Example : The amount of rainfall recorded at a US weather station in January is r.v. X and the amount in February at the same weather station is r.v. Y. Suppose (X, Y) has a $BVN(6, 4, 1, 0.25, 0.1)$.

Find $P(X \leq 5)$, $P(Y \leq 5 \mid X = 5)$.

Solution : $X \sim N(6, 1)$

$$P(X \leq 5) = P\left(\frac{X-6}{1} \leq \frac{5-6}{1}\right) =$$

$$= P(Z \leq -1) = \Phi(-1) = 0.1587$$

The conditional distn of $Y / X = 5$

is normal with mean

$$= 4 + \frac{0.1 \times 0.5}{1}(5-6) = 3.95$$

and variance $0.25(1-0.01) = 0.2475$

So $P(Y \leq 5 / X=5) = P\left(Z \leq \frac{5-3.95}{\sqrt{.2475}}\right)$

$$= \Phi(2.11) = 0.9826.$$

2. $X_1 \to$ life of a tube (hrs)

$X_2 \to$ filament diameter (inches)

$(X_1, X_2) \sim BVN(2000, 0.1, 2500, 0.01, 0.87)$

If a filament diameter is 0.98, what is the probability that the tube will last 1950 hrs?

<u>Sol.^n</u> The conditional dist^n of X_1 given $X_2 = 0.098$ is normal with mean

$$2000 + \frac{0.87 \times 50}{0.1}(0.098 - 0.1)$$

$$= 2000.87$$

& variance $\quad 2500\left(1-(0.87)^2\right]$

$$= 607.25$$

$$P\left(X_1 > 1950 \;\middle|\; X_2 = 0.098\right)$$

$$= P\left(Z > \frac{1950-2000.87}{\sqrt{607.25}}\right) = \Phi(-2.06)$$

$$= 0.9803$$

The Joint M.G.F of a BVN

$$M_{X,Y}(s,t) = E\left(e^{sX+tY}\right)$$

$$= E^{Y}\left\{E\left(e^{sX+tY} \,\middle|\, Y\right)\right\}$$

$$= E\left\{e^{tY} \underbrace{E\left(e^{sX} \,\middle|\, Y\right)}\right\}$$

The inner expectation is the mgf of

the conditional distⁿ of X given Y at the point s.

$$= E\left[e^{tY} M_{X|Y}(s)\right]$$

$$= E\left(e^{tY} e^{\left\{\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(Y-\mu_2)\right\}s + \frac{1}{2}\sigma_1^2(1-\rho^2)s^2}\right)$$

$$= e^{\left\{\mu_1 s - \frac{\rho\sigma_1\mu_2}{\sigma_2} s + \frac{1}{2}\sigma_1^2(1-\rho^2)s^2\right\}} E\left[e^{Y\left(t + \frac{\rho\sigma_1}{\sigma_2} s\right)}\right]$$

$$= e^{\left\{\mu_1 s - \frac{\rho\sigma_1\mu_2 s}{\sigma_2} + \frac{1}{2}\sigma_1^2(1-\rho^2)s^2\right\}} M_Y\left(t + \frac{\rho\sigma_1}{\sigma_2} s\right)$$

$$= e^{\left[\mu_1 s - \cancel{\frac{\rho\sigma_1\mu_2 s}{\sigma_2}} + \frac{1}{2}\sigma_1^2 s^2 - \frac{1}{2}\sigma_1^2\rho^2 s^2 + \mu_2\left(t + \cancel{\frac{\rho\sigma_1}{\sigma_2} s}\right) + \frac{1}{2}\sigma_2^2\left(t + \frac{\rho\sigma_1}{\sigma_2} s\right)^2\right]}$$

$$= e^{[\mu_1 s + \mu_2 t + \frac{1}{2}\sigma_1^2 s^2 + \frac{1}{2}\sigma_2^2 t^2 + \rho\sigma_1\sigma_2 st]}$$

<u>Theorem</u> : Let $(X,Y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$.
Then X and Y are independent if and only if $\rho = 0$.

<u>Pf</u> $M_{X,Y}(s,t) = M_X(s) M_Y(t)$

$\Leftrightarrow \rho = 0$

Linearity Property of a BVN

Theorem: $(X, Y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$

$$\iff aX + bY \sim N(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\rho\sigma_1\sigma_2)$$

for all $a, b \in \mathbb{R}$ (a, b are not zero together)

Random Vectors:

$$\underline{X} = (X_1, \ldots, X_k) : \Omega \to \mathbb{R}^k$$

The joint cdf of $\underline{X}$ is

$$F_{\underline{X}}(\underline{x}) = P(X_1 \le x_1, \ldots\ldots, X_k \le x_k)$$

$$\underline{x} = (x_1, \ldots\ldots, x_k) \in \mathbb{R}^k.$$

Properties : i) $\lim\limits_{x_i \to -\infty} F_{\underline{X}}(\underline{x}) = 0$

$\forall \; i = 1, \ldots k.$

(ii) If we take limit as $x_i \to +\infty$ for some i, we get marginal cdf of $(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_k)$

(iii) $F_{\underline{X}}(\underline{x})$ is non-decreasing in each of its arguments

(iv) $F_{\underline{X}}(\underline{x})$ is continuous from right in each of its arguments.

The condition of independence is

$$F_{\underline{X}}(\underline{x}) = \prod_{i=1}^{k} F_{X_i}(x_i) \quad \forall \ \underline{x} \in \mathbb{R}^k$$

If $(X_1, \ldots, X_k)$ is discrete then we have $p_{\underline{X}}(\underline{x})$ as the joint pmf of $\underline{X}$

satisfying

(i) $p_{\underline{X}}(\underline{x}) \geq 0 \quad \forall \ \underline{x} \in \mathbb{R}^k$

(ii) $$\sum_{\underline{x} \in \mathcal{X}} \cdots \sum p_{\underline{X}}(\underline{x}) = 1$$

(iii) $p_{\underline{X}}(\underline{x}) = P(X_1 = x_1, \ldots, X_k = x_k)$.

The marginal distributions of any subset $(X_{r_1}, \ldots X_{r_m})$ of $(X_1 \ldots, X_k)$ $(m < k)$ can be obtained by summing over remaining subscripts.

If $\underline{X} = (X_1, \ldots, X_k)$ is jointly continuous with pdf $f_{\underline{X}}(\underline{x})$ satisfying

(i) $f_{\underline{X}}(\underline{x}) \geq 0 \quad \forall \; \underline{x} \in \mathbb{R}^k$

(ii) $$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\underline{X}}(\underline{x})\, dx_1 \cdots dx_k = 1$$

(iii) $\forall \quad A \subseteq \mathbb{R}^K$ (measurable)

$$P(\underline{X} \in A) = \int_{\underline{x} \in A} \cdots \int f_{\underline{X}}(\underline{x})\, d\underline{x}$$

The joint mgf of $\underline{X} = (X_1, \ldots, X_k)$ is defined as $\quad t = (t_1, \ldots, t_k)$

$$M_{\underline{X}}(\underline{t}) = E\left(e^{\underline{t}^T \underline{X}}\right) = E\left(e^{\sum_{i=1}^{k} t_i X_i}\right)$$

provided it exists in a neighbourhood of $\underline{0}$.

<u>Theorem</u> : $X_1, \ldots, X_k$ are independent

$$\Leftrightarrow \quad M_{\underline{X}}(\underline{t}) = \prod_{i=1}^{k} M_{X_i}(t_i) \quad \forall\, \underline{t} \in R^k$$

Theorem: If $X_1, \ldots, X_k$ are independent r.v's then

$$M_{\sum_{i=1}^{k} X_i}(t) = \prod_{i=1}^{k} M_{X_i}(t)$$

Additive Property of Binomial Distⁿ

Let $X_1, \ldots, X_k$ be independent r.v's and $X_i \sim Bin(n_i, p)$.

$i = 1 \ldots k$. Then $Y = \sum_{i=1}^{k} X_i \sim Bin(n, p)$

where $n = \sum_{i=1}^{k} n_i$

Pf. $M_Y(t) = \prod_{i=1}^{k} M_{X_i}(t)$

$= \prod_{i=1}^{k} (q + pe^t)^{n_i} = (q + pe^t)^{\sum n_i}$

which is mgf of $Bin(\sum n_i, p)$.

Additive Property of a Poisson Dist^n

Let $X_1 \ldots X_k$ be independent r.v's

with $X_i \sim \mathcal{P}(\lambda_i)$, $i=1 \ldots k$.

Then $Y = \sum_{i=1}^{k} X_i \sim \mathcal{P}(\lambda)$, where

$$\lambda = \sum_{i=1}^{k} \lambda_i .$$

Pf. $M_Y(t) = \prod_{i=1}^{k} M_{X_i}(t)$

$$= \prod_{i=1}^{k} \left[e^{\lambda_i (e^t - 1)} \right]$$

$$= e^{(\sum \lambda_i)(e^t - 1)} = e^{\lambda(e^t - 1)}$$

which is mgf of $P(\lambda)$ distn.

Theorem: Sum of independent and identical geometric r.v's is negative binomial.

Pf. Let $X_1 \ldots\ldots X_k$ be independent $Geo(p)$ r.v's. Let $Y = \sum_{i=1}^{k} X_i$

$$M_Y(t) = \prod_{i=1}^{k} M_{X_i}(t) = \left(\frac{pe^t}{1-qe^t}\right)^k$$

which is mgf of $NB(k, p)$.

Additive Property of Negative Binomial

Dist^n: If $X_1 \dots X_k$ are independent r.vs with $X_i \sim NB(r_i, p)$, $i=1 \dots k$, then $Y = \sum_{i=1}^{k} X_i \sim NB(\sum r_i, p)$.

Theorem: Let $X_1 \dots X_k$ be independent $Exp(\lambda)$ r.v's. Then $Y = \sum X_i$ has $Gamma(k, \lambda)$ dist^n.

Pf. $M_Y(t) = \prod_{i=1}^{k} M_{X_i}(t)$

$$= \left(\frac{\lambda}{\lambda - t}\right)^k$$ which is mgf

of Gamma (k, λ) distⁿ.

<u>Additive Property of a Gamma Distⁿ</u>

Let $X_1, \ldots, X_k$ be independent

r.v's and $X_i \sim \text{Gamma}(r_i, \lambda)$.

Then $Y = \sum_{i=1}^{k} X_i \sim \text{Gamma}\left(\sum r_i, \lambda\right)$.

Linearity Property of Normal Distⁿ

Let $X_1, \ldots, X_k$ be independent r.v.s & $X_i \sim N(\mu_i, \sigma_i^2)$, $i=1 \ldots k$. Let $Y = \sum_{i=1}^{k} (a_i X_i + b_i)$

Then $Y \sim N\left(\sum_{i=1}^{k} (a_i \mu_i + b_i), \sum_{i=1}^{k} a_i^2 \sigma_i^2 \right)$

Pf. $M_Y(t) = E\left(e^{tY}\right) = E\left[e^{t(\Sigma a_i X_i + b_i)}\right]$

$$= e^{t \Sigma b_i} \, E\left(e^{t \Sigma a_i X_i}\right)$$

$$= e^{t \Sigma b_i} \, E\left(\prod_{i=1}^{k} e^{(a_i t) X_i}\right)$$

$$= e^{t \Sigma b_i} \prod_{i=1}^{k} E\left\{e^{(a_i t) X_i}\right\}$$

$$= e^{(t\sum b_i)} \prod_{i=1}^{k} M_{X_i}(a_i t)$$

$$= e^{t\sum b_i} \cdot \prod_{i=1}^{n} e^{a_i t \mu_i + \frac{1}{2} a_i^2 t^2 \sigma_i^2}$$

$$= e^{t\sum_{i=1}^{k}(a_i\mu_i + b_i) + \frac{1}{2} t^2 \left(\sum a_i^2 \sigma_i^2\right)}$$

which is mgf of $N\left(\sum(a_i\mu_i+b_i), \sum a_i^2\sigma_i^2\right)$

Properties of Sums of Expectations / Variances / Covariances etc.

$$E\left[\sum_{i=1}^{n}(a_i X_i + b_i)\right] = \sum_{i=1}^{n}\left[a_i E(X_i) + b_i\right]$$

In particular $E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i)$

$$\mathrm{Var}\left(\sum_{i=1}^{n}(a_i X_i + b_i)\right)$$

$$= \sum_{i=1}^{n} a_i^2 \operatorname{Var}(X_i)$$

$$+ 2 \sum_{i<j} \sum a_i a_j \operatorname{Cov}(X_i, X_j)$$

In particular

$$\operatorname{Var}\left(\sum_{i=1}^{n} X_i \right) = \sum_{i=1}^{n} \operatorname{Var}(X_i) + 2 \sum_{i<j} \sum \operatorname{Cov}(X_i, X_j)$$

If $X_1, \dots X_n$ are independent, then

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) .$$

Transformations of Random Vectors

$\underline{X} = (X_1, \dots X_k)$ is a r.v. $\mathcal{Z}$

we define $y_i = g_i(\underline{x})$, $i = 1 \dots k$

Then $\underline{Y} = (Y_1, \dots Y_k)$ is a r.v.

We can find distⁿ of $\underline{Y}$ using

(i) Direct CDF, (ii) Direct PMF in case of discrete, (iii) M.G.F.

In case $\underline{X}$ & $\underline{Y}$ both are continuous

we have Jacobian approach.

Theorem: Let $\underline{X} = (X_1, \ldots, X_n)$ be continuous random vector with joint pdf $f_{\underline{X}}(\underline{x})$, $\underline{x} = (x_1, \ldots, x_n)$.

(a) Let $u_i = g_i(\underline{x})$, $i = 1 \ldots n$ and $\underline{U} = (U_1, \ldots, U_n) : \mathbb{R}^n \to \mathbb{R}^n$ be one-to-one transformation so that inverse transformation $x_i = h_i(\underline{u})$, $i = 1 \ldots n$ can be defined over the range of transformation.

(b) Assume that the transformation & its inverse are both continuous

(c) Assume that $\dfrac{\partial x_i}{\partial u_j}$, $i, j = 1 \ldots n$

exist & are continuous.

(d) Let Jacobian of transformations

$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial u_1} & \dfrac{\partial x_1}{\partial u_2} & \cdots & \dfrac{\partial x_1}{\partial u_n} \\ \vdots & & & \\ \dfrac{\partial x_n}{\partial u_1} & \dfrac{\partial x_n}{\partial u_2} & \cdots & \dfrac{\partial x_n}{\partial u_n} \end{vmatrix}$$

Let $J \neq 0$ in the range of transformation.

Then $\underline{U} = (U_1, \dots, U_n)$ is a continuous random vector with the joint pdf given by

$$f_{\underline{U}}(\underline{u}) = f_{\underline{X}}\left(h_1(\underline{u}), \dots, h_n(\underline{u})\right) |J|$$

<u>Example</u>: Let X_1, X_2, X_3 i.i.d. $\mathrm{Exp}(1)$.

$$Y_1 = X_1 + X_2 + X_3, \quad Y_2 = \frac{X_1+X_2}{X_1+X_2+X_3}, \quad Y_3 = \frac{X_1}{X_1+X_2}$$

$Y_1 \sim \text{Gamma}(3,1)$

This is a one-to-one transformation and the inverse transformation is

$$x_1 = y_1 y_2 y_3$$

$$x_2 = y_1 y_2 (1-y_3)$$

$$x_3 = y_1 (1-y_2)$$

$$J = \begin{vmatrix} y_2 y_3 & y_1 y_3 & y_1 y_2 \\ y_2(1-y_3) & y_1(1-y_3) & -y_1 y_2 \\ 1-y_2 & -y_1 & 0 \end{vmatrix}$$

$$= -y_1^2 y_2$$

The joint pdf of $\underline{X} = (X_1, X_2, X_3)$ is

$$f_{\underline{X}}(\underline{x}) = \prod_{i=1}^{3} f_{X_i}(x_i) = \begin{cases} e^{-\sum x_i}, & x_i > 0 \\ 0, & \text{ew} \end{cases}$$

So the joint pdf of $\underline{Y} = (Y_1, Y_2, Y_3)$ is

$$f_{\underline{Y}}(\underline{y}) = \begin{cases} e^{-y_1} y_1^2 y_2, & y_1 > 0,\ 0 < y_2 < 1,\ 0 < y_3 < 1 \\ 0, & \text{ew} \end{cases}$$

The marginal pdf of (Y_1, Y_2) is

$$f_{Y_1, Y_2}(y_1, y_2) = \begin{cases} e^{-y_1} y_1^2 y_2, & y_1 > 0,\ 0 < y_2 < 1 \\ 0, & \text{ew} \end{cases}$$

The marginal pdf of Y_1 is

$$f_{Y_1}(y_1) = \begin{cases} \frac{1}{2} y_1^2 e^{-y_1}, & y_1 > 0 \\ 0, & \text{ew} \end{cases}$$

Gamma (3, 1)

The marginal pdf of Y_2 is

$$f_{Y_2}(y_2) = \begin{cases} 2y_2, & 0 < y_2 < 1 \\ 0, & \text{ew} \end{cases} \sim \text{Beta}(2,1)$$

The marginal pdf of Y_3 is

$$f_{Y_3}(y_3) = \begin{cases} 1, & 0 < y_3 < 1 \\ 0, & \text{ew} \end{cases} \sim U(0,1)$$

We observe here that

$$f_{\underline{Y}}(\underline{y}) = \prod_{i=1}^{3} f_{Y_i}(y_i) \quad \forall \ \underline{y} \in \mathbb{R}^3.$$

So Y_1, Y_2, Y_3 are independently distributed.

<u>Example</u>: Let $X, Y \overset{i.i.d.}{\sim} U(0,1)$

$U = X+Y$, $V = X-Y$. Find joint and marginal dist^{ns} of U and V.

<u>Sol^n</u> The joint density of (X,Y) is

$$f_{X,Y}(x,y) = \begin{cases} 1, & 0<x<1,\ 0<y<1 \\ 0, & \text{ow} \end{cases}$$

$$x = \frac{u+v}{2}, \quad y = \frac{u-v}{2}$$

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

So the joint pdf of (U,V) is

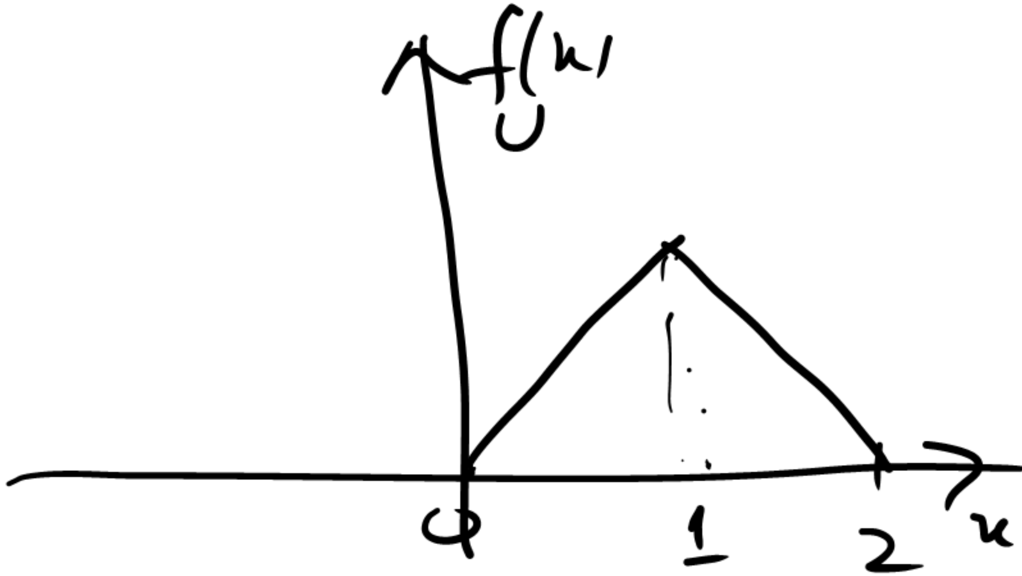
$$f_{U,V}(u,v) = \begin{cases} \frac{1}{2}, & 0<u+v<2,\ 0<u-v<2 \\ & 0<u<2,\ -1<v<1 \\ 0, & \text{ew} \end{cases}$$

The marginal pdf of U is

$$f_U(u) = \begin{cases} \frac{1}{2}\int_{-u}^{u} dv , & \text{if } 0 < u \leq 1 \\ \frac{1}{2}\int_{u-2}^{2-u} dv & 1 < u < 2 \\ 0, & \text{ew} \end{cases}$$

So

$$f_U(u) = \begin{cases} u, & 0 < u \leq 1 \\ 2-u, & 1 < u < 2 \end{cases}$$



The marginal pdf of V is

$$f_V(v) = \begin{cases} \frac{1}{2} \int_{-v}^{v+2} du \,, & -1 < v \le 0 \\ \frac{1}{2} \int_{v}^{2-v} du \,, & 0 < v < 1 \end{cases}$$

$$= \begin{cases} 1+v, & -1 < v \le 0 \\ 1-v, & 0 < v < 1 \end{cases}$$