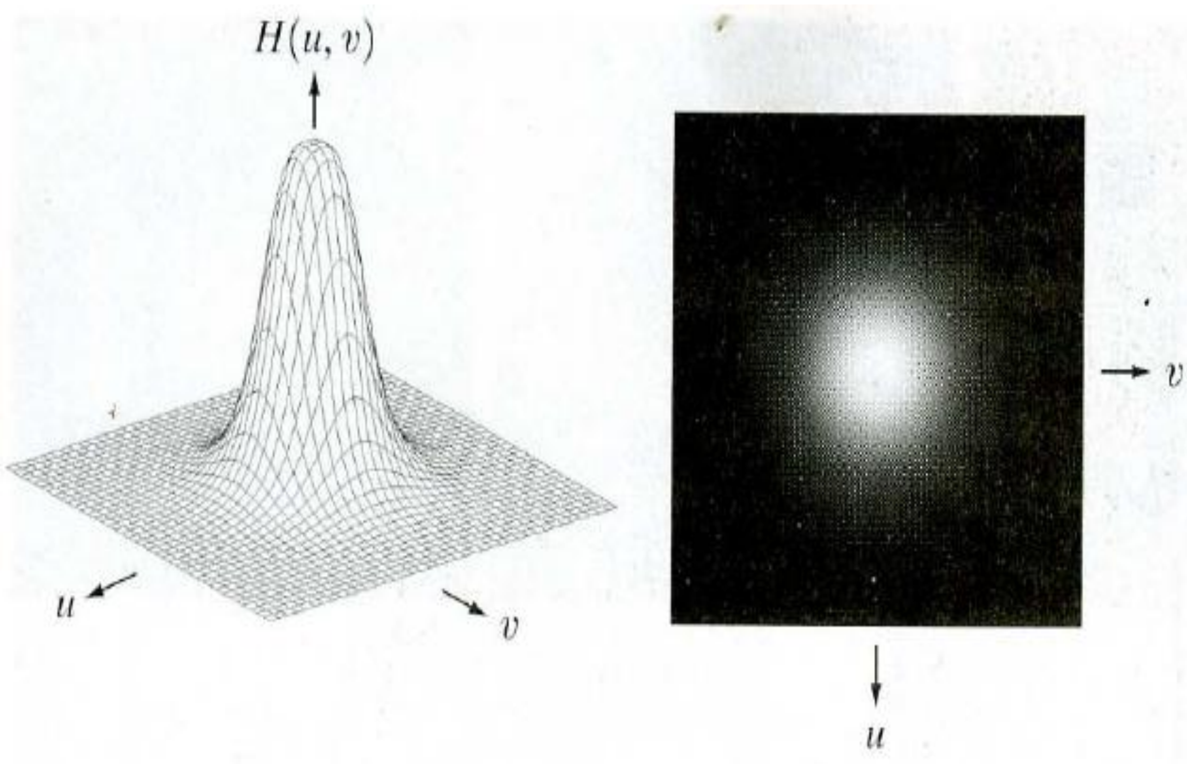
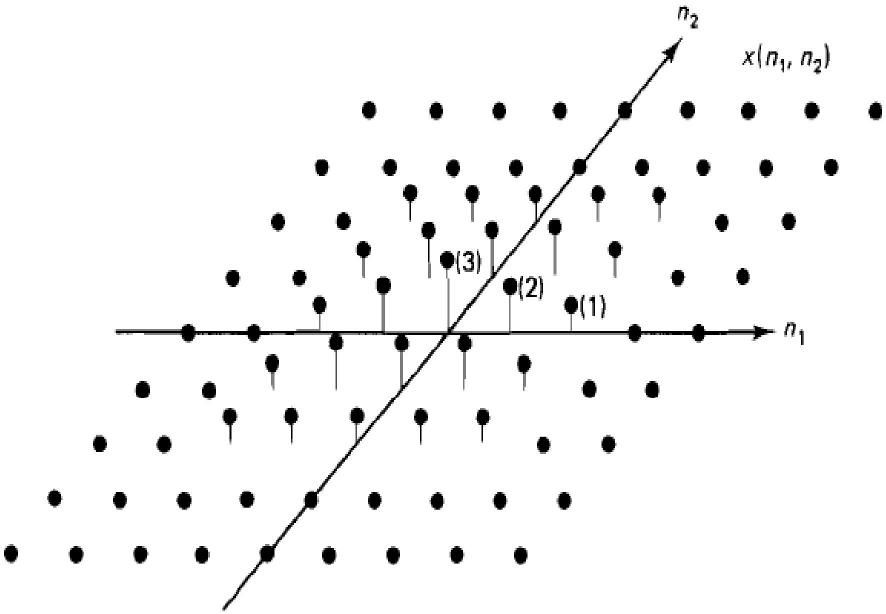
Procesamiento digital de Imágenes

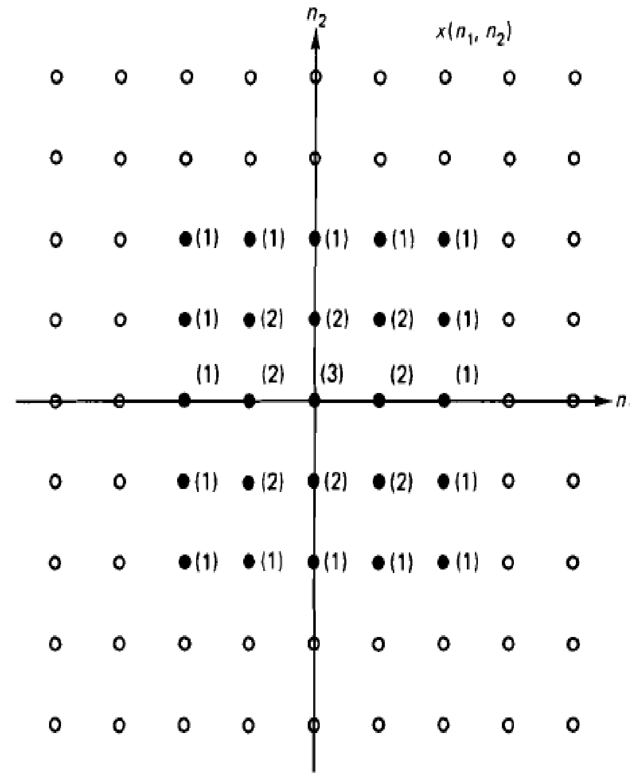
Señales y sistemas



Señales y sistemas

Señales discretas en el espacio





$$x(n_1, n_2)$$

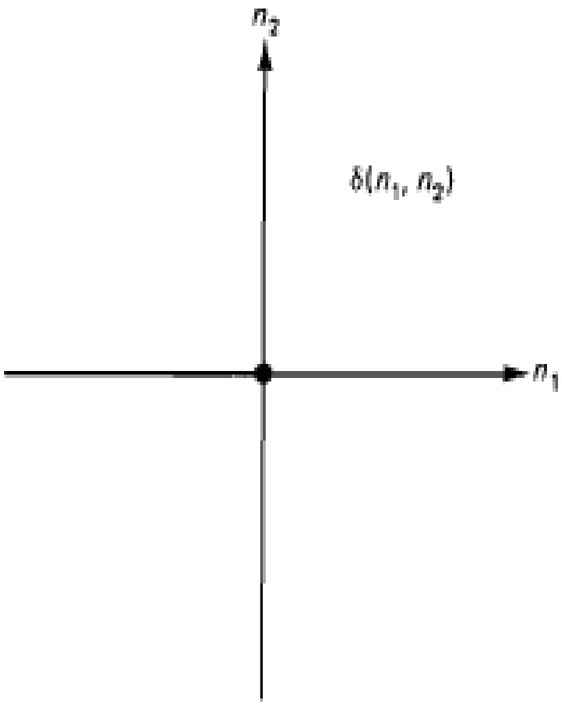
Señales y sistemas

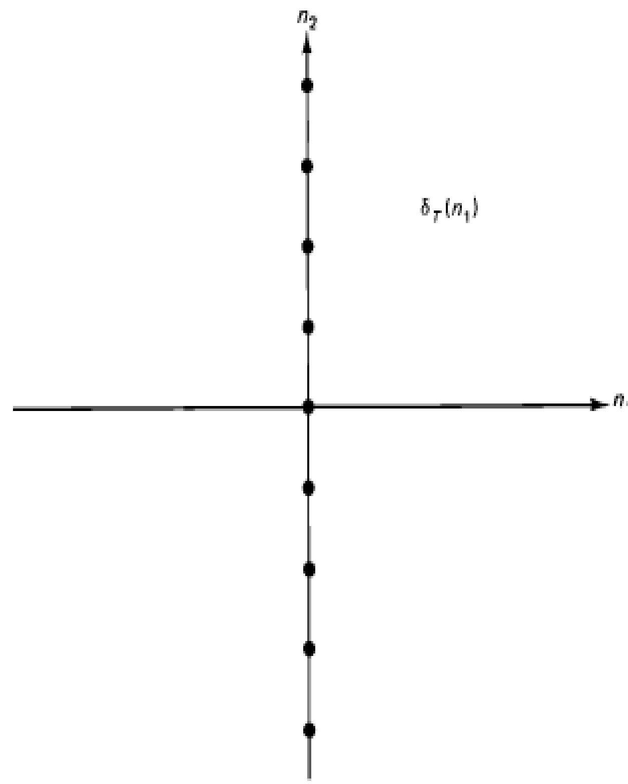
Secuencias en 2D

$$
\begin{aligned}
x(n_1,n_2) = \cdots &+ x(-1,-1)\,\delta(n_1+1,n_2+1) + x(0,-1)\,\delta(n_1,n_2+1) \\
&+ x(1,-1)\,\delta(n_1-1,n_2+1) + \cdots + x(-1,0)\,\delta(n_1+1,n_2) \\
&+ x(0,0)\,\delta(n_1,n_2) + x(1,0)\,\delta(n_1-1,n_2) \\
&+ \cdots + x(-1,1)\,\delta(n_1+1,n_2-1) \\
&+ x(0,1)\,\delta(n_1,n_2-1) + x(1,1)\,\delta(n_1-1,n_2-1) + \cdots \\
= &\boxed{\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} x(k_1,k_2)\,\delta(n_1-k_1,n_2-k_2).}
\end{aligned}
$$

Señales y sistemas

Ejemplos: delta y línea de impulsos





$$\delta(n_1, n_2) = \begin{cases} 1, & n_1 = n_2 = 0 \\ 0, & \text{otherwise.} \end{cases}$$

$$x(n_1, n_2) = \delta_T(n_1) = \begin{cases} 1, & n_1 = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Señales y sistemas

Secuencias separables

Una secuencia es separable si:

$$x(n_1, n_2) = f(n_1)g(n_2)$$

Ejemplos:

$$\delta(n_1, n_2) = \delta(n_1)\,\delta(n_2)$$

$$u(n_1, n_2) = u(n_1)u(n_2)$$

Señales y sistemas

Secuencias periódicas

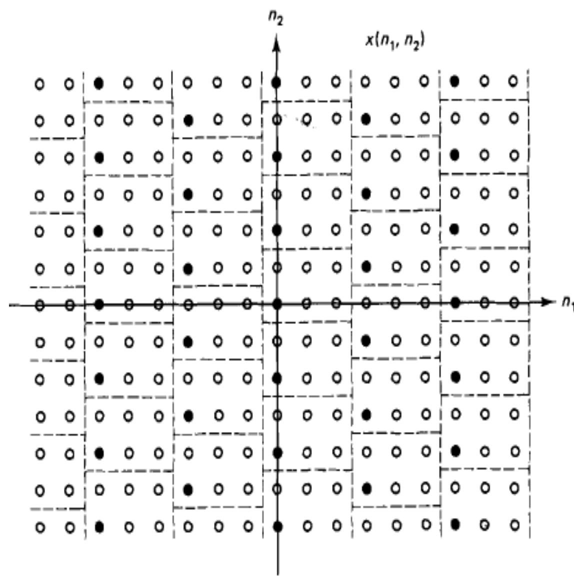
Una secuencia es periódica con periodo N_1 x N_2 si:

$$x(n_1, n_2) = x(n_1 + N_1, n_2) = x(n_1, n_2 + N_2) \quad \text{for all } (n_1, n_2)$$

Para todo $N_1 \neq 0$ y $N_2 \neq 0$

Ejemplos:

$x(n_1, n_2) = \cos(\pi n_1 + (\pi/2) n_2)$ con periodo 2 x 4



Periodo 6 x 2

Señales y sistemas

Sistemas lineales e invariantes

Todo Sistema aplica una transformación sobre la entrada

$$y(n_1, n_2) = T[x(n_1, n_2)].$$

Si la transformación cumple con la siguiente propiedad es lineal:

$$\text{Linearity} \Longleftrightarrow T[ax_1(n_1, n_2) + bx_2(n_1, n_2)] = ay_1(n_1, n_2) + by_2(n_1, n_2)$$

Si además se cumple con la siguiente propiedad es invariante

$$\text{Shift invariance} \Longleftrightarrow T[x(n_1 - m_1, n_2 - m_2)] = y(n_1 - m_1, n_2 - m_2)$$

En general :

$$y(n_1, n_2) = T[x(n_1, n_2)] = T\left[\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} x(k_1, k_2)\delta(n_1 - k_1, n_2 - k_2)\right]$$

$$= \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} x(k_1, k_2)T[\delta(n_1 - k_1, n_2 - k_2)].$$

$$h(n_1, n_2) = T[\delta(n_1, n_2)].$$

Señales y sistemas
Convolución

En el caso que el sistema sea lineal e invariante tenemos:

$$h(n_1, n_2) = T[\delta(n_1, n_2)].$$

$$y(n_1, n_2) = x(n_1, n_2) * h(n_1, n_2)$$

$$= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1, k_2)h(n_1 - k_1, n_2 - k_2).$$

Propiedades

Commutativity

$$x(n_1, n_2) * y(n_1, n_2) = y(n_1, n_2) * x(n_1, n_2)$$

Associativity

$$(x(n_1, n_2) * y(n_1, n_2)) * z(n_1, n_2) = x(n_1, n_2) * (y(n_1, n_2) * z(n_1, n_2))$$

Distributivity

$$x(n_1, n_2) * (y(n_1, n_2) + z(n_1, n_2))$$

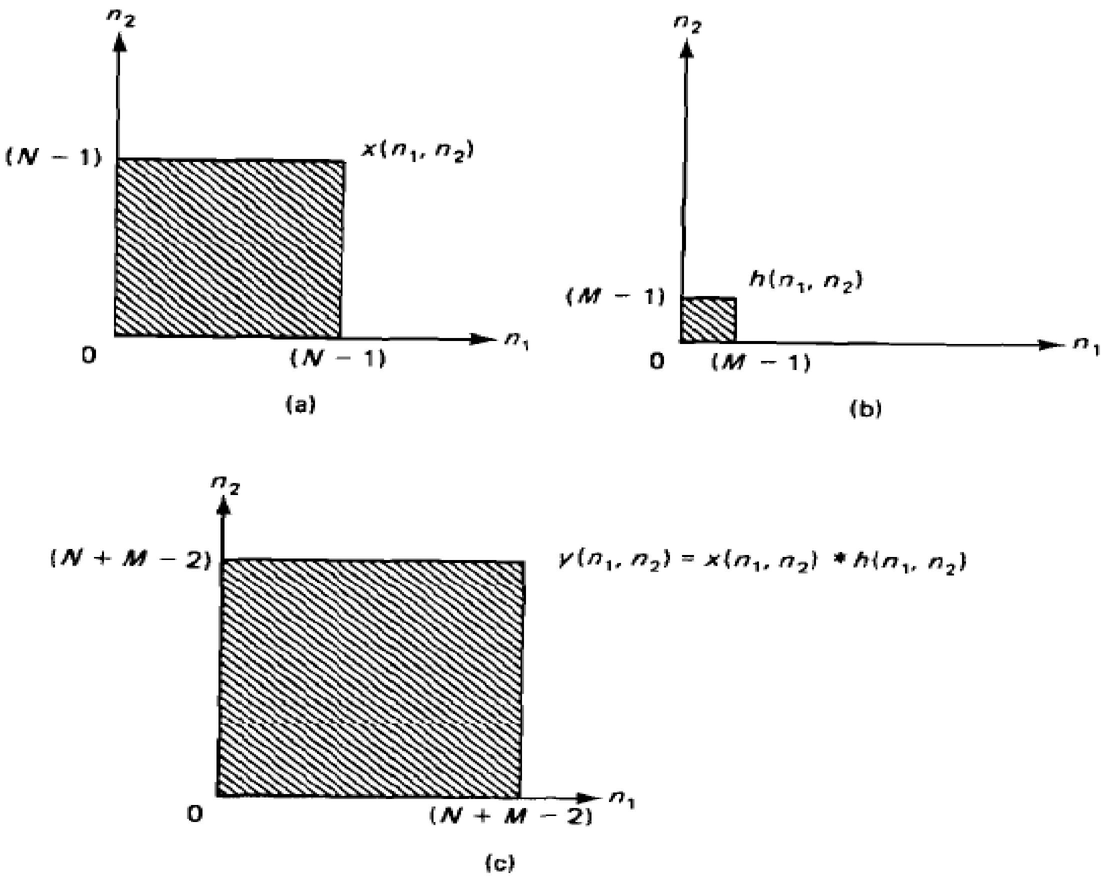
$$= (x(n_1, n_2) * y(n_1, n_2)) + (x(n_1, n_2) * z(n_1, n_2))$$

Convolution with Shifted Impulse

$$x(n_1, n_2) * \delta(n_1 - m_1, n_2 - m_2) = x(n_1 - m_1, n_2 - m_2)$$

Señales y sistemas
Convolución

Notar que las dimensiones de la matriz resultante es N + M - 2



Señales y sistemas
Transformada de Fourier

Discrete-Space Fourier Transform Pair

$$X(\omega_1, \omega_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1, n_2) e^{-j\omega_1 n_1} e^{-j\omega_2 n_2}$$

$$x(n_1, n_2) = \frac{1}{(2\pi)^2} \int_{\omega_1=-\pi}^{\pi} \int_{\omega_2=-\pi}^{\pi} X(\omega_1, \omega_2) e^{j\omega_1 n_1} e^{j\omega_2 n_2}\, d\omega_1\, d\omega_2$$

2D-DSFT

$$F(u, v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}.$$

$$f(x, y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)}$$

2D-DFT

Señales y sistemas

Transformada de Fourier Propiedades

$$x(n_1, n_2) \longleftrightarrow X(\omega_1, \omega_2)$$
$$y(n_1, n_2) \longleftrightarrow Y(\omega_1, \omega_2)$$

Property 1. *Linearity*

$$ax(n_1, n_2) + by(n_1, n_2) \longleftrightarrow aX(\omega_1, \omega_2) + bY(\omega_1, \omega_2)$$

Property 2. *Convolution*

$$x(n_1, n_2) * y(n_1, n_2) \longleftrightarrow X(\omega_1, \omega_2)Y(\omega_1, \omega_2)$$

Property 3. *Multiplication*

$$x(n_1, n_2)y(n_1, n_2) \longleftrightarrow X(\omega_1, \omega_2) \circledast Y(\omega_1, \omega_2)$$

$$= \frac{1}{(2\pi)^2} \int_{\theta_1=-\pi}^{\pi} \int_{\theta_2=-\pi}^{\pi} X(\theta_1, \theta_2)Y(\omega_1 - \theta_1, \omega_2 - \theta_2)\, d\theta_1\, d\theta_2$$

Property 4. *Separable Sequence*

$$x(n_1, n_2) = x_1(n_1)x_2(n_2) \longleftrightarrow X(\omega_1, \omega_2) = X_1(\omega_1)X_2(\omega_2)$$

Property 5. *Shift of a Sequence and a Fourier Transform*

(a) $x(n_1 - m_1, n_2 - m_2) \longleftrightarrow X(\omega_1, \omega_2)e^{-j\omega_1 m_1}e^{-j\omega_2 m_2}$

(b) $e^{j\nu_1 n_1}e^{j\nu_2 n_2}x(n_1, n_2) \longleftrightarrow X(\omega_1 - \nu_1, \omega_2 - \nu_2)$

Señales y sistemas

Transformada de Fourier Propiedades

Property 6. Differentiation

$$\text{(a)} \quad -jn_1 x(n_1, n_2) \longleftrightarrow \frac{\partial X(\omega_1, \omega_2)}{\partial \omega_1}$$

$$\text{(b)} \quad -jn_2 x(n_1, n_2) \longleftrightarrow \frac{\partial X(\omega_1, \omega_2)}{\partial \omega_2}$$

Property 7. Initial Value and DC Value Theorem

$$\text{(a)} \quad x(0, 0) = \frac{1}{(2\pi)^2} \int_{\omega_1 = -\pi}^{\pi} \int_{\omega_2 = -\pi}^{\pi} X(\omega_1, \omega_2)\, d\omega_1\, d\omega_2$$

$$\text{(b)} \quad X(0, 0) = \sum_{n_1 = -\infty}^{\infty} \sum_{n_2 = -\infty}^{\infty} x(n_1, n_2)$$

Señales y sistemas

Transformada de Fourier Propiedades

Property 8. Parseval's Theorem

(a) $$\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} x(n_1, n_2)y^*(n_1, n_2) = \frac{1}{(2\pi)^2}\int_{\omega_1=-\pi}^{\pi}\int_{\omega_2=-\pi}^{\pi} X(\omega_1, \omega_2)Y^*(\omega_1, \omega_2)\, d\omega_1\, d\omega_2$$

(b) $$\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} |x(n_1, n_2)|^2 = \frac{1}{(2\pi)^2}\int_{\omega_1=-\pi}^{\pi}\int_{\omega_2=-\pi}^{\pi} |X(\omega_1, \omega_2)|^2\, d\omega_1\, d\omega_2$$

Property 9. Symmetry Properties

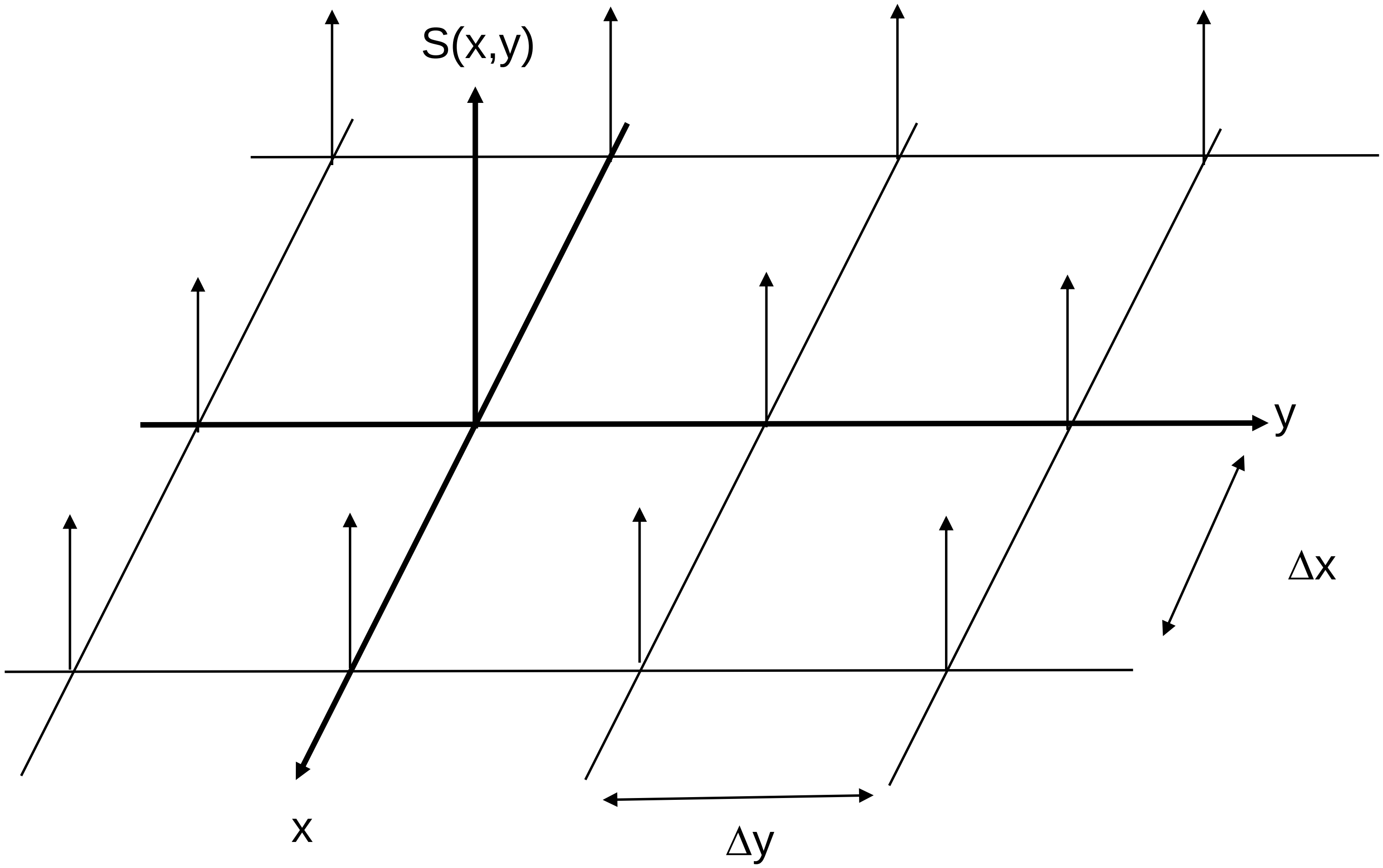
(a) $x(-n_1, n_2) \longleftrightarrow X(-\omega_1, \omega_2)$
(b) $x(n_1, -n_2) \longleftrightarrow X(\omega_1, -\omega_2)$
(c) $x(-n_1, -n_2) \longleftrightarrow X(-\omega_1, -\omega_2)$
(d) $x^*(n_1, n_2) \longleftrightarrow X^*(-\omega_1, -\omega_2)$
(e) $x(n_1, n_2)$: real $\longleftrightarrow X(\omega_1, \omega_2) = X^*(-\omega_1, -\omega_2)$
$X_R(\omega_1, \omega_2)$, $|X(\omega_1, \omega_2)|$: even (symmetric with respect to the origin)
$X_I(\omega_1, \omega_2)$, $\theta_x(\omega_1, \omega_2)$: odd (antisymmetric with respect to the origin)
(f) $x(n_1, n_2)$: real and even $\longleftrightarrow X(\omega_1, \omega_2)$: real and even
(g) $x(n_1, n_2)$: real and odd $\longleftrightarrow X(\omega_1, \omega_2)$: pure imaginary and odd

Property 10. Uniform Convergence

For a stable $x(n_1, n_2)$, the Fourier transform of $x(n_1, n_2)$ uniformly converges.

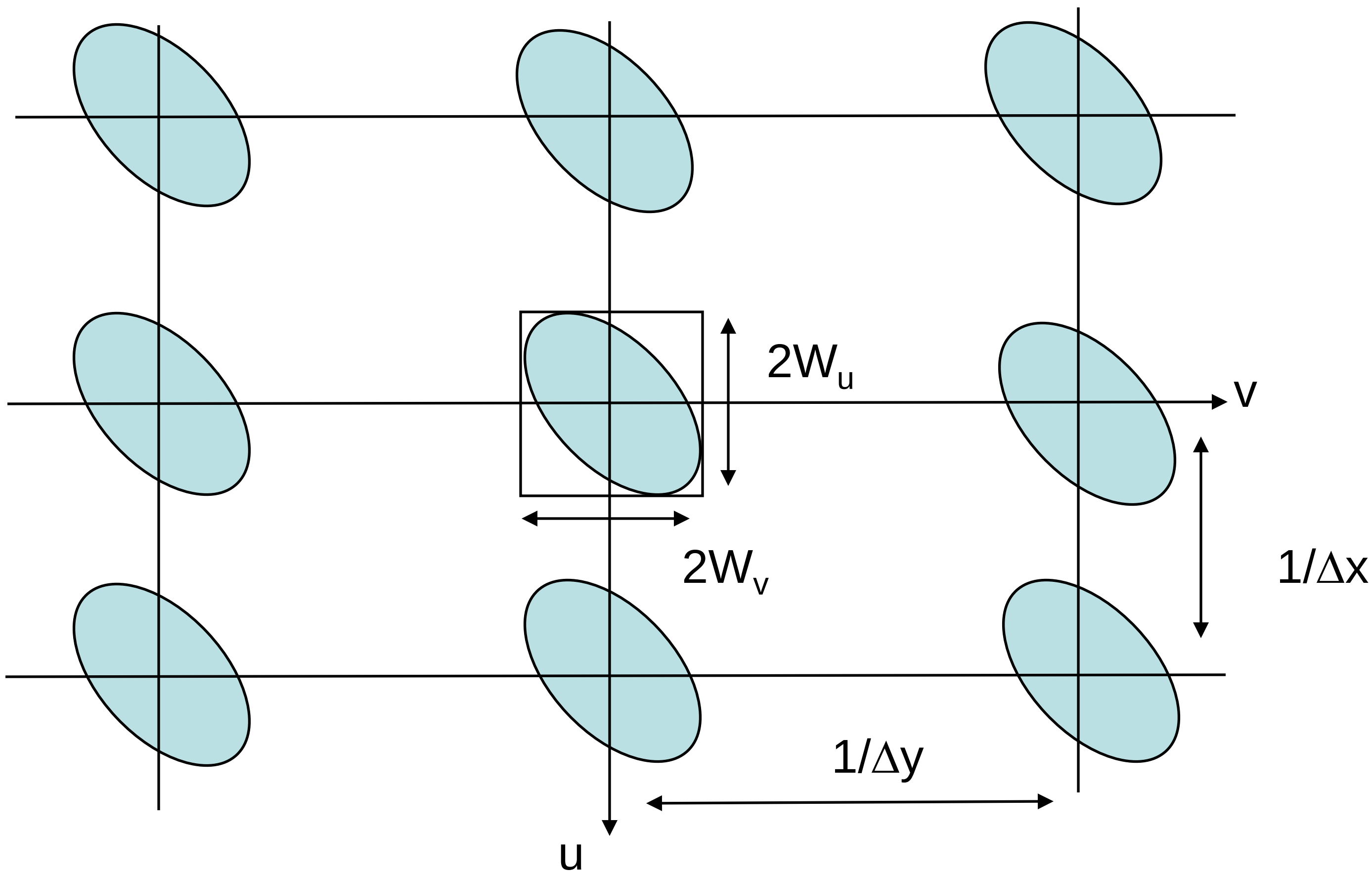
Señales y sistemas

Transformada de Fourier Muestreo



Señales y sistemas

Transformada de Fourier Muestreo



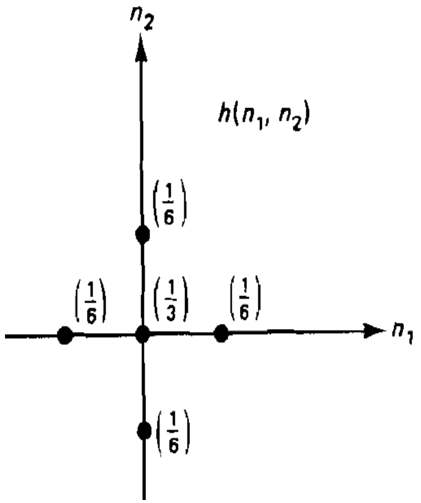
Señales y sistemas
Transformada de Fourier

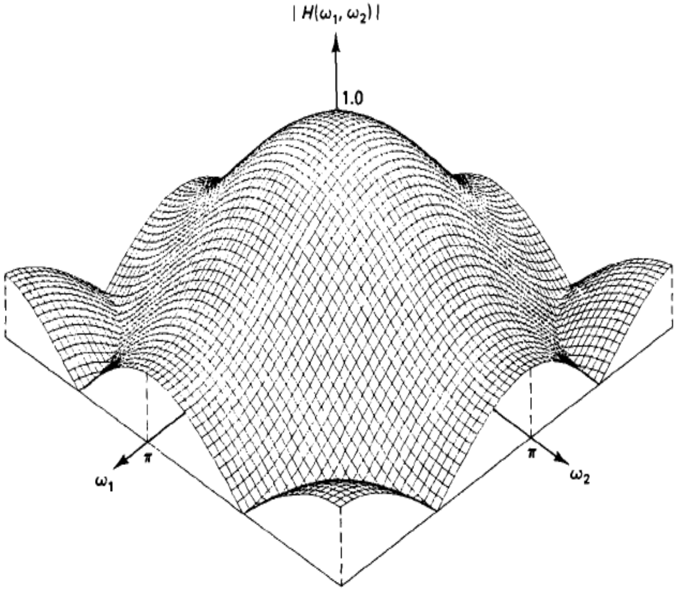
Ejemplo:

$$H(\omega_1, \omega_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} h(n_1, n_2) e^{-j\omega_1 n_1} e^{-j\omega_2 n_2}$$

$$= \tfrac{1}{3} + \tfrac{1}{6} e^{-j\omega_1} + \tfrac{1}{6} e^{-j\omega_2} + \tfrac{1}{6} e^{j\omega_1} + \tfrac{1}{6} e^{j\omega_2}$$

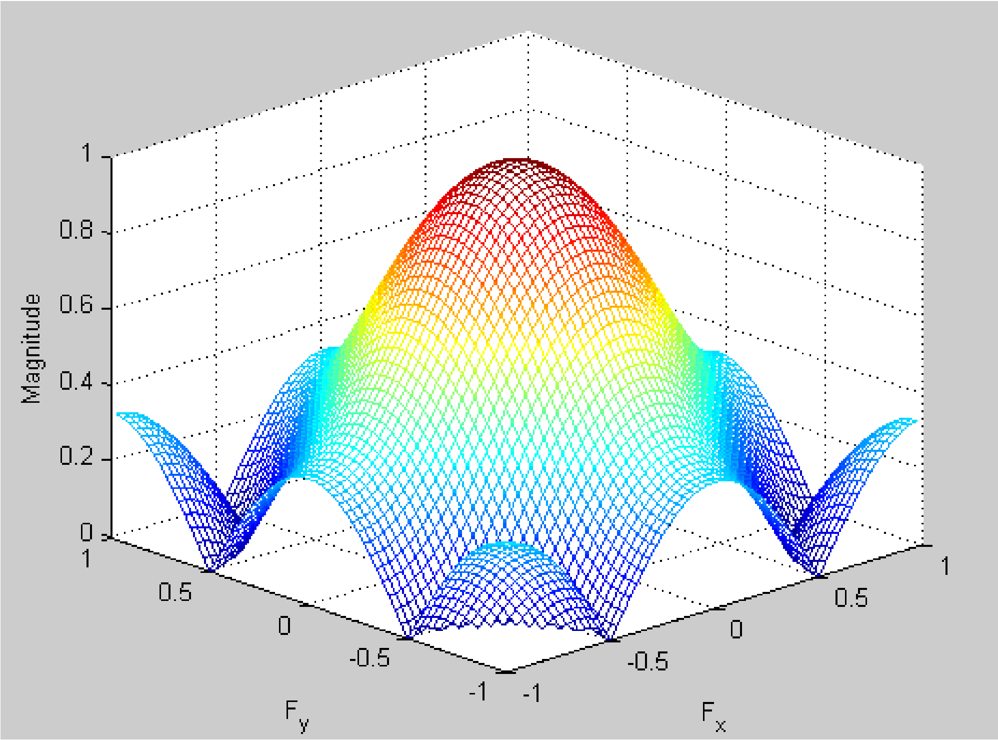
$$= \tfrac{1}{3} + \tfrac{1}{3}\cos\omega_1 + \tfrac{1}{3}\cos\omega_2.$$

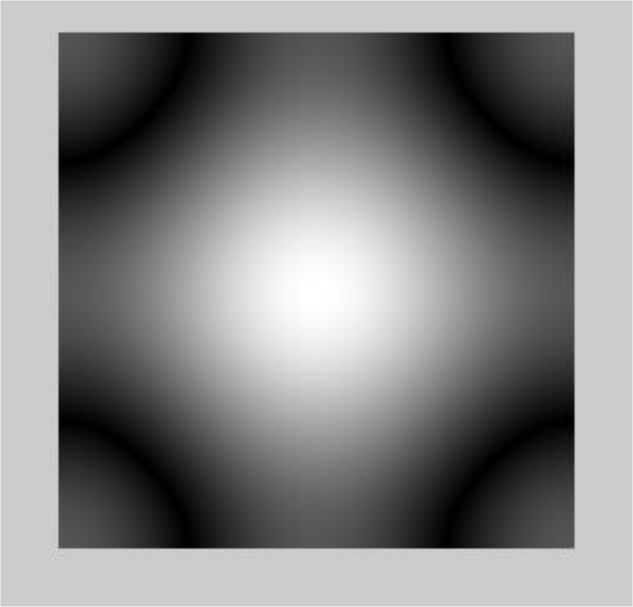


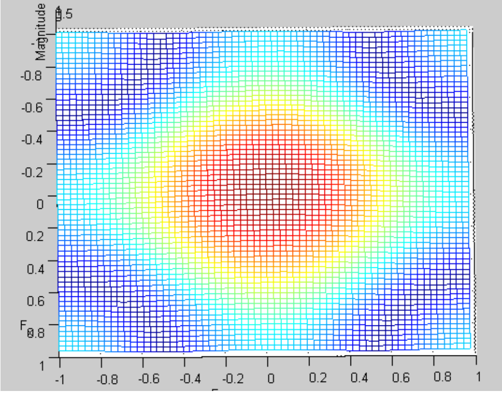


Señales y sistemas
Transformada de Fourier

Ejemplo previo en matlab:







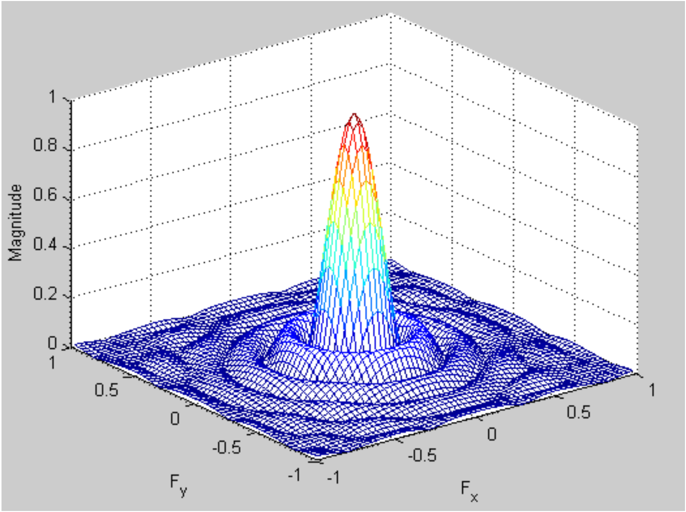
Matlab support

h=[0 1/6 0 ; 1/6 1/3 1/6 ; 0 1/6 0];

freqz2(h);

Señales y sistemas
Transformada de Fourier

Ejemplo:



Matlab support

freqz2

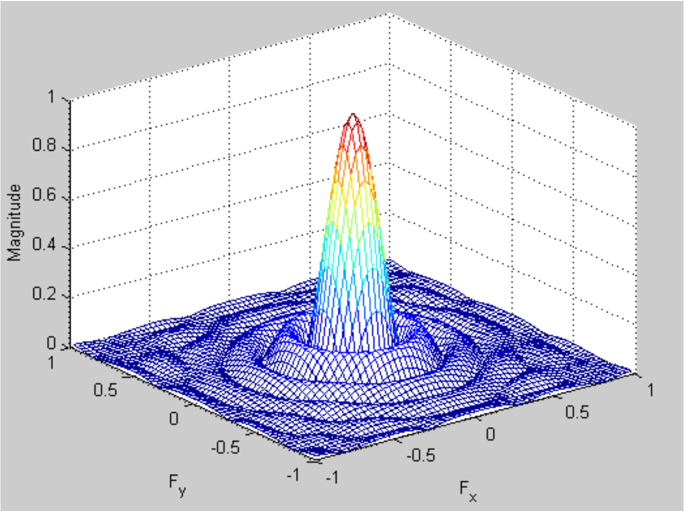
fft2

conv2

imfilter

Señales y sistemas
Transformada de Fourier

Ejemplo:



Matlab support

h = fspecial('disk'); -- low pass

b=imread('barbara.gif'); -- image

h1=imfilter(b,h); -- Filtered image