Random Experiment

An experiment is observing something happening / conducting something resulting in some outcome.

Deterministic Expt. If after fixing the equipment / conditions the outcome of expt is priorly determined, then it is called deterministic expt.

If even after fixing the initial set up or conditions, the outcome of the expt is not fixed, it is called a random expt.

<u>Sample Space</u> : The set of all possible outcomes of a random expt is called

a sample space.

<u>Examples</u>: Coin Toss : $\{H, T\} = \Omega$ S

Dice Toss : $\Omega = \{1, 2, 3, 4, 5, 6\}$

Life of a headphone (in hours) $\Omega = [0, \infty)$ or $(0, \infty)$

Time taken to complete 100 mt race Olympic final 10 sprints :
$S_1, S_2, \ldots, S_{10}$

Winner's time $\Omega_1 = (9.5, 10.5]$

Winner : $\Omega_2 = \{ S_1, S_2, \ldots, S_{10} \}$

Order of Sprinters

$$\Omega_3 = \{ (S_1 S_2 \ldots S_{10}), (S_2, S_3 \ldots S_{10}, S_1) \\ \ldots \ldots \}$$

10! elements

Rainfall in a monsoon season in Odisha (cm)

$$\Omega = (100, 300)$$

Events : Any subset of a sample space is an event.

$E = (9.8, 10.0) \longrightarrow$ winning time is between 9.8 sec to 10 sec.

$F = \{150\}$ cm. $\longrightarrow$ rainfall in monsoon in Odisha is 150 cm.

Types of Events : Since events are sets, the set theoretic operations lead to various combinations of events.

<u>Union</u>: $E \cup F \rightarrow$ occurrence of either E or F or both

$E_1 \cup E_2 \cup \dots \cup E_n \rightarrow$ occurrence of at least one E_i, $\rightarrow \bigcup_{i=1}^{n} E_i$

$$\bigcup_{i=1}^{\infty} E_i$$

Intersection : $E \cap F \to$ simultaneous occurrence of E and F

$$\bigcap_{i=1}^{n} E_i = E_1 \cap E_2 \cap \cdots \cap E_n$$

$\to$ simultaneous occurrence of $E_1, \ldots, E_n$

$$\bigcap_{i=1}^{\infty} E_i$$

$E^c \rightarrow$ not occurrence of event E

$A - B = A \cap B^c$

$\rightarrow$ occurrence of A but not B

If $E = \phi$, then E is called an impossible event

If $E = \Omega$, then E is called a sure event.

If $E \cap F = \phi$ then E and F are called disjoint or mutually exclusive events. ie occurrence of one excludes the possibility of occurrence of other event.

If events $A_1, A_2, \ldots,$ are such that

$$\bigcup A_i = \Omega,$$

then $A_1, A_2, \ldots$ are called exhaustive events.

As the subject of probability evolved since seventeenth century, there have been several methods to evaluate probability. When these methods were developed, they were considered as defenitions. However, later they were found to be inadequate in some aspects and so called these as definitions is not proper.

Classical Method of Computing Probability

(Laplace - 1813) Suppose a random expt has N possible outcomes which are mutually exclusive, exhaustive and equally likely. Let M of these be favourable to the occurrence of an event E. Then the prob of E is defined as

$$P(E) = \frac{M}{N}.$$

<u>Drawbacks of this method</u> :

1. In actual expt N need not be finite.

2. It may not be possible to enumerate all possible outcomes.

3. The definition is circular in nature. It uses the term 'equally likely' which means that outcomes are with equal prob.

Relative frequency Method / Empirical Method of Calculation of Probability (Von Mises)

Suppose a random expt is conducted a large number of times independently under identical conditions. Let a_n denote the number of times the event E occurs in n trials of the expt. Then

$$P(E) = \lim_{n \to \infty} \frac{a_n}{n} \quad \text{(provided the limit exists)}$$

<u>Example</u>: H H T H H T H H T . . . .

$E = \{H\}$

$$\frac{a_n}{n} \rightarrow \frac{1}{1}, \frac{2}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{4}{6}, \dots$$

$$= \begin{cases} \dfrac{2k-1}{3k-2} & n = 3k-2 \\ \dfrac{2k}{3k-1} & n = 3k-1 \\ \dfrac{2k}{3k}, & n = 3k \end{cases}, k = 1, 2 \dots$$

$$\underset{n \to \infty}{Lt} \frac{a_n}{n} = \frac{2}{3} = P(E)$$

<u>Limitations</u> : 1. Actual data may not be available sometimes.

2. The event may not be an impossible event, but the prob may be zero through this definition.

eg. $a_n = n^{1/3}$, $\frac{a_n}{n} \to 0$

The event may not be sure, but the prob may turn out to be one.

eg. $a_n = n - \sqrt{n}$. $\frac{a_n}{n} \to 1$

but event is not sure.

Axiomatic Set up — Kolmogorov (1933)

Let Ω be a space

Let $\mathcal{B}$ be a class of subsets of Ω.

We say that $\mathcal{B}$ is a σ-field (σ-algebra) if it satisfies the following two criteria:

(i) $E \in \mathcal{B} \Rightarrow E^c \in \mathcal{B}$

(ii) For any seq. $E_1, E_2, \ldots \in \mathcal{B}$,

$$\bigcup_{i=1}^{\infty} E_i \in \mathcal{B}.$$

This structure allows for inclusion of all relevant set theoretic operations: unions, intersections, differences, complement etc.

Let Ω be a sample space.

Let $\mathcal{B}$ be a σ-field of subsets of Ω

Then $(\Omega, \mathcal{B})$ is called a measureable

Space.

Axiomatic Definition of Probability

Let $(\Omega, \mathcal{B})$ be a measureable space.

A set function $P : \mathcal{B} \to \mathbb{R}$ is said to be probability function if it satisfies the following three axioms:

P_1 : $P(E) \geq 0 \quad \forall E \in \mathcal{B}$ (axiom of non-negativity)

P_2 : $P(\Omega) = 1$

P_3 : For any seq. of pairwise disjoint events $E_i \in \mathcal{B}$,

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

→ axiom of countable additivity

$(\Omega, \mathcal{B}, P)$ is called a prob. space.

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

$$\mathcal{B}_1 = \{ \{1,2\}, \{3,4\} \}$$

$$\mathcal{B}_2 = \{ \{1,2\}, \{3,4,5,6\}, \{3,4\}, \{1,2,5,6\} \}$$

not σ-fields.

$$\mathcal{C} = \{\phi, \{1,2\}, \{3,4,5,6\}, \Omega\}, \ \sigma\text{-field.}$$