Random Variable and Distribution Functions

Lesson 5: Joint Distribution: Continuous Random Variable

Let the two random variables X and Y are continuous random variables and the function $f(x, y)$ is defined as

$$\text{(i) } f(x, y) \geq 0 \ \forall \ x, y$$

$$\text{(ii) } \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$$

then $f(x, y)$ is known as joint probability density function of X and Y. In this case, the probability that the two variables will fall in the finite intervals $[a, b]$ and $[c, d]$ is given by

$$P(a < X < b, c < Y < d) = \int_{a}^{b} \int_{c}^{d} f(x, y) dx dy$$

Marginal Probabilities:

(i) For X:

The marginal probability density function for X is given as

$$f(x) = \int_{-\infty}^{\infty} f(x, y) dy$$

(i) For Y:

The marginal probability density function for Y is given as

$$g(y) = \int_{-\infty}^{\infty} f(x, y) dx$$

Note: Two continuous random variables X and Y are called independent iff

$$f(x, y) = f(x) g(y) \ \forall \ (x, y)$$

Joint Probability Distribution Function:

Let (X, Y) be a two dimensional continuous random variable. Then their joint distribution function is denoted by $F(x, y)$ and is defined by

$$F(x,y) = P(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x,y)dxdy$$

where $f(x,y)$ is the joint probability density function for X and Y. The relationship between density function and distribution function in a joint distribution is given by

$$f(x,y) = \frac{\partial^2 F}{\partial x \partial y}$$

provided the density function $f(x,y)$ is continuous at (x, y).

Problems:

Ex.1. The joint probability density function is given by:

$$f(x,y) = cxy, \qquad 0 < x < 4, 1 < y < 5$$
$$= 0 \text{ , otherwise}$$

Evaluate

(i) c

(ii) $P(X \geq 3, Y \leq 2)$

(iii) $P(1 < X < 2,\ 2 < Y < 3)$

Solution:

(i) We know that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)dxdy = 1$$

$$\text{or,}\qquad c\int_0^4\int_1^5 xydxdy = 1 \rightarrow c = \frac{1}{96}$$

(ii) $P(X \geq 3, Y \leq 2) = \frac{1}{96}\int_3^4\int_1^2 xydxdy = \frac{7}{128}$

(iii) $P(1 < X < 2,\ 2 < Y < 3) = \frac{1}{96}\int_1^2\int_2^3 xydxdy = \frac{5}{128}$

Ans.

Ex.2. For the joint probability density function

$$f(x, y) = 4xye^{-(x^2+y^2)}, x, y \geq 0$$

Find the marginal distribution for X and Y and test whether X and Y are independent or not.

Solution: The marginal distribution for X is given as

$$f(x) = \int_{-\infty}^{\infty} f(x, y)dy$$

$$= 4xe^{-x^2}\int_0^{\infty} ye^{-y^2}dy$$

$$= 4xe^{-x^2} \times \frac{1}{2} = 2xe^{-x^2}$$

Similarly, the marginal distribution for Y is given as

$$g(y) = \int_{-\infty}^{\infty} f(x, y)dx$$

$$= 4ye^{-y^2}\int_0^{\infty} xe^{-x^2}dx$$

$$= 2ye^{-y^2}$$

Now, we have

$$f(x) \times g(y) = 4xye^{-(x^2+y^2)} = f(x, y)$$

Hence the two random variables X and Y are independent.

Ans.

Ex.3. The joint probability density function of X and Y is given by

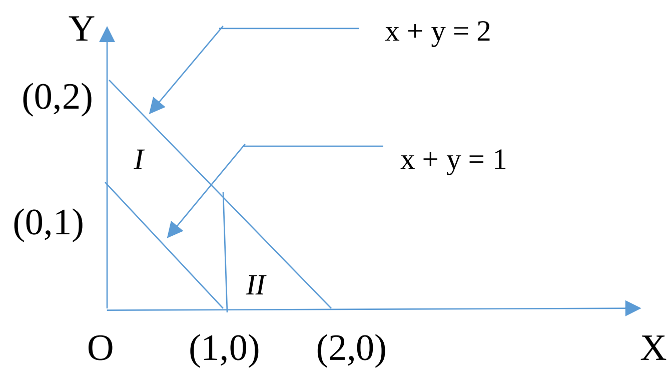
$$f(x, y) = e^{-(x+y)}, 0 \leq x < \infty, 0 \leq y < \infty$$

Evaluate $P(1 < X + Y < 2)$

Solution: The required probability can be written as

$$P(1 < X + Y < 2) = \iint_{I} f(x, y) dxdy + \iint_{II} f(x, y) dxdy - - - (1)$$

where the two regions I and II are indicated in the following diagram.



Hence from equation (1), we get

$$P(1 < X + Y < 2) = \int_{0}^{1} \left(\int_{1-x}^{2-x} e^{-(x+y)} dy \right) dx + \int_{1}^{2} \left(\int_{0}^{2-x} e^{-(x+y)} dy \right) dx$$

$$= \frac{2}{e} - \frac{3}{e^2}$$

Ans.