Special Probability Distribution

Lesson 3: Normal/Gaussian Distribution

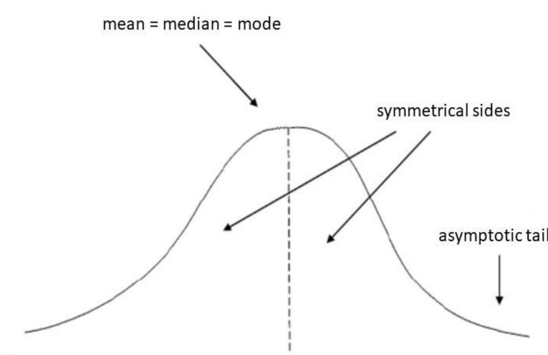
A continuous random variable X is said to follow a normal distribution if its probability density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right], -\infty < x < \infty$$

Here μ and σ are mean and standard deviation of the distribution and are called the parameters of the distribution. The random variable X is called normal random variable or Gaussian random variable and is denoted as

$$X \sim N(\mu, \sigma^2)$$

In this case, the curve y = f(x) is called a normal probability curve or normal curve or Gaussian curve.



Properties:

(i) The distribution is symmetric about the mean of the distribution μ

(ii) Mean = Median = Mode = μ

(iii) The distribution approaches the horizontal axis asymptotically, the horizontal axis being the asymptote to the probability curve

(iv) No portion of the distribution lies below the horizontal axis

(v) Total area under the curve and above the horizontal axis is unity

(vi) Skewness = Kurtosis = 0

(vii) As x increases numerically, $f(x)$ decreases rapidly, the maximum probability occurring at $x = \mu$ and is given by $\frac{1}{\sigma\sqrt{2\pi}}$

(viii) The linear combination of independent normal variates is also a normal variate

Standard Normal Distribution:

If a variable X is distributed normally with mean μ and standard deviation σ, then $Z = \frac{X-\mu}{\sigma}$ is called a standard normal variate and its probability density function given by

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, -\infty < z < \infty$$

is called the standard normal distribution with mean 0 and variance 1. We can write this symbolically,

$$Z \sim N(0, 1)$$

The corresponding distribution function will be given by

$$F(z) = P(Z \leq z) = \int_{-\infty}^{z} f(t)dt$$

Note:

(a) $F(-z) = 1 - F(z)$

Problems:

Ex.1. The continuous random variable Z has a standard normal distribution. Calculate the probability of the following:

(i) $Z < 1.3$ (ii) $Z > 1.3$ (iii) $Z < -1.3$ (iv) $Z > -1.3$

(v) $-1.37 \leq Z \leq 2.01$ (vi) $|Z| \leq 0.5$ (vii) $-1.79 \leq Z \leq -0.54$

Solution:

(i) $P(Z < 1.3) = F(1.3) = 0.90320$

(ii) $P(Z > 1.3) = 1 - P(Z < 1.3) = 1 - F(1.3) = 1 - 0.9032 = 0.0968$

(iii) $P(Z < -1.3) = F(-1.3) = 1 - F(1.3) = 0.0968$

(iv) $P(Z > -1.3) = 1 - P(Z < -1.3) = 1 - F(-1.3) = 1 - 0.0968 = 0.9032$

$[= P(Z < 1.3)]$

(v) $P(-1.37 \leq Z \leq 2.01) = P(Z < 2.01) - P(Z < -1.37)$

$= F(2.01) - F(-1.37)$ = 0.97778 – 0.08534 = 0.89244

(vi) $P(|Z| \leq 0.5) = P(-0.5 \leq Z \leq 0.5) = P(Z < 0.5) - P(Z < -0.5)$

= 0.69146 – 0.30854 = 0.38292

(vii) $P(-1.79 \leq Z \leq -0.54) = P(Z < -0.54) - P(Z < -1.79)$

= 0.96327 – 0.70540 = 0.25787. Ans.

Ex.2. Weights of 500 students in a college is normally distributed with average weight 95 lbs. and standard deviation 7.5 lbs. Find how many students will have the weight between 100 and 110 lbs.

Solution: Let us consider that X = weight of a student. Then we have

$$X \sim N(95, 7.5^2)$$

Therefore the probability of a student having weight between 100 and 110 lbs. is

$$P(100 < X < 110) = P\left(\frac{100-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{110-\mu}{\sigma}\right)$$

$$= P\left(\frac{100-95}{7.5} < Z < \frac{110-95}{7.5}\right)$$

$$= P(0.67 < Z < 2)$$

$$= P(Z < 2) - P(Z < 0.67)$$

= 0.97725 – 0.74857 = 0.22868

Hence, the number of students having this weight distribution = 500 × 0.22868

$\cong 114.$ Ans.

Ex.3. The compressive strength of samples of cement can be modeled by a normal distribution with a mean 6000 kg/cm^2 and standard deviation of 100 kg/cm^2.

(i) What is the probability that a sample's strength is less than 6250 kg/cm^2?

(ii) What is the probability of sample's strength in between 5800 and 5900 kg/cm^2?

(iii) What strength is exceeded by 95% of the samples?

Solution: Let us consider that X = strength of a cement sample. Then we have

$$X \sim N(6000, 100^2)$$

(i) The probability that a sample's strength is less than 6250 kg/cm^2 is given by

$$P(X < 6250) = P\left(\frac{X-\mu}{\sigma} < \frac{6250-\mu}{\sigma}\right)$$

$$= P\left(Z < \frac{6250-6000}{100}\right)$$

$$= P(Z < 2.5) = 0.99379$$

(ii) The probability of sample's strength to be between 5800 and 5900 kg/cm^2 is given by

$$P(5800 < X < 5900) = P\left(\frac{5800-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{5900-\mu}{\sigma}\right)$$

$$= P\left(\frac{5800-6000}{100} < Z < \frac{5900-6000}{100}\right)$$

$$= P(-2 < Z < -1)$$

$$= P(Z < -1) - P(Z < -2)$$

$$= 0.15866 - 0.02275$$

$$= 0.13591$$

(iii) Let S denotes the strength which is exceeded by 95% of the sample's strength. Then it is mathematically given as

$$P(X > S) = 0.95$$

$$\text{or, } P\left(\frac{X-\mu}{\sigma} > \frac{S-\mu}{\sigma}\right) = 0.95$$

$$\text{or, } P\left(Z > \frac{S-6000}{100}\right) = 0.95$$

$$\text{or, } 1 - P\left(Z < \frac{S-6000}{100}\right) = 0.95$$

$$\text{or, } P\left(Z < \frac{S-6000}{100}\right) = 1 - 0.95 = 0.05$$

$$\text{or, } \frac{S-6000}{100} = -1.64$$

$$\text{or, } S = 5836 \text{ kg/cm}^2$$

Ans.

Ex.4. In a distribution exactly normal, 10.03% of the items are under 25 kg weight and 89.97% of the items are under 70 kg weight. What are the mean and standard deviation of the distribution?

Solution: Let us consider that X = weight (in kg) of the items. Then we have

$$P(X < 25) = 0.1003 \quad \text{and} \quad P(X < 70) = 0.8997$$

$$\rightarrow P\left(Z < \frac{25-\mu}{\sigma}\right) = 0.1003 \quad \text{and} \quad P\left(Z < \frac{70-\mu}{\sigma}\right) = 0.8997$$

$$\rightarrow \frac{25-\mu}{\sigma} = -1.28 \quad \text{and} \quad \frac{70-\mu}{\sigma} = 1.28$$

Solving these two equations, we get $\mu = 47.5$ kg and $\sigma = 17.578$ kg.

Ans.

Ex.5. The local authorities in a certain city install 10000 electric lamps in the streets of a city. If these lamps have an average life of 1000 burning hours with a standard deviation of 200 hours, assuming normality, what number of lamps might be expected to fail (i) in the first 800 burning hours? (ii) between 800 and 1200 burning hours?

After what period of burning hours would you expect that (a) 10% of the lamps would fail? (b) 10% of the lamps would be still burning?

Solution: Let us consider that X = life of an electric lamp in burning hours. Then we have

$$X \sim N(1000, 200^2)$$

(i) A lamp will fail in the first 800 burning hours, if its life is less than 800 hours. The probability for this to happen for a lamp is given by

$$P(X < 800) = P\left(Z < \frac{800 - 1000}{200}\right) = P(Z < -1) = 0.15866$$

∴ The number of such lamps = 10000×0.15866 = 1586.6 ≅ 1587.

(ii) The probability that a lamp will fail between 800 and 1200 burning hours is

$$P(800 < X < 1200) = P\left(\frac{800 - 1000}{200} < Z < \frac{1200 - 1000}{200}\right)$$

$$= P(-1 < Z < 1) = 0.6826$$

∴ The number of such lamps = 10000×0.6826 = 6826.

(a) Let 10% of the lamps would fail after x_1 hours of burning. Then we may write

$$P(X < x_1) = 0.1$$

$$\text{or, } P\left(Z < \frac{x_1 - 1000}{200}\right) = 0.1$$

$$\text{or, } \frac{x_1 - 1000}{200} = -1.28$$

$$\text{or, } x_1 = 744 \text{ hours}$$

(b) Let 10% of the lamps would be still burning after x_2 hours. Then

$$P(X > x_2) = 0.1$$

$$\text{or, } P\left(Z > \frac{x_2 - 1000}{200}\right) = 0.1$$

$$\text{or, } 1 - P\left(Z < \frac{x_2 - 1000}{200}\right) = 0.1$$

$$\text{or, } \frac{x_2 - 1000}{200} = 1.29$$

$$\text{or, } x_2 = 1258 \text{ hours}$$

Ans.