Special Probability Distribution

Lesson 4: Exponential Distribution

The exponential distribution is used to model time intervals (T) between random events. In other words, it describes waiting time between Poisson occurrences. The number of events in a Poisson occurrence are discreet variables but the time intervals between such occurrences is a continuous random variable T. The distribution has the probability density function as

$$f(t) = \lambda e^{-\lambda t}, \qquad t \geq 0, \lambda > 0$$
$$= 0, \qquad \text{otherwise}$$

Here λ is called the rate parameter of the distribution.

Note:

(i) If times between random events follow the exponential distribution with rate λ, then the total number of events in a time period of length t follows the Poisson distribution with parameter λt

(ii) $f(t) \geq 0 \;\; \forall \, t \geq 0$

(iii) Mean of the distribution = $\mu = E(T) = \frac{1}{\lambda}$ and variance = $\sigma^2 = \frac{1}{\lambda^2}$

(iv) The corresponding distribution function will be given by

$$F(t) = P(T \leq t) = \int_0^t f(z)dz = 1 - e^{-\lambda t}$$

Problems:

Ex.1. The lifetime of an alkaline battery is exponentially distributed with $\lambda = 0.05$ per hour. (i) What are the mean and standard deviation of battery lifetime?

(ii) What are the probabilities for the battery to last between 10 and 15 hours and to last more than 20 hours?

Solution:

(i) Mean = standard deviation = $\frac{1}{\lambda} = \frac{1}{0.05} = 20$ hours

(ii) Let T be the lifetime (in hours) of the battery. Then probability for the battery to last between 10 and 15 hours is given by

$$P(10 < T < 15) = \int_{10}^{15} f(t)dt = \int_{10}^{15} \lambda e^{-\lambda t} dt = \int_{10}^{15} 0.05 e^{-0.05t} dt = 0.1341$$

Similarly, the probability for the battery to last more than 20 hours is given as

$$P(T > 20) = \int_{20}^{\infty} f(t)dt = \int_{20}^{\infty} 0.05 e^{-0.05t} dt = 0.3679$$

Ans.

Ex.2. Accidents occur with a Poisson distribution at an average rate of 2 per week.

(i) Calculate the probability of more than 3 accidents in any one week

(ii) What is the probability that at least two weeks will elapse between accidents?

Solution:

Let X = Number of accidents in a week. This is a discrete random variable following the Poisson distribution with mean = λ = 2

(i) Hence the probability that there will be more than 3 accidents in a week will be

$$P(X > 3) = 1 - P(X \leq 3)$$

$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)]$$

$$= 1 - \left[\frac{e^{-2}2^0}{0!} + \frac{e^{-2}2^1}{1!} + \frac{e^{-2}2^2}{2!} + \frac{e^{-2}2^3}{3!}\right]$$

= 0.14288.

(ii) Let T = waiting time (in weeks) between two consecutive accidents. Then T follows an exponential distribution with rate parameter $\lambda = 2$

Then probability that at least two weeks will elapse between accidents is given by

$P(T \geq 2) = \int_2^{\infty} 2e^{-2t} dt = 0.01832$ Ans.

Ex.3. The arrival rate of cars at a gas station is 40 customers per hour.

(i) What is the probability of having no arrival of cars in a 5 minute interval?

(ii) What are the mean and variance of number of arrivals of cars in 5 minutes?

(iii) What is the probability for having 3 arrivals of cars in a 5 minute interval?

Solution: Let T = time (in hours) between arrivals of two consecutive cars. It is given that $\lambda = 40$

(i) Probability of having no arrival of cars in a 5 minute interval is given by

$$P\left(T > \frac{5}{60}\right) = \int_{\frac{5}{60}}^{\infty} 40e^{-40t}dt = 0.03567$$

(ii) The number of arrivals of cars will follow the Poisson distribution with parameter = $\lambda t = 40 \times \frac{5}{60} = 3.333$. Thus the mean and variance of this Poisson distribution will be 3.333.

(iii) As the arrivals of cars follow the Poisson distribution, hence the probability of having 3 arrivals in a 5 minute interval = $P(X = 3) = \frac{e^{-3.333}(3.333)^3}{3!} = 0.2202$

Ans.

Ex.4. The lifetime T (years) of an electronic component is a continuous random variable with probability density function given by $f(t) = e^{-t}, t \geq 0$. Find the lifetime L which a typical component is 60% certain to exceed. If 5 components are sold to a manufacturer, find the probability that at least one of them will have a lifetime less than L years.

Solution: We have been given $P(T > L) = 0.6 \rightarrow \int_L^{\infty} e^{-t}dt = 0.6$

or, $e^{-L} = 0.6 \rightarrow L = 0.5108$ years

Let X = number of components sold to manufacturer with lifetime less than L years

Then the probability that there will be at least one of them having lifetime less than L years is given by $P(X \geq 1) = 1 - P(X = 0)$

$= 1 - {}^{n}C_x p^x q^{n-x}$, where $n = 5, x = 0,\ p = 0.4$ and $q = 0.6$

$= 1 - {}^{5}C_0 (0.4)^0 (0.6)^{5-0} = 1 - (0.6)^5 = 0.92$ Ans.