Notes on Mathematics

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Updated on: July 4, 2022

Contents

PART I
BASICS

I.1 Elementary Set Theory

A set S is a collection of objects. The objects of a set are called the elements. The union of two sets S, T contains elements of both sets, and is written as $S \cup U$. The intersection of two sets contains the common elements of the two sets, and is represented $S \cap U$. The complement of a set $A(\subset S)$ with respect to some set S is represented as A'. The difference of set S from T is defined as $T - S = T \cap S'$. The algebra of these operations is as follows

- *Commutativity,*

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

- *Associativity,*

$$A \cap (B \cap C) = (A \cap B) \cap C$$
$$A \cup (B \cup C) = (A \cup B) \cup C$$

- *Distribution,*

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- *De Morgan's rules,*

$$(A \cap B)' = A' \cup B'$$
$$(A \cup B)' = A' \cap B'$$

Definition I.1.1 The cartesian product of two sets A and B, denoted $A \times B$, is the set of ordered pairs $\{(a, b) | \forall a \in A \quad \text{and} \quad b \in B\}$.

Other than these operations one can define a set operation called disjoint union, denoted $S \sqcup T$, which is loosely constructed in the following way: we first make copies S' and T' such that $S' \cap T' = \emptyset$ and then take their ordinary union. A more rigourous definition would be constructed later.

I.2 Relations

Definition I.2.1 A relation on a set A is a subset C of the cartesian product $A \times A$. If $(x, y) \in C$ then it is denoted as xCy.

Definition I.2.2 An equivalence relation on a set A is a relation C on A such that:

- It is reflexive, i.e. $xCx \; \forall \; x \in A$.
- It is symmetric, i.e. if xCy then yCx.
- It is transitive, i.e. if xCy and yCz then xCz.

Generally the symbol $\sim$ is used to denote an equivalence relation. For a given element $x \in A$ we also define a set called the equivalence class as:

$$E = \{y \mid y \sim x\}$$

Proposition I.2.3 Two equivalence classes E and E' are either disjoint or equal.

Proof | Let E be the equivalence class of x and E' be the equivalence class of x'. Assuming that $E \cap E'$ is non-empty, for all $y \in E \cap E'$ it follows that $y \sim x'$ and $y \sim x$. From symmetry and transitivity it follows that $x' \sim x$. Hence every element similar to x' will be similar to x. Hence $E' = E$, whenever $E \cap E'$ is non-empty. ∎

Definition I.2.4 A partition of a set A is a collection of disjoint nonempty subsets of A whose union is all of A.

Proposition I.2.5 Given any partition $\mathscr{D}$ of A, there is a unique equivalence relation C on A such that each element of $\mathscr{D}$ is an equivalence class of C.

Proof | Consider a relation C defined as: xCy if both x and y belong to the same element of $\mathscr{D}$. Since x is always in the same element as itself, xCx is true for all x. If xCy, which means x is in the same subset as y. Since the converse is also true, yCx. If x is in the same subset as y and y is in the subset as z, then x is in the same subset as z. Hence xCy and yCz imply xCz. This means that C is an equivalent relation. Each element of $\mathscr{D}$ can be viewed as an equivalence class of C.

Assume that there exist two equivalence relations C_1 and C_2 such that the set of each their equivalence classes is $\mathscr{D}$. Let E_1 and E_2 be equivalence classes of x with respect to relations C_1 and C_2. E_1 and E_2 must be the same since we are claiming that both relations generate the identical collection of sets. Hence if yC_1x then yC_2x which implies that $C_1 = C_2$. ∎

Definition I.2.6 The quotient of the set S, denoted $S/\sim$ with respect to the equivalence relation $\sim$ is the set of equivalence classes of S with respect to $\sim$.

I.3 Functions

Definition I.3.1 A rule of assignment is a subset r of the cartesian product $C \times D$ of two sets, having the property that each element of C appears as the first ordinate of at most one ordered pair in r.

From this definition one can easily conclude that, if $r \subset C \times D$ and $(c, d), (c, d') \in r$ then $d = d'$. One can think of r as assigning an element $c \in C$, the element $d \in D$. The set C is called the domain of r and D is called the image set.

Definition I.3.2 A function f is a rule of assignment r, along with a set B which contains the image set of r. The domain of r is also the domain of f. The set B is called the range.

A function having a domain A and range B is written as $f : A \to B$. Given an element $a \in A$, $f(a)$ denotes a unique element in B, hence $(a, f(a)) \in r$.

Definition I.3.3 Given a function $f : A \to B$ and a subset $A_0 \subset A$, then a restriction of f to A_0 is the mapping $f|A_0 : A_0 \to B$ with rule:

$$\{(a, f(a))|a \in A_0\}$$

Definition I.3.4 Given functions $f : A \to B$ and $g : B \to C$, the composite function is defined as $g \circ f : A \to C$, such that $g \circ f(a) = g(f(a))$. More formally, the rule of the function $g \circ f : A \to C$ is:

$$\{(a, c)|\forall b \in B, \quad f(a) = b \quad \text{and} \quad g(b) = c\}$$

Definition I.3.5 A function $f : A \to B$ is said to be injective if,

$$f(a) = f(a') \implies a = a'.$$

The function is called surjective if for each $b \in B$ there exists an $a \in A$ such that $b = f(a)$. If f is both injective and surjective it is said to be bijective.

Proposition I.3.6 For each bijective function $f : A \to B$, there exists a unique function, called the inverse function, $f^{-1} : B \to A$ such that $f \circ f^{-1}$ and $f^{-1} \circ f$ are both identity functions.

Proof | Since f is bijective for every $a \in A$ there exists a unique $b \in B$ (from injection), and for every $b \in B$ also there exists an $a \in A$ (from surjectivity). This implies that every $b \in B$ has a unique pre-image in A. Denote this pre-image by $f^{-1}(b)$. The rule of the inverse function is given by:

$$\{(b, f^{-1}(b))|\forall b \in B\}$$

This proves the existence of inverse. Using the definition of composite function, the rule of the composite function $f \circ f^{-1}$ will be:

$$\{(b, b)|\forall b \in B\}.$$

Hence the composite is the identity function. Similarly the composite function $f^{-1} \circ f$ is also identity.

For proving the uniqueness, consider there exist two inverse functions, f^{-1} and $\tilde{f}^{-1}$, of f. Hence,

$$f(f^{-1}(b)) = b,$$

$$\implies \tilde{f}^{-1}(f(f^{-1}(b))) = \tilde{f}^{-1}(b),$$

But since $\tilde{f}^{-1}(f(a)) = a$,

$$f^{-1}(b) = \tilde{f}^{-1}(b) \quad \forall b \in B$$

Hence the inverse is unique. ■

Proposition I.3.7 The inverse of a bijective function $f : A \to B$ is also bijective.

Proof | Let the inverse be f^{-1}. Let $b, b' \in B$ such that

$$f^{-1}(b) = f^{-1}(b')$$

$$\implies f(f^{-1}(b)) = f(f^{-1}(b'))$$

$$\implies b = b'$$

This shows that f^{-1} is injective. For proof of surjectivity, we can show that for each $a \in A$ there exists a $b(= f(a)) \in B$ such that $a = f^{-1}(b)$. This shows that f^{-1} is also bijective. ∎

Proposition I.3.8 Let $f : A \to B$. If there are functions $g : B \to A$ and $h : B \to A$ such that $g(f(a)) = a\, \forall a \in A$ and $f(h(b)) = b\, \forall b \in B$, then f is bijective and $g = h = f^{-1}$.

Proof | Let $a, a' \in A$, such that

$$f(a) = f(a')$$

Using the function g,

$$g(f(a)) = g(f(a')),$$
$$\implies a = a'$$

hence f is an injective function. Now coming to surjectivity. Using the existance of h, we can show that for each $b \in B$ there exists $a(= h(b)) \in A$ such that $b = f(a)$. Hence f is a bijective function. For the final part of the proposition, since,

$$f(h(b)) = b,$$
$$\implies g(f(h(b))) = g(b),$$
$$\implies h(b) = g(b) \quad \forall b \in B \quad .$$

And since the inverse is unique, they must also be equal to f^{-1}. ∎

When there exists a bijection $f : A \to B$ then A and B are called *isomorphic*. This is sometimes represented as $A \simeq B$. Using the concept of isomorphism the notion of disjoint union can be made more rigourous.

Definition I.3.9 The disjoint union of two sets A and B is detemined by constructing sets $A' \simeq A$ and $B' \simeq B$ such that $A' \cup B' = \emptyset$, and then determining the union $A' \cup B'$. Such sets can always be constructed for every set since $\{0\} \times A \simeq A$ and $\{1\} \times B \simeq B$ and $(\{0\} \times A) \cap (\{1\} \times B) = \emptyset$.

A less restrictive notion of invertibility is defined in terms of *left-invertible* and *right-invertible* functions. If for a function $f : A \to B$ there exists a $g : B \to A$ such that $g \circ f : A \to A$ is id_A then f is said to be left invertible. Similarly if there exists $h : B \to A$ such that $f \circ h : B \to B$ is id_B then f is called right invertible. The following is more general statement to proposition I.3.8.

Proposition I.3.10 Let $f : A \to B$ be a function then:

1) f is injective if and only if it is left invertible.
2) f is surjective if and only if it is right invertible.

Proof | For statement 1, the foward implication follows from the fact that if f injective then on can construct a $g : B \to A$ as follows: let $p \in B$ be a fixed point and

$$g(b) = \begin{cases} a, \text{ where } f(a) = b \\ p, \text{ when } b \text{ not in image of A.} \end{cases} \tag{I.3.1}$$

Clearly the function $g \circ f(a) = id_A(a) = a$. The backward implication for statement 1, is true because if a g exists such that:

$$g \circ f(a) = a\, \forall\, a \in A$$

then,
$$g \circ f(a) = a \neq a' = g \circ f(a') \implies f(a) \neq f(a')$$
which is the contrapositive of the statement required to prove.

Statement 2 can be proven in a similar way. Assuming that f is surjective, it follows that for each b there is atleast one $a \in A$ such that $f(a) = b$. Choosing anyone of these a for each b we can construct the map $h : B \to A, h(b) = a$. Hence it follows that $f \circ h(b) = f(a) = b = id_B(b)$. For the backward implication, assuming that there is an $h : B \to A$ such that $\forall\, b \in B, f \circ h(b) = id_B(b) = b$. Since $h(b) = a$ for some $a \in A$, it follows $f(a) = b$ for some $a \in A$. Hence f is surjective. ∎

There is another way to look at bijective functions using the concept of monomorphisms and epimorphisms. This is a more fundamental and equivalent approach to defining bijections.

DEFINITION I.3.11 A function $f : A \to B$ is said to be a monomorphism if the following holds:
$$\forall\, Z, \forall\, \alpha', \alpha'' : Z \to A,\ f \circ \alpha' = f \circ \alpha'' \implies \alpha' = \alpha''$$

PROPOSITION I.3.12 A function $f : A \to B$ is a monomorphism if and only if it is injective.

Proof | Consider first the forward implication. Assuming f is a monomorphism, we know that for all sets Z and $\alpha', \alpha'' : Z \to A$,
$$f \circ \alpha' = f \circ \alpha'' \implies \alpha' = \alpha''$$
if $\alpha'(z) = a$ and $\alpha''(z) = a'$ then the above condition reduces to,
$$f(a) = f(a') \implies a = a'$$
Hence f is injective.

For the backward implication we assume that f is an injective function. Then we know that f is left-invertible, with inverse g. If,
$$\begin{aligned} & f \circ \alpha' = f \circ \alpha'' \\ \implies & g \circ f \circ \alpha' = g \circ f \circ \alpha'' \\ \implies & \alpha' = \alpha'' \end{aligned}$$
∎

DEFINITION I.3.13 A function $f : A \to B$ is said to be an epimorphism if,
$$\forall\, Z, \forall\, \beta', \beta'' : B \to Z,\ \beta' \circ f = \beta'' \circ f \implies \beta' = \beta''$$

PROPOSITION I.3.14 A function $f : A \to B$ is an epimorphism if and only if it is surjective.

Proof | Let's first consider the forward implication:
$$(\forall\, Z,\ \beta', \beta'' : B \to Z\ \beta' \circ f = \beta'' \circ f \implies \beta' = \beta'') \implies (\forall\, b \in B\ \exists\, a \in A \text{ such that } b = f(a))$$

The contraposition of this statment is:

$$(\exists\, b \in B,\, \forall\, a \in A,\, b \neq f(a)) \implies (\exists Z,\, \beta', \beta'' : B \to Z \text{ such that } \beta' \neq \beta'' \;\;\&\;\; \beta' \circ f = \beta'' \circ f)$$

Assuming there exists $b \in B$ such that b is not in the image of f, let $Z = \{0, 1\}$, $\beta'(b) = 0,\ \forall\, b \in B$, and

$$\beta''(b) = \begin{cases} 0, & \text{if } b \text{ is in image of } f \\ 1, & \text{if } b \text{ is not in image of } f \end{cases}$$

Clearly $\beta' \neq \beta''$ but we have $\beta' \circ f = \beta'' \circ f$. Hence the contrapositive is true, therefore proving the forward implication.

For the backward implication, assuming the funciton is surjective we also know that it would be right invertible. Let h be the right inverse then,

$$\begin{aligned} \beta' \circ f &= \beta'' \circ f \\ \implies \beta' \circ f \circ h &= \beta'' \circ f \circ h \\ \implies \beta' &= \beta'' \end{aligned}$$

Hence completing the proof. ∎

Diagrams. Diagrams are graphical representations of a collection of sets and how they are operated on by functions. A diagram is said to be *commutative* if taking different paths between sets result in the same function. For example if $f : A \to B$ and $g : B \to C$, then here is a commutative diagram of A, B, C:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

For injective functions a $\hookrightarrow$ is used, for surjective functions $\twoheadrightarrow$ is used, and isomorphisms are represented by $\xrightarrow{\sim}$.

Canonical Decomposition. Let $f : A \to B$ be a function on A. Define the equivalence relation $\sim$ on A as $a \sim a'$ iff $f(a) = f(a')$. A surjection $g : A \twoheadrightarrow A/\!\sim$ can be defined as $g(a) = [a]_\sim$. Also it is possible to find an injection $h : \operatorname{im} f \hookrightarrow B$, given by $h(b) = b$. Hence we have a diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, two heads, "g"'] & B \\ A/\!\sim & \operatorname{im} f \arrow[u, hook, "h"'] \end{tikzcd}$$

If we can find an isomorphism $i : A/\!\sim\ \xrightarrow{\sim} \operatorname{im} f$ then the above diagram will commute. Consider the following proposition:

> **Proposition I.3.15** The function $i : A/\!\sim\, \to \operatorname{im} f$ given by $i([a]_\sim) = f(a)$ is an isomorphism.

Proof | First we must check if i is a function. Let $[a]_\sim, [a']_\sim \in A/\sim$ then, $[a] = [a'] \implies f(a) = f(a') \implies i([a]_\sim) = i([a']_\sim)$. This means for each $[a]_\sim$ there is a unique image.

Injective. If $i([a]_\sim) = i([a']_\sim)$ then $f(a) = f(a')$ by definition of i. Further it implies that $a \sim a'$ by definition of the equivalence relation. Hence a and a' are in the same equivalence class, or $[a]_\sim = [a']_\sim$.

Surjective. Let $b \in \operatorname{im} f$. Then there exists an $a \in A$ such that $f(a) = b$. Hence there exists $[a]_\sim$ such that $b = i([a]_\sim)$. ■

As a result of this proposition we have shown that any funciton f can be decomposed according to the commutative diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, two heads, "g"'] & B \\ A/\sim \arrow[r, "\sim", "i"'] & \operatorname{im} f \arrow[u, hook, "h"'] \end{tikzcd}$$

This shows that any function can be written as a composition of injections, surjections, and isomorphisms. This decomposition is called the canonical decompositions.

DEFINITION I.3.16 Let $f : A \to B$ be a function, and $A_0 \subset A$. Then define,

$$f(A_0) = \{b \mid b = f(a),\ a \in A_0\},$$

and

$$f^{-1}(B_0) = \{a \mid f(a) \in B_0\}.$$

Note that this definition is for all functions, not just bijective functions.

PROPOSITION I.3.17 Let $f : A \to B$ be a function, and let $A_0 \subset A$, $B_0 \subset B$ then,

$$A_0 \subset f^{-1}(f(A_0)) \quad \text{and} \quad f(f^{-1}(B_0)) \subset B_0$$

Proof | For the first statement, let $a \in A_0$. Then $f(a) \in f(A_0)$. Which further implies, by definition, that $a \in f^{-1}(f(A_0))$. Hence,

$$\implies A_0 \subset f^{-1}(f(A_0))$$

For the second part of the proposition, let $b \in f(f^{-1}(B_0))$. This means that there exists $a \in f^{-1}(B_0)$ such that $b = f(a)$. Since $a \in f^{-1}(B_0)$, again by definition, $f(a) \in B_0$. Hence $b \in B_0$. Since this is true for any $b \in f(f^{-1}(B_0))$, we conclude that $f(f^{-1}(B_0)) \subset B_0$. ■

PROPOSITION I.3.18 Let $f : A \to B$, $A_0,\ A_1 \subset A$, and $B_0,\ B_1 \subset B$. Then f^{-1} preserves:

1) inclusions
2) unions
3) intersections
4) differences

Proof Preservation of inclusion: let $B_0 \subset B_1$. From the definition it follows that $f^{-1}(B_0) = \{a \mid f(a) \in B_0\}$. Since $B_0 \subset B_1$, if $f(a) \in B_0$ then $f(a) \in B_1$. Hence if $a \in f^{-1}(B_0)$ then $a \in f^{-1}(B_1)$. Hence $f^{-1}(B_0) \subset f^{-1}(B_0)$.

Proof of preservation of unions: the set $f^{-1}(B_0 \cup B_1) = \{a \mid f(a) \in B_0 \cup B_1\}$. While $f^{-1}(B_i) = \{a \mid f(a) \in B_i$. The union $f^{-1}(B_0) \cup f^{-1}(B_1) = \{a \mid f(a) \in B_0$ or $f(a) \in B_1\}$, which is the same as $\{a \mid f(a) \in B_0 \cup B_1\}$. Hence the two sides are equivalent.

Proof for intersections and differences is very similar to the one for unions. ■

Unlike it's inverse f only preserves inclusions and unions. Showing this is pretty easy. Also another property of functions is that $(g \circ f)^{-1}(C_0)$ is equivalent to $f^{-1}(g^{-1}(C_0))$ for functions $f : A \to B$, $g : B \to C$, and set $C_0 \subset C$.

I.4 Categories

A category is essentially a collection of 'objects' and of 'morphisms' between these objects, satisfying a list of natural conditions. These objects might be sets, groups, vector spaces, etc. Since there is simply no set of all sets (due to Russell's paradox), this collection of objects is just too 'big' to be called a set. The formal term used is a *class of objects*. The formal definition of categories is as follows.

Definition I.4.1 A category C consists of:

1) a class Obj(C) of *objects* of the category.
2) for every two objects A, B of C, a set $\mathsf{Hom}_{\mathsf{C}}(A, B)$ of morphisms satisfying the following properties:
   i) for every object A of C there exists (at least) one morphism $1_A \in \mathsf{H}_{\mathsf{C}}(A, A)$. This is the identity on A.
   ii) one can compose morphisms: two morphisms $f \in \mathsf{Hom}_{\mathsf{C}}(A, B)$ and $\mathsf{Hom}_{\mathsf{C}}(B, C)$ determine a morphism $gf \in \mathsf{Hom}_{\mathsf{C}}(A, C)$. For every triplet of objects A, B, C of C there is a function (of sets)

$$\mathsf{Hom}_{\mathsf{C}}(A, B) \times \mathsf{Hom}_{\mathsf{C}}(B, C) \to \mathsf{Hom}_{\mathsf{C}}(A, C)$$

   iii) this composition law is associative.
   iv) the identity morphisms are identities with respect to composition, i.e. if $f \in \mathsf{Hom}_{\mathsf{C}}(A, B)$ then

$$f 1_A = f,\ 1_B f = f$$

   v) the sets $\mathsf{Hom}_{\mathsf{C}}(A, B)$ and $\mathsf{Hom}_{\mathsf{C}}(C, D)$ are disjoint unless $A = C$ and $B = D$.

One can make morphism diagrams similar to those of set functions. The set of morphisms from an object to itself are known as endomorphisms and are denoted $\mathsf{End}(A)$. The subscript C will be dropped from now on, unless it is necassary to use it.

Example I.4.2 (Sets) As a first example consider the category Set defined as $\mathsf{Obj}(\mathsf{Set})$ = the class of all sets, and $\mathsf{Hom}(A, B)$ = the set of all set-functions from A to B. We must verify if this is a category. For every A there is an indentity function $1_A : A \to A$, $1_A(a) = a$. Composition of set-functions is possible, the composition is known to be associative, and the identity function is identity with respect to the composition. The last property is also triially true. Hence Set is indeed a category.

Example I.4.3 Consider this example.

Suppose S is a set and $\sim$ is a reflexive and transitive relation. Then define a category C as:

- objects are elements of S;
- $\mathsf{Hom}(a, b)$, where a, b are objects, is the set consisting $(a, b) \in S \times S$ if $a \sim b$, and let $\mathsf{Hom}(a, b) = \emptyset$ otherwise.

Verification that this is a category:

1) Since $a \sim a$ using reflexive property, $1_a = (a, a) \in \mathsf{Hom}(a, a)$.
2) Given two morphisms $f = (a, b) \in \mathsf{Hom}(a, b)$ and $g = (b, c) \in \mathsf{Hom}(b, c)$ then using transitivity we know that $a \sim c$ and hence $gf = (a, c) \in \mathsf{Hom}(a, c)$. Hence a composition exists.
3) The composition is clearly associative.
4) Let $f = (a, b) \in \mathsf{Hom}(a, b)$, and we know that $1_a = (a, a)$ and $1_b = (b, b)$. Clearly $f 1_a = (a, b)$ and $1_b f = (a, b)$.
5) Since each $\mathsf{Hom}(a, b)$ has either one element (a, b) or is empty, any two set of morphisms will be disjont.

As an example of this kind of category consider the set $\{1, 2, 3\}$ along with the ordering $\leq$. The following is a commutative diagram of this category:

$$
\begin{tikzcd}
1 \arrow[loop left, "1_1"] \arrow[r, "f"] \arrow[dr, "gf"'] & 2 \arrow[loop right, "1_2"] \arrow[d, "g"] \\
& 3 \arrow[loop right, "1_3"]
\end{tikzcd}
$$

PART II

ABSTRACT ALGEBRA

PART III

TOPOLOGY

PART IV

MEASURE THEORY

IV.1 Intuition of Measure

Consider any interval of $\mathbb{R}$, $(a,b]$. Intuitively we define the length of this interval as $\lambda((a,b]) = b-a$. Now is it possible to extend this concept of length to any subset of $\mathbb{R}$? For that we would wish to find a function λ such that the following properties are true:

1) $\lambda : \mathfrak{P}(\mathbb{R}) \to \mathbb{R}^+$ is a set function.
2) If $I = (a,b]$ is any interval in $\mathbb{R}$ then $\lambda(I) = b-a$.
3) Length of union of two disjoint subsets $A, B \in \mathfrak{P}(\mathbb{R})$ must be the sum of their individual lengths, i.e. $\lambda(A+B) = \lambda(A) + \lambda(B)$. This can be extended to any countable union of pairwise disjoint subsets.
4) The translation of any subset $A \in \mathfrak{P}(\mathbb{R})$ must have the same length, i.e. $\forall\, x \in \mathbb{R}$, $\lambda(A+x) = \lambda(A)$.

Conjecture IV.1.1 The function λ as described above does not exist.

To prove this we must first prove some other propositions. We use the following notations: let $\sim$ be an equivalence relation on $\mathbb{R}$ defined as:

$$x \sim y \text{ if } x - y \in \mathbb{Q}$$

Let $[x]$ denote the equivalence classes of x, and let $\Lambda = \mathbb{R}/\sim$.

Proposition IV.1.2 The set Λ is uncouontable.

Proof | Let $\alpha \in \Lambda$ be an equivalence class. Let $x \in \alpha$ be a fixed point. Then for each $y \in \alpha$ it is possible to find a unique rational number given by $x - y$. Hence α is a countable set. Since the countable union of countable sets is countable, but $\mathbb{R}$ is uncountable, it follows that Λ must be uncountable. ∎

Let $\Omega \subset \mathbb{R}$ be set constructed in the following way: for each $\alpha \in \Lambda$ we know that a point can be found between $(0,1)$; so take one such point from each α and put it in the set Ω. From this construction it is easy to see that $\Omega \subset (0,1)$.

Proposition IV.1.3 Let $p, q \in \mathbb{Q}$, then either $\Omega + q = \Omega + p$ or $\Omega + q \cap \Omega + p = \emptyset$.

Proof | Let's say $\Omega + q \cap \Omega + p \neq \emptyset$. Then for $a, b \in \Omega$ we can find an $x \in \Omega + q \cap \Omega + p$ such that $x = a + p = b + q$. Hence $a - b = q - p$ for all $a, b \in \Omega$. Since $q - p$ is rational, $a - b$ will also be rational. Hence a and b belong to the same equivalence class. But since we only chose one element from each equivalence class in the construction of Ω, we must have $a = b$. Hence $q = p$, making the two sets in question equal. ∎

Hence from this proposition we can say that if $q \neq p$ then $\Omega + q \cap \Omega + p = \emptyset$.

Proposition IV.1.4 Let λ be a length function as defined above. If $A \subset B \subset \mathbb{R}$ then $\lambda(A) \leq \lambda(B)$.

Proof | Since $B = A \cup (B - A)$, and A and $B - A$ are disjoint,

$$\begin{aligned}\lambda(B) &= \lambda(A \cup (B - A)) \\ &= \lambda(A) + \lambda(B - A) \\ &\geq \lambda(A)\end{aligned}$$

Hence proving our claim. ■

Now we are ready to prove conjecture IV.1.1.

proof of conjecture IV.1.1 | Consider the union of sets $\Omega + q$:

$$\bigcup_{\substack{q \in \mathbb{Q} \\ -1<q<1}} \Omega + q$$

From proposition IV.1.3 we know that this is a union of disjoint sets. Hence,

$$\lambda \left(\bigcup_{\substack{q \in \mathbb{Q} \\ -1<q<1}} \Omega + q \right) = \sum_{\substack{q \in \mathbb{Q} \\ -1<q<1}} \lambda\left(\Omega + q\right)$$

Using property 4 of λ,

$$\lambda \left(\bigcup_{\substack{q \in \mathbb{Q} \\ -1<q<1}} \Omega + q \right) = \sum_{\substack{q \in \mathbb{Q} \\ -1<q<1}} \lambda\left(\Omega\right) = 0$$

Let $x \in (0, 1)$. Let $a \in \Omega \cap [x]$. Then we know that $x - a = q$ for some rational q. Since $a \in \Omega$ implies $a \in (0, 1)$, the range of q must be $-1 < q < 1$. Hence $x = a + q$ for some $-1 < q < 1$ implying that

$$x \in \bigcup_{\substack{q \in \mathbb{Q} \\ -1<q<1}} \Omega + q$$

further implying that,

$$(0, 1) \subset \bigcup_{\substack{q \in \mathbb{Q} \\ -1<q<1}} \Omega + q$$

Using proposition IV.1.4,

$$1 \leq \lambda \left(\bigcup_{\substack{q \in \mathbb{Q} \\ -1<q<1}} \Omega + q \right)$$

Hence we have arrived at a contradiction. This shows that a function λ with the properties 1,2,3,4 as given above does not exist. ■

Hence this shows that to construct a general notion of length (called the *measure*) we must let go of one of the four properties: 1,2,3, or 4. Since $2, 3, 4$ are essential for a notion of length, we change 1 to be the following:

1) $\lambda : \mathscr{B}(\subset \mathfrak{P}(\mathbb{R})) \to \mathbb{R}^+$ is a set function.

This means that we are discarding the notion that all subsets of $\mathbb{R}$ can be assigned a length.

IV.2 Formal Notion of Measure

Definition IV.2.1 A class of subsets, $\mathscr{A}$, of a set Ω is said to be a semi-algebra if:

1) $\Omega \in \mathscr{A}$,
2) closed under finite intersections,
3) The compliment of any set in $\mathscr{A}$ can be expressed as unions of finite pairwise disjoint sets in $\mathscr{A}$.

Definition IV.2.2 A class of subsets, $\mathscr{A}$, of a set Ω is said to be an algebra if:

1) $\Omega \in \mathscr{A}$,
2) closed under finite intersections,
3) Closed under compliment.

Definition IV.2.3 A class of subsets, $\mathscr{A}$, of a set Ω is said to be a σ-algebra if:

1) $\Omega \in \mathscr{A}$,
2) closed under countable intersections,
3) Closed under compliment.

Proposition IV.2.4 Let Ω be a set and $\mathscr{A}_\alpha \subset \mathfrak{P}(\Omega)$ be algebras, where $\alpha \in I$ (no assumptions have been made on I). Then

$$\mathscr{A} = \bigcap_{\alpha \in I} \mathscr{A}_\alpha$$

is also an algebra.

Proof | Since $\Omega \in \mathscr{A}_\alpha$, $\forall\, \alpha \in I$, implies that $\Omega \in \mathscr{A}$. If $A_1, ..., A_n \in \mathscr{A}$ then $A_1, ..., A_n \in \mathscr{A}_\alpha$ for any $\alpha \in I$. Since $\mathscr{A}_\alpha$ is an algebra, it follows that $\bigcap_{j=1}^n A_j$ is in $\mathscr{A}_\alpha$ for any $\alpha \in I$; hence it is also in $\mathscr{A}$. If $A \in \mathscr{A}$ then it is in every $\mathscr{A}_\alpha$ and hence its compliment is in every $\mathscr{A}_\alpha$. ■

The above proposition also applies to σ-algebras as well. Essentially the same argument applies, just that instead of finite sets we have countable intersection, i.e. $n \to \infty$. To denote that something applies to both algebras and σ-algebras we use the notation $(\sigma-)$algebra.

Definition IV.2.6 A class $\mathscr{C}$ of subsets of set Ω is said to *generate* an $(\sigma-)$algebra $\mathscr{A}$ if $\mathscr{C} \subset \mathscr{A}$ and if for any $(\sigma-)$alegbra $\mathscr{A}' \supset \mathscr{C}$ implies that $\mathscr{A} \subset \mathscr{A}'$.

Proposition IV.2.7 Every class $\mathscr{C} \subset \mathfrak{P}(\Omega)$ generates an $(\sigma-)$algebra.

Proof | Let $\mathscr{A}_\alpha$, $\alpha \in I$ be all the $(\sigma-)$algebras which contain the class $\mathscr{C}$. Then we know that,

$$\mathscr{A} = \bigcap_{\alpha \in I} \mathscr{A}_\alpha$$

is also an $(\sigma-)$algebra, and it will contain $\mathscr{C}$. From the definition of intersection it follows that $\mathscr{A} \subset \mathscr{A}_\alpha$. Hence $\mathscr{A}$ is the $(\sigma-)$algebra generated by $\mathscr{C}$. ■

Lemma IV.2.8 If $\mathscr{S}$ is a semi-algebra and $\mathscr{A}$ is the algebra generated by $\mathscr{S}$ then

$$A \in \mathscr{A} \iff \exists \text{ pairwise disjoint } E_1, ..., E_n \in \mathscr{S} \text{ such that } A = \bigcup_{j=1}^{n} E_j$$

Proof | $(\Longleftarrow)$ Assuming that A is finite union of disjoint sets $E_1, ..., E_n \in \mathscr{S}$ we need to show that $A \in \mathscr{A}$. Since $E_1, ..., E_n$ are in $\mathscr{S}$ it follows that they are also in $\mathscr{A}$. It further follows that the compliment of each $E_j \in \mathscr{A}$. Since

$$\left(\bigcap_{j=1}^{n} E_j^c\right)^c = \bigcup_{j=1}^{n} E_j,$$

and algebras are closed under finite intersections, $A \in \mathscr{A}$.

$(\Longrightarrow)$ Let $\mathscr{B}$ be the class defined as:

$$\mathscr{B} = \{B \mid \text{where } B = \bigcup_{j=1}^{n} F_j,\ F_j \in \mathscr{S} \text{ are pairwise disjoint.}\}$$

If we can show that $\mathscr{B}$ is an algebra containing $\mathscr{S}$ then by definition of generated algebras $\mathscr{A} \subset \mathscr{B}$. This shows that any element of $\mathscr{A}$ can be expressed as a finite union of disjoint sets. Hence all that remains is to show that $\mathscr{B}$ is an algebra containing $\mathscr{S}$.

1) Clearly by the definition, any element of $\mathscr{S}$ is also in $\mathscr{B}$. Hence $\mathscr{S} \subset \mathscr{B}$ and hence $\Omega \in \mathscr{B}$.
2) Let $B_1, ..., B_n \in \mathscr{B}$ then

$$\begin{aligned}
\bigcap_{j=1}^{n} B_j &= \bigcap_{j=1}^{n} \bigcup_{i=1}^{m} F_{ji} \\
&= \bigcup_{i=1}^{m} \bigcap_{j=1}^{n} F_{ji}, \text{ using definition of } \mathscr{B} \\
&= \bigcup_{i=1}^{m} E_i, \text{ where, } E_i = \bigcap_{j=1}^{n} F_{ji}
\end{aligned}$$

Since $\mathscr{S}$ is closed under finite intersections, this shows that $\mathscr{B}$ is closed under finite intersections.
3) Let $B \in \mathscr{B}$. Then,

$$\begin{aligned}
B^c &= \left(\bigcup_{i=1}^{m} F_i\right)^c \\
&= \bigcap_{i=1}^{m} F_i^c \\
&= \bigcap_{i=1}^{m} \bigcup_{j=1}^{n} E_{ij} \text{ (using property 3 of semi-algebras)} \\
&= \bigcup_{j=1}^{n} E_j, \text{ where } E_j = \bigcap_{i=1}^{m} E_{ij}
\end{aligned}$$

Since $\mathscr{S}$ is closed under finite intersections, this shows that $\mathscr{B}$ is closed under compliment. $\mathscr{B}$ is indeed an algebra containing $\mathscr{S}$, hence completing our proof. ■

DEFINITION IV.2.9 Let $\mathscr{C}$ be a class of subsets of Ω such that $\emptyset \in \mathscr{C}$, and let $\mu : \mathscr{C} \to \mathbb{R}^+$ be a function such that:

1) $\mu(\emptyset) = 0$,
2) If $E_1, ..., E_n \in \mathscr{C}$ are pairwise disjoint and if $\bigcup_{j=1}^n E_j \in \mathscr{C}$ then $\mu(\bigcup_{j=1}^n E_j) = \sum_{j=1}^n \mu(E_j)$.

then μ is said to be an additive measure.

Observe that if we have a $A \in \mathscr{C}$ such that $\mu(A) < \infty$ then:

$$\begin{aligned} \mu(A \cup \emptyset) &= \mu(A) + \mu(\emptyset) \\ \mu(A) &= \mu(A) + \mu(\emptyset) \\ \implies \mu(\emptyset) &= 0 \end{aligned}$$

Hence the first condition is just a consequence of the second if a subset with finite measure exists. Secondly observe that if $E \subset F \in \mathscr{C}$ and $F - E \in \mathscr{C}$ then:

$$\mu(E \cup F - E) = \mu(F) = \mu(E) + \mu(F - E)$$

this means that $\mu(E) \leq \mu(F)$, the equality being true when $\mu(E) = \infty$. In the case where $\mu(E) < \infty$ we have the identity $\mu(F - E) = \mu(F) - \mu(E)$.

EXAMPLE IV.2.11 Let Ω be any non-empty set and let $X_1, X_2, ... \in \Omega$. Also let $a_1, a_2, ... \geq 0$ be some constants. Then define a measure $\mu : \mathscr{C} \subset (\mathfrak{P}(\Omega)) \to \mathbb{R}^+$ as:

$$\mu(A) = \sum_{j \geq 1} a_j 1\{X_j \in A\}$$

where,

$$1\{X_j \in A\} = \begin{cases} 1, \text{ if } X_j \in A \\ 0, \text{ if } X_j \notin A \end{cases}$$

It is easy to see that this measure is indeed additive.

DEFINITION IV.2.12 Let $\mathscr{C}$ be a class of subsets of Ω such that $\emptyset \in \mathscr{C}$, and let $\mu : \mathscr{C} \to \mathbb{R}^+$ be a function such that:

1) $\mu(\emptyset) = 0$,
2) If $E_1, E_2, ... \in \mathscr{C}$ are pairwise disjoint and if $\bigcup_{j \geq 1} E_j \in \mathscr{C}$ then $\mu(\bigcup_{j \geq 1} E_j) = \sum_{j \geq 1} \mu(E_j)$.

then μ is said to be a $\sigma-$additive measure.

EXAMPLE IV.2.13 Let $\Omega = (0, 1)$ and $\mathscr{C} = \{(a, b] \mid 0 \leq a < b < 1\} \cup \{\emptyset\}$. Define a function $\mu : \mathscr{C} \to \mathbb{R}^+$ as:

$$\mu(a, b] = \begin{cases} \infty, \text{ if } a = 0 \\ b - a, \text{ if } a \neq 0 \end{cases}$$

Clearly since a subset with finite measure exists $\mu(\emptyset) = 0$. Also since,

$$(a, b] = \bigcup_{j=1}^{n} (a_j, a_{j+1}], \text{ where } a_1 = a \ \& \ a_n = b$$

when $a = 0$, $a_1 = 0$ and hence applying the measure on both sides we get ∞. When $a \neq 0$, so are none of the a_j and hence:

$$\mu(a, b] = b - a = (a_2 - a_1] + ... + (a_n - a_{n-1}] = \sum_{j=1}^{n} \mu(a_j, a_{j+1}]$$

Hence μ is additive. But it is possible to show that μ is not $\sigma-$additive. Consider for example the interval $(0, 1/2]$, and let $x_1 = 1/2, x_2, ...$ be a monotonic decreasing sequence in $(0, 1)$ which converges to 0. Then

$$(0, 1/2] = \bigcup_{j \geq 1} (x_{j+1}, x_j]$$

Clearly $\mu(0, 1/2] = \infty$, but $\mu(x_{j+1}, x_j] = x_{j+1} - x_j$ which is finite.

DEFINITION IV.2.14 Let $\mathscr{C}$ be a class of subsets of Ω and $\mu : \mathscr{C} \to \mathbb{R}^+$ be any set function. Then,

1) μ is said to be *continuous from below* at $E \in \mathscr{C}$ if $\forall \ (E_n)_{n \geq 1} \in \mathscr{C}$, $E_n \uparrow E \implies \lim \mu(E_n) = \mu(E)$.
2) μ is said to be *continuous from above* at $E \in \mathscr{C}$ if $\forall \ (E_n)_{n \geq 1} \in \mathscr{C}$, $E_n \downarrow E$ and $\exists \ n_0$ such that $\mu(E_{n_0}) < \infty$ implies that $\lim \mu(E_n) = \mu(E)$.

If the condition of existence of n_0 such that $\mu(E_{n_0}) < \infty$ is removed then some unwanted cases arise. For example consider a measure on some class of $\mathbb{R}$. Consider the sequence of intervals $I_n = [n, \infty)$. Clearly $\bigcup_{n \geq 1} [n, \infty) = \emptyset$, but $\mu(\emptyset) = 0$ while $\mu(I_n) = \infty$, $\forall \ n$. This shows that no measure can be continuous from above on $\mathbb{R}$. This leads us to add the condition of existence of some set in the sequence which has finite measure.

LEMMA IV.2.16 Let $\mathscr{A}$ be an algebra and let $\mu : \mathscr{A} \to \mathbb{R}^+$ be an additive measure, then:

1) μ is $\sigma-$additive $\implies$ μ is continuous.
2) μ is continuous from below $\implies$ μ is $\sigma-$additive.
3) μ is continuous from above at $\emptyset$ and μ is a finite measure $\implies$ μ is $\sigma-$additive.

Proof | 1) Assume μ is $\sigma-$additive, let $E \in \mathscr{C}$, and let $(E_n)_{n \geq 1} \in \mathscr{C}$ such that $E_n \uparrow E$. Let $F_1 = E_1$ and $F_n = E_n - E_{n-1}$. Clearly by this definition $\bigcup_{j \geq 1} F_j = \bigcup_{j \geq 1} E_j = E$. Then,

$$\mu\left(\bigcup_{j \geq 1} F_j\right) = \sum_{j \geq 1} \mu(F_j) = \lim_{n \to \infty} \sum_{j \geq 2}^{n} \left(\mu(E_j) - \mu(E_{j-1})\right) + \mu(E_1) = \lim \mu(E_n)$$

Hence μ is continuous from below.

For proving continuity from above, let $(E_n)_{n \geq 1} \in \mathscr{C}$ such that some $\mu(E_{n_0}) < \infty$ and $E_n \downarrow E$. Let $G_m = E_{n_0} - E_{n_0+m}$ be a sequence of sets, $\bigcup_{m \geq n_0} G_m = E_{n_0} - E$. Using the fact the μ is continuous from below,

$$\lim_{m \to \infty} \mu(G_m) = \mu(E_{n_0}) - \mu(E)$$

hence,

$$\lim_{m\to\infty} \mu(E_{n_0}) - \lim_{m\to\infty} \mu(E_{n_0+m}) = \mu(E_{n_0}) - \mu(E)$$
$$\lim_{m\to\infty} \mu(E_{n_0+m}) = \mu(E)$$

This is the same as $\lim \mu(E_n) = \mu(E)$.

2) Assume that μ is continuous from below. Let $E \in \mathscr{C}$ be represented as the union of pairwise disjoint sets $E_1, E_2, ...$ Let $F_1, F_2, ...$ be a sequence defined as:

$$F_k = \bigcup_{j=1}^{k} E_j$$

Clearly F_k is a sequence that conerges to E from below. Using the fact that μ is additive and continuous from below:

$$\mu(E) = \lim_{n\to\infty} \mu(F_n) = \lim_{n\to\infty} \mu\left(\bigcup_{j=1}^{n} E_j\right) = \lim_{n\to\infty} \sum_{j=1}^{n} \mu(E_j) = \sum_{j\geq 1} \mu(E_j)$$

Hence μ is $\sigma-$additive.

3) Assume that μ is continuous from above at $\emptyset$. Let $A \in \mathscr{C}$ and let $A_1, A_2, ...$ be pairwise disjoints sets whose union is A. Define the sets $E_1, E_2, ...$ as

$$E_n = A - \bigcup_{j=1}^{n} A_j$$

Clearly $E_n \downarrow \emptyset$. Using finiteness, additivity, and continuity from above at $\emptyset$,

$$\lim_{n\to\infty} \mu(E_n) = 0$$
$$\implies \lim_{n\to\infty} \mu\left(A - \bigcup_{j=1}^{n} A_j\right) = 0$$
$$\implies \mu(A) = \sum_{j\geq 1} \mu(A_j)$$

This completes the proof. ∎

Theorem IV.2.17 (Extension Theorem) Let $\mathscr{S}$ be a semi-algebra, $\mu : \mathscr{S} \to \mathbb{R}^+$ be an additive measure, and let $\mathscr{A}$ be the algebra generated by $\mathscr{S}$. Then there exists a $\nu : \mathscr{A} \to \mathbb{R}^+$, called the *extension* of μ, such that:

1) $\nu(A) = \mu(A)$, $\forall\, A \in \mathscr{S}$.
2) ν is additive.

In addition such a measure on $\mathscr{A}$ is unique.

Proof | Let $\nu : \mathscr{A} \to \mathbb{R}^+$ be a function defined in the following way. Using lemma IV.2.8 we know that for any $A \in \mathscr{A}$ we can find disjoint $E_1, ..., E_n \in \mathscr{S}$ such that $A = \bigcup_{j=1}^{n} E_n$; then define ν as,

$$\nu(A) = \sum_{j=1}^{n} \mu(E_j)$$

First we must show that ν is well defined, since there can be more than one sequence of pairwise disjoint sets whose union is A. Let $E_1, ..., E_n$ and $F_1, ..., F_m$ be two sequences of pairwise disjoint sets in $\mathscr{S}$ whose union is A. Then,

$$\nu(A) = \sum_{j=1}^{n} \mu(E_j)$$

and

$$\nu(A) = \sum_{j=1}^{m} \mu(F_j).$$

Since $A = \bigcup_{k=1}^{m} F_k$

$$\implies E_j = \bigcup_{k=1}^{m} F_k \cap E_j$$

$$\implies \mu(E_j) = \sum_{k=1}^{m} \mu(F_k \cap E_j)$$

Hence,

$$\nu(A) = \sum_{j=1}^{n} \sum_{k=1}^{m} \mu(F_k \cap E_j)$$

Similarly it can shown that,

$$\mu(F_j) = \sum_{k=1}^{n} \mu(E_k \cap F_j)$$

and therefore

$$\sum_{j=1}^{m} \mu(F_j) = \sum_{j=1}^{n} \mu(E_j).$$

Hence ν is well defined.

Clearly for $A \in \mathscr{S}$ we have $\nu(A) = \mu(A)$. For additivity, let $A_1, A_2, ..., A_n$ be pairwise disjoint sets in $\mathscr{A}$ whose union is A. Again from lemma IV.2.8 for each $A_j = \bigcup_{k=1}^{n_j} E_{jk}$ where $E_{j1}, ..., E_{jn_j} \in \mathscr{S}$ are pairwise disjoint. Let $F_1, ..., F_N$, where $N = \sum_{j=1}^{n} n_j$, be defined as $F_1 = E_{11}, F_2 = E_{12}$ and so on. Then,

$$A = \bigcup_{j=1}^{N} F_j$$

$$\implies \nu(A) = \sum_{j=1}^{N} \mu(F_j) = \sum_{j=1}^{n} \sum_{k=1}^{n_j} \mu(E_{jk})$$

since

$$\nu(A_j) = \sum_{j=1}^{n_j} \mu(E_{jk}),$$

$$\implies \nu(A) = \sum_{j=1}^{n} \nu(A_j)$$

Therefore ν is additive.

For uniqueness, let's assume that two such functions ν_1 and ν_2 exist. From property 1 we know that $\nu_1(A) = \nu_2(A) \forall\ A \in \mathscr{S}$. Let $A \in \mathscr{A}$ and let $A_1, ..., A_n \in \mathscr{S}$ be pairwise disjoint with union A.

Then using additivity

$$\nu_1(A) = \sum_{j=1}^{n} \nu_1(A_j) = \sum_{j=1}^{n} \nu_2(A_j) = \nu_2(A)$$

This completes the proof. ■

This is theorem can be easily generalised for $\sigma-$additive measures. The only change in the proof would be considering countably many A_j in the proof of additivity.

IV.3 Caratheodory Theorem

Until now we have shown that extension ν of $\sigma-$additive measure μ on semi-algebra $\mathscr{S}$ is also $\sigma-$additive on algebra $\mathscr{A}$ generated by $\mathscr{S}$. The goal of this section is to show that the extension $\pi : \mathscr{F} \to \mathbb{R}^+$, where $\mathscr{F}$ is the $\sigma-$algebra generated by $\mathscr{S}$ is $\sigma-$additive and unique. In order to do this we follow the following steps:

1) Define a $\pi^* : \mathfrak{P}(\Omega) \to \mathbb{R}^+$ and show that it is something called an *outer measure*.
2) Define a class $\mathscr{M} \subset \mathfrak{P}(\Omega)$, and show that it is a $\sigma-$algebra.
3) Show that $\mathscr{A} \subset \mathscr{M}$. This has the implication that $\mathscr{F} \subset \mathscr{M}$.
4) Show that $\pi^*|_{\mathscr{M}}$ is $\sigma-$additive and $\pi^*|_{\mathscr{A}} = \nu$. Hence $\pi^*|_{\mathscr{M}}$ is an extension.
5) Finally show that this extension is unique.

Definition IV.3.1 Let $A \subset \Omega$ for some set Ω. Then the collection $\{E_i \subset \Omega \mid i \geq 1\}$ is said to be a covering of A if $A \subset \bigcup_{j\geq 1} E_j$. Note that at least one covering exists for every subset and that is $\{\Omega\}$.

Definition IV.3.2 Let $\pi^* : \mathfrak{P}(\Omega) \to \mathbb{R}^+$ for some set Ω defined in the following way: let $A \subset \Omega$ and let $\{E_i \in \mathscr{A} \mid i \geq 1\}$ be a covering of A then

$$\pi^*(A) = \inf_{\{E_i\}} \sum_{i\geq 1} \nu(E_i)$$

This is to be read as infimum of $\sum_{i\geq 1} \nu(E_1)$ over all coverings of A which are in the algebra $\mathscr{A}$.

Definition IV.3.3 Let $\mathscr{C}$ be a class of subsets of Ω such that $\emptyset \in \mathscr{C}$, and let $\mu : \mathscr{C} \to \mathbb{R}^+$ be a function such that:

1) $\mu(\emptyset) = 0$,
2) μ is monotone, i.e. $E \subset F$ where $E, F \in \mathscr{C} \implies \mu(E) \leq \mu(F)$,
3) μ is sub-additive, i.e. $E \in \mathscr{C}$ and $\{E_i \in \mathscr{C} \mid i \geq 1\}$ is a covering of E then $\mu(E) \leq \sum_{i\geq 1} \mu(E_i)$.

Then μ is said to be an outer measure.

Proposition IV.3.4 The function π^* as defined above is an outer measure.

Proof | Since $\emptyset \subset \Omega$, and it is a subset of every possible covering, clearly for the covering $\{E_i = \emptyset \forall\, i \geq 1\}$,

$$\sum_{i\geq 1} \nu(E_i) = 0$$

and hence $\pi^*(\emptyset) = 0$.

Let $E \subset F$ where $E, F \in \mathscr{C}$. Let $\{F_j \mid j \geq 1\}$ be a covering of F. Since $E \subset F$ any covering of F is also a covering of E. If $E_j = F \cap F_j$ then $E_j \subset F_j$ and $\bigcup_{j \geq 1} E_j = F \supset E$. Hence $\{E_j\}$ is a covering E. Since ν is a $\sigma-$additive measure,

$$\nu(E_j) \leq \nu(F_j) \text{ and hence, } \sum_{i \geq 1} \nu(E_i) \leq \sum_{i \geq 1} \nu(F_i).$$

Since for every covering of F a covering of E can be constructed such that the above is true, π^* is monotone. ∎