# Introduction to Hyperbolic Surfaces

IDC452 Presentation

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## Outline

# Riemannian Manifold

## Smooth Manifold

A smoth manifold $X$ is a Huasdorff, second countable topological space equipped with a set

$$\mathcal{A} = \{(U, \phi) \mid U \subset X \text{ is open, } \phi : U \to \mathbb{R}^n \text{ is a homeomorphism}\}$$

such that for any $(U, \phi)$ and $(V, \psi)$ with $U \cap V \neq \emptyset$, the map $\psi \circ \phi^{-1}$ is a smooth map on the appropriate domain and range.

## Riemannian Manifold (cont.)

A Riemannian metric $g$ on a manifold $X$ is a smooth assignment of inner product $\langle \circ, \circ \rangle_p$ to the tangent space $T_pX$ at each point $p \in X$. We write $g(v, u) = \langle v, u \rangle_p$. The pair $X, g$ is called a Riemannian manifold.

# Riemannian Manifold (cont.)

Given a Riemann metric we can define the length of a curve $C : [0, 1] \to X$ as

$$\ell(C) = \int_0^1 \|\dot{C}(t)\|_p dt$$

where the norm is induced by the inner product $g$ at $p$. Given two points $p, q \in X$ we define the distance $\rho(p, q)$ between them as $\inf \ell(C)$ over all curves $C$ between $p$ and $q$.

## Isometry

We say that two Riemannian surfaces $X, g$ and $Y, h$ are isometric if there exists a diffeomorphism $\phi : X \to Y$ such that

$$g(v, u) = h(\phi_*(v), \phi_*(u))$$

for all $v, u \in T_pX$ for all $p$.

# Geodesics

## Geodesics

A geodesic curve $C$ on a Riemannian manifold $X$ is a curve with unit speed that realises locally the distnace between two points; i.e. for every $t \in [0,1]$ has a closed neighborhood $[t_0, t_1]$ such that $\ell(C\big|_{[t_0,t_1]}) = (t_1 - t_0)$.

In Riemannian geometry the geodesic curve can also defined using a PDE involving the *Connections*.

## Proposition

Isometries map geodesics to geodesics.

# Outline

# Hyperbolic Surfaces

## Hyperbolic Plane

Let $\mathbb{H}$ be the open unit disk. With the Riemann metric $ds^2 = 4(dr^2 + r^2 d\theta^2)/(1-r)^2$, $\mathbb{H}$ can be given Reimann metric structure.

## Geodesics on the Hyperbolic Plane

The geodesic curves in $\mathbb{H}$ are just semi-circles which are perpendicular to the boundary circle and the diameters.

## Hyperbolic Manifold

A surface with a Riemannian metric is said to be a hyperbolic manifold if it is locally isometric to the hyperbolic plane. This means that for every point $p \in X$ there is an open set $U$ in the atlas which is isometric to $\mathbb{H}$.

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# Hopf-Rinow Theorem (half of it)

**Hopf-Rinow Theorem**

In a complete hyperbolic surface all geodesics can be extended indefinitely.

**Proof.**

Suppose that $\gamma : (-\epsilon, \epsilon) \to X$ is a bounded geodesic in $X$. Then consider a sequence of points $\gamma(t_n)$ where $t_n \to \epsilon$. Since $X$ is complete the Cauchy sequence $\gamma(t_n)$ converges to a unique point, say $x_1$. Let $U, \phi$ be some chart centered around $x_1$. The image $\phi \circ \gamma$ is a geodesic by previous proposition. Extend this geodesic in $\mathbb{H}$. Then the pull back of this extension extends $\gamma$ at $x_1$ till a new end point $x_2$. Repeating this process one can indefnitely extend $\gamma$. $\square$

# Outline

# Main Result

**Theorem**

Every complete, connected, and simply connected hyperbolic surface is isometric to $\mathbb{H}$.

Let $X$ be a hyperbolic manifold that satisfies these conditions. Fix a point $a \in X$ and a chart $(U, \phi)$ centered at $a$. The proof of this is long so let's break it into three parts.

1. There exists a map $E : \mathbb{H} \to X$ (dependent on our choice $a$ and the chart). We call this the exponential.
2. There exists a local isometry $D : X \to \mathbb{H}$ such that $D\big|_U = \phi$. This is called the developing map.
3. The two maps above our inverses of each other, and thus each of them are isometries.

# Step 1: The Exponential

## The Exponential map

Let $z \in \mathbb{H}$ and $\gamma$ be the unique geodesic from 0 and $z$. Then $\phi^{-1}(\gamma \cap \phi(U))$ in $U$ can be extended to a geodesic from $a$ to the point $x \in X$ such that the distance between $a$ and $x$ is the same as 0 and $z$. Define $E(z) = x$.

# Step 2: Developing

Consider any point $b \in X$. Let $\gamma$ be any path from $a$ to $b$. By compactness there is a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that you can cover the curve $\gamma$ by finitely many coordinate balls $U_i$ containing $\gamma([t_i, t_{i+1}])$, each of which is isometric to an open set $\tilde{U}_i$ by some chart $\phi_i$. We can assume that $U_0 = U$ and $\phi_0 = \phi$. Also without loss of generality assume that $\phi_i = \phi_{i+1}$ on the intersection $U_i \cap U_{i+1}$.

**The Developing map**

Define the developing map $D(b) = \phi_n(b)$.

We need to show the well definedness of this map first.

# Developing (cont.)

1. If we choose a different partition then we construct a refinement with $\phi_i$ just being the restricitions of the previous one. This does not change the value of $D(b)$.
2. If we choose a different curve $\gamma'$ between $a$ and $b$ then there is a homotopy between $\gamma'$ and $\gamma$. Since the image of the homotopy is compact there exists finitely many coordinate balls $V_i$ with charts $\psi_i$ (all of which glue up w.l.o.g). By Lebesgue number lemma there is an $N$ such that image of each $1/N \times 1/N$ square grid in $[0,1] \times [0,1]$ is contained in some $V_i$. Finally $D(b)$ will independent of the curve.

Also note that by definition $D$ is a local isometry.

# Step 3: Finishing the proof

It is easy to see that $D \circ E = 1_{\mathbb{H}}$: let $x \in \mathbb{H}$ then $E(z)$ lies on a geodesic $\gamma$ from $a$ to $E(z)$ such that $\phi \circ \gamma$ is part of the geodesic connecting 0 and $z$. Let $U_i$ be any minial cover of the geodesic from $a$ to $E(z)$. Then $\phi_n(E(z))$ lies on the extension of the geodesic $\phi \circ \gamma$ and $\rho(0, D(E(z))) = \rho(0, z)$ since $D$ is an isometry, but there is only one such point on the geodesic: $x$. Hence $D \circ E(x) = x$.
Note that on the image of $E$ in $X$ the map $E \circ D$ is identity. $E(\mathbb{H})$ is closed and open (since $E$ is local injection it follows by Invariance of domain theorem). Since $X$ is connected the only non-trivial clopen subset is $X$ itself. Thus $E(\mathbb{H}) = X$. Hence $E \circ D = 1_X$. This completes the proof.

# Outline

# Teichmuller Space

## Marked Spaces

Let $S$ be a topological surface then define the pair $(X, \phi)$ to be a marked space if $X$ is a hyperbolic surface and $\phi : S \to X$ is a homeomorphism.

We can define an equivalence relation on the set of all marked spaces of $S$ as follows: let $(X, \phi) \sim (Y, \psi)$ iff there exists an isometry $I : X \to Y$ such that $I \circ \phi$ is isotopic to $\psi$.

## Teichmuller Space

Define the Teichmuller space $Teich(S)$ to be the set of all equivalence classes of all marked surfaces of $S$.

In other words the Teichmuller space is the collection of all hyperbolic structures on $S$ upto isotopy. We can give a metrizable topology on the Teichmuller space.

# References

Casson and Bleiler, *Automorphisms of surfaces after nielson and thurston*, 1993.

Bruno Martelli, *An introduction to geometric topology*, 2016.

Thank You!