Chapter 4
Solar System

4.1 Kepler's Law

Our goal is to calculate the position of Earth as a function of time.

Earth and Sun model: $F_G = \frac{G M_S M_E}{r^2}$, where Ms and Mg are the masses of the Sun and Earth, r the distance between them, and G is the gravitational constant.

Units:
Distance: AU (astronomical unit)
Time: years
The euler cromer method steps and equations:

Calculate the distance r_i from the sun: $r_i = (x_i^2 + y_i^2)^{1/2}$.
Compute $v_{x,i+1} = v_{x,i} - \frac{4\pi^2 x_i}{r_i^3}\Delta t$ and $v_{y,i+1} = v_{y,i} - \frac{4\pi^2 y_i}{r_i^3}\Delta t$.
The Euler-Cromer step: calculate x_{i+1} and y_{i+1} using $v_{x,i+1}$ and $v_{y,i+1}$:
$x_{i+1} = x_i + v_{x,i+1}\Delta t$, $y_{i+1} = y_i + v_{y,i+1}\Delta t$.

Motion of the Earth:
- r = 1AU
- x0 = 1AU
- y0 = 0AU
- v0x = 0AU/yr
- v0y = 2πAU/yr

All planets move in elliptical orbits, with the Sun at one focus.

The line joining a planet to the Sun sweeps out equal areas in equal times.

If T is the period and a the semimajor axis of the orbit, then T^3/a^2 is a constant (i.e., the same constant for all of the planets).

4.2 The inverse-square law and the stability of planetary orbits

Kepler's Laws are a direct consequence of Newton's inverse-square gravitational force:
$F(r) \propto 1/r^2$

A two-body system can be treated as a one-body problem using the reduced mass concept:

$$\mu \equiv m_1 m_2/(m_1 + m_2)$$

The equation for an orbit in polar coordinates is: $r = \left(\frac{L^2}{\mu G M_S M_P}\right) \frac{1}{1 - e\cos\theta}$

- e = 0 → Circle
- 0 < e < 1 → Ellipse (Kepler's first law)
- e = 1 → Parabola
- e > 1 → Hyperbola

Key Orbital Parameters

- Perihelion (closest point): rmin = a(1−e)
- Aphelion (farthest point): rmax = a(1+e)
- Orbital Speed $$v_{\max} = \sqrt{GM_S}\sqrt{\frac{(1+e)}{a(1-e)}\left(1+\frac{M_P}{M_S}\right)}$$ $$v_{\min} = \sqrt{GM_S}\sqrt{\frac{(1-e)}{a(1+e)}\left(1+\frac{M_P}{M_S}\right)}$$

Deviations from inverse-square law

- β = 3.00 → Unstable orbit, planet ejected.
- β = 2.50 → Ellipse rotates significantly.
- β = 2.01 → Small deviations still cause noticeable precession.

Even slight deviations from inverse-square law cause instability, making it unlikely that real gravity deviates from this law.

4.3 Precession of the perihelion of mercury

Mercury's orbit deviates from a perfect ellipse; its perihelion (closest point to the Sun) shifts over time.

The modified force law includes a tiny extra term: $F_G \approx \frac{G\, M_S\, M_M}{r^2}\left(1 + \frac{\alpha}{r^2}\right)$, where $\alpha \approx 1.1 \times 10^{-8} AU^2$.

Numerical Simulation Approach:

Direct simulation of Mercury's orbit would take 230,000 years → instead, use extrapolation.

Steps:

1. Modify the planetary motion program to include the relativistic correction.
2. Simulate precession for artificially large values of α.

3. Measure precession rate as a function of α.
4. Extrapolate results to the real-world value of α.

Least-squares method: Used to fit a best-fit line to precession data.

Linear extrapolation: Used to estimate precession at realistic values of α.

Final result: The extrapolated precession rate is 43 arcseconds per century, matching experimental observations.

4.4 The three-body problem and the effect of jupiter on earth

Unlike the two-body problem (which has exact solutions), the three-body problem has no general analytical solution.
Even adding just one more planet (Jupiter) makes the system significantly more complex.
The gravitational effect of Jupiter on Earth's orbit is studied numerically.

Jupiter's gravitational force: $F_{E,J} = \frac{G M_J M_E}{r_{EJ}^2}$

The total force on Earth includes contributions from both the Sun and Jupiter.
Euler-Cromer method is used to update planetary positions iteratively:

Calculate the distances among Earth, Jupiter, and the Sun:
$r_e(i) = \sqrt{x_e(i)^2 + y_e(i)^2}$,
$r_j(i) = \sqrt{x_j(i)^2 + y_j(i)^2}$,
$r_{EJ} = \sqrt{[x_e(i) - x_j(i)]^2 + [y_e(i) - y_j(i)]^2}$.
Compute the new velocity of Earth
$v_{e,x}(i+1) = v_{e,x}(i) - \frac{4\pi^2 x_e(i)}{r_e(i)^3}\Delta t - \frac{4\pi^2 (M_J/M_S)[x_e(i) - x_j(i)]}{r_{EJ}(i)^3}\Delta t$,
and similarly for the y-component, $v_{e,y}(i+1)$.
Compute the new velocity of Jupiter
$v_{j,x}(i+1) = v_{j,x}(i) - \frac{4\pi^2 x_j(i)}{r_j(i)^3}\Delta t - \frac{4\pi^2 (M_E/M_S)[x_j(i) - x_e(i)]}{r_{EJ}(i)^3}\Delta t$,
and similarly for the y-component, $v_{j,y}(i+1)$.
Use the Euler-Cromer method to calculate the new positions of Earth and Jupiter:
$x_e(i+1) = x_e(i) + v_{e,x}(i+1)\Delta t$ // $y_e(i+1) = y_e(i) + v_{e,y}(i+1)\Delta t$ //
and similarly for Jupiter.

Increasing Jupiter's mass:

- 10 × mass of Jupiter → Still negligible effect.
- 100 × mass of Jupiter → Slight perturbations but still stable.
- 1000 × mass of Jupiter (~mass of Sun) → Earth's orbit becomes unstable; Earth is eventually ejected from the Solar System.