Chapter 3
Oscillatory motion and chaos

Examples of oscillatory phenomena: motion of electrons in atoms, electronic circuits, and planetary orbits.
Simple Pendulum: A mass swinging on a string under gravity; in ideal conditions (small angles, no friction), it follows simple harmonic motion (SHM).
Real Pendulum: When friction, external forces, or large angles are considered, the motion becomes more complex and can lead to chaotic behavior.

3.1 Simple Harmonic Motion

A mass on a massless string swinging under gravity, with forces acting parallel and perpendicular to the string.
The parallel force adds to 0, as the string does not stretch or break.
The perpendicular force is: $F_\theta = -m\,g\,\sin\theta$
General solution: $\theta = \theta_0 \sin(\Omega\,t + \phi)$, where $\Omega = \sqrt{\frac{g}{l}}$.
Using the euler method:

$$\omega_{i+1} = \omega_i - (g/\ell)\theta_i \Delta t$$
$$\theta_{i+1} = \theta_i + \omega_i \Delta t$$
$$t_{i+1} = t_i + \Delta t$$

- Causes oscillation amplitude to grow over time, which is incorrect.
- Does not conserve total energy, making it unsuitable for long-term oscillatory motion.
- Errors persist regardless of how small the time step (Δt) is.

Energy Conservation in SHM:

- Total energy $E = \frac{1}{2}m\ell^2\omega^2 + mg\ell(1-\cos\theta)$ should remain constant.
- Euler method incorrectly increases energy over time: $E_{i+1} = E_i + \frac{1}{2}mg\ell\left(\omega_i^2 + \frac{g}{\ell}\theta_i^2\right)(\Delta t)^2$

Euler-Cromer Method:

$$\omega_{i+1} = \omega_i - (g/\ell)\theta_i \Delta t$$
$$\theta_{i+1} = \theta_i + \omega_{i+1}\Delta t \text{ (Note that } \omega_{i+1} \text{ is used to calculate } \theta_{i+1}.)$$
$$t_{i+1} = t_i + \Delta t$$

- A small adjustment where velocity (ω) is updated first, leading to energy conservation.

3.2 Making the pendulum more interesting: adding dissipation, nonlinearity, and driving force

Damped Pendulum:

- Introduces frictional force proportional to velocity.
- Equation of motion: $\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\theta - q\frac{d\theta}{dt}$
- Three regimes of damping:
    1. Underdamped – small friction and amplitude decays with time:

       $$\theta(t) = \theta_0 e^{-qt/2}\sin(\sqrt{\Omega^2 - q^2/4}\,t + \phi)$$

    2. Overdamped – large friction, monotonic, exponential decay of theta:

       $$\theta(t) = \theta_0 e^{-(q/2 \pm \sqrt{q^2/4 - \Omega^2})\,t}$$

    3. Critically damped :

       $$\theta(t) = (\theta_0 + Ct)e^{-qt/2}$$

Driven Pendulum:

- External sinusoidal force with amplitude FD and frequency ΩD.
- Equation of motion: $\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\theta - q\frac{d\theta}{dt} + F_D \sin(\Omega_D t)$
- Steady state solution: $\theta(t) = \theta_0 \sin(\Omega_D t + \phi)$
- Where the amplitude is: $\theta_0 = \frac{F_D}{\sqrt{(\Omega^2 - \Omega_D^2)^2 + (q\,\Omega_D)^2}}$
- Resonance: When ΩD matches natural frequency Ω, amplitude grows significantly.

Large Angle Motion:

- Equation of motion: $\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin\theta$, considering a nonlinear pendulum without friction and without a driving force.
- Since there is no means of adding to or removing energy from this system, the total mechanical energy is conserved and the pendulum executes a periodic motion.
- Period depends on amplitude – larger swings take longer.

3.3 Chaos in the driven nonlinear pendulum

Physical pendulum: nonlinear + damped + driven pendulum

- Nonlinearity (no small-angle approximation).

- Damping (frictional force −qdθ/dt).
- Driving Force (FDsin(ΩDt)).

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin\theta - q\frac{d\theta}{dt} + F_D\sin(\Omega_D t)$$

Euler-cromer:

$\omega_{i+1} = \omega_i - [\,(g/\ell)\sin\theta_i - q\omega_i + F_D\sin(\Omega_D\, t_i)]\,\Delta t$
$\theta_{i+1} = \theta_i + \omega_{i+1}\Delta t$
If θ_{i+1} is out of the range $[-\pi, \pi]$, add or subtract 2π to keep it in this range.

Different Behaviors Based on Driving Force FD:

Low Drive (FD=0.5):

- After initial transients decay, motion settles into steady oscillations at the driving frequency.
- Predictable, periodic behavior.

High Drive (FD=1.2):

- Motion becomes chaotic, non-repeating, and unpredictable.
- Pendulum swings “over the top” with irregular jumps.

Chaos and Sensitivity to Initial Conditions:

Two pendulums with nearly identical initial conditions diverge exponentially in motion.

Lyapunov Exponent (λ) measures divergence:

- λ>0 ⟶ chaotic behavior (exponential divergence).
- λ<0 ⟶ predictable, nonchaotic motion.
- Transition to chaos occurs at λ=0.

Phase-Space Representation:

Instead of plotting θ(t), we plot angular velocity (ω) vs. angle (θ).

Chaotic regime:

- No simple trajectory, but not completely random.
- Forms a strange attractor, a fractal-like structure that characterizes chaotic systems.