Random Variable and Distribution Functions

Lesson 1: Discrete Random Variable

Random Variable:

For a mathematical definition of the random variable, let us consider the probability space (S, $\mathscr{B}$, P), where S = sample space, $\mathscr{B}$ = collection of all subsets of the sample space S of events and P = the probability function defined on $\mathscr{B}$

A random variable is a function $X(\omega)$ with domain S and range $\mathbb{R}$ such that for every real number a, the event $[\omega: X(\omega) \le a] \in \mathscr{B}$.

Properties:

(i) If X_1 and X_2 are random variables and c_1, c_2 are constants, then $c_1X_1 \pm c_2X_2$ are also random variables.

(ii) If X is a random variable, then $|X|$ and $\frac{1}{X}$ are also random variables

(iii) If X is a random variable and f is a continuous and increasing function, then $f(X)$ is also a random variable.

Distribution Function/Cumulative Distribution Function:

Let X be a random variable. The function F defined for all real x by

$$F(x) = P(X \le x) \quad -\infty < x < \infty$$

is called the distribution function of the random variable.

Properties:

1. If $F(x)$ is the distribution function of the random variable X and if $a < b$, then

   (i) $P(a < X \le b) = F(b) - F(a)$

(ii) $P(a \leq X \leq b) = P(X = a) + F(b) - F(a)$

(iii) $P(a < X < b) = F(b) - F(a) - P(X = b)$

(iv) $P(a \leq X < b) = P(X = a) + F(b) - F(a) - P(X = b)$

Note: If $P(X = a) = P(X = b) = 0$, then all four events have the same probability

2. $0 \leq F(x) \leq 1$

3. If $x < y$, then $F(x) \leq F(y)$

4. $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$ and $F(\infty) = \lim_{x \to \infty} F(x) = 1$

Type of Random Variable:

There are two types of random variables

- Discrete
- Continuous

Discrete Random Variable:

A real valued function defined on a discrete sample space is called a discrete random variable.

Probability Mass Function:

If X is a discrete random variable with distinct values $x_1, x_2, x_3, \ldots\ldots, x_n, \ldots.$ with probabilities $p_1, p_2, p_3, \ldots\ldots, p_n, \ldots.$ such that $\sum_{i=1}^{\infty} p_i = 1$, then the function $p(x)$ defined as

$$p(x) = \text{Probability that } X \text{ assumes the value } x$$

is called the probability mass function of the random variable X. The set of ordered pairs $\{(x_1, p_1), (x_2, p_2), (x_3, p_3), \ldots., (x_n, p_n), \ldots.\}$ is known as the probability distribution of the random variable X.

Note: The numbers p_i, i = 1, 2, … must satisfy the following conditions

a) $p_i \geq 0 \; \forall \; i$ and b) $\sum_{i=1}^{\infty} p_i = 1$

Discrete Distribution Function:

In this case, there are a countable number of points $x_1, x_2, ...$ and numbers $p_i \geq 0, \sum_{i=1}^{\infty} p_i = 1$ such that $F(x) = P(x_i \leq x)$.

Problems:

Ex.1. A random variable X has the following probability function:

$X = x$	0	1	2	3	4	5	6	7
$p(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

(i) Find k

(ii) Evaluate $P(X < 6),\; P(X \geq 6),\; P(0 < X < 5)$

(iii) If $P(X \leq a) > \frac{1}{2}$, find the minimum value of a

(iv) Write the distribution function of X

Solution: (i) We know that $\sum_{x=0}^{7} p(x) = 1 \rightarrow 10k^2 + 9k - 1 = 0$

$$\rightarrow k = \frac{1}{10} \text{ or } -1$$

But $p(x) \neq$ negative, so $k = \frac{1}{10}$

(ii) $P(X < 6) = P(X = 0) + P(X = 1) + \cdots + P(X = 5)$

$$= 0 + k + 2k + \ldots + k^2 = \frac{81}{100}$$

$P(X \geq 6) = 1 - P(X < 6) = \frac{19}{100}$

$P(0 < X < 5) = P(X = 1) + \cdots + P(X = 4) = \frac{4}{5}$

(iii) We know that $P(X \leq a) > \frac{1}{2}$. Let us substitute different values of a as:

If $a = 0 \to P(X \leq 0) = 0 < \frac{1}{2}$

$a = 1 \to P(X \leq 1) = 0 + \frac{1}{10} < \frac{1}{2}$

$a = 2 \to P(X \leq 2) = 0 + \frac{1}{10} + \frac{2}{10} = \frac{3}{10} < \frac{1}{2}$

$a = 3 \to P(X \leq 3) = 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} = \frac{5}{10} = \frac{1}{2}$

$a = 4 \to P(X \leq 4) = 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} = \frac{8}{10} = \frac{4}{5} > \frac{1}{2}$

This shows that $a = 4$.

(iv) The distribution function of X is given by

X	0	1	2	3	4	5	6	7
$F(x) = P(X \leq x)$	0	$k = \frac{1}{10}$	$3k = \frac{3}{10}$	$5k = \frac{5}{10}$	$8k = \frac{8}{10}$	$8k + k^2 = \frac{81}{100}$	$8k + 3k^2 = \frac{83}{100}$	$9k + 10k^2 = 1$

Ans.

Ex.2. Two dice are rolled. Let X denotes the random variable which counts the sum of the numbers on the upturned faces. Construct a table giving the non-zero values of the probability mass function. Also find the distribution function of X.

Solution: If both dice are unbiased and the two rolls are independent, then each sample point of the sample space S has probability $\frac{1}{36}$. Then

$$p(2) = P(X = 2) = P\{(1,1)\} = \frac{1}{36}$$

$$p(3) = P(X = 3) = P\{(1,2),(2,1)\} = \frac{2}{36}$$

$$p(4) = P(X = 4) = P\{(1,3),(3,1),(2,2)\} = \frac{3}{36}$$

$$p(5) = P(X = 5) = P\{(1,4),(4,1),(2,3),(3,2)\} = \frac{4}{36}$$

$$p(6) = P(X = 6) = P\{(1,5),(5,1),(2,4),(4,2),(3,3)\} = \frac{5}{36}$$

$$p(7) = P(X = 7) = P\{(1,6),(6,1),(2,5),(5,2),(3,4),(4,3))\} = \frac{6}{36}$$

$$p(8) = P(X = 8) = P\{(2,6),(6,2),(3,5),(5,3),(4,4)\} = \frac{5}{36}$$

$$p(9) = P(X = 9) = P\{(3,6),(6,3),(4,5),(5,4)\} = \frac{4}{36}$$

$$p(10) = P(X = 10) = P\{(4,6),(6,4),(5,5)\} = \frac{3}{36}$$

$$p(11) = P(X = 11) = P\{(5,6),(6,5)\} = \frac{2}{36}$$

$$p(12) = P(X = 12) = P\{(6,6)\} = \frac{1}{36}$$

Therefore the table for the probability mass function $p(x)$ and the probability distribution function $F(x)$ is given by:

X	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
$F(x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{15}{36}$	$\frac{21}{36}$	$\frac{26}{36}$	$\frac{30}{36}$	$\frac{33}{36}$	$\frac{35}{36}$	$\frac{36}{36}$

Ans.