Random Variable and Distribution Functions

Lesson 2: Continuous Random Variable

Continuous Random Variable:

A random variable X is said to be continuous if it can take all possible values (integral as well as fractional) between certain limits.

Probability Density Function:

The probability density function of a random variable X is defined as

$$f(x) = \lim_{\delta x \to 0} \frac{P(x \leq X \leq x + \delta x)}{\delta x}$$

The probability for a variate value to fall in the interval dx is $f(x)dx$ and hence the probability for a variate value to fall in the finite interval $[x_1, x_2]$ is

$$P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f(x)dx - - - - (1)$$

which represents the area between the curves y = f (x), x-axis and the ordinates at $x = x_1$ and $x = x_2$. Further, since total probability is unity, we have

$$\int_a^b f(x)dx = 1$$

where [a, b] is the range of the random variable X.

Note:

(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

(iii) For a continuous random variable X, the probability at a point c is always zero, i.e, $P(X = c) = 0$ $\forall$ possible values of c. This leads to an important result:

$$P(x_1 \leq X \leq x_2) = P(x_1 \leq X < x_2) = P(x_1 < X \leq x_2) = P(x_1 < X < x_2)$$

Continuous Distribution Function:

If X is a continuous random variable with density function $f(x)$, then the function $F(x)$ defined as

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt\,, \quad -\infty < x < \infty$$

is called the distribution function of the random variable X.

Properties:

(i) $0 \leq F(x) \leq 1, -\infty < x < \infty$

(ii) $F(x)$ is non decreasing function of x

(iii) $P(a \leq X \leq b) = \int_a^b f(x)dx = F(b) - F(a)$

$$= P(a < X < b)$$

$$= P(a < X \leq b)$$

$$= P(a \leq X < b)$$

(iv) The relationship between density and distribution function is given by

$$f(x) = F'(x)$$

Problems:

Ex.1. The probability density function of a random variable X is $f(x) = k\sin\left(\frac{\pi x}{5}\right), 0 \leq x \leq 5$. Determine the value of k.

Solution: We know that $\int_{-\infty}^{\infty} f(x)dx = 1$. For the given case this means

$$\int_0^5 f(x)dx = 1 \to k\int_0^5 \sin\left(\frac{\pi x}{5}\right)dx = 1 \to k = \frac{\pi}{10}$$

Ans.

Ex.2. Verify that the following function is a distribution function

$$F(x) = \begin{cases} 0 & x < -a \\ \frac{1}{2}\left(\frac{x}{a} + 1\right), & -a \le x \le a \\ 1 & x > a \end{cases}$$

Solution: We know that $0 \le F(x) \le 1$. Also $F(x)$ is continuous at $x = a$ and $x = -a$. Now the derivative of $F(x)$ is given by

$$\frac{d}{dx}[F(x)] = \begin{cases} \frac{1}{2a}, & -a \le x \le a \\ 0 & \text{otherwise} \end{cases} = f(x), \text{(say)}$$

In order that $F(x)$ is a distribution function, $f(x)$ must be a probability density function. This means we have to show that $\int_{-\infty}^{\infty} f(x)dx = 1$.

Now,

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-a}^{a} f(x)dx = \frac{1}{2a}\int_{-a}^{a} dx = 1$$

Hence $F(x)$ is a distribution function.

Ans.

Ex.3. Let X be a random variable with probability density function $f(x) = 6x(1-x),\ 0 \le x \le 1$. Determine a number b such that $P(X < b) = P(X > b)$. Also find the distribution function of X.

Solution: We have $\quad P(X < b) = P(X > b)$

This means $\quad \int_0^b f(x)dx = \int_b^1 f(x)dx$

or, $\quad 6\int_0^b x(1-x)dx = 6\int_b^1 x(1-x)dx$

or, $\ 4b^3 - 6b^2 + 1 = 0$

or, $(2b-1)(2b^2 - 2b - 1) = 0$

or, $b = \frac{1}{2}\ ,\ \frac{1 \pm \sqrt{3}}{2}$

Since $0 \le x \le 1$, therefore $b = \frac{1}{2}$ is the required answer.

Now let us calculate the distribution function of X.

For $-\infty < x < 0$, $F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{x} 0\ dt = 0$

For $0 \le x < 1$, $F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{0} f(t)dt + \int_{0}^{x} f(t)dt$

$$= \int_{-\infty}^{0} 0\ dt + \int_{0}^{x} 6t(1-t)dt$$

$$= 3x^2 - 2x^3$$

For $1 \le x < \infty$, $F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt$

$$= \int_{-\infty}^{0} f(t)dt + \int_{0}^{1} f(t)dt + \int_{1}^{x} f(t)dt$$

$$= \int_{-\infty}^{0} 0\ dt + \int_{0}^{1} 6t(1-t)dt + \int_{1}^{x} 0\ dt$$

$$= 1$$

Hence the distribution function of X is

$$F(x) = \begin{cases} 0 & -\infty < x < 0 \\ 3x^2 - 2x^3, & 0 \le x < 1 \\ 1 & 1 \le x < \infty \end{cases}$$

Ans.

Ex.4. The life (in hours) of a certain part of a radio tube is a continuous random variable X with density function

$$f(x) = \begin{cases} \dfrac{100}{x^2}, & x \ge 100 \\ 0, & \text{otherwise} \end{cases}$$

(i) What is the probability that all of three such tubes in a given radio set will have to be replaced during the first 150 hours of operation?

(ii) What is the probability that none of three of the original tubes will have to be replaced during the first 150 hours of operation?

(iii) What is the probability that a tube will last less than 200 hours if it is known that the tube is still functioning after 150 hours of service?

Solution: (i) A tube needs to be replaced within 150 hours if its life is less than or equal to 150 hours, i.e,1

$$P(X \leq 150) = \int_{100}^{150} f(x)dx = \int_{100}^{150} \frac{100}{x^2} dx = \frac{1}{3}$$

Therefore, probability that three such tubes need to be replaced within 150 hours of operation = $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$

(ii) Probability that a tube needs not to be replaced within 150 hours is given by

$$P(X > 150) = 1 - P(X \leq 150) = 1 - \frac{1}{3} = \frac{2}{3}$$

Hence, probability that none of three of the original tubes will have to be replaced during the first 150 hours of operation = $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$

(iii) Probability that a tube will last less than 200 hours is $P(X < 200)$ but when the knowledge of its lasting more than 150 hours is already known, then it becomes

$$P(X < 200 | X > 150) = \frac{P(150 < X < 200)}{P(X > 150)} = \frac{\int_{150}^{200} \frac{100}{x^2} dx}{\frac{2}{3}} = \frac{1}{4}$$

Ans.