Testing of Hypothesis: Small Sample Tests

Lesson 7: Test of Hypothesis for Difference of Two Population Means μ_1 and μ_2 (a common population variance σ^2 is unknown)

Let $\overline{x_1}$ and $\overline{x_2}$ are the two means of two independent random samples of sizes n_1 and n_2 (< 30) drawn from two normal populations with means μ_1 and μ_2 and having a common unknown variance. Then the unbiased estimator of this common unknown population variance is given by

$$U^2 = \frac{n_1 {s_1}^2 + n_2 {s_2}^2}{n_1 + n_2 - 2} \ (s_1,\ s_2 = \text{standard deviations of the two samples})$$

We know that the z statistic for difference of two means is given by

$$z = \frac{(\overline{x_1} - \overline{x_2}) - (\mu_1 - \mu_2)}{\sqrt{\frac{{\sigma_1}^2}{n_1} + \frac{{\sigma_2}^2}{n_2}}}$$

If we now consider σ_1 and σ_2 to be equal and replace them by the estimator U, then the test statistic thus formed is given by

$$\mathrm{t} = \frac{(\overline{x_1} - \overline{x_2}) - (\mu_1 - \mu_2)}{U\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} ------- (1)$$

and is known as **Fisher's t**. It follows the **Students' t-distribution** with $\nu = (n_1 + n_2 - 2)$ degrees of freedom.

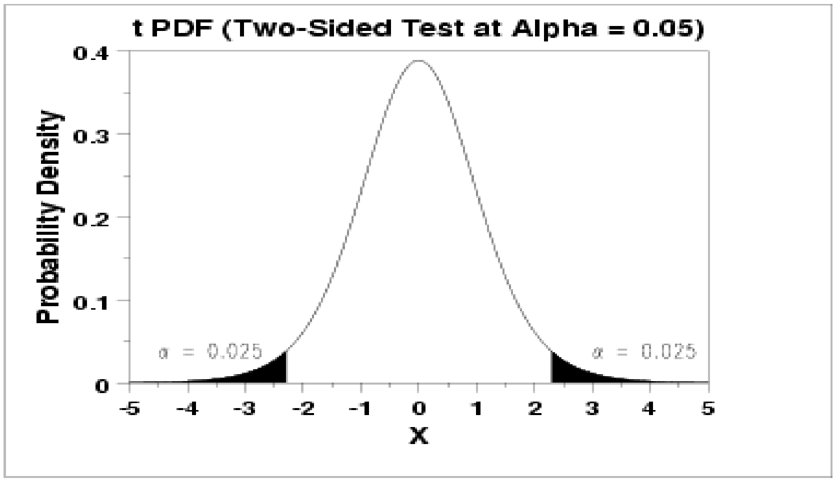
The steps for the testing are:

(i) Formulate null hypothesis, H_0 : $\mu_1 = \mu_2$: There is no significant difference between the two population means

(ii) Formulate alternative hypothesis, H_1 : $\mu_1 \neq \mu_2$ (or otherwise) : There is a significant difference between the two population means

(iii) Level of significance is : α (say 5%)

(iv) From the two tailed test, the critical region for $\nu = (n_1 + n_2 - 2)$ degrees of freedom is :



(v) Compute the test statistic t

(vi) Reject H_0 if the computed value of t falls in the critical region, otherwise accept H_0

Problems:

Ex.1. Two types of batteries are tested for their length of life and the following data are obtained.

	Type A	Type B
Number of samples	9	8
Mean life (hours)	600	640
Variance	121	144

Is there a significant difference in the two means? (At 5% significance level)

Solution:

Here the sample sizes are $n_1 = 9,\ n_2 = 8\ (< 30)$, so small sample test is to be applied.

The sample means are $\overline{x_1} = 600,\ \overline{x_2} = 640$

The sample standard deviations are $s_1 = 11,\ s_2 = 12$

Degrees of freedom is ν = (9 + 8 – 2) = 15

Let us consider that the lengths of lives of both types of batteries are normally distributed with a common unknown variance. Then the unbiased estimator of this common unknown population variance is given by

$$U^2 = \frac{n_1 {s_1}^2 + n_2 {s_2}^2}{n_1 + n_2 - 2} = 149.4 \text{ or, } U = 12.2$$

Now, we have

(i) Null hypothesis, $H_0 : \mu_1 = \mu_2$

(ii) Alternative hypothesis, $H_1 : \mu_1 \neq \mu_2$

(iii) $\alpha = 0.05$

(iv) The critical region at 5% level of significance for 15 degrees of freedom is $|t| \geq 2.131$

(v) $t = \frac{(\overline{x_1} - \overline{x_2}) - (\mu_1 - \mu_2)}{U\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{(600 - 640) - 0}{12.2\sqrt{\frac{1}{9} + \frac{1}{8}}} = -6.73$

(vi) As computed $|t| > 2.131$, therefore H_0 is rejected

Hence, we conclude that there is a significant difference in the two means.

Ans.

Ex.2. Two samples of 6 and 5 items gave the following data. Test at 5% level of significance if the difference of means is significant.

	Population 1	Population 2
Sample size	6	5
Sample mean	40	50
Sample standard deviation	8	10

Solution: Here the sample sizes are $n_1 = 6,\ n_2 = 5\ (< 30)$, so small sample test is to be applied

The sample means are $\overline{x_1} = 40,\ \overline{x_2} = 50$

The sample standard deviations are $s_1 = 8,\ s_2 = 10$

Degrees of freedom is ν = (6 + 5 – 2) = 9

Let us consider that the two populations are normally distributed with a common unknown variance. Then the unbiased estimator of this common unknown population variance is given by

$$U^2 = \frac{n_1 {s_1}^2 + n_2 {s_2}^2}{n_1 + n_2 - 2} = 98.22 \text{ or, } U = 9.91$$

(i) Null hypothesis, $H_0 : \mu_1 = \mu_2$

(ii) Alternative hypothesis, $H_1 : \mu_1 \neq \mu_2$

(iii) $\alpha = 0.05$

(iv) The critical region at 5% level of significance for 9 degrees of freedom is $|t| \geq 2.262$

(v) $t = \frac{(\overline{x_1} - \overline{x_2}) - (\mu_1 - \mu_2)}{U\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{(40-50)-0}{9.91\sqrt{\frac{1}{6}+\frac{1}{5}}} = -1.667$

(vi) As computed $|t| < 2.262$, therefore H_0 is accepted

Hence, we conclude that the difference between the two means is not significant.

Ans.

Ex.3. A group of 5 patients from a hospital is treated with medicine A and weigh 42, 39, 48, 60 and 41 kg; a second group of 7 patients from the same hospital is treated with medicine B and weigh 38, 42, 56, 64, 68, 69, 62 kg. Do you agree with the claim that medicine B increases the weight significantly? (Test at 5% significance level)

Solution:

Here the sample sizes are $n_1 = 5,\ n_2 = 7\ (< 30)$, so small sample test is to be applied.

The sample means are $\overline{x_1} = \frac{1}{5}\sum_{i=1}^{5} x_i = 46,\ \overline{x_2} = \frac{1}{7}\sum_{i=1}^{7} y_i = 57$

The sample variances are ${s_1}^2 = 58,\ {s_2}^2 = \frac{926}{7}$

Degrees of freedom is $\nu = (5 + 7 - 2) = 10$

Let us consider that the two populations are normally distributed with a common unknown variance. Then the unbiased estimator of this common unknown population variance is given by

$$U^2 = \frac{n_1 {s_1}^2 + n_2 {s_2}^2}{n_1 + n_2 - 2} = 121.6 \text{ or, } U = 11.03$$

(i) Null hypothesis, $H_0 : \mu_1 = \mu_2$

(ii) Alternative hypothesis, $H_1 : \mu_1 < \mu_2$

(iii) $\alpha = 0.05$

(iv) The critical region at 5% level of significance for 10 degrees of freedom is $t < -1.812$

(v) $t = \frac{(\overline{x_1} - \overline{x_2}) - (\mu_1 - \mu_2)}{U\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{(46-57)-0}{11.03\sqrt{\frac{1}{5}+\frac{1}{7}}} = -1.7$

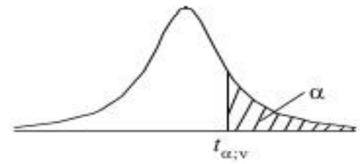
(vi) As computed t > -1.812, therefore H_0 is accepted

Hence, we conclude that, we do not agree with the claim that medicine B increases the weight significantly.

Ans.

Table of the Student's *t*-distribution

The table gives the values of $t_{\alpha;\nu}$ where
$\Pr(T_\nu > t_{\alpha;\nu}) = \alpha$, with ν degrees of freedom



ν \ α	0.1	0.05	0.025	0.01	0.005	0.001	0.0005
1	3.078	6.314	12.076	31.821	63.657	318.310	636.620
2	1.886	2.920	4.303	6.965	9.925	22.326	31.598
3	1.638	2.353	3.182	4.541	5.841	10.213	12.924
4	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	1.319	1.714	2.069	2.500	2.807	3.485	3.767
24	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	1.296	1.671	2.000	2.390	2.660	3.232	3.460
120	1.289	1.658	1.980	2.358	2.617	3.160	3.373
∞	1.282	1.645	1.960	2.326	2.576	3.090	3.291