Special Probability Distribution

Lesson 5: Gamma Distribution

Let us consider a system consisting of one original and (n – 1) spare components such that in the case of failure of the original component, one of the (n – 1) spare components can be used. When the last component fails, the whole system fails. Let $X_1, X_2, \ldots, X_n$ be the lifetimes of the n components. Then each of $X_1, X_2, \ldots, X_n$ will be random variables following identical exponential distribution with parameter λ and will be independent to each other. Then the total lifetime (time until total failure) of the entire system is

$$T = \sum_{i=1}^{n} X_i$$

This new random variable will have the following probability density function

$$f(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{\Gamma(n)}, t \geq 0, \quad \lambda, n > 0 - - - - - - - (1)$$

which is known as the gamma distribution with two parameters λ and n. Here λ is called the rate parameter and n is called the shape parameter of the distribution. If $\lambda = 1$, then from equation (1), we get

$$f(t) = \frac{t^{n-1} e^{-t}}{\Gamma(n)}, \qquad t \geq 0, \quad n > 0 - - - - - (2)$$

This distribution is called the standard gamma distribution with single parameter n.

Note:

(i) If n = 1, then from equation (1), we get $f(t) = \lambda e^{-\lambda t}, \ \ t \geq 0$, which is the density function of the exponential distribution

(ii) For large n, gamma distribution tends to normal distribution

(iii) Mean of the distribution = $\mu = \frac{n}{\lambda}$ and variance = $\sigma^2 = \frac{n}{\lambda^2}$

Problems:

Ex.1. Suppose that the reaction time X has a standard gamma distribution with n = 2. Evaluate (i) $P(3 \leq X \leq 5)$ (ii) $P(X > 4)$

Solution:

The associated density function to the random variable X is given by

$$f(x) = \frac{x^{n-1}e^{-x}}{\Gamma(n)} \text{ for } n = 2 \quad \text{i.e,} \quad f(x) = \frac{xe^{-x}}{\Gamma(2)} = xe^{-x}, x \geq 0$$

(i) $P(3 \leq X \leq 5) = \int_3^5 xe^{-x}dx = 0.15872$

(ii) $P(X > 4) = \int_4^\infty xe^{-x}dx = 0.09158$

Ans.

Ex.2. The survival time in weeks of an animal when subjected to certain exposure of gamma radiation has a gamma distribution with n = 5 and $\lambda = \frac{1}{10}$

(i) What is the mean survival time of a randomly selected animal?

(ii) What is the probability that an animal survives more than 30 weeks?

Solution:

Let X = survival time of the animal in weeks.

(i) Mean survival time of the animal = $\frac{n}{\lambda} = \frac{5}{1/10} = 50$ weeks

(ii) The probability that the animal survives more than 30 weeks is

$P(X > 30) = \int_{30}^\infty \frac{\lambda^n x^{n-1}e^{-\lambda x}}{\Gamma(n)}dx$, with n = 5 and $\lambda = \frac{1}{10}$

$$= \int_{30}^\infty \frac{\left(\frac{1}{10}\right)^5 x^4 e^{-\frac{x}{10}}}{\Gamma(5)}dx$$

$= 0.8155$

Ans.

Ex.3. The daily consumption of electric power (in millions of KW hours) in a certain city is a random variable X following gamma distribution with parameters $n = 2$ and $\lambda = \frac{1}{3}$. Find the probability that the power supply is inadequate on any given day if the daily capacity of the power plant is 12 million KW hours.

Solution:

The power supply will be inadequate on any given day when the daily consumption of the city will be more than the daily capacity of the power plant.

The probability for the event is given by

$$P(X > 12) = \int_{12}^{\infty} \frac{\lambda^n x^{n-1} e^{-\lambda x}}{\Gamma(n)} dx \ \text{ with } n = 2 \text{ and } \lambda = \frac{1}{3}$$

$$= \int_{12}^{\infty} \frac{\left(\frac{1}{3}\right)^2 x^{2-1} e^{-\frac{x}{3}}}{\Gamma(2)} dx$$

$= 0.09158.$

Ans.