Introduction to Kahler Geometry

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1 Manifolds

1.1 Basic Definitions

Definition 1.1 A topological manifold M of dimension n is a Hausdorff topological space with an open covering $\mathscr{U}$ such that for all $U \in \mathscr{U}$ there exists a homeomorphism $\phi_U : U \to \tilde{U} \subset \mathbb{R}^n$.

Definition 1.2 The set $\{\phi_U \mid U \in \mathscr{U}\}$ as defined above is called an atlas. Each element of an atlas is called a chart.

Definition 1.3 A topological manifold M is said to be smooth if for any two $U, V \in \mathscr{U}$ with $U \cap V \neq \emptyset$ the map $\phi_{UV} := \phi_U \circ \phi_V^{-1} : \phi_V(U \cap V) \to \phi_U(U \cap V)$ is a diffeomorphism. The atlas is said to be orientable if the Jacobian determinant of ϕ_{UV} is everywhere positive. An oriented manifold is a smooth manifold with an oriented atlas.

Definition 1.4 Let M be a smooth manifold. A continuous function $f : M \to \mathbb{R}$ is said to be smooth if at every $x \in M$ there is some $U \in \mathscr{U}$ containing x such that

$$f_U = f \circ \phi_U^{-1}$$

is smooth.

The smoothness of a function is well defined since if f_U is smooth then f_V is also smooth where V is another open set containing x. This is becasue

$$f_V = f \circ \phi_V^{-1} = f \circ \phi_U^{-1} \circ \phi_U \circ \phi_V^{-1} = f_U \circ \phi_{UV}.$$

Definition 1.6 Let M, N be smooth manifolds with atlas $\{\phi_U\}_{U \in \mathscr{U}}$ and $\{\psi_V\}_{V \in \mathscr{V}}$. A continuous map $f : M \to N$ is said to be smooth if $\psi_V \circ f \circ \phi_U^{-1}$ is a smooth map in the usual sense for all $U \in \mathscr{U}$ and $V \in \mathscr{V}$. If the map is invertible and the inverse is smooth then it is called a diffeomorphism.

Definition 1.7 A local coordinate system around $x \in M$ is a diffeomorphism from an open neighborhood around x to an open set in $\mathbb{R}^n$.

Unless otherwise stated all manifolds will be assumed to be connected.

1.2 Tangent Space

Definition 1.8 If $f : U \to \mathbb{R}^m$ is a smooth function on $\mathbb{R}^n$ then its differential at any point $x \in U$ is a linear map $\mathrm{d}f_x : \mathbb{R}^n \to \mathbb{R}^m$ whose matrix in the standard basis is given by

$$\mathrm{d}f_x = \begin{pmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_x & \left.\frac{\partial f_1}{\partial x_2}\right|_x & \cdots & \left.\frac{\partial f_1}{\partial x_n}\right|_x \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \left.\frac{\partial f_m}{\partial x_1}\right|_x & \left.\frac{\partial f_m}{\partial x_2}\right|_x & \cdots & \left.\frac{\partial f_m}{\partial x_n}\right|_x \end{pmatrix}$$

Proposition 1.9 (Chain Rule) Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$. The $f : U \to \mathbb{R}^n$ and $g : V \to \mathbb{R}^k$ be smooth maps. Then for every $x \in U \cap f^{-1}(V)$ we have

$$\mathrm{d}(g \circ f)_x = \mathrm{d}g_{f(x)} \circ \mathrm{d}f_x$$

proof by Rudin. Let $y = f(x)$, $A = \mathrm{d}f_x$ and $B = \mathrm{d}g_{f(x)}$. By definition of the derivative

$$\begin{aligned} u(h) &= f(x+h) - f(x) - Ah \\ v(k) &= g(y+k) - g(y) - Bk \end{aligned}$$

where $|u(h)/h|, |v(k)/k| \to 0$ as $|h|, |k| \to 0$ respectively. Let $k = f(x+h) - f(x)$, then

$$|k| = |u(h) + Ah| \leq \left(\left|\frac{u(h)}{h}\right| + \|A\| \right) |h|$$

where $\|A\|$ is the operator norm. Substituting this we get

$$\begin{aligned} v(u(h) + Ah) &= g(f(x+h)) - g(f(x)) - B(u(h) + Ah) \\ &= g \circ f(x+h) - g \circ f(x) - BAh - Bu(h) \end{aligned}$$

hence

$$\begin{aligned} \left| \frac{g \circ h(x+h) - g \circ f(x) - BAh}{h} \right| &= \left| \frac{v(k) + Bu(h)}{h} \right| \\ &\leq \left| \frac{v(k)}{h} \right| + \|B\| \left| \frac{u(h)}{h} \right| \\ &= \left| \frac{v(k)}{k} \right| \left(\left| \frac{u(h)}{h} \right| + \|A\| \right) + \|B\| \left| \frac{u(h)}{h} \right| \\ &\to 0, \text{ as } |h| \to 0 \end{aligned}$$

Thus the derivatie of $g \circ f$ is BA which was what we had to prove. ∎

Definition 1.10 Let $x \in M$ and define $\mathscr{I}_x = \{U \in \mathscr{U} \mid x \in U\}$. Let $\sim_x$ be a relation on $\mathscr{I}_x \times \mathbb{R}^n$ defined as

$$(U, u) \sim_x (V, v) \iff u = (\mathrm{d}\phi_{UV})_{\phi_V(x)}(v)$$

Proposition 1.11 The relation $\sim_x$ is an equivalence relation.

Proof. It is reflexive since

$$\phi_{UU} = \mathrm{id} \implies \mathrm{d}\phi_{UU} = I_n \implies (\mathrm{d}\phi_{UU})_{\phi_U(x)} u = I_n u = u.$$

Symmetry can be proved using the fact that $\mathrm{d}f_x^{-1} = (\mathrm{d}f_x)^{-1}$ (which follows from the chain rule), and transitivity also follows from the chain rule. ∎

Consider the space $\mathscr{I}_x \times \mathbb{R}^n / \sim_x$. Consider any two elements $[(U, u)]$ and $[(V, v)]$. Think of u, v as vectors in $\mathbb{R}^n$ and $(\mathrm{d}\phi_{UV})_{\phi_V(x)}$ as an invertible linear transformation on $\mathbb{R}^n$. In a sense the equivalence relation defined above is saying that if the two vectors u, v are related by the change of basis generated by $(\mathrm{d}\phi_{UV})_{\phi_V(x)}$ then $(U, u) \sim_x (V, v)$. Given any $[(V, v)]$ and a $U \in \mathscr{I}_x$ one can find a u', given by $u' = (\mathrm{d}\phi_{UV})_{\phi_V(x)} v$, such that $[(V, v)] = [(U, u')]$. This motivates a natural definition of addition on the space $\mathscr{I}_x \times \mathbb{R}^n / \sim_x$ in the following way:

$$[(U, u)] + [(V, v)] = [(U, u + u')],$$

where u' is as defined above. This addition is well defined since

$$\phi_{UW} = \phi_{UV} \circ \phi_{VW}.$$

So if we have $(W, w) \in [(V, v)]$ then

$$(\mathrm{d}\phi_{WU})_{\phi_W(x)} w = (\mathrm{d}\phi_{UV})_{\phi_{VW}(\phi_W(x))} (\mathrm{d}\phi_{VW})_{\phi_W(x)} w$$
$$= (\mathrm{d}\phi_{UV})_{\phi_V(x)} v = u'.$$

Also the commutativity follows from the fact that

$$u + u' = (\mathrm{d}\phi_{UV})_{\phi_V(x)} \left((\mathrm{d}\phi_{VU})_{\phi_U(x)} u + v \right).$$

Associativity can also be similarly verified. Similarly in a natural fashion one can define scalar multiplication in the following way:

$$a[(U, u)] = [(U, au)],\ a \in \mathbb{R}.$$

PROPOSITION 1.12 The space $\mathscr{I}_x \times \mathbb{R}^n / \sim_x$ is an $n-$dimensional vector space.

Proof. The vector space part should be clear from the above discussion. The zero vector is $[(U, 0)]$ (it's unique cause $(\mathrm{d}\phi_{UV})_{\phi_V(x)}$ is a linear transformation). Consider any $[(U, u)]$. Then $u = \sum_{i=1}^n a_i e_i$, where e_i are the standard basis on $\mathbb{R}^n$. Thus it follows that

$$[(U, u)] = \sum_{i=1}^n a_i [(U, e_i)].$$

Thus the vectors $\{[(U, e_i)]\}$ span the vector space. This is a linearly independent set since

$$\begin{aligned} & \sum_{i=1}^n a_i [(U, e_i)] = [(U, 0)] \\ \implies & [(U, \sum_{i=1}^n a_i e_i)] = [(U, 0)] \\ \implies & \sum_{i=1}^n a_i e_i) = 0 \implies a_i = 0\ \forall\, i. \end{aligned}$$

Thus $\mathscr{I}_x \times \mathbb{R}^n / \sim_x$ is an n dimensional vector space. ■

NOTATION 1.13 From here on represent the vector space $\mathscr{I}_x \times \mathbb{R}^n / \sim_x$ by $T_x M$ and the vectors as $X, Y, \cdots$. This space is called the tangent space and the vectors in the space are called tangent vectors at x.

PROPOSITION 1.14 Given any $X \in T_x M$ and $U \in \mathscr{I}_x$ then there is a unique u such that $X = [(U, u)]$.

Proof. Suppose $X = [(U, u)] = [(U, u')]$. Hence

$$u' = (\mathrm{d}\phi_{UU})_{\phi_U(x)} u = I_n u = u.$$

Thus u is unique. ■

We think of X as an abstract tangent vector at x and u as it's concrete representation in the chart ϕ_U.

Definition 1.16 Define the tangent bundle as the disjoint union $TM = \bigsqcup_{x \in M} T_x M$. Later on we will see that the tangent bundle is a vector bundle, and moreover a smooth manifold of dimension $2n$.