Example: The lung cancer failure rate of a t-year old male smoker is given by

$$Z(t) = 0.027 + 0.00025\,(t-40)^2, \quad t \geq 40$$

Assuming that a (40+) old male smoker survives other hazards, find the density function of the life.

Find the prob that he survives to age 50.
If he survives to age 50, find the prob that he survives to age 60.

$$R(t) = e^{-\int_{40}^{t} Z(s)\,ds}$$

$$= e^{-\left[0.027(t-40) + \frac{0.00025}{3}(t-40)^3\right]}$$

$$f(t) = -\frac{d}{dt} R(t)$$

(shifted Weibull distn)

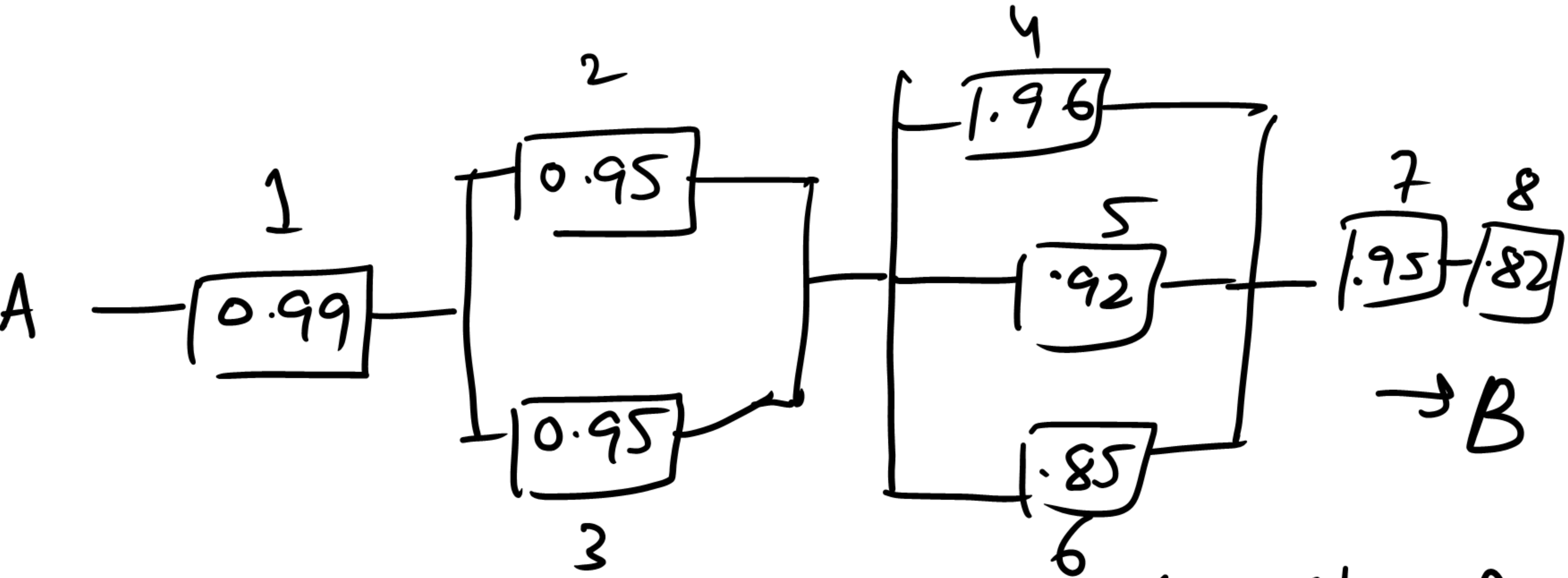
$$P(\text{survives to age } 50) = P(X > 50)$$

$$= R(50) = e^{-0.35\,33} \simeq 0.70$$

$$P(X > 60 \mid X > 50) = \frac{P(X > 60)}{P(X > 50)}$$

$$= \frac{R(60)}{R(50)} \cong 0.426$$

<u>Example</u>: A system consisting of components in series and parallel



The figures in squares denote reliabilities of components. Find the system reliability

$$R_x(t) = 0.99 \times \left\{ 1 - (1 - .95)^2 \right\}$$

$$\left\{1-(1-.96)(1-.92)(1-.85)\right\}\times 0.95 \times 0.82$$

$$= 0.99 \times 0.9975 \times 0.99952 \times 0.95 \times 0.82$$

$$= 0.7689.$$

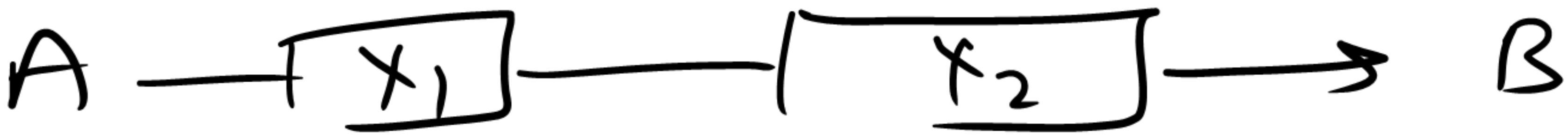
<u>Example</u>: A system consists of two independent components connected in a series. The lifespan of the first component follows a Weibull dist^n with

$\alpha = 0.006$ & $\beta = 0.5$. The second has a lifespan that follows exponential dist^n with mean 25000 (hrs).

(a) Find the system reliability at 2500 hrs

(b) find the prob that the system will fail before 2000 hrs.

(c) If the two components are connected in parallel, what is the system reliability at 2500 hrs. ?



$$f_{X_1}(x_1) = \alpha \beta x^{\beta-1} e^{-\alpha x^{\beta}}, \quad \alpha = 0.006, \ \beta = 0.5$$

$$R_{X_1}(t) = e^{-\alpha t^{\beta}}$$

$$f_{X_2}(x_2) = \frac{1}{25000} e^{-x_2/25000}, \quad x_2 > 0$$

$$R_{X_2}(t) = e^{-t/25000}$$

(a) $R_X(t) = R_{X_1}(t)\, R_{X_2}(t)$

$$R_X(2500) = e^{-0.006(2500)^{0.5}} \, e^{-\frac{2500}{25000}}$$

$$= e^{-0.4} \approx 0.67$$

(b) $P(X < 2000) = 1 - P(X \geq 2000)$

$$= 1 - R_X(2000)$$

$$= 1 - e^{-0.006(2000)^{0.5}} \, e^{-\frac{2000}{25000}} \approx$$

(c) $R_X(t) = 1 - (1 - R_{X_1}(t))(1 - R_{X_2}(t))$

$$\approx 0.98$$

Normal Distribution :

A continuous r.v X is said to have a normal distribution with mean μ and variance σ^2 ($N(\mu, \sigma^2)$) if it has pdf given by

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad x \in \mathbb{R}$$

$$\mu \in \mathbb{R}, \ \sigma > 0$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$z = \frac{x-\mu}{\sigma}, \quad dz = \frac{1}{\sigma} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-z^2/2} dz$$

$$= \frac{\cancel{2}}{\sqrt{\cancel{2}\pi}} \int_0^\infty \frac{t^{-1/2}}{\sqrt{\cancel{2}}} e^{-t}$$

$$\frac{z^2}{2} = t$$

$$z = \sqrt{2t}$$

$$dz = \frac{1}{\sqrt{2t}} dt$$

$$= \frac{1}{\sqrt{\pi}} \Gamma\left(\tfrac{1}{2}\right) = 1$$

$$E\left(\frac{X-\mu}{\sigma}\right)^k = \int_{-\infty}^{\infty} \left(\frac{x-\mu}{\sigma}\right)^k \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \int_{-\infty}^{\infty} z^k \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz$$

If $k=1$, then this integral is an odd function and so it will vanish.

That is $E\left(\frac{X-\mu}{\sigma}\right) = 0 \Rightarrow E(X) = \mu$

So $E\left(\frac{X-\mu}{\sigma}\right)^{2m+1} = 0 \Rightarrow E(X-\mu)^{2m+1} = 0$

$m = 0, 1, 2, \dots$

So all odd ordered central moments of a normal dist^n vanish.

In particular $\mu_3 = 0$ and so $\beta_1 = 0$

$$E\left(\frac{x-\mu}{\sigma}\right)^2 = \int_{-\infty}^{\infty} z^2 \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} z^2 e^{-z^2/2} dz$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty \frac{2t\, e^{-t}}{\sqrt{2t}}\, dt$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty t^{1/2} e^{-t}\, dt = \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right)$$

$$= \frac{\cancel{2}}{\cancel{\sqrt{\pi}}} \frac{1}{\cancel{2}} \cdot \cancel{\Gamma\left(\frac{1}{2}\right)} = 1$$

$$\Rightarrow \quad E(X-\mu)^2 = \sigma^2 = Var(X)$$

$$E\left(\frac{X-\mu}{\sigma}\right)^4 = \frac{2}{\sqrt{2\pi}}\int_0^\infty z^4 e^{-z^2/2}\,dz$$

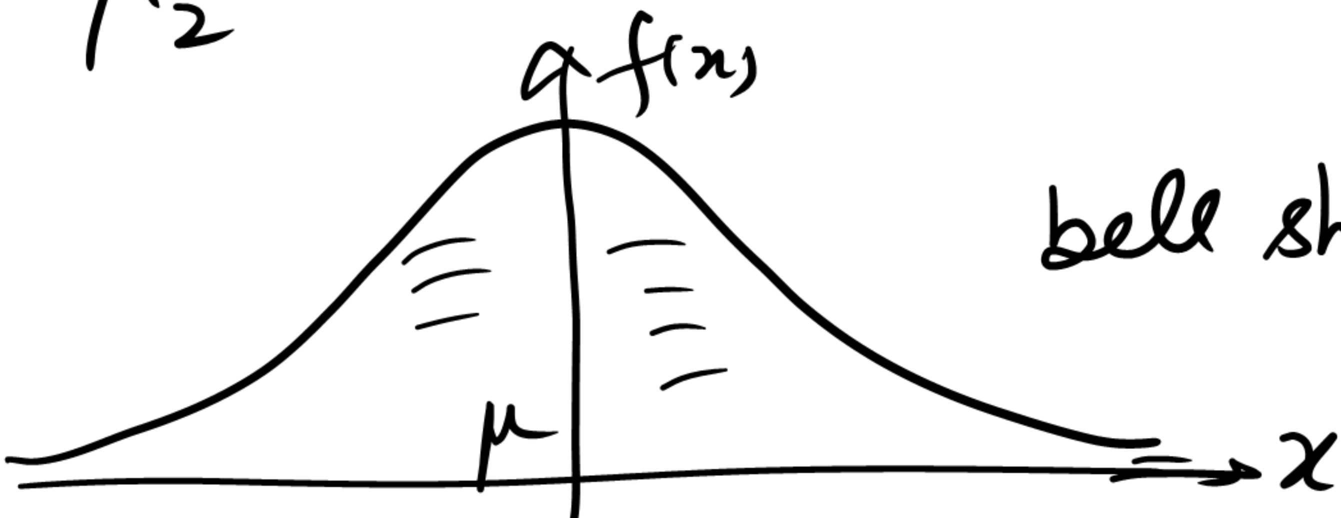
$$= \frac{2}{\sqrt{2\pi}}\int_0^\infty \frac{4t^2}{\sqrt{2t}} e^{-t}\,dt$$

$$= \frac{4}{\sqrt{\pi}}\int_0^\infty t^{3/2} e^{-t}\,dt = \frac{4}{\sqrt{\pi}}\Gamma\left(\frac{5}{2}\right)$$

$$= \frac{\cancel{4}}{\sqrt{\cancel{\pi}}} \cdot \frac{3}{\cancel{2}} \cdot \frac{1}{\cancel{2}} \cdot \cancel{\sqrt{\pi}} = 3$$

$$\Rightarrow \quad \mu_4 = E(X-\mu)^4 = 3\sigma^4$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} - 3 = \frac{3\sigma^4}{\sigma^4} - 3 = 0$$



bell shaped

$$\text{Mean}(X) = \text{Median}(X) = \text{Mode}(X)$$
$$= \mu.$$

$$\text{MGF.} = M_X(t) = E\left(e^{tX}\right)$$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\sigma z+\mu) - z^2/2} \cdot dz$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma t)^2} \, dz}_{1}$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

<u>Linearity Property of a Normal</u>

<u>Distribution</u> : Let $X \sim N(\mu, \sigma^2)$

and let $Y = aX + b$.

Then $Y \sim N(a\mu + b, a^2\sigma^2)$

$$M_Y(t) = E\left(e^{tY}\right) = E\left[e^{t(aX+b)}\right]$$

$$= e^{bt} \, E\left(e^{(at)X}\right)$$

$$= e^{bt} \, M_X(at)$$

$$= e^{bt} \cdot e^{\mu(at) + \frac{1}{2}\sigma^2 (at)^2}$$

$$= e^{(a\mu+b)t + \frac{1}{2} a^2\sigma^2 t^2}$$

which is mgf of $N(a\mu+b, a^2\sigma^2)$.

So if $X \sim N(\mu, \sigma^2)$. Then

$$Z = \frac{X-\mu}{\sigma} \sim N(0, 1)$$

This is called standard normal distribution.

The pdf of Z is denoted by

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad -\infty < z < \infty$$

The cdf of Z is denoted by

$$\Phi(z) = \int_{-\infty}^{z} \phi(t)\, dt$$

The tables of standard normal cdf Φ are widely available.

The cdf of a general normal distribution can be evaluated using cdf of a standard normal distn.

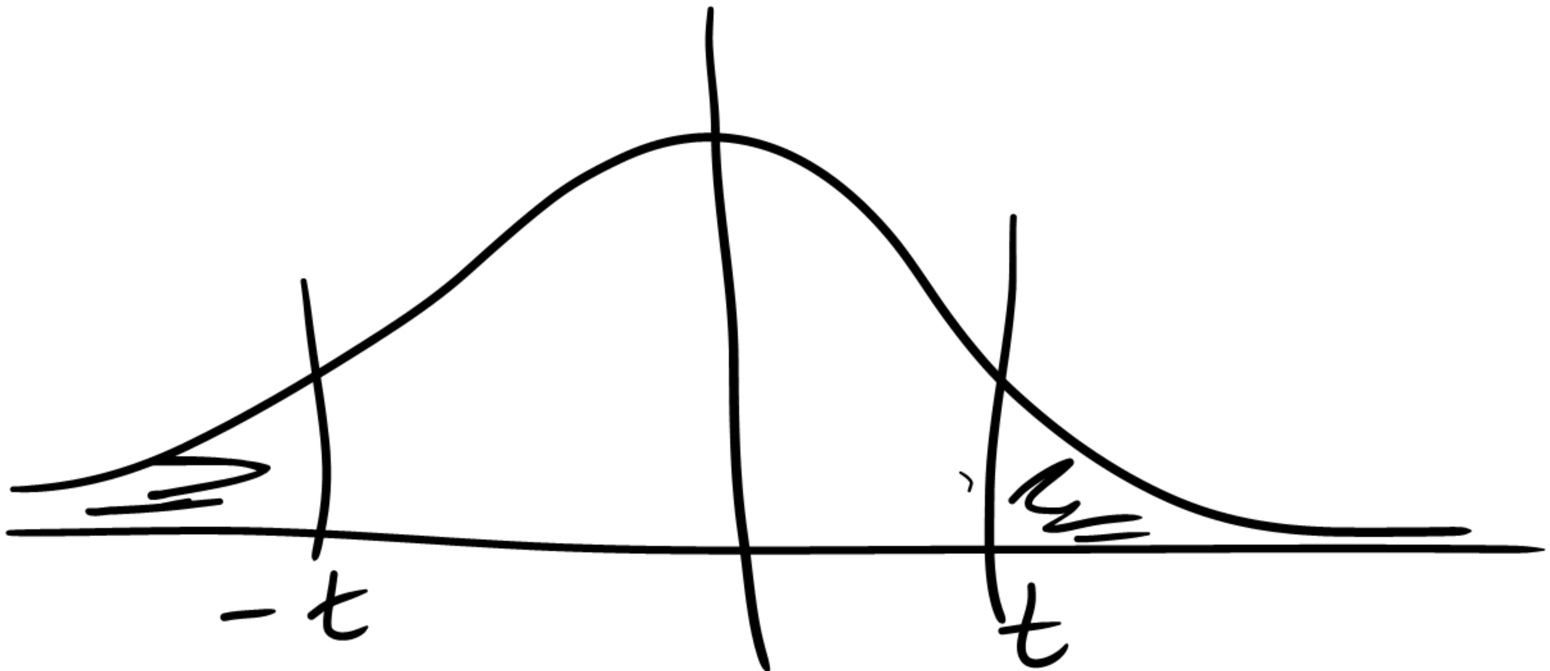
$$X \sim N(\mu, \sigma^2)$$

$$F_X(x) = P(X \leq x) = P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right)$$

$$= P\left(Z \le \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

In particular $\Phi(0) = \frac{1}{2}$

$$\Phi(t) = 1 - \Phi(-t)$$



$$P(a < X \leq b) = P(X \leq b) - P(X \leq a)$$

$$= F_X(b) - F_X(a) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

<u>Example</u>: The time that a sprinter takes to complete a mile race is normally dist^d with mean 241 sec. & s.d. 2 sec. Find the prob that he will take less than 4 mins.?

more than 3 min 55 sec ?

$$X \sim N(241, 4)$$

$$P(X < 240) = \Phi\left(\frac{240-241}{2}\right)$$

$$= \Phi(-0.5) = 0.3085$$

$$P(X > 235) = P\left(Z > \frac{235-241}{2}\right)$$

$$= P(Z > -3) = \Phi(3) = 0.9987$$

<u>Example</u>: Steel rods have diameters dist^d normally with mean 3 inches and s.d. σ. We want specification $(2.99, 3.01)$ for diameter.

It is observed that 5% are rejected as undersized/oversized

$$P(X < 2.99) = 0.05$$

$$P\left(\frac{X-3}{\sigma} < \frac{2.99-3}{\sigma}\right) = 0.05$$

$$\Rightarrow \Phi\left(-\frac{1}{\sigma}\right) = 0.05$$

$$-\frac{1}{\sigma} = -1.65 \Rightarrow \sigma = \frac{1}{1.65}$$

$$\approx 0.606$$