\ q = 0.4,\ n = 6$.

Let X : the number of candidates passing the examination

Then the probability of having x successes is given by the probability mass function

$$P(X = x) = p(x) = {}^6C_x p^x q^{6-x},\ \ x = 0, 1, 2, 3, 4, 5, 6$$

Then the probability that at least 4 candidates passed the examination is given by

$P(X \geq 4) = P(X = 4) + P(X = 5) + P(X = 6)$

$= 0.311 + 0.1866 + 0.0466$

$= 0.544.$

Ans.

Ex.5. In 10 independent throws of a defective die, the probability that an even number will appear 5 times is twice the probability that an even number will appear 4 times. Find the probability that an even number will not appear at all in 10 independent throws of the die.

Solution: Let us denote the event that 'an even number will appear' is a success

Then p = probability of success in a single trial

q = probability of failure

n = total number of throws = 10

We also know that the probability of having x successes in 10 throws is given by the probability mass function

$$P(X = x) = p(x) = {}^{10}C_x p^x q^{10-x}, \quad x = 0, 1, 2, \ldots\ldots, 10$$

It is given that

$$P(X = 5) = 2P(X = 4)$$

$$\text{or,} \quad {}^{10}C_5 p^5 q^{10-5} = 2\,{}^{10}C_4 p^4 q^{10-4}$$

$$\text{or,} \quad 3p = 5q = 5(1 - p)$$

$$\text{or,} \quad p = \frac{5}{8} \rightarrow q = \frac{3}{8}$$

Hence, the probability that an even number will not appear at all in 10 independent throws of the die = $P(X = 0) = {}^{10}C_0 p^0 q^{10-0} = \left(\frac{3}{8}\right)^{10}$.

Ans.