Random Variable and Distribution Functions

Lesson 6: Chebyshev's Inequality

Bounds on Probability:

If X is a random variable with mean μ and a finite variance σ^2, then for any positive number k, we have

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

OR ------------------ (1)

$$P(|X-\mu| < k\sigma) \geq 1-\frac{1}{k^2}$$

A convenient form of the above statement can be found by taking $k\sigma = \rho$ with $\rho > 0$, i.e,

$$P(|X-\mu| \geq \rho) \leq \frac{\sigma^2}{\rho^2}$$

OR ------------------ (2)

$$P(|X-\mu| < \rho) \geq 1-\frac{\sigma^2}{\rho^2}$$

The above inequality gives us bounds on probability that how far a random variable X is deviated when both mean μ and variance σ^2 of the distribution are known. Since the standard deviation σ gives us the idea of variability of data around the mean, thus for smaller values of σ, there will be higher probability that the data will be concentrated around the mean. The inequality also helps when the probability distribution is unknown, but better bounds are found for known distributions.

Problems:

Ex.1. A random variable X is normally distributed with mean μ and variance σ^2. Compute $P(|X-\mu| \geq 2\sigma)$

Solution: We know that

$$P(|X-\mu| \geq \rho) \leq \frac{\sigma^2}{\rho^2}$$

From this, we can write

$$P(|X-\mu| \geq 2\sigma) \leq \frac{\sigma^2}{(2\sigma)^2} = \frac{1}{4} = 0.25$$

Ans.

Ex.2. Let X be a random variable such that $E(X) = 3$ and $E(X^2) = 13$. Calculate a lower bound for the probability that X lies between (- 2) and 8 using Chebyshev's inequality.

Solution: We need to evaluate a lower bound of $P(-2 < X < 8)$. We know that

$\mu = E(X) = 3$ and $\sigma^2 = V(X) = E(X^2) - \{E(X)\}^2 = 13 - 9 = 4 \rightarrow \sigma = 2$

As a lower bound is required to be found, we shall use

$$P(|X-\mu| < \rho) \geq 1 - \frac{\sigma^2}{\rho^2}$$

$$\text{i.e,} \quad P(-\rho < X - 3 < \rho) \geq 1 - \frac{4}{\rho^2}$$

$$\text{or,} \quad P(3-\rho < X < 3+\rho) \geq 1 - \frac{4}{\rho^2}$$

By taking $\rho = 5$, we can write

$$P(-2 < X < 8) \geq 1 - \frac{4}{25} = 0.84$$

Hence the lower bound on the required probability is 0.84.

Ans.

Ex.3. The number of items cleared by an assembly line during a week is a random variable with mean 50 and variance 25. What can be said about the probability that this week's clearance will be between 40 and 60?

Solution: It is given that $\mu = 50$ and $\sigma^2 = V(X) = 25$. We also know that

$$P(|X-\mu| < \rho) \geq 1 - \frac{\sigma^2}{\rho^2}$$

$$\text{i.e,}\quad P(-\rho < X - 50 < \rho) \geq 1 - \frac{25}{\rho^2}$$

$$\text{or,}\quad P(50-\rho < X < 50+\rho) \geq 1 - \frac{25}{\rho^2}$$

By taking $\rho = 10$, we can write

$$P(40 < X < 60) \geq 1 - \frac{25}{100} = 0.75$$

We can therefore conclude that the probability that this week's clearance will be between 40 and 60 is more than 75%.

Ans.

Ex.4. A random variable X with unknown probability distribution has a mean 8 and standard deviation 3. Evaluate the bounds on

(i) $P(-4 < X < 20)$

(ii) $P(|X-8| \geq 6)$

Solution:

(i) We know that $\mu = 8$ and $\sigma = 3$. We also know that

$$P(|X-\mu| < \rho) \geq 1 - \frac{\sigma^2}{\rho^2}$$

$$\text{i.e,}\quad P(-\rho < X - 8 < \rho) \geq 1 - \frac{9}{\rho^2}$$

$$\text{or,}\quad P(8-\rho < X < 8+\rho) \geq 1 - \frac{9}{\rho^2}$$

Let us consider $\rho = 12$. Then we can write

$$P(-4 < X < 20) \geq 1 - \frac{9}{144} = 0.9375$$

(ii) We also know that

$$P(|X - \mu| \geq \rho) \leq \frac{\sigma^2}{\rho^2} - - - - (1)$$

We can compare the required probability $P(|X - 8| \geq 6)$ with equation (1) and consequently write

$$P(|X - 8| \geq 6) \leq \frac{3^2}{6^2} = 0.25$$

Ans.

Ex.5. Determine the smallest value of k in the Chebyshev's inequality for which the probability is at least 0.95.

Solution: From the Chebyshev's inequality, we may write

$$P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

$$\text{or,} \quad 0.95 = 1 - \frac{1}{k^2}$$

$$\text{or,} \quad k^2 = 20 \rightarrow k = 2\sqrt{5}$$

Hence the smallest value of k will be $2\sqrt{5}$.

Ans.