k^{th} absolute moment about origin

$$\beta_k' = E|X|^k, \quad k = 1, 2 \ldots$$

k^{th} absolute moment about mean

$$\beta_k = E|X - \mu|^k, \quad k = 1, 2 \ldots$$

Factorial moment

$$\alpha_k = E\{X(X-1) \ldots (X-k+1)\}$$

$$k = 0, 1, 2 \ldots$$

Let X be continuous r.v. with pdf

$$f(x) = \begin{cases} \frac{2}{x^3}, & x \geq 1 \\ 0, & x < 1 \end{cases}$$

$$E(X) = \int_1^\infty \frac{2}{x^2}\,dx = 2$$

$$E(X^2) = \int_1^\infty \frac{2}{x}\,dx \rightarrow \text{does not exist}$$

<u>Example</u>. Suppose a car showroom has 10 cars out of which 3 have some defects. A customer buys two at random.

$X \rightarrow$ no of defective cars in purchase

$\rightarrow 0, 1, 2$

$$p_X(0) = \frac{\binom{7}{2}}{\binom{10}{2}} = \frac{7}{15}$$

$$p_X(1) = \binom{7}{1}\binom{3}{1} \Big/ \binom{10}{2} = \frac{7}{15}$$

$$p_X(2) = \binom{3}{2} \Big/ \binom{10}{2} = \frac{1}{15}$$

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 7/15 & 0 \leq x < 1 \\ 14/15, & 1 \leq x < 2 \\ 1, & x \geqslant 2 \end{cases}$$

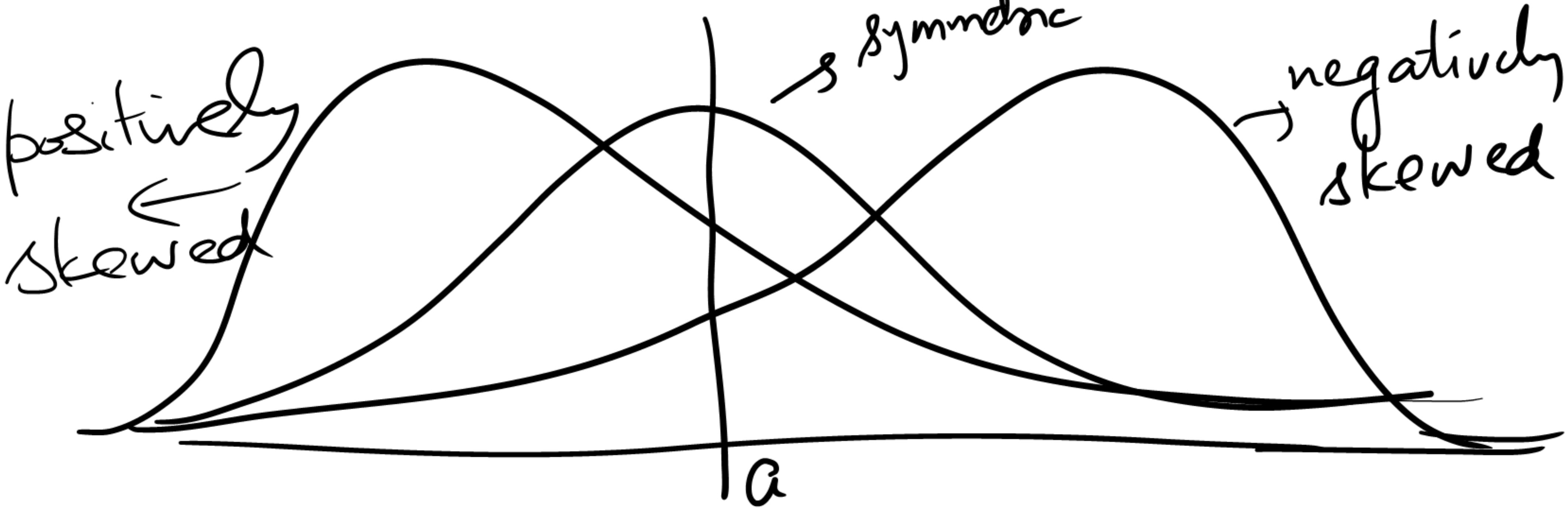
$$E(X) = 0 \cdot \frac{7}{15} + 1 \cdot \frac{7}{15} + 2 \cdot \frac{1}{15} = \frac{3}{5}$$

$$E(X^2) = 0^2 \cdot \frac{7}{15} + 1^2 \cdot \frac{7}{15} + 2^2 \cdot \frac{1}{15} = \frac{11}{15}$$

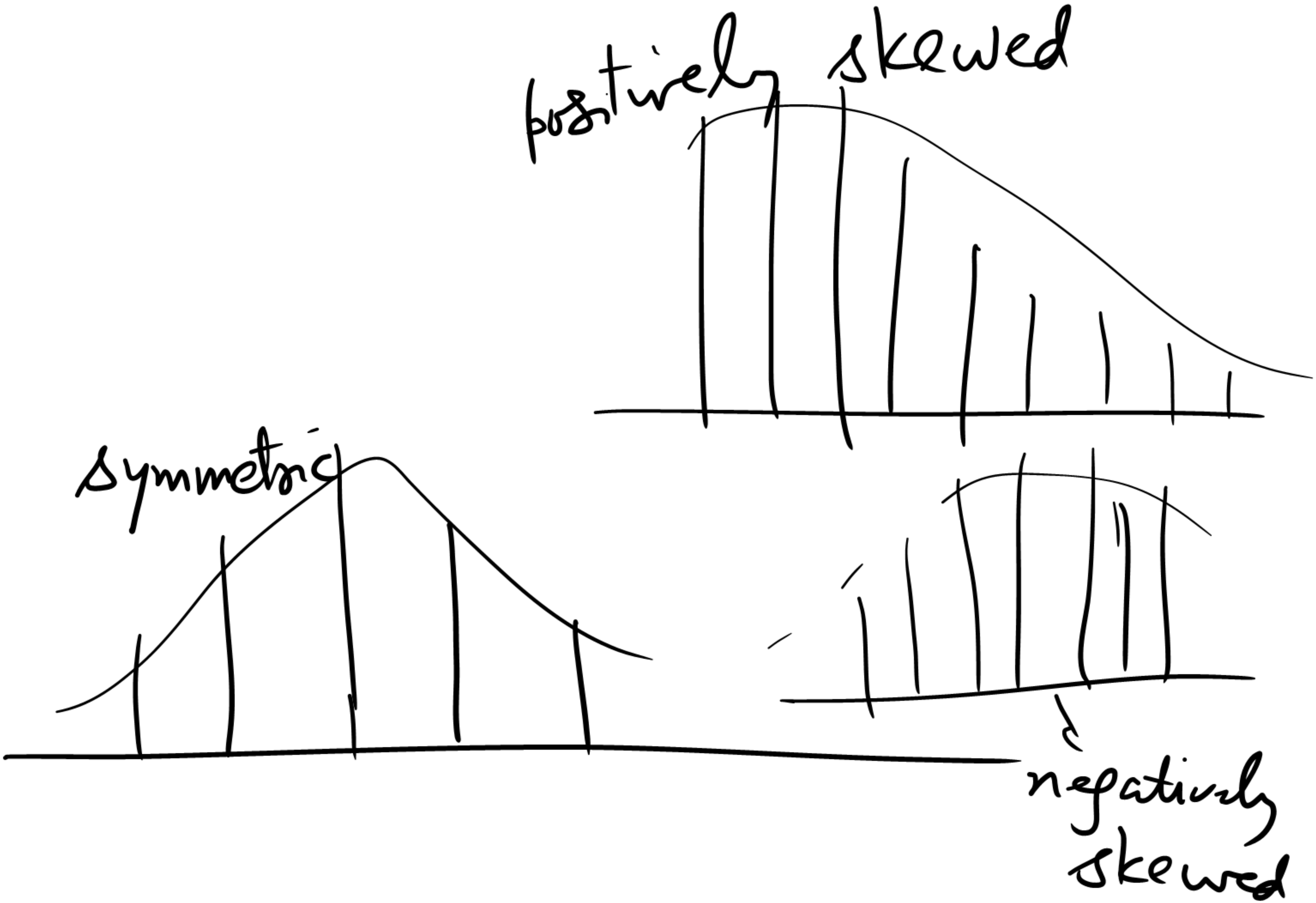
$$V(X) = \mu_2' - \mu_1'^2 = \frac{11}{15} - \frac{9}{25} = \frac{28}{75}$$

$$= 0.37$$

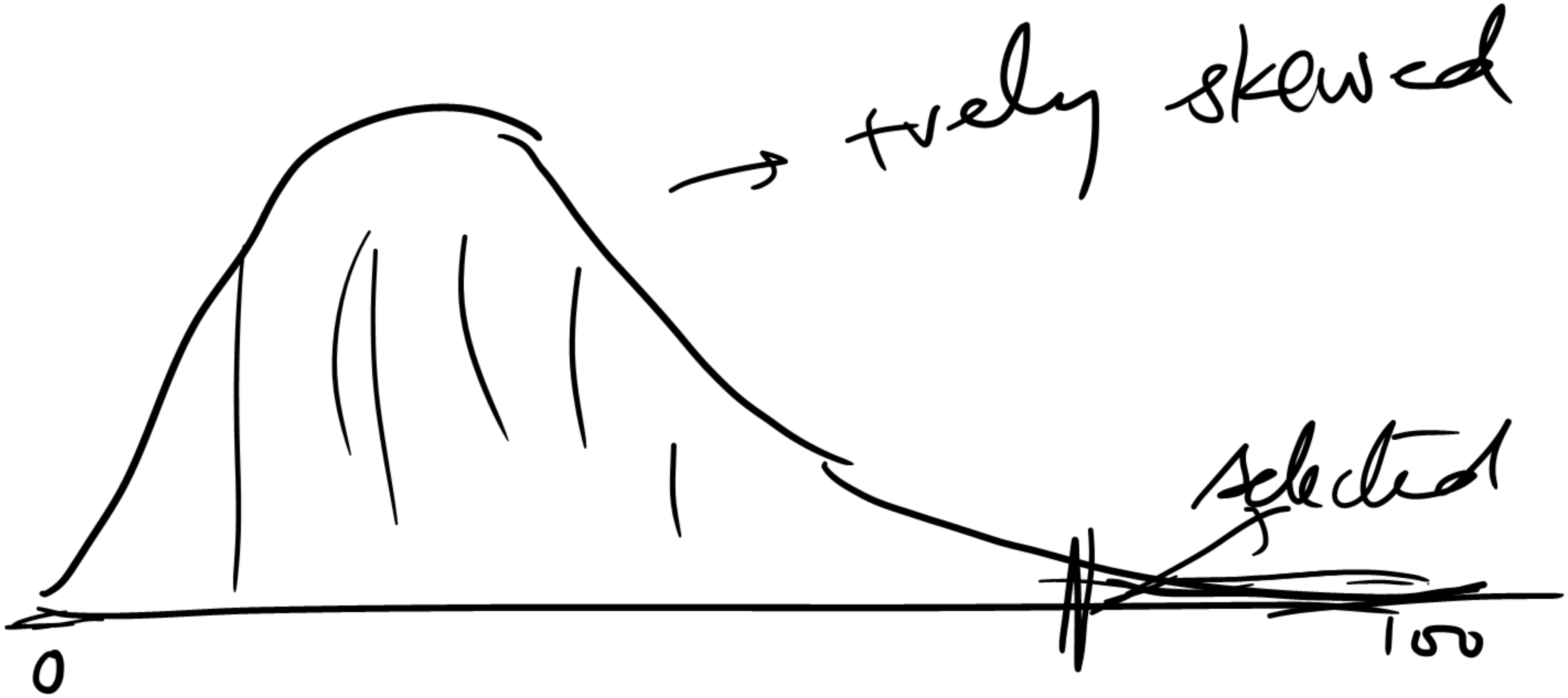
$$\text{s.d.} \simeq \boxed{0.6}$$



If a curve/distribution is not symmetric, it is called skewed


positively skewed
symmetric
negatively skewed

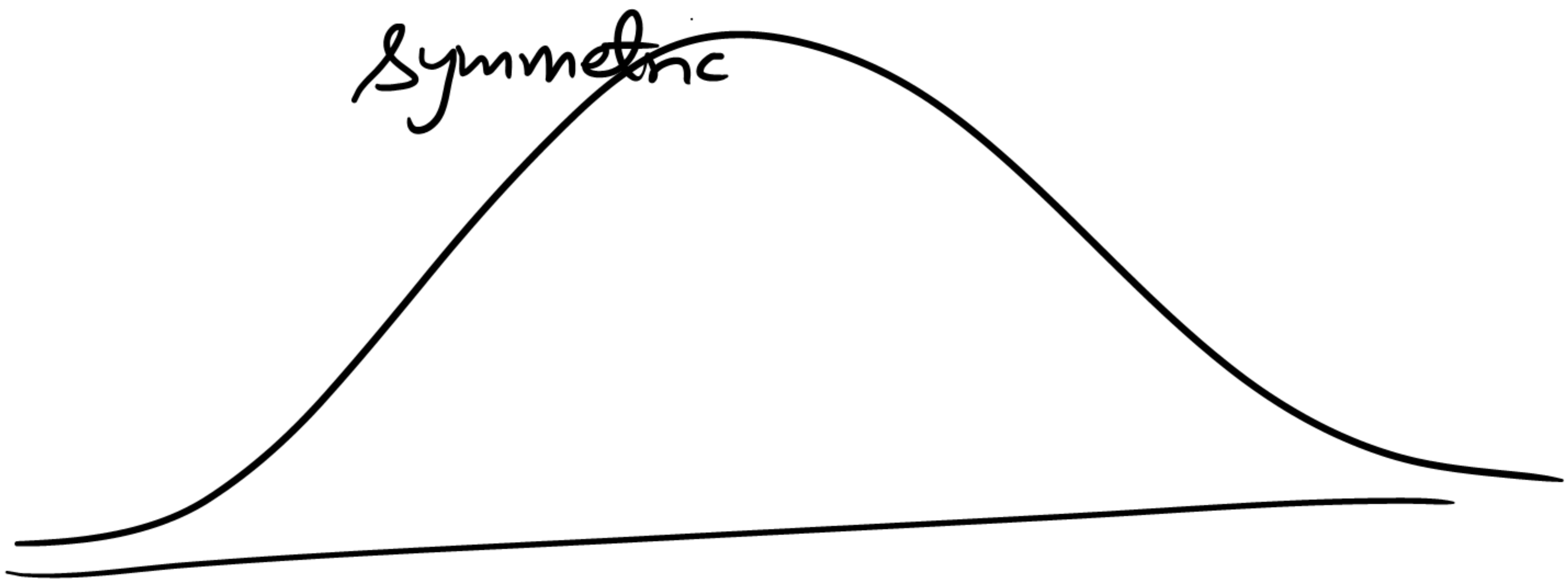
Examples: Marks of candidates in a competetive exam



Marks of students in a Secondary/ Higher Secondary exam

→ negatively skewed distn

heights of adult females in an ethnic group



A random variable X is symmetric about a poin α if

$$P(X \geq \alpha + x) = P(X \leq \alpha - x) \quad \forall\, x \in \mathbb{R}$$

or

$$F(\alpha - x) = 1 - F(\alpha + x) + P(X = \alpha + x) \quad \forall\, x \in \mathbb{R}$$

If $\alpha = 0$, then

$$F(-x) = 1 - F(x) + P(X = x)$$

If X is continuous with pdf $f(x)$,

$$f(\alpha - x) = f(\alpha + x) \quad \forall\ x$$

For $\alpha = 0$

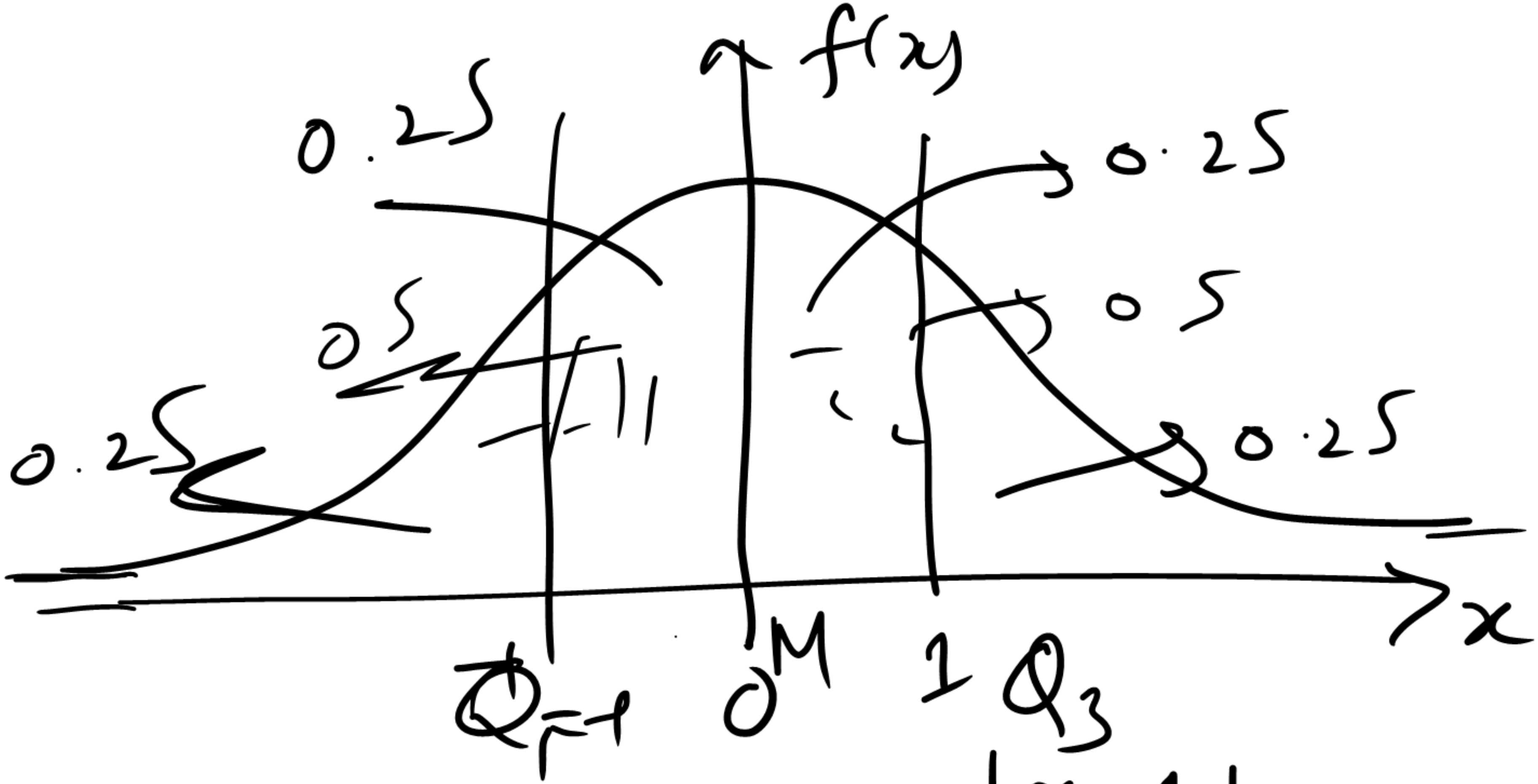
$$f(-x) = f(x) \quad \forall\ x$$

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad x \in \mathbb{R}$$

$\to$ Cauchy distn

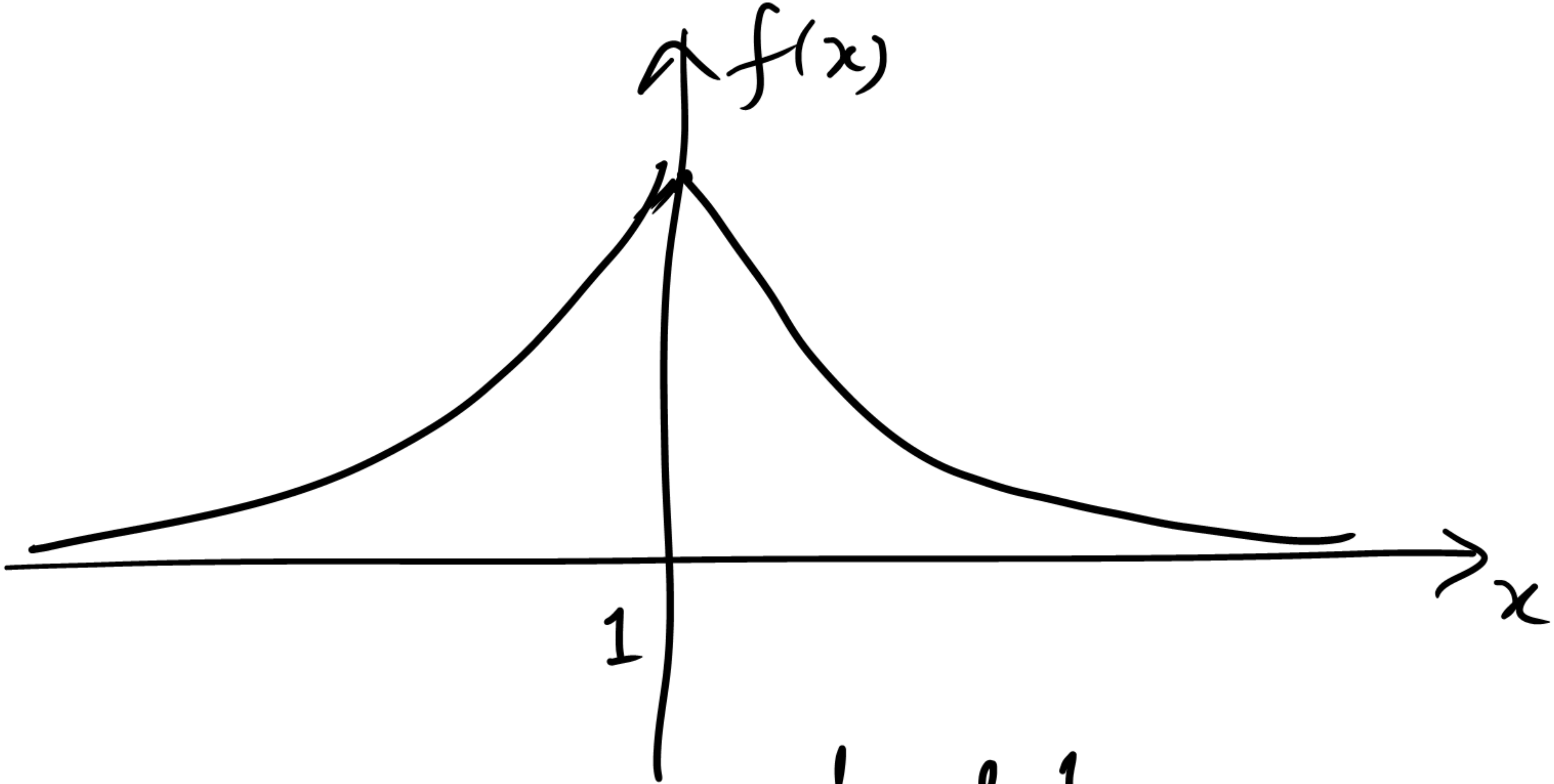
$$f(-x) = f(x) \quad \forall\ x$$

So f is symmetric about 0



$$f(x) = \frac{1}{2} e^{-|x-1|}, \quad x \in \mathbb{R}$$

double explonential or Laplace



symmetric about 1

$p_X(1) = \frac{1}{4}, \quad p_X(2) = \frac{1}{2}, \quad p_X(3) = \frac{1}{4}$

Then X is symmetric about 2.

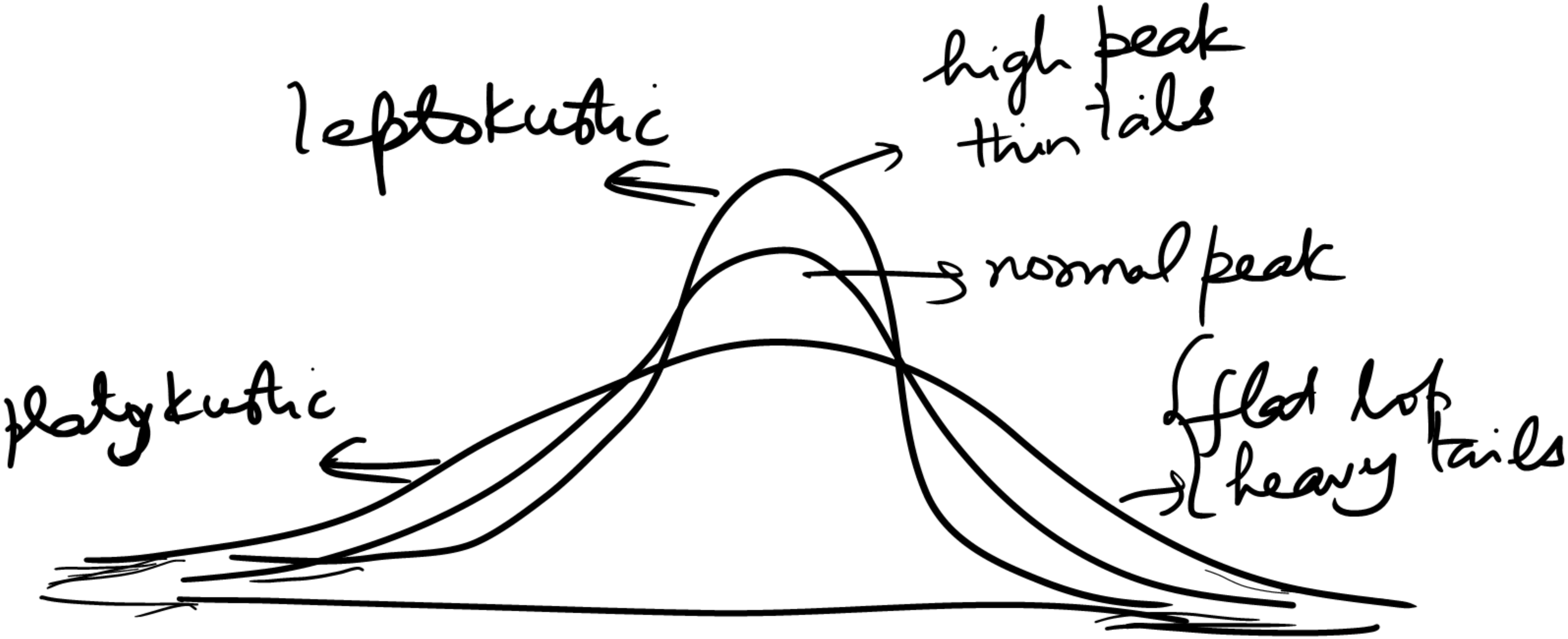
A measure of skewness is

$$\beta_1 = \frac{\mu_3}{\mu_2^{3/2}} = \frac{\mu_3}{\sigma^3}$$

For symmetric distn $\beta_1 = 0$

For +vely skewed distn. $\beta_1 > 0$

For −vely skewed distn $\beta_1 < 0$



Kurtosis → peakedness

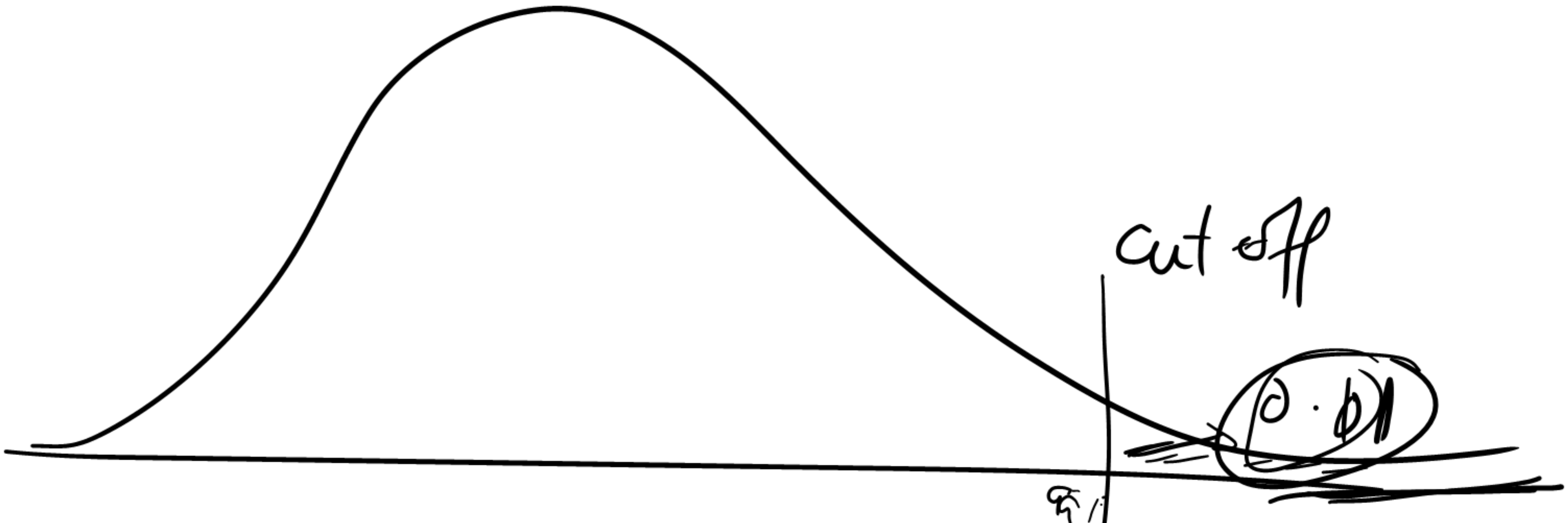
A measure of Kurtosis is defined as

$$\beta_2 = \left(\frac{\mu_4}{\sigma^4} - 3 \right)$$

$\beta_2 = 0 \rightarrow$ normal peak

$> 0 \rightarrow$ leptokurtic

$< 0 \rightarrow$ platikurtic

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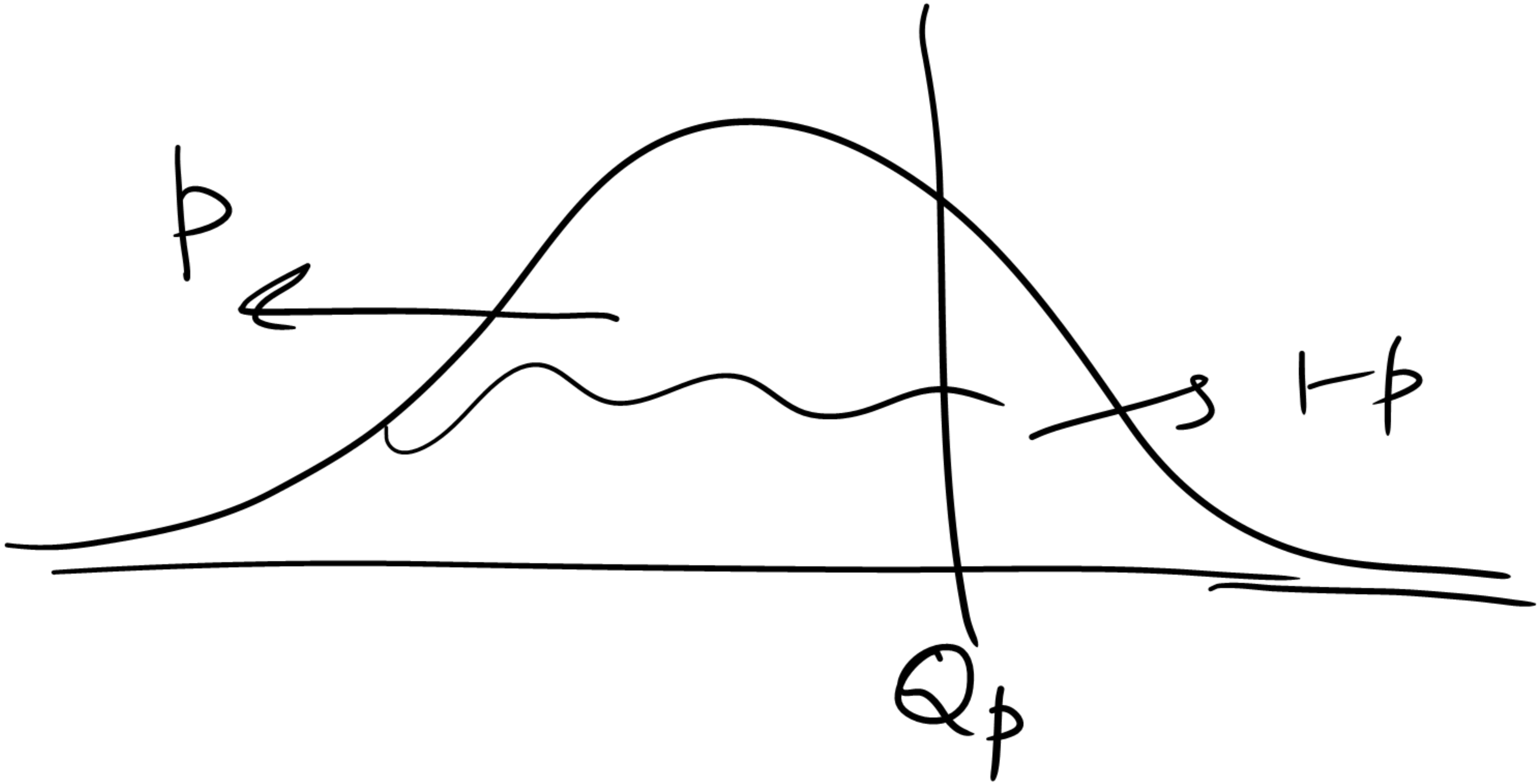
Quantiles : A number Q_p which satisfies

$$P(X \leq Q_p) = p \qquad 0 < p < 1,$$

$$P(X \geq Q_p) = 1 - p$$

is called p^{th} quantile of

distribution of X



If X is continuous then the condition is $F_X(Q_p) = p$

i.e. a unique quantile

<u>Example</u>

Consider Cauchy dist^n

$$f(x) = \frac{1}{\pi} \cdot \frac{1}{(1+x^2)}, \quad x \in \mathbb{R}$$

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\pi} \frac{1}{1+t^2}\, dt$$

$$= \frac{1}{\pi}\left(\tan^{-1} x + \frac{\pi}{2}\right)$$

$$= \frac{1}{2} + \frac{1}{\pi} \cdot \tan^{-1} x$$

$$p = \frac{1}{2} \rightarrow Q_{1/2} = \text{Median} = M$$

$F(0) = \frac{1}{2} \Rightarrow$ Median $= 0$
for Cauchy distⁿ.

Quartiles

$Q_1 \rightarrow$ for $p = 1/4$

$Q_3 \rightarrow$ for $p = 3/4$

Deciles $p = 1/10, 2/10, \ldots, 9/10$

Percentiles $p = 1/100, 2/100, \ldots, \frac{99}{100}$

For Cauchy dist^n

$$F(x) = \frac{1}{2} + \frac{1}{\pi} \cdot \tan^{-1}(x)$$

$x = -1$ $\qquad F(-1) = \frac{1}{4}$

$Q_1 = -1 \rightarrow$ first quartile

$x = 1$, $F(1) = \frac{3}{4}$, third quartile

<u>Example</u>: $p_X(1) = \frac{1}{4}$, $p_X(2) = \frac{1}{2}$,

$p_X(3) = \frac{1}{4}$

For median $P(X \le M) \geq \frac{1}{2}$ (1)

$P(X \geq M) \geq \frac{1}{2}$ (2)

$$\left.\begin{array}{ll} (1) \rightarrow & M \geq 2 \\ (2) & M \leq 2 \end{array}\right\} \Longrightarrow M = 2$$

Example $p_X(-2) = \frac{1}{4}$, $p_X(0) = \frac{1}{4}$

$p_X(1) = \frac{1}{3}$, $p_X(2) = \frac{1}{6}$

Here $0 \le M \le 1$

(not unique median)

Moment Generating Function

mgf

For a r.v. X, mgf is defined as

$$M_X(t) = E\left(e^{tX}\right)$$

provided the expectation exists for some $t \neq 0$.

$$E(e^{tX}) = E\left[1 + \frac{tX}{1!} + \frac{t^2X^2}{2!} + \cdots\right]$$

$$= 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots$$

i.e in mgf, the coefficient of $t^k/k!$ is μ'_k, $k = 1, 2 \ldots$

Also substituting $t = 0$ in k^{th} derivative of $M_X(t)$ wrt t, we get μ'_k.

Theorem: The mgf uniquely determines

a dist^n, and if the mgf exists, it is unique.

Example: $p_X(1) = \frac{1}{4}$, $p_X(2) = \frac{1}{2}$, $p_X(3) = \frac{1}{4}$

$$M_X(t) = E(e^{tX}) = \left(\frac{1}{4}e^{t} + \frac{1}{2}e^{2t} + \frac{1}{4}e^{3t}\right)$$

$$\frac{d}{dt} M_X(t) = \frac{1}{4} e^t + e^t + \frac{3}{4} e^t$$

Put $t=0$, $\mu_1' = \frac{1}{4} + 1 + \frac{3}{4} = 2$

$$Y = aX + b$$

$$M_Y(t) = E\left(e^{tY}\right) = E\left(e^{t(aX+b)}\right)$$

$$= e^{bt} E\left(e^{atX}\right) = e^{bt} M_X(at)$$