Basic Probability

Lesson 3: Conditional Probability; Baye's Theorem

Conditional Probability:

For any two events A and B, we have

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

-------------- (1)

Here $P(A|B)$ represents the conditional probability of occurrence of A when the event B has already happened. Similarly one can define

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Note: (i) The conditional probabilities $P(A|B)$ and $P(B|A)$ exist only when $P(B) \neq 0$ and $P(A) \neq 0$.

(ii) From equation (1), we can write for any two events A and B,

$$P(A \cap B) = P(A|B)P(B), \; P(B) > 0$$

$$= P(B|A)P(A), \; P(A) > 0$$

This is known as Multiplication Theorem.

Independent Events:

Two events A and B are called independent events, if and only if,

$$P(A \cap B) = P(A)P(B)$$

This means, the conditional probabilities would become,

$$P(A|B) = P(A) \text{ and } P(B|A) = P(B)$$

Th.: If A and B are independent events, then

(i) A and $\bar{B}$ are also independent
(ii) $\bar{A}$ and B are also independent
(iii) $\bar{A}$ and $\bar{B}$ are also independent

Problems:

Ex.1. A problem in Statistics is given to three students A, B and C whose chances of solving it are ½, ¾ and ¼. What is the probability that the problem will be solved if all of them try independently?

Solution: Let A, B and C denote the events that the problem will be solved by the students A, B and C respectively. Then $P(A) = \frac{1}{2}$, $P(B) = \frac{3}{4}$ and $P(C) = \frac{1}{4}$.

The problem will be solved if any one of them solves it. Hence we need to find $P(A \cup B \cup C)$. Now

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

$$= P(A) + P(B) + P(C) - P(A)P(B) - P(A)P(C) - P(B)P(C) + P(A)P(B)P(C)$$

[since the events A, B and C are independent to each other]

$$= \frac{1}{2} + \frac{3}{4} + \frac{1}{4} - \left(\frac{1}{2} \times \frac{3}{4}\right) - \left(\frac{1}{2} \times \frac{1}{4}\right) - \left(\frac{3}{4} \times \frac{1}{4}\right) + \left(\frac{1}{2} \times \frac{3}{4} \times \frac{1}{4}\right) = \frac{29}{32}.$$

Ans.

Ex.2. The odds against manager X settling the wage dispute with the workers are given by 8 : 6. The odds in favor of manager Y settling the same dispute are given by 14 : 16.

(i) What is the chance that neither settles the dispute, if they both try independently of each other?
(ii) What is the probability that the dispute will be settled?

Solution: Let X : event that manager X settles the dispute And Y : event that manager Y settles the dispute. Then $P(X) = \frac{6}{14}$ and $P(Y) = \frac{14}{30}$

(i) The dispute will not be settled when neither X nor Y can settle it. The event for this is given by $(\bar{X} \cap \bar{Y})$. Hence

$P(\bar{X} \cap \bar{Y}) = P(\bar{X})P(\bar{Y})$ [since X and Y are independent, so are $\bar{X}$ and $\bar{Y}$]

$$= [1 - P(X)] \times [1 - P(Y)] = \left(1 - \frac{6}{14}\right)\left(1 - \frac{7}{15}\right) = \frac{32}{105}.$$

(ii) The dispute will be settled if any of them settles it. The event for this is $P(X \cup Y)$. Hence

$$P(X \cup Y) = 1 - P(\overline{X \cup Y}) = 1 - P(\bar{X} \cap \bar{Y}) = 1 - \frac{32}{105} = \frac{73}{105}$$

Ans.

Ex.3. Given that $P(A) = \frac{3}{8}$, $P(B) = \frac{5}{8}$ and $P(A \cup B) = \frac{3}{4}$. Find $P(A|B)$ and $P(B|A)$. Are A and B independent?

Solution: We know that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

or, $\frac{3}{4} = \frac{3}{8} + \frac{5}{8} - P(A \cap B)$

or, $P(A \cap B) = \frac{1}{4}$

Then, $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{5}{8}} = \frac{2}{5}$ and $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{4}}{\frac{3}{8}} = \frac{2}{3}$

However, $P(A) \times P(B) = \frac{3}{8} \times \frac{5}{8} = \frac{15}{64} \neq P(A \cap B) = \frac{1}{4}$. Hence the events A and B are not independent.

Ans.

Baye's Theorem:

If $E_1, E_2, \ldots, E_n$ are mutually disjoint events with $P(E_i) \neq 0$ (i = 1, 2, …., n), then for any arbitrary event A which is a subset of $\cup_{i=1}^{n} E_i$ such that $P(A) > 0$, we have

$$P(E_i|A) = \frac{P(E_i)P(A|E_i)}{\sum_{i=1}^{n} P(E_i)P(A|E_i)} = \frac{P(E_i)P(A|E_i)}{P(A)}$$

Problems:

Ex.1. The probabilities of X, Y and Z becoming managers are 4/9, 2/9 and 1/3 respectively. The probabilities that a Bonus Scheme will be introduced if X, Y and Z become managers are 3/10, ½ and 4/5 respectively.

(i) What is the probability that the Bonus Scheme will be introduced?
(ii) If the Bonus Scheme has been introduced, what is the probability that the manager appointed was X?

Solution: Let X, Y and Z : events that X, Y and Z become managers respectively

B : Bonus Scheme is introduced

Then we have $P(X) = \frac{4}{9}$, $P(Y) = \frac{2}{9}$ and $P(Z) = \frac{1}{3}$

We also have $P(B|X) = \frac{3}{10}$, $P(B|Y) = \frac{1}{2}$ and $P(B|Z) = \frac{4}{5}$

(i) The probability that the Bonus Scheme will be introduced is given by

$$P(B) = P(X)P(B|X) + P(Y)P(B|Y) + P(Z)P(B|Z)$$

$$= \frac{23}{45} = 0.511$$

(ii) We need to find $P(X|B)$ which is given by Baye's theorem

$$P(X|B) = \frac{P(X)P(B|X)}{P(B)} = \frac{6}{23} = 0.261$$

Ans.

Ex.2. The chances that Dr. A will diagnose a disease correctly is 60%. The chances that a patient will die by his treatment after correct diagnosis is 40% and the chance of death by wrong diagnosis is 70%. A patient of Dr. A died who had the disease. What is the chance that the disease was diagnosed correctly?

Solution: Let C: event that the disease was diagnosed correctly

W: event that the disease was diagnosed wrongly

and D: event that the patient having the disease, dies

We have, $P(C) = 0.6$, $P(W) = 1 - P(C) = 0.4$, $P(D|C) = 0.4$, $P(D|W) = 0.7$

We need to find $P(C|D)$. According to Baye's theorem,

$$P(C|D) = \frac{P(C)P(D|C)}{P(D)} \text{ ------------ (1)}$$

Probability of death, $P(D) = P(C)P(D|C) + P(W)P(D|W) = 0.52$

Then from equation (1), we have,

$$P(C|D) = \frac{0.6 \times 0.4}{0.52} = 0.46$$

Ans.

Ex.3. In a bolt factory, machines A, B and C manufacture respectively 25%, 35% and 40% of the total output. Of their output, 5%, 4% and 2% are defective bolts. A bolt is drawn at random from the product output and is found to be defective. What are the probabilities that it was manufactured by machines A, B and C?

Solution: Let A, B and C: events that the bolt selected at random was manufactured by the machines A, B and C respectively

D: event that a defective bolt is chosen

Then we have, $P(A) = 0.25$, $P(B) = 0.35$ and $P(C) = 0.4$. Further we know that $P(D|A) = 0.05$, $P(D|B) = 0.04$ and finally $P(D|C) = 0.02$.

The probability that a defective bolt is chosen is given by

$$P(D) = P(A)P(D|A) + P(B)P(D|B) + P(C)P(D|C)$$

$$= 0.25 \times 0.05 + 0.35 \times 0.04 + 0.4 \times 0.02$$

$$= 0.0125 + 0.014 + 0.008 = 0.0345$$

Then the probability that it came from the machine A will be given by Baye's theorem as

$$P(A|D) = \frac{P(A)P(D|A)}{P(D)} = \frac{0.25 \times 0.05}{0.0345} = 0.36$$

Similarly, the probabilities that it was manufactured from machines B and C are

$$P(B|D) = \frac{P(B)P(D|B)}{P(D)} = \frac{0.35 \times 0.04}{0.0345} = 0.4$$

and

$$P(C|D) = \frac{P(C)P(D|C)}{P(D)} = \frac{0.4 \times 0.02}{0.0345} = 0.232$$

Ans.