Example: Let (X, Y) be jointly distributed continuous random variables with pdf

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{y}, & 0 < x < y < 1 \\ 0, & \text{ew} \end{cases}$$

The marginal pdf of X is

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \int_x^1 \frac{1}{y}\,dy = \begin{cases} -\log_e x, & 0<x<1 \\ 0, & \text{ew} \end{cases}$$

The marginal pdf of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_0^y \frac{1}{y}\,dx$$

$$= \begin{cases} 1, & 0<y<1 \\ 0, & \text{ew} \end{cases}$$

The conditional pdf of X given $Y=y$

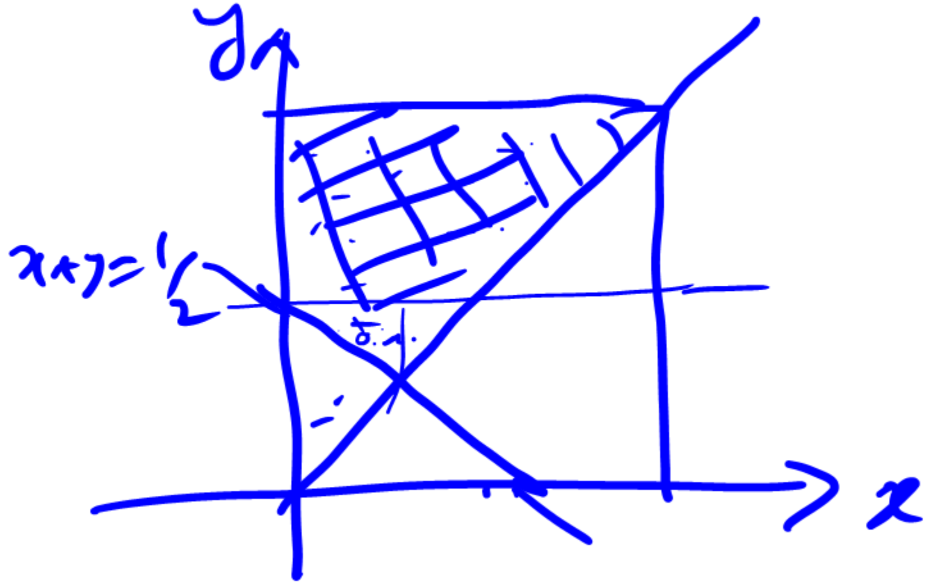
$$f_{X|Y=y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{1}{y}, & 0<x<y \\ 0, & \text{ew} \end{cases}$$

ie $X/_{Y=y} \sim U(0, y)$

The conditional pdf of Y given $X=x$ is

$$f_{Y/X=x}(y/x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} -\dfrac{1}{y \log_e x}, & x<y<1 \\ 0, & \text{ew} \end{cases}$$

$$P\left(X+Y > \frac{1}{2}\right)$$



$$= \iint_A f(x,y)\,dx\,dy \qquad A \to \text{shaded region}$$

$$= \int_{1/4}^{1/2} \int_{\frac{1}{2}-y}^{y} \frac{1}{y}\,dx\,dy + \int_{1/2}^{1} \int_{0}^{y} \frac{1}{y}\,dx\,dy$$

$$= 1 - \frac{1}{2} \log_e 2$$

$$P\left(X < \frac{1}{2} \,\middle|\, Y = 3/4\right) = \int_0^{1/2} \frac{4}{3}\,dx = \frac{2}{3}$$

$$X\big|_{Y=3/4} \sim U(0, 3/4)$$

$$P(Y > 3/4 \mid X = 1/2) = \frac{1}{\ln_e 2} \int_{3/4}^{1} \frac{1}{y}\,dy$$

$$f_{Y|X=1/2}(y) = + \frac{1}{y \ln_e 2}, \quad \frac{1}{2} < y < 1$$

$$= \frac{-\ln(3/4)}{\ln_e 2} = \frac{\ln 4 - \ln 3}{\ln 2} = \cdots$$

<u>Ex.</u> $X \rightarrow$ amount of oil in a tank (in thousands of litres)

at the beginning of the day

$Y \to$ sale during the day (proportions)

$$f_{X,Y}(x,y) = \begin{cases} 2, & 0 < y < x < 1 \\ 0, & \text{ew} \end{cases}$$

Find (i) marginal densities of X & Y

(ii) Conditional densities of $X \mid Y = y$ &

$Y \mid X = x$.

(iii) $P\left(X-Y>\frac{1}{2}\right)$, (iv) $P\left(X-Y>\frac{1}{4}\right)$

(v) $P\left(Y>\frac{1}{2} \,\middle|\, X=\frac{3}{4}\right)$, (vi) $P\left(X>2Y\right)$

Independence of Random Variables

We say that r.v's X & Y are independent if

$$F_{X,Y}(x,y) = F_X(x)\,F_Y(y)$$

for all $(x,y) \in \mathbb{R}^2$

or if $P((X,Y) \in Q) = P(X \in A_1)\, P(Y \in A_2)$ $\forall$ all

where $Q = A_1 \times A_2$

$Q \downarrow \in \mathbb{R}^2$, $A_1 \downarrow \mathbb{R}^1$, $A_2 \downarrow \mathbb{R}^1$

If both X and Y are discrete then the definition of independence reduces to

$$p_{X,Y}(x,y) = p_X(x)\, p_Y(y)$$

$\forall \ (x,y) \in \mathcal{X} \times \mathcal{Y}.$

In case (X,Y) are continuous, the condition becomes

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) \quad \forall (x,y) \in \mathbb{R}^2.$$

Examples: $f_{X,Y}(x,y) = \begin{cases} 2y e^{-x}, & x>0,\ 0<y<1 \\ 0, & \text{ew} \end{cases}$

$$f_X(x) = \int_0^1 2y e^{-x} \, dy = \begin{cases} e^{-x}, & 0<x \\ 0, & \text{ew} \end{cases}$$

$$f_Y(y) = \int_0^\infty 2y e^{-x}\, dx = \begin{cases} 2y, & 0<y<1 \\ 0, & ew \end{cases}$$

So $f_{X,Y}(x,y) = f_X(x) f_Y(y) \quad \forall (x,y) \in \mathbb{R}^2$

So X & Y are independent r.v's here.

2. Let (X,Y) be discrete with joint pmf

$$p_{X,Y}(i,j) = \left(\frac{e^{-2}\, 2^i}{i!}\right)\left(\frac{1}{4}\right)^{j-1}\left(\frac{3}{4}\right),$$

$i = 0, 1, 2 \cdots$

$j = 1, 2 \cdots$

The marginal pmf of X is

$$p_X(i) = \frac{e^{-2}\, 2^i}{i!} \sum_{j=1}^{\infty} \left(\frac{1}{4}\right)^{j-1} \left(\frac{3}{4}\right)$$

$$= \frac{e^{-2}\, 2^i}{i!}, \quad i = 0, 1, 2 \cdots$$

$X \sim P(2)$

The marginal pmf of Y is

$$p_Y(j) = \left(\frac{1}{4}\right)^{j-1}\left(\frac{3}{4}\right)\sum_{i=0}^{\infty}\frac{e^{-2}\,2^{i}}{i!}$$

$$= \left(\frac{1}{4}\right)^{j-1}\left(\frac{3}{4}\right), \quad j=1,2\ldots$$

So $Y \sim \mathrm{Geo}(3/4)$.

So $p_{X,Y}(i,j) = p_X(i)\, p_Y(j)$ $\forall (i,j)$

X & Y are independent here.

Let $g: \mathbb{R}^2 \to \mathbb{R}$ be a measurable fn.

We define

$$E\,g(X,Y) = \sum_{(x_i, y_j) \in \mathcal{X} \times \mathcal{Y}} \sum g(x_i, y_j)\, p_{X,Y}(x_i, y_j)$$

if (X,Y) is jointly discrete & the sum on the right is absolutely convergent.

If (X,Y) is jointly continuous,

$$E\, g(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\, \underbrace{dx\,dy}_{dy\,dx}$$

provided the integral is absolutely convergent.

Let $g(x,y) = x^r y^s$,

$r = 0, 1, 2, \ldots$

$s = 0, 1, 2, \ldots$

The $(r,s)^{th}$ noncentral product

moment of (X, Y) is defined as

$$\mu'_{r,s} = E(X^r Y^s)$$

$$\mu'_{1,1} = E(XY)$$ $\longrightarrow$ the first noncentral product moment

$\mu'_{1,0} = E(X)$, $\quad \mu'_{0,1} = E(Y)$.

$= \mu_X \qquad\qquad = \mu_Y$

We define $(r, s)^{th}$ central product

moment as

$$\mu_{r,s} = E\left[(X-\mu_X)^r (Y-\mu_Y)^s\right]$$

$$\mu_{1,1} = E\left[(X-\mu_X)(Y-\mu_Y)\right]$$

is called Covariance between X and Y

$$= E\left[XY - X\mu_Y - \mu_X Y + \mu_X\mu_Y\right]$$

$$= E(XY) - \mu_X\mu_Y - \cancel{\mu_X\mu_Y} + \cancel{\mu_X\mu_Y}$$

So $\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$

Suppose $g: \mathbb{R} \to \mathbb{R}$, $h: \mathbb{R} \to \mathbb{R}$ be measurable functions. If X and Y are independent r.v.'s then

$$E[g(X)h(Y)] = E\{g(X)\}E\{h(Y)\}$$

<u>Proof</u>: (continuous case) Suppose the joint pdf of (X,Y) is $f_{X,Y}(x,y)$

and the marginal pdf's of X and Y are $f_X(x)$ & $f_Y(y)$ respectively.

Then

$$E[g(X)h(Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y) f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y) f_X(x) f_Y(y)\,dx\,dy$$

$$= \left(\int_{-\infty}^{\infty} g(x) f_X(x)\,dx\right)\left(\int_{-\infty}^{\infty} h(y) f_Y(y)\,dy\right)$$

$$= \quad E\{g(X)\}\, E\{h(Y)\}$$

So if X and Y are independent,

then $\quad E(X^r Y^s) = E(X^r)\, E(Y^s)$

& $\quad \mathrm{Cov}(X,Y) = 0$

For r.v.'s X & Y, we define the coefficient of correlation $\rho_{X,Y}$ as

$$\rho_{X,Y} = \mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\mathrm{s.d.}(X)\ \mathrm{s.d.}(Y)}$$

If we use the notation

$\sigma_X^2 = \mathrm{Var}(X)$, $\quad \sigma_Y^2 = \mathrm{Var}(Y)$

$\sigma_{XY} = \mathrm{Cov}(X,Y)$, then

$$\rho_{X,Y} = \frac{\sigma_{XY}}{\sigma_X \ \sigma_Y}$$

Theorem: For any r.v.s X & Y

$$-1 \leq \rho_{X,Y} \leq 1$$

provided $\rho_{X,Y}$ exists.

Proof: Let $U = \dfrac{X-\mu_X}{\sigma_X}$, $V = \dfrac{Y-\mu_Y}{\sigma_Y}$.

$E(U)=0$, $E(V)=0$, $E(U^2)=1$,

$$E(V^2) = 1$$

Consider $E(U-V)^2 \geqslant 0$

$$\Rightarrow E(U^2) + E(V^2) - 2E(UV) \geqslant 0$$

$$\Rightarrow \quad E(UV) \leqslant 1 \quad \cdots (1)$$

Similarly consider $E(U+V)^2 \geq 0$

$$\Rightarrow E(U^2) + E(V^2) + 2E(UV) \geq 0$$

$$\Rightarrow \quad E(UV) \geq -1 \quad \cdots (2)$$

Now

$$E(UV) = E\left(\frac{X-\mu_X}{\sigma_X}\right)\left(\frac{Y-\mu_Y}{\sigma_Y}\right)$$

$$= \frac{\operatorname{Cov}(X,Y)}{\sigma_X \sigma_Y} = \rho_{X,Y}$$

Combining (1) & (2) we get

$$-1 \leq \rho_{X,Y} \leq 1$$

We now consider the cases where

equality at one end may be achieved. In (1) equality is achieved if $P(U = V) = 1$

or

$$P\left(\frac{X-\mu_X}{\sigma_X} = \sigma_X \frac{(Y-\mu_Y)}{\sigma_Y}\right) = 1$$

or $\quad P(X = bY + c) = 1$

where $b > 0$, $c \in \mathbb{R}$

ie X and Y are perfectly linearly related in positive direction

ie when X & Y are perfectly linearly related in +ve direction then

$$\rho_{X,Y} = 1$$

The equality in (2) is attained if

$$P(U = -V) = 1$$

ie $P(X = bY + c) = 1$

where $b < 0$, $c \in \mathbb{R}$.

ie $\rho_{X,Y} = -1$ if X & Y are perfectly linearly related in negative direction.

So $\rho_{X,Y}$ can be considered to be a measure of linear relationship between X and Y.

If $\rho_{X,Y} = 0$, we say that X and Y are uncorrelated.

If X and Y are independent, then $\text{Cov}(X,Y) = 0 \Rightarrow \rho_{X,Y} = 0$.

However, if $\rho_{X,Y} = 0$, it does not imply that X and Y are independent.

Let X and Y be discrete &

X \ Y	-1	0	1	$p_X(x)$
0	0	1/3	0	1/3
1	1/3	0	1/3	2/3
$p_Y(y)$	1/3	1/3	1/3	

$$E(XY) = 0\cdot(-1)\cdot 0 + 0\cdot 0\cdot \frac{1}{3} + 0\cdot 1\cdot 0$$

$$+ 1\cdot(-1)\cdot\frac{1}{3} + 1\cdot 0\cdot 0 + 1\cdot 1\cdot \frac{1}{3}$$

$$= 0$$

$$E(X) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = \frac{2}{3}$$

$$E(Y) = -1 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} = 0$$

So $\mathrm{Cov}(X,Y) = 0 \Rightarrow \rho_{X,Y} = 0$

But X and Y are not independent here.

<u>Example</u>: Let (X,Y) be jointly

distributed continuous r.v:s with joint pdf

$$f_{X,Y}(x,y) = \begin{cases} x+y, & 0<x<1, \\ & 0<y<1 \\ 0, & \text{ew} \end{cases}$$

$$E(XY) = \int_0^1 \int_0^1 xy\,(x+y)\,dx\,dy$$

$$= \frac{1}{3}$$

$$f_X(x) = \int_0^1 (x+y)\,dy = \begin{cases} x+\frac{1}{2}, & 0<x<1 \\ 0, & \text{ew} \end{cases}$$

$$f_Y(y) = \int_0^1 (x+y)\,dx = \begin{cases} y + \frac{1}{2}, & 0 < y < 1 \\ 0, & ew \end{cases}$$

$$E(X) = \int_0^1 x\left(x + \frac{1}{2}\right)dx = \frac{7}{12} = E(Y)$$

$$E(X^2) = \int_0^1 x^2\left(x + \frac{1}{2}\right)dx = \frac{5}{12} = E(Y^2)$$

$$V(X) = \frac{11}{144} = V(Y), \qquad Cov(X, Y) = -\frac{1}{144}$$

So $\rho_{X,Y} = -\frac{1}{11}$.

Ex. Find $\rho_{X,Y}$ for all examples of jointly discrete & continuous r.v's

Joint Moment Generating Function

$$M_{X,Y}(s,t) = E\left(e^{sX+tY}\right)$$

provided it exists for some $(s,t) \neq (0,0)$

Theorem: X & Y are independent

$$\Longleftrightarrow \quad M_{X,Y}(s,t) = M_X(s)\,M_Y(t)$$

$$\forall\, (s,t)$$

Theorem: If X and Y are independent

then $$M_{X+Y}(t) = M_X(t)\,M_Y(t)$$

$$\forall\, t \in \mathbb{R}.$$

Pf. $M_{X+Y}(t) = E\left(e^{t(X+Y)}\right)$

$= E\left(e^{tX} \cdot e^{tY}\right) = E\left(e^{tX}\right)\left(Ee^{tY}\right)$

$= M_X(t)\, M_Y(t)$

Bivariate Normal Distribution

A continuous jointly dist^d r.v. (X, Y) is said to have bivariate normal distribution if it has pdf

$$f_{X,Y}(x,y) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \operatorname{Exp}\left[-\frac{1}{2(1-\rho^2)} \left\{ \left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) \right\} \right]$$

$x, y, \mu_1, \mu_2 \in R$, $\sigma_1 > 0$, $\sigma_2 > 0$, $-1 < \rho < 1$

To find marginal pdf of X, we write $f(x,y)$ as

$$= \left[\frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2} \right] \underbrace{\left(\frac{1}{\sqrt{2\pi}\,\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2\sigma_2^2(1-\rho^2)}\left[\underbrace{y - \left\{ \mu_2 + \frac{\rho\sigma_2(x-\mu_1)}{\sigma_1} \right\}} \right]^2} \right)}$$

So integrating wrt y over $\mathbb{R}$, the term in parenthesis integrates to 1 and we get the marginal pdf of X

as $$f_X(x) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}$$

ie $X \sim N(\mu_1, \sigma_1^2)$

Also we get the conditional distn of Y given $X = x$ as

$$N\left(\mu_2 + \rho\,\sigma_2\left(\frac{x-\mu_1}{\sigma_1}\right),\ \sigma_2^2(1-\rho^2)\right)$$

Similarly, we can get marginal dist^n of $Y \sim N(\mu_2, \sigma_2^2)$ and the conditional dist^n of X given $Y=y$ as $N\left(\mu_1 + \rho\sigma_1\left(\frac{y-\mu_2}{\sigma_2}\right), \sigma_1^2(1-\rho^2)\right)$

Theorem: If $(X,Y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ then the marginal and conditional pdf's of X and Y & $X|Y=y$

$Y|X=x$ are all normal.

Conversely, if the marginal & conditional dist^{ns} are normal then

(X,Y) has BVN.

Thus if $(X,Y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$, then

$E(X) = \mu_x$, $V(X) = \sigma_1^2$, $E(Y) = \mu_2$

$V(Y) = \sigma_2^2$

$$E(X \mid Y=y) = \mu_1 + \rho\sigma_1\left(\frac{y-\mu_2}{\sigma_2}\right)$$

$$V(X \mid Y=y) = \sigma_1^2(1-\rho^2)$$

$$E(Y \mid X=x) = \mu_2 + \rho\,\sigma_2\left(\frac{x-\mu_1}{\sigma_1}\right)$$

$$V(Y \mid X=x) = \sigma_2^2(1-\rho^2).$$

Theorem: Let (X,Y) have a joint dist^n and $g: \mathbb{R}^2 \to \mathbb{R}$ be a measurable function. Then

$$E\,g(X,Y) = E^{Y}\left[E^{X|Y}\left\{g(X,Y)\,\middle|\,Y\right\}\right]$$

or

$$= E^{X}\left[E^{Y|X}\left\{g(X,Y)\,\middle|\,X\right\}\right]$$

Proof for continuous case $\to$

Suppose the notations are standard.

$$E\{g(X,Y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, \frac{f_{X,Y}(x,y)}{f_Y(y)}\, f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(x,y)\, f_{X|Y=y}(x|y)\, dx \right) f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \underbrace{E\{g(X,Y) \mid Y=y\}}\, f_Y(y)\, dy$$

$$= E^{Y}\left[E^{X|Y}\{g(X,Y) \mid Y\} \right]$$

For bivariate normal distn

$$Cov(X,Y) = E[(X-\mu_1)(Y-\mu_2)]$$

$$= E\left\{ E\left[(X-\mu_1)(Y-\mu_2) \,\middle|\, Y \right] \right\}$$

$$= E\left[(Y-\mu_2) \; E\left\{ (X-\mu_1) \,\middle|\, Y \right\} \right]$$

$$= E\left[(Y-\mu_2)\, \rho\, \sigma_1 \frac{(Y-\mu_2)}{\sigma_2} \right]$$

$$= \frac{\rho \sigma_1}{\sigma_2} E(Y-\mu_2)^2 = \frac{\rho \sigma_1 \sigma_2^2}{\sigma_2}$$

$$= \rho \sigma_1 \sigma_2$$

So $$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma_1 \sigma_2} = \rho.$$