Notes on Mathematics

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PART I

BASICS

1 Elementary Set Theory

A set S is a collection of objects. The objects of a set are called the elements. The union of two sets S, T contains elements of both sets, and is written as $S \cup U$. The intersection of two sets contains the common elements of the two sets, and is represented $S \cap U$. The complement of a set $A(\subset S)$ with respect to some set S is represented as A'. The difference of set S from T is defined as $T - S = T \cap S'$. The algebra of these operations is as follows

- *Commutativity,*

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

- *Associativity,*

$$A \cap (B \cap C) = (A \cap B) \cap C$$
$$A \cup (B \cup C) = (A \cup B) \cup C$$

- *Distribution,*

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- *De Morgan's rules,*

$$(A \cap B)' = A' \cup B'$$
$$(A \cup B)' = A' \cap B'$$

Definition 1.1 The cartesian product of two sets A and B, denoted $A \times B$, is the set of ordered pairs $\{(a,b)|\forall a \in A \quad \text{and} \quad b \in B\}$.

Other than these operations one can define a set operation called disjoint union, denoted $S \sqcup T$, which is loosely constructed in the following way: we first make copies S' and T' such that $S' \cap T' = \emptyset$ and then take their ordinary union. A more rigourous definition would be constructed later.

2 Relations

Definition 2.1 A relation on a set A is a subset C of the cartesian product $A \times A$. If $(x,y) \in C$ then it is denoted as xCy.

Definition 2.2 An equivalence relation on a set A is a relation C on A such that:

- It is reflexive, i.e. $xCx \ \forall \ x \in A$.
- It is symmetric, i.e. if xCy then yCx.
- It is transitive, i.e. if xCy and yCz then xCz.

Generally the symbol $\sim$ is used to denote an equivalence relation. For a given element $x \in A$ we also define a set called the equivalence class as:

$$E = \{y \mid y \sim x\}$$

Proposition 2.3 Two equivalence classes E and E' are either disjoint or equal.

Proof | Let E be the equivalence class of x and E' be the equivalence class of x'. Assuming that $E \cap E'$ is non-empty, for all $y \in E \cap E'$ it follows that $y \sim x'$ and $y \sim x$. From symmetry and transitivity it follows that $x' \sim x$. Hence every element similar to x' will be similar to x. Hence $E' = E$, whenever $E \cap E'$ is non-empty. ■

Definition 2.4 A partition of a set A is a collection of disjoint nonempty subsets of A whose union is all of A.

Proposition 2.5 Given any partition $\mathscr{D}$ of A, there is a unique equivalence relation C on A such that each element of $\mathscr{D}$ is an equivalence class of C.

Proof | Consider a relation C defined as: xCy if both x and y belong to the same element of $\mathscr{D}$. Since x is always in the same element as itself, xCx is true for all x. If xCy, which means x is in the same subset as y. Since the converse is also true, yCx. If x is in the same subset as y and y is in the subset as z, then x is in the same subset as z. Hence xCy and yCz imply xCz. This means that C is an equivalent relation. Each element of $\mathscr{D}$ can be viewed as an equivalence class of C.

Assume that there exist two equivalence relations C_1 and C_2 such that the set of each their equivalence classes is $\mathscr{D}$. Let E_1 and E_2 be equivalence classes of x with respect to relations C_1 and C_2. E_1 and E_2 must be the same since we are claiming that both relations generate the identical collection of sets. Hence if yC_1x then yC_2x which implies that $C_1 = C_2$. ■

Definition 2.6 The quotient of the set S, denoted $S/\sim$ with respect to the equivalence relation $\sim$ is the set of equivalence classes of S with respect to $\sim$.

3 Functions

Definition 3.1 A rule of assignment is a subset r of the cartesian product $C \times D$ of two sets, having the property that each element of C appears as the first ordinate of at most one ordered pair in r.

From this definition one can easily conclude that, if $r \subset C \times D$ and $(c, d), (c, d') \in r$ then $d = d'$. One can think of r as assigning an element $c \in C$, the element $d \in D$. The set C is called the domain of r and D is called the image set.

Definition 3.2 A function f is a rule of assignment r, along with a set B which contains the image set of r. The domain of r is also the domain of f. The set B is called the range.

A function having a domain A and range B is written as $f : A \to B$. Given an element $a \in A$, $f(a)$ denotes a unique element in B, hence $(a, f(a)) \in r$.

Definition 3.3 Given a function $f : A \to B$ and a subset $A_0 \subset A$, then a restriction of f to A_0 is the mapping $f|A_0 : A_0 \to B$ with rule:

$$\{(a, f(a))|a \in A_0\}$$

Definition 3.4 Given functions $f : A \to B$ and $g : B \to C$, the composite function is defined as $g \circ f : A \to C$, such that $g \circ f(a) = g(f(a))$. More formally, the rule of the function $g \circ f : A \to C$ is:

$$\{(a, c)|\forall b \in B, \quad f(a) = b \quad \text{and} \quad g(b) = c\}$$

Definition 3.5 A function $f : A \to B$ is said to be injective if,

$$f(a) = f(a') \implies a = a'.$$

The function is called surjective if for each $b \in B$ there exists an $a \in A$ such that $b = f(a)$. If f is both injective and surjective it is said to be bijective.

Proposition 3.6 For each bijective function $f : A \to B$, there exists a unique function, called the inverse function, $f^{-1} : B \to A$ such that $f \circ f^{-1}$ and $f^{-1} \circ f$ are both identity functions.

Proof Since f is bijective for every $a \in A$ there exists a unique $b \in B$ (from injection), and for every $b \in B$ also there exists an $a \in A$ (from surjectivity). This implies that every $b \in B$ has a unique pre-image in A. Denote this pre-image by $f^{-1}(b)$. The rule of the inverse function is given by:

$$\{(b, f^{-1}(b))|\forall b \in B\}$$

This proves the existence of inverse. Using the definition of composite function, the rule of the composite function $f \circ f^{-1}$ will be:

$$\{(b, b)|\forall b \in B\}.$$

Hence the composite is the identity function. Similarly the composite function $f^{-1} \circ f$ is also identity.

For proving the uniqueness, consider there exist two inverse functions, f^{-1} and $\tilde{f}^{-1}$, of f. Hence,

$$f(f^{-1}(b)) = b,$$

$$\implies \tilde{f}^{-1}(f(f^{-1}(b))) = \tilde{f}^{-1}(b),$$

But since $\tilde{f}^{-1}(f(a)) = a$,

$$f^{-1}(b) = \tilde{f}^{-1}(b) \quad \forall b \in B$$

Hence the inverse is unique. ■

Proposition 3.7 The inverse of a bijective function $f : A \to B$ is also bijective.

Proof Let the inverse be f^{-1}. Let $b, b' \in B$ such that

$$f^{-1}(b) = f^{-1}(b')$$

$$\implies f(f^{-1}(b)) = f(f^{-1}(b'))$$

$$\implies b = b'$$

This shows that f^{-1} is injective. For proof of surjectivity, we can show that for each $a \in A$ there exists a $b(= f(a)) \in B$ such that $a = f^{-1}(b)$. This shows that f^{-1} is also bijective. ∎

Proposition 3.8 Let $f : A \to B$. If there are functions $g : B \to A$ and $h : B \to A$ such that $g(f(a)) = a\ \forall a \in A$ and $f(h(b)) = b\ \forall b \in B$, then f is bijective and $g = h = f^{-1}$.

Proof | Let $a, a' \in A$, such that

$$f(a) = f(a')$$

Using the function g,

$$g(f(a)) = g(f(a')),$$
$$\implies a = a'$$

hence f is an injective function. Now coming to surjectivity. Using the existance of h, we can show that for each $b \in B$ there exists $a(= h(b)) \in A$ such that $b = f(a)$. Hence f is a bijective function. For the final part of the proposition, since,

$$f(h(b)) = b,$$
$$\implies g(f(h(b))) = g(b),$$
$$\implies h(b) = g(b) \quad \forall b \in B \quad .$$

And since the inverse is unique, they must also be equal to f^{-1}. ∎

When there exists a bijection $f : A \to B$ then A and B are called *isomorphic*. This is sometimes represented as $A \simeq B$. Using the concept of isomorphism the notion of disjoint union can be made more rigourous.

Definition 3.9 The disjoint union of two sets A and B is detemined by constructing sets $A' \simeq A$ and $B' \simeq B$ such that $A' \cup B' = \emptyset$, and then determining the union $A' \cup B'$. Such sets can always be constructed for every set since $\{0\} \times A \simeq A$ and $\{1\} \times B \simeq B$ and $(\{0\} \times A) \cap (\{1\} \times B) = \emptyset$.

A less restrictive notion of invertibility is defined in terms of *left-invertible* and *right-invertible* functions. If for a function $f : A \to B$ there exists a $g : B \to A$ such that $g \circ f : A \to A$ is id_A then f is said to be left invertible. Similarly if there exists $h : B \to A$ such that $f \circ h : B \to B$ is id_B then f is called right invertible. The following is more general statement to proposition 3.8.

Proposition 3.10 Let $f : A \to B$ be a function then:

1) f is injective if and only if it is left invertible.
2) f is surjective if and only if it is right invertible.

Proof | For statement 1, the foward implication follows from the fact that if f injective then on can construct a $g : B \to A$ as follows: let $p \in B$ be a fixed point and

$$g(b) = \begin{cases} a, & \text{where } f(a) = b \\ p, & \text{when } b \text{ not in image of A.} \end{cases} \tag{3.1}$$

Clearly the function $g \circ f(a) = id_A(a) = a$. The backward implication for statement 1, is true because if a g exists such that:

$$g \circ f(a) = a \,\forall\, a \in A$$

then,

$$g \circ f(a) = a \neq a' = g \circ f(a') \implies f(a) \neq f(a')$$

which is the contrapositive of the statement required to prove.

Statement 2 can be proven in a similar way. Assuming that f is surjective, it follows that for each b there is atleast one $a \in A$ such that $f(a) = b$. Choosing anyone of these a for each b we can construct the map $h : B \to A, h(b) = a$. Hence it follows that $f \circ h(b) = f(a) = b = id_B(b)$. For the backward implication, assuming that there is an $h : B \to A$ such that $\forall\, b \in B, f \circ h(b) = id_B(b) = b$. Since $h(b) = a$ for some $a \in A$, it follows $f(a) = b$ for some $a \in A$. Hence f is surjective. ∎

There is another way to look at bijective functions using the concept of monomorphisms and epimorphisms. This is a more fundamental and equivalent approach to defining bijections.

Definition 3.11 A function $f : A \to B$ is said to be a monomorphism if the following holds:

$$\forall\, Z, \forall\, \alpha', \alpha'' : Z \to A,\ f \circ \alpha' = f \circ \alpha'' \implies \alpha' = \alpha''$$

Proposition 3.12 A function $f : A \to B$ is a monomorphism if and only if it is injective.

Proof | Consider first the forward implication. Assuming f is a monomorphism, we know that for all sets Z and $\alpha', \alpha'' : Z \to A$,

$$f \circ \alpha' = f \circ \alpha'' \implies \alpha' = \alpha''$$

if $\alpha'(z) = a$ and $\alpha''(z) = a'$ then the above condition reduces to,

$$f(a) = f(a') \implies a = a'$$

Hence f is injective.

For the backward implication we assume that f is an injective function. Then we know that f is left-invertible, with inverse g. If,

$$\begin{aligned} f \circ \alpha' &= f \circ \alpha'' \\ \implies g \circ f \circ \alpha' &= g \circ f \circ \alpha'' \\ \implies \alpha' &= \alpha'' \end{aligned}$$

∎

Definition 3.13 A function $f : A \to B$ is said to be an epimorphism if,

$$\forall\, Z, \forall\, \beta', \beta'' : B \to Z,\ \beta' \circ f = \beta'' \circ f \implies \beta' = \beta''$$

Proposition 3.14 A function $f : A \to B$ is an epimorphism if and only if it is surjective.

Proof | Let's first consider the forward implication:

$$(\forall\, Z,\ \beta', \beta'' : B \to Z\ \beta' \circ f = \beta'' \circ f \implies \beta' = \beta'') \implies (\forall\, b \in B\ \exists\, a \in A \text{ such that } b = f(a))$$

The contraposition of this statment is:

$$(\exists\, b \in B,\ \forall\, a \in A,\ b \neq f(a)) \implies (\exists Z,\ \beta', \beta'' : B \to Z \text{ such that } \beta' \neq \beta'' \ \ \&\ \ \beta' \circ f = \beta'' \circ f)$$

Assuming there exists $b \in B$ such that b is not in the image of f, let $Z = \{0, 1\}$, $\beta'(b) = 0,\ \forall\, b \in B$, and

$$\beta''(b) = \begin{cases} 0, & \text{if } b \text{ is in image of } f \\ 1, & \text{if } b \text{ is not in image of } f \end{cases}$$

Clearly $\beta' \neq \beta''$ but we have $\beta' \circ f = \beta'' \circ f$. Hence the contrapositive is true, therefore proving the forward implication.

For the backward implication, assuming the funciton is surjective we also know that it would be right invertible. Let h be the right inverse then,

$$\begin{aligned} \beta' \circ f &= \beta'' \circ f \\ \implies \beta' \circ f \circ h &= \beta'' \circ f \circ h \\ \implies \beta' &= \beta'' \end{aligned}$$

Hence completing the proof. ∎

Diagrams. Diagrams are graphical representations of a collection of sets and how they are operated on by functions. A diagram is said to be *commutative* if taking different paths between sets result in the same function. For example if $f : A \to B$ and $g : B \to C$, then here is a commutative diagram of A, B, C:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

For injective functions a $\hookrightarrow$ is used, for surjective functions $\twoheadrightarrow$ is used, and isomorphisms are represented by $\xrightarrow{\sim}$.

Canonical Decomposition. Let $f : A \to B$ be a function on A. Define the equivalence relation $\sim$ on A as $a \sim a'$ iff $f(a) = f(a')$. A surjection $g : A \twoheadrightarrow A/\!\sim$ can be defined as $g(a) = [a]_\sim$. Also it is possible to find an injection $h : \operatorname{im} f \hookrightarrow B$, given by $h(b) = b$. Hence we have a diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, two heads, "g"'] & B \\ A/\!\sim & \operatorname{im} f \arrow[u, hook, "h"'] \end{tikzcd}$$

If we can find an isomorphism $i : A/\!\sim \xrightarrow{\sim} \operatorname{im} f$ then the above diagram will commute. Consider the following proposition:

> **Proposition 3.15** The function $i : A/\!\sim \to \operatorname{im} f$ given by $i([a]_\sim) = f(a)$ is an isomorphism.

Proof | First we must check if i is a function. Let $[a]_\sim, [a']_\sim \in A/\sim$ then, $[a] = [a'] \implies f(a) = f(a') \implies i([a]_\sim) = i([a']_\sim)$. This means for each $[a]_\sim$ there is a unique image.

Injective. If $i([a]_\sim) = i([a']_\sim)$ then $f(a) = f(a')$ by definition of i. Further it implies that $a \sim a'$ by definition of the equivalence relation. Hence a and a' are in the same equivalence class, or $[a]_\sim = [a']_\sim$.

Surjective. Let $b \in \operatorname{im} f$. Then there exists an $a \in A$ such that $f(a) = b$. Hence there exists $[a]_\sim$ such that $b = i([a]_\sim)$. ∎

As a result of this proposition we have shown that any funciton f can be decomposed according to the commutative diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, two heads, "g"'] & B \\ A/\sim \arrow[r, "\sim", "i"'] & \operatorname{im} f \arrow[u, hook, "h"'] \end{tikzcd}$$

This shows that any function can be written as a composition of injections, surjections, and isomorphisms. This decomposition is called the canonical decompositions.

DEFINITION 3.16 Let $f : A \to B$ be a function, and $A_0 \subset A$. Then define,

$$f(A_0) = \{b \mid b = f(a),\ a \in A_0\},$$

and

$$f^{-1}(B_0) = \{a \mid f(a) \in B_0\}.$$

Note that this definition is for all functions, not just bijective functions.

PROPOSITION 3.17 Let $f : A \to B$ be a function, and let $A_0 \subset A$, $B_0 \subset B$ then,

$$A_0 \subset f^{-1}(f(A_0)) \quad \text{and} \quad f(f^{-1}(B_0)) \subset B_0$$

Proof | For the first statement, let $a \in A_0$. Then $f(a) \in f(A_0)$. Which further implies, by definition, that $a \in f^{-1}(f(A_0))$. Hence,

$$\implies A_0 \subset f^{-1}(f(A_0))$$

For the second part of the proposition, let $b \in f(f^{-1}(B_0))$. This means that there exists $a \in f^{-1}(B_0)$ such that $b = f(a)$. Since $a \in f^{-1}(B_0)$, again by definition, $f(a) \in B_0$. Hence $b \in B_0$. Since this is true for any $b \in f(f^{-1}(B_0))$, we conclude that $f(f^{-1}(B_0)) \subset B_0$. ∎

PROPOSITION 3.18 Let $f : A \to B$, $A_0,\ A_1 \subset A$, and $B_0,\ B_1 \subset B$. Then f^{-1} preserves:

1) inclusions
2) unions
3) intersections
4) differences

Proof | Preservation of inclusion: let $B_0 \subset B_1$. From the definition it follows that $f^{-1}(B_0) = \{a \mid f(a) \in B_0\}$. Since $B_0 \subset B_1$, if $f(a) \in B_0$ then $f(a) \in B_1$. Hence if $a \in f^{-1}(B_0)$ then $a \in f^{-1}(B_1)$. Hence $f^{-1}(B_0) \subset f^{-1}(B_0)$.

Proof of preservation of unions: the set $f^{-1}(B_0 \cup B_1) = \{a \mid f(a) \in B_0 \cup B_1\}$. While $f^{-1}(B_i) = \{a \mid f(a) \in B_i$. The union $f^{-1}(B_0) \cup f^{-1}(B_1) = \{a \mid f(a) \in B_0$ or $f(a) \in B_1\}$, which is the same as $\{a \mid f(a) \in B_0 \cup B_1\}$. Hence the two sides are equivalent.

Proof for intersections and differences is very similar to the one for unions. ■

Unlike it's inverse f only preserves inclusions and unions. Showing this is pretty easy. Also another property of functions is that $(g \circ f)^{-1}(C_0)$ is equivalent to $f^{-1}(g^{-1}(C_0))$ for functions $f : A \to B$, $g : B \to C$, and set $C_0 \subset C$.

4 Categories

A category is essentially a collection of 'objects' and of 'morphisms' between these objects, satisfying a list of natural conditions. These objects might be sets, groups, vector spaces, etc. Since there is simply no set of all sets (due to Russell's paradox), this collection of objects is just too 'big' to be called a set. The formal term used is a *class of objects*. The formal definition of categories is as follows.

Definition 4.1 A category C consists of:

1) a class Obj(C) of *objects* of the category.
2) for every two objects A, B of C, a set $\mathsf{Hom}_\mathsf{C}(A, B)$ of morphisms satisfying the following properties:
    i) for every object A of C there exists (at least) one morphism $1_A \in \mathsf{H}_\mathsf{C}(A, A)$. This is the identity on A.
    ii) one can compose morphisms: two morphisms $f \in \mathsf{Hom}_\mathsf{C}(A, B)$ and $\mathsf{Hom}_\mathsf{C}(B, C)$ determine a morphism $gf \in \mathsf{Hom}_\mathsf{C}(A, C)$. For every triplet of objects A, B, C of C there is a function (of sets)
$$\mathsf{Hom}_\mathsf{C}(A, B) \times \mathsf{Hom}_\mathsf{C}(B, C) \to \mathsf{Hom}_\mathsf{C}(A, C)$$
    iii) this composition law is associative.
    iv) the identity morphisms are identities with respect to composition, i.e. if $f \in \mathsf{Hom}_\mathsf{C}(A, B)$ then
$$f1_A = f,\ 1_B f = f$$
    v) the sets $\mathsf{Hom}_\mathsf{C}(A, B)$ and $\mathsf{Hom}_\mathsf{C}(C, D)$ are disjoint unless $A = C$ and $B = D$.

One can make morphism diagrams similar to those of set functions. The set of morphisms from an object to itself are known as endomorphisms and are denoted $\mathsf{End}(A)$. The subscript C will be dropped from now on, unless it is necassary to use it.

Example 4.2 (Sets) As a first example consider the category Set defined as $\mathsf{Obj}(\mathsf{Set}) =$ the class of all sets, and $\mathsf{Hom}(A, B) =$ the set of all set-functions from A to B. We must verify if this is a category. For every A there is an indentity function $1_A : A \to A$, $1_A(a) = a$. Composition of set-functions is possible, the composition is known to be associative, and the identity function is identity with respect to the composition. The last property is also triially true. Hence Set is indeed a category.

Example 4.3 Consider this example.

Suppose S is a set and $\sim$ is a reflexive and transitive relation. Then define a category C as:

- objects are elements of S;
- $\mathsf{Hom}(a,b)$, where a,b are objects, is the set consisting $(a,b) \in S \times S$ if $a \sim b$, and let $\mathsf{Hom}(a,b) = \emptyset$ otherwise.

Verification that this is a category:

1) Since $a \sim a$ using reflexive property, $1_a = (a,a) \in \mathsf{Hom}(a,a)$.
2) Given two morphisms $f = (a,b) \in \mathsf{Hom}(a,b)$ and $g = (b,c) \in \mathsf{Hom}(b,c)$ then using transitivity we know that $a \sim c$ and hence $gf = (a,c) \in \mathsf{Hom}(a,c)$. Hence a composition exists.
3) The composition is clearly associative.
4) Let $f = (a,b) \in \mathsf{Hom}(a,b)$, and we know that $1_a = (a,a)$ and $1_b = (b,b)$. Clearly $f1_a = (a,b)$ and $1_b f = (a,b)$.
5) Since each $\mathsf{Hom}(a,b)$ has either one element (a,b) or is empty, any two set of morphisms will be disjont.

As an example of this kind of category consider the set $\{1,2,3\}$ along with the ordering $\leq$. The following is a commutative diagram of this category:

$$\begin{tikzcd}
1 \arrow[loop left, "1_1"] \arrow[r, "f"] \arrow[dr, "gf"'] & 2 \arrow[loop right, "1_2"] \arrow[d, "g"] \\
& 3 \arrow[loop right, "1_3"]
\end{tikzcd}$$

PART II

ABSTRACT ALGEBRA

PART III
TOPOLOGY

PART IV
MEASURE THEORY

1 Intuition of Measure

Consider any interval of $\mathbb{R}$, $(a, b]$. Intuitively we define the length of this interval as $\lambda((a, b]) = b - a$. Now is it possible to extend this concept of length to any subset of $\mathbb{R}$? For that we would wish to find a function λ such that the following properties are true:

1) $\lambda : \mathfrak{P}(\mathbb{R}) \to \mathbb{R}^+$ is a set function.
2) If $I = (a, b]$ is any interval in $\mathbb{R}$ then $\lambda(I) = b - a$.
3) Length of union of two disjoint subsets $A, B \in \mathfrak{P}(\mathbb{R})$ must be the sum of their individual lengths, i.e. $\lambda(A + B) = \lambda(A) + \lambda(B)$. This can be extended to any countable union of pairwise disjoint subsets.
4) The translation of any subset $A \in \mathfrak{P}(\mathbb{R})$ must have the same length, i.e. $\forall\, x \in \mathbb{R}$, $\lambda(A + x) = \lambda(A)$.

?

Conjecture 1.1 The function λ as described above does not exist.

To prove this we must first prove some other propositions. We use the following notations: let $\sim$ be an equivalence relation on $\mathbb{R}$ defined as:

$$x \sim y \text{ if } x - y \in \mathbb{Q}$$

Let $[x]$ denote the equivalence classes of x, and let $\Lambda = \mathbb{R}/\sim$.

Proposition 1.2 The set Λ is uncouontable.

Proof | Let $\alpha \in \Lambda$ be an equivalence class. Let $x \in \alpha$ be a fixed point. Then for each $y \in \alpha$ it is possible to find a unique rational number given by $x - y$. Hence α is a countable set. Since the countable union of countable sets is countable, but $\mathbb{R}$ is uncountable, it follows that Λ must be uncountable. ■

Let $\Omega \subset \mathbb{R}$ be set constructed in the following way: for each $\alpha \in \Lambda$ we know that a point can be found between $(0, 1)$; so take one such point from each α and put it in the set Ω. From this construction it is easy to see that $\Omega \subset (0, 1)$.

Proposition 1.3 Let $p, q \in \mathbb{Q}$, then either $\Omega + q = \Omega + p$ or $\Omega + q \cap \Omega + p = \emptyset$.

Proof | Let's say $\Omega + q \cap \Omega + p \neq \emptyset$. Then for $a, b \in \Omega$ we can find an $x \in \Omega + q \cap \Omega + p$ such that $x = a + p = b + q$. Hence $a - b = q - p$ for all $a, b \in \Omega$. Since $q - p$ is rational, $a - b$ will also be rational. Hence a and b belong to the same equivalence class. But since we only chose one element from each equivalence class in the construction of Ω, we must have $a = b$. Hence $q = p$, making the two sets in question equal. ■

Hence from this proposition we can say that if $q \neq p$ then $\Omega + q \cap \Omega + p = \emptyset$.

Proposition 1.4 Let λ be a length function as defined above. If $A \subset B \subset \mathbb{R}$ then $\lambda(A) \leq \lambda(B)$.

Proof | Since $B = A \cup (B - A)$, and A and $B - A$ are disjoint,

$$\begin{aligned}\lambda(B) &= \lambda(A \cup (B - A)) \\ &= \lambda(A) + \lambda(B - A) \\ &\geq \lambda(A)\end{aligned}$$

Hence proving our claim. ■

Now we are ready to prove conjecture 1.1.

proof of conjecture 1.1 | Consider the union of sets $\Omega + q$:

$$\bigcup_{\substack{q \in \mathbb{Q} \\ -1<q<1}} \Omega + q$$

From proposition 1.3 we know that this is a union of disjoint sets. Hence,

$$\lambda\left(\bigcup_{\substack{q \in \mathbb{Q} \\ -1<q<1}} \Omega + q\right) = \sum_{\substack{q \in \mathbb{Q} \\ -1<q<1}} \lambda(\Omega + q)$$

Using property 4 of λ,

$$\lambda\left(\bigcup_{\substack{q \in \mathbb{Q} \\ -1<q<1}} \Omega + q\right) = \sum_{\substack{q \in \mathbb{Q} \\ -1<q<1}} \lambda(\Omega) = 0$$

Let $x \in (0,1)$. Let $a \in \Omega \cap [x]$. Then we know that $x - a = q$ for some rational q. Since $a \in \Omega$ implies $a \in (0,1)$, the range of q must be $-1 < q < 1$. Hence $x = a + q$ for some $-1 < q < 1$ implying that

$$x \in \bigcup_{\substack{q \in \mathbb{Q} \\ -1<q<1}} \Omega + q$$

further implying that,

$$(0,1) \subset \bigcup_{\substack{q \in \mathbb{Q} \\ -1<q<1}} \Omega + q$$

Using proposition 1.4,

$$1 \leq \lambda\left(\bigcup_{\substack{q \in \mathbb{Q} \\ -1<q<1}} \Omega + q\right)$$

Hence we have arrived at a contradiction. This shows that a function λ with the properties 1,2,3,4 as given above does not exist. ■

Hence this shows that to construct a general notion of length (called the *measure*) we must let go of one of the four properties: 1,2,3, or 4. Since $2, 3, 4$ are essential for a notion of length, we change 1 to be the following:

1) $\lambda : \mathscr{B}(\subset \mathfrak{P}(\mathbb{R})) \to \mathbb{R}^+$ is a set function.

This means that we are discarding the notion that all subsets of $\mathbb{R}$ can be assigned a length.

2 Formal Notion of Measure

Definition 2.1 A class of subsets, $\mathscr{A}$, of a set Ω is said to be a semi-algebra if:

1) $\Omega \in \mathscr{A}$,
2) closed under finite intersections,
3) The compliment of any set in $\mathscr{A}$ can be expressed as unions of finite pairwise disjoint sets in $\mathscr{A}$.

Definition 2.2 A class of subsets, $\mathscr{A}$, of a set Ω is said to be an algebra if:

1) $\Omega \in \mathscr{A}$,
2) closed under finite intersections,
3) Closed under compliment.

Definition 2.3 A class of subsets, $\mathscr{A}$, of a set Ω is said to be a σ-algebra if:

1) $\Omega \in \mathscr{A}$,
2) closed under countable intersections,
3) Closed under compliment.

Proposition 2.4 Let Ω be a set and $\mathscr{A}_\alpha \subset \mathfrak{P}(\Omega)$ be algebras, where $\alpha \in I$ (no assumptions have been made on I). Then

$$\mathscr{A} = \bigcap_{\alpha \in I} \mathscr{A}_\alpha$$

is also an algebra.

Proof Since $\Omega \in \mathscr{A}_\alpha$, $\forall\, \alpha \in I$, implies that $\Omega \in \mathscr{A}$. If $A_1, ..., A_n \in \mathscr{A}$ then $A_1, ..., A_n \in \mathscr{A}_\alpha$ for any $\alpha \in I$. Since $\mathscr{A}_\alpha$ is an algebra, it follows that $\bigcap_{j=1}^{n} A_j$ is in $\mathscr{A}_\alpha$ for any $\alpha \in I$; hence it is also in $\mathscr{A}$. If $A \in \mathscr{A}$ then it is in every $\mathscr{A}_\alpha$ and hence its compliment is in every $\mathscr{A}_\alpha$. ■

The above proposition also applies to σ-algebras as well. Essentially the same argument applies, just that instead of finite sets we have countable intersection, i.e. $n \to \infty$. To denote that something applies to both algebras and σ-algebras we use the notation $(\sigma-)$algebra.

Definition 2.6 A class $\mathscr{C}$ of subsets of set Ω is said to *generate* an $(\sigma-)$algebra $\mathscr{A}$ if $\mathscr{C} \subset \mathscr{A}$ and if for any $(\sigma-)$alegbra $\mathscr{A}' \supset \mathscr{C}$ implies that $\mathscr{A} \subset \mathscr{A}'$.

Proposition 2.7 Every class $\mathscr{C} \subset \mathfrak{P}(\Omega)$ generates an $(\sigma-)$algebra.

Proof Let $\mathscr{A}_\alpha$, $\alpha \in I$ be all the $(\sigma-)$algebras which contain the class $\mathscr{C}$. Then we know that,

$$\mathscr{A} = \bigcap_{\alpha \in I} \mathscr{A}_\alpha$$

is also an $(\sigma-)$algebra, and it will contain $\mathscr{C}$. From the definition of intersection it follows that $\mathscr{A} \subset \mathscr{A}_\alpha$. Hence $\mathscr{A}$ is the $(\sigma-)$algebra generated by $\mathscr{C}$. ■

Lemma 2.8 If $\mathscr{S}$ is a semi-algebra and $\mathscr{A}$ is the algebra generated by $\mathscr{S}$ then

$$A \in \mathscr{A} \iff \exists \text{ pairwise disjoint } E_1, ..., E_n \in \mathscr{S} \text{ such that } A = \bigcup_{j=1}^{n} E_j$$

Proof | ($\Longleftarrow$) Assuming that A is finite union of disjoint sets $E_1, ..., E_n \in \mathscr{S}$ we need to show that $A \in \mathscr{A}$. Since $E_1, ..., E_n$ are in $\mathscr{S}$ it follows that they are also in $\mathscr{A}$. It further follows that the compliment of each $E_j \in \mathscr{A}$. Since

$$\left(\bigcap_{j=1}^{n} E_j^c\right)^c = \bigcup_{j=1}^{n} E_j,$$

and algebras are closed under finite intersections, $A \in \mathscr{A}$.

($\Longrightarrow$) Let $\mathscr{B}$ be the class defined as:

$$\mathscr{B} = \{B \mid \text{where } B = \bigcup_{j=1}^{n} F_j,\ F_j \in \mathscr{S} \text{ are pairwise disjoint.}\}$$

If we can show that $\mathscr{B}$ is an algebra containing $\mathscr{S}$ then by definition of generated algebras $\mathscr{A} \subset \mathscr{B}$. This shows that any element of $\mathscr{A}$ can be expressed as a finite union of disjoint sets. Hence all that remains is to show that $\mathscr{B}$ is an algebra containing $\mathscr{S}$.

1) Clearly by the definition, any element of $\mathscr{S}$ is also in $\mathscr{B}$. Hence $\mathscr{S} \subset \mathscr{B}$ and hence $\Omega \in \mathscr{B}$.
2) Let $B_1, ..., B_n \in \mathscr{B}$ then

$$\begin{aligned}
\bigcap_{j=1}^{n} B_j &= \bigcap_{j=1}^{n} \bigcup_{i=1}^{m} F_{ji} \\
&= \bigcup_{i=1}^{m} \bigcap_{j=1}^{n} F_{ji}, \text{ using definition of } \mathscr{B} \\
&= \bigcup_{i=1}^{m} E_i, \text{ where, } E_i = \bigcap_{j=1}^{n} F_{ji}
\end{aligned}$$

Since $\mathscr{S}$ is closed under finite intersections, this shows that $\mathscr{B}$ is closed under finite intersections.
3) Let $B \in \mathscr{B}$. Then,

$$\begin{aligned}
B^c &= \left(\bigcup_{i=1}^{m} F_i\right)^c \\
&= \bigcap_{i=1}^{m} F_i^c \\
&= \bigcap_{i=1}^{m} \bigcup_{j=1}^{n} E_{ij} \text{ (using property 3 of semi-algebras)} \\
&= \bigcup_{j=1}^{n} E_j, \text{ where } E_j = \bigcap_{i=1}^{m} E_{ij}
\end{aligned}$$

Since $\mathscr{S}$ is closed under finite intersections, this shows that $\mathscr{B}$ is closed under compliment.

$\mathscr{B}$ is indeed an algebra containing $\mathscr{S}$, hence completing our proof. ■

Definition 2.9 Let $\mathscr{C}$ be a class of subsets of Ω such that $\emptyset \in \mathscr{C}$, and let $\mu : \mathscr{C} \to \mathbb{R}^+$ be a function such that:

1) $\mu(\emptyset) = 0$,
2) If $E_1, ..., E_n \in \mathscr{C}$ are pairwise disjoint and if $\bigcup_{j=1}^n E_j \in \mathscr{C}$ then $\mu(\bigcup_{j=1}^n E_j) = \sum_{j=1}^n \mu(E_j)$.

then μ is said to be an additive measure.

Observe that if we have a $A \in \mathscr{C}$ such that $\mu(A) < \infty$ then:

$$\begin{aligned} \mu(A \cup \emptyset) &= \mu(A) + \mu(\emptyset) \\ \mu(A) &= \mu(A) + \mu(\emptyset) \\ \implies \mu(\emptyset) &= 0 \end{aligned}$$

Hence the first condition is just a consequence of the second if a subset with finite measure exists. Secondly observe that if $E \subset F \in \mathscr{C}$ and $F - E \in \mathscr{C}$ then:

$$\mu(E \cup F - E) = \mu(F) = \mu(E) + \mu(F - E)$$

this means that $\mu(E) \leq \mu(F)$, the equality being true when $\mu(E) = \infty$. In the case where $\mu(E) < \infty$ we have the identity $\mu(F - E) = \mu(F) - \mu(E)$. This property is called monotonicity.

Observe that if A, B are any sets in $\mathscr{C}$ and $A \cup B \in \mathscr{C}$ then additivity implies that $\mu(A \cup B) \leq \mu(A) + \mu(B)$, since

$$\mu(A \cup B) = \mu(A \cup (B - A)) = \mu(A) + \mu(B - A) \leq \mu(A) + \mu(B). \textit{(using monotonicity)}$$

Example 2.12 Let Ω be any non-empty set and let $X_1, X_2, ... \in \Omega$. Also let $a_1, a_2, ... \geq 0$ be some constants. Then define a measure $\mu : \mathscr{C} \subset (\mathfrak{P}(\Omega)) \to \mathbb{R}^+$ as:

$$\mu(A) = \sum_{j \geq 1} a_j 1\{X_j \in A\}$$

where,

$$1\{X_j \in A\} = \begin{cases} 1, \text{ if } X_j \in A \\ 0, \text{ if } X_j \notin A \end{cases}$$

It is easy to see that this measure is indeed additive.

Definition 2.13 Let $\mathscr{C}$ be a class of subsets of Ω such that $\emptyset \in \mathscr{C}$, and let $\mu : \mathscr{C} \to \mathbb{R}^+$ be a function such that:

1) $\mu(\emptyset) = 0$,
2) If $E_1, E_2, ... \in \mathscr{C}$ are pairwise disjoint and if $\bigcup_{j \geq 1} E_j \in \mathscr{C}$ then $\mu(\bigcup_{j \geq 1} E_j) = \sum_{j \geq 1} \mu(E_j)$.

then μ is said to be a $\sigma-$additive measure.

Example 2.14 Let $\Omega = (0,1)$ and $\mathscr{C} = \{(a,b] \mid 0 \leq a < b < 1\} \cup \{\emptyset\}$. Define a function $\mu : \mathscr{C} \to \mathbb{R}^+$ as:

$$\mu(a,b] = \begin{cases} \infty, \text{ if } a = 0 \\ b - a, \text{ if } a \neq 0 \end{cases}$$

Clearly since a subset with finite measure exists $\mu(\emptyset) = 0$. Also since,

$$(a,b] = \bigcup_{j=1}^{n} (a_j, a_{j+1}], \text{ where } a_1 = a \;\; \& \;\; a_n = b$$

when $a = 0$, $a_1 = 0$ and hence applying the measure on both sides we get ∞. When $a \neq 0$, so are none of the a_j and hence:

$$\mu(a,b] = b - a = (a_2 - a_1] + ... + (a_n - a_{n-1}] = \sum_{j=1}^{n} \mu(a_j, a_{j+1}]$$

Hence μ is additive. But it is possible to show that μ is not $\sigma-$additive. Consider for example the interval $(0, 1/2]$, and let $x_1 = 1/2, x_2, ...$ be a monotonic decreasing sequence in $(0,1)$ which converges to 0. Then

$$(0, 1/2] = \bigcup_{j \geq 1} (x_{j+1}, x_j]$$

Clearly $\mu(0, 1/2] = \infty$, but $\mu(x_{j+1}, x_j] = x_{j+1} - x_j$ which is finite.

Definition 2.15 Let $\mathscr{C}$ be a class of subsets of Ω and $\mu : \mathscr{C} \to \mathbb{R}^+$ be any set function. Then,

1) μ is said to be *continuous from below* at $E \in \mathscr{C}$ if $\forall \; (E_n)_{n \geq 1} \in \mathscr{C}$, $E_n \uparrow E \implies \lim \mu(E_n) = \mu(E)$.
2) μ is said to be *continuous from above* at $E \in \mathscr{C}$ if $\forall \; (E_n)_{n \geq 1} \in \mathscr{C}$, $E_n \downarrow E$ and $\exists \; n_0$ such that $\mu(E_{n_0}) < \infty$ implies that $\lim \mu(E_n) = \mu(E)$.

If the condition of existence of n_0 such that $\mu(E_{n_0}) < \infty$ is removed then some unwanted cases arise. For example consider a measure on some class of $\mathbb{R}$. Consider the sequence of intervals $I_n = [n, \infty)$. Clearly $\bigcup_{n \geq 1} [n, \infty) = \emptyset$, but $\mu(\emptyset) = 0$ while $\mu(I_n) = \infty$, $\forall \; n$. This shows that no measure can be continuous from above on $\mathbb{R}$. This leads us to add the condition of existence of some set in the sequence which has finite measure.

Lemma 2.17 Let $\mathscr{A}$ be an algebra and let $\mu : \mathscr{A} \to \mathbb{R}^+$ be an additive measure, then:

1) μ is $\sigma-$additive $\implies$ μ is continuous.
2) μ is continuous from below $\implies$ μ is $\sigma-$additive.
3) μ is continuous from above at $\emptyset$ and μ is a finite measure $\implies$ μ is $\sigma-$additive.

Proof | 1) Assume μ is $\sigma-$additive, let $E \in \mathscr{C}$, and let $(E_n)_{n \geq 1} \in \mathscr{C}$ such that $E_n \uparrow E$. Let $F_1 = E_1$ and $F_n = E_n - E_{n-1}$. Clearly by this definition $\bigcup_{j \geq 1} F_j = \bigcup_{j \geq 1} E_j = E$. Then,

$$\mu\left(\bigcup_{j \geq 1} F_j\right) = \sum_{j \geq 1} \mu(F_j) = \lim_{n \to \infty} \sum_{j \geq 2}^{n} \left(\mu(E_j) - \mu(E_{j-1})\right) + \mu(E_1) = \lim \mu(E_n)$$

Hence μ is continuous from below.

For proving continuity from above, let $(E_n)_{n\geq 1} \in \mathscr{C}$ such that some $\mu(E_{n_0}) < \infty$ and $E_n \downarrow E$. Let $G_m = E_{n_0} - E_{n_0+m}$ be a sequence of sets, $\bigcup_{m\geq n_0} G_m = E_{n_0} - E$. Using the fact the μ is continuous from below,

$$\lim_{m\to\infty} \mu(G_m) = \mu(E_{n_0}) - \mu(E)$$

hence,

$$\lim_{m\to\infty} \mu(E_{n_0}) - \lim_{m\to\infty} \mu(E_{n_0+m}) = \mu(E_{n_0}) - \mu(E)$$
$$\lim_{m\to\infty} \mu(E_{n_0+m}) = \mu(E)$$

This is the same as $\lim \mu(E_n) = \mu(E)$.

2) Assume that μ is continuous from below. Let $E \in \mathscr{C}$ be represented as the union of pairwise disjoint sets $E_1, E_2, ...$ Let $F_1, F_2, ...$ be a sequence defined as:

$$F_k = \bigcup_{j=1}^{k} E_j$$

Clearly F_k is a sequence that conerges to E from below. Using the fact that μ is additive and continuous from below:

$$\mu(E) = \lim_{n\to\infty} \mu(F_n) = \lim_{n\to\infty} \mu\left(\bigcup_{j=1}^{n} E_j\right) = \lim_{n\to\infty} \sum_{j=1}^{n} \mu(E_j) = \sum_{j\geq 1} \mu(E_j)$$

Hence μ is $\sigma-$additive.

3) Assume that μ is continuous from above at $\emptyset$. Let $A \in \mathscr{C}$ and let $A_1, A_2, ...$ be pairwise disjoints sets whose union is A. Define the sets $E_1, E_2, ...$ as

$$E_n = A - \bigcup_{j=1}^{n} A_j$$

Clearly $E_n \downarrow \emptyset$. Using finiteness, additivity, and continuity from above at $\emptyset$,

$$\lim_{n\to\infty} \mu(E_n) = 0$$
$$\implies \lim_{n\to\infty} \mu\left(A - \bigcup_{j=1}^{n} A_j\right) = 0$$
$$\implies \mu(A) = \sum_{j\geq 1} \mu(A_j)$$

This completes the proof. ■

THEOREM 2.18 (Extension Theorem) Let $\mathscr{S}$ be a semi-algebra, $\mu : \mathscr{S} \to \mathbb{R}^+$ be an additive measure, and let $\mathscr{A}$ be the algebra generated by $\mathscr{S}$. Then there exists a $\nu : \mathscr{A} \to \mathbb{R}^+$, called the *extension* of μ, such that:

1) $\nu(A) = \mu(A)$, $\forall\, A \in \mathscr{S}$.
2) ν is additive.

In addition such a measure on $\mathscr{A}$ is unique.

Proof | Let $\nu : \mathscr{A} \to \mathbb{R}^+$ be a function defined in the following way. Using lemma 2.8 we know that for any $A \in \mathscr{A}$ we can find disjoint $E_1, ..., E_n \in \mathscr{S}$ such that $A = \bigcup_{j=1}^n E_n$; then define ν as,

$$\nu(A) = \sum_{j=1}^{n} \mu(E_j)$$

First we must show that ν is well defined, since there can be more than one sequence of pairwise disjoint sets whose union is A. Let $E_1, ..., E_n$ and $F_1, ..., F_m$ be two sequences of pairwise disjoint sets in $\mathscr{S}$ whose union is A. Then,

$$\nu(A) = \sum_{j=1}^{n} \mu(E_j)$$

and

$$\nu(A) = \sum_{j=1}^{m} \mu(F_j).$$

Since $A = \bigcup_{k=1}^m F_k$

$$\implies E_j = \bigcup_{k=1}^{m} F_k \cap E_j$$

$$\implies \mu(E_j) = \sum_{k=1}^{m} \mu(F_k \cap E_j)$$

Hence,

$$\nu(A) = \sum_{j=1}^{n} \sum_{k=1}^{m} \mu(F_k \cap E_j)$$

Similarly it can shown that,

$$\mu(F_j) = \sum_{k=1}^{n} \mu(E_k \cap F_j)$$

and therefore

$$\sum_{j=1}^{m} \mu(F_j) = \sum_{j=1}^{n} \mu(E_j).$$

Hence ν is well defined.

Clearly for $A \in \mathscr{S}$ we have $\nu(A) = \mu(A)$. For additivity, let $A_1, A_2, ..., A_n$ be pairwise disjoint sets in $\mathscr{A}$ whose union is A. Again from lemma 2.8 for each $A_j = \bigcup_{k=1}^{n_j} E_{jk}$ where $E_{j1}, ..., E_{jn_j} \in \mathscr{S}$ are pairwise disjoint. Let $F_1, ..., F_N$, where $N = \sum_{j=1}^n n_j$, be defined as $F_1 = E_{11}, F_2 = E_{12}$ and so on. Then,

$$A = \bigcup_{j=1}^{N} F_j$$

$$\implies \nu(A) = \sum_{j=1}^{N} \mu(F_j) = \sum_{j=1}^{n} \sum_{k=1}^{n_j} \mu(E_{jk})$$

since

$$\nu(A_j) = \sum_{j=1}^{n_j} \mu(E_{jk}),$$

$$\implies \nu(A) = \sum_{j=1}^{n} \nu(A_j)$$

Therefore ν is additive.

For uniqueness, let's assume that two such functions ν_1 and ν_2 exist. From property 1 we know that $\nu_1(A) = \nu_2(A) \forall\ A \in \mathscr{S}$. Let $A \in \mathscr{A}$ and let $A_1, ..., A_n \in \mathscr{S}$ be pairwise disjoint with union A. Then using additivity

$$\nu_1(A) = \sum_{j=1}^{n} \nu_1(A_j) = \sum_{j=1}^{n} \nu_2(A_j) = \nu_2(A)$$

This completes the proof. ■

This is theorem can be easily generalised for $\sigma-$additive measures. The only change in the proof would be considering countably many A_j in the proof of additivity.

3 Caratheodory Theorem

Until now we have shown that extension ν of $\sigma-$additive measure μ on semi-algebra $\mathscr{S}$ is also $\sigma-$additive on algebra $\mathscr{A}$ generated by $\mathscr{S}$. The goal of this section is to show that the extension $\pi : \mathscr{F} \to \mathbb{R}^+$, where $\mathscr{F}$ is the $\sigma-$algebra generated by $\mathscr{S}$ is $\sigma-$additive and unique. In order to do this we follow the following steps:

1) Define a $\pi^* : \mathfrak{P}(\Omega) \to \mathbb{R}^+$ and show that it is something called an *outer measure*.
2) Define a class $\mathscr{M} \subset \mathfrak{P}(\Omega)$, and show that it is a $\sigma-$algebra.
3) Show that $\mathscr{A} \subset \mathscr{M}$. This has the implication that $\mathscr{F} \subset \mathscr{M}$.
4) Show that $\pi^*|_{\mathscr{M}}$ is $\sigma-$additive and $\pi^*|_{\mathscr{A}} = \nu$. Hence $\pi^*|_{\mathscr{M}}$ is an extension.
5) Finally show that this extension is unique.

Definition 3.1 Let $A \subset \Omega$ for some set Ω. Then the collection $\{E_i \subset \Omega \mid i \geq 1\}$ is said to be a covering of A if $A \subset \bigcup_{j\geq 1} E_j$. Note that at least one covering exists for every subset and that is $\{\Omega\}$.

Definition 3.2 Let $\pi^* : \mathfrak{P}(\Omega) \to \mathbb{R}^+$ for some set Ω defined in the following way: let $A \subset \Omega$ and let $\{E_i \in \mathscr{A} \mid i \geq 1\}$ be a covering of A then

$$\pi^*(A) = \inf_{\{E_i\}} \sum_{i\geq 1} \nu(E_i)$$

This is to be read as infimum of $\sum_{i\geq 1} \nu(E_1)$ over all coverings of A which are in the algebra $\mathscr{A}$.

Definition 3.3 Let $\mathscr{C}$ be a class of subsets of Ω such that $\varnothing \in \mathscr{C}$, and let $\mu : \mathscr{C} \to \mathbb{R}^+$ be a function such that:

1) $\mu(\varnothing) = 0$,
2) μ is monotone, i.e. $E \subset F$ where $E, F \in \mathscr{C} \implies \mu(E) \leq \mu(F)$,
3) μ is sub-additive, i.e. $E \in \mathscr{C}$ and $\{E_i \in \mathscr{C} \mid i \geq 1\}$ is a covering of E then $\mu(E) \leq \sum_{i\geq 1} \mu(E_i)$.

Then μ is said to be an outer measure.

Proposition 3.4 The function π^* as defined above is an outer measure.

Proof | Since $\emptyset \subset \Omega$, and it is a subset of every possible covering, clearly for the covering $\{E_i = \emptyset \mid \forall\, i \geq 1\}$,

$$\sum_{i\geq 1} \nu(E_i) = 0$$

and hence $\pi^*(\emptyset) = 0$.

Let $E \subset F$ where $E, F \in \mathscr{C}$. Let $\{F_j \mid j \geq 1\}$ be a covering of F. Since $E \subset F$ any covering of F is also a covering of E. If $E_j = F \cap F_j$ then $E_j \subset F_j$ and $\bigcup_{j\geq 1} E_j = F \supset E$. Hence $\{E_j\}$ is a covering E. Since ν is a $\sigma-$additive measure,

$$\nu(E_j) \leq \nu(F_j) \text{ and hence, } \sum_{i\geq 1} \nu(E_i) \leq \sum_{i\geq 1} \nu(F_i).$$

Since for every covering of F a covering of E can be constructed in the above manner such that the above inequality is true, hence π^* is monotone.

For sub-additivity let $E \subset \Omega$ and let $\{E_i \in \mathscr{A} \mid i \geq 1\}$ be a covering of E. In the case when $\pi^*(E_i) = \infty$, clearly $\pi^*(E) \leq \pi^*(E_i)$. In the case when $\pi^*(E_i) < \infty\, \forall\, i \geq 1$, for each $\epsilon > 0$ we can find a covering of E_i, say $\{F_{ij} \in \mathscr{A} \mid j \geq 1\}$ such that,

$$\pi^*(E_i) \leq \sum_{j\geq 1} \nu(F_{ij}) \leq \pi^*(E_i) + \frac{\epsilon}{2^i}$$

Hence,

$$\sum_{i\geq 1} \pi^*(E_i) \leq \sum_{j\geq 1} \nu(\bigcup_{j\geq 1} F_{ij}) = \sum_{i\geq 1} \nu(E_i) \leq \sum_{i\geq 1} \pi^*(E_i) + \epsilon$$

Using $\pi^*(E) \leq \sum_{i\geq 1} \nu(E_i)$, we get

$$\pi^*(E) \leq \sum_{i\geq 1} \nu(E_i) \leq \sum_{i\geq 1} \pi^*(E_i) + \epsilon$$

Since this is true for arbitrary ϵ we conclude that π^* is sub-additive. ■

Definition 3.5 A set $A \subset \Omega$ is said to be measurable if $\forall\, E \subset \Omega$,

$$\pi^*(E) = \pi^*(E \cap A) + \pi^*(E \cap A^c)$$

Define $\mathscr{M}$ to be the set of all measurable subsets of Ω.

Using sub-additivity of π^* it is possible to prove that

$$\pi^*(E) \leq \pi^*(E \cap A) + \pi^*(E \cap A^c)$$

since $E = (E \cap A) \cup (E \cap A^c)$. Hence showing that A is measurable just boils down to showing

$$\pi^*(E) \geq \pi^*(E \cap A) + \pi^*(E \cap A^c)$$

Proposition 3.7 The alegebra $\mathscr{A}$ of subsets of Ω is a subset of $\mathscr{M}$.

Proof | Let $A \in \mathscr{A}$ and let $E \in \Omega$. If we can show that A is measurable, we prove the proposition. Let $\{E_i \in \mathscr{A}\}$ be a covering of E, and let $\epsilon > 0$. In the case when $\pi^*(E_i) = \infty$ even for a single i, it is clear that the inequality

$$\pi^*(E) \geq \pi^*(E \cap A) + \pi^*(E \cap A^c) \tag{3.1}$$

holds. In the case when $\pi^*(E_i) < \infty$ forall $i \geq 1$ let $\epsilon > 0$. Then

$$\pi^*(E) \leq \sum_{i \geq 1} \nu(E_i) \leq \pi^*(E) + \epsilon$$

Since $E \cap A \subset \bigcup_{i \geq 1} E_i \cap A$,

$$\pi^*(E \cap A) \leq \sum_{i \geq 1} \nu(E_i \cap A)$$

Using similar arguments for A^c

$$\pi^*(E \cap A^c) \leq \sum_{i \geq 1} \nu(E_i \cap A^c)$$

Adding these two inequalities

$$\pi^*(E \cap A^c) + \pi^*(E \cap A) \leq \sum_{i \geq 1} \nu(E_i)$$

Here I have used the additivity of ν since $E_i \cap A$ and $E_i \cap A^c$ are in the algebra. Using inequality (1):

$$\pi^*(E \cap A^c) + \pi^*(E \cap A) \leq \sum_{i \geq 1} \nu(E_i) \leq \pi^*(E) + \epsilon$$

Since this is true for arbitrary ϵ, we have

$$\pi^*(E \cap A^c) + \pi^*(E \cap A) \leq \pi^*(E)$$

Hence we have shown that A is measurable, completing the proof. ■

Proposition 3.8 $\mathscr{M}$ is a $\sigma-$algebra.

Proof | Since every algebra is a subset of $\mathscr{M}$ clearly $\Omega \in \mathscr{M}$. Also it is easy to see that if $A \in \mathscr{M}$ then $A^c \in \mathscr{M}$ since replcaing A by A^c in the condition of measurable set does not change the inequality. The only condition that remains to be checked is closure under countable union. First consider the finite case. Let $A, B \in \mathscr{M}$. We are required to show that $\forall\ E \subset \Omega$,

$$\pi^*(E) \geq \pi^*(E \cap (A \cup B)) + \pi^*(E \cap (A \cup B)^c).$$

Since

$$\pi^*(E) = \pi^*(E \cap A) + \pi^*(E - A),$$
$$\pi^*(E) = \pi^*(E \cap B) + \pi^*(E - B)$$

Thus

$$\begin{aligned}\pi^*(E - A) &= \pi^*((E - A) \cap B) + \pi^*((E - A) - B) \\ &= \pi^*(E \cap A^c \cap B) + \pi^*(E - (A \cup B)^c),\end{aligned}$$

implying that

$$\pi^*(E) = \pi^*(E \cap A) + \pi^*(E \cap A^c \cap B) + \pi^*(E - (A \cup B)^c)$$

$$\geq \pi^*(E \cap A \cup B) + \pi^*(E - (A \cup B)^c).$$

The final inequality comes from the sub-additivity of π^* and the fact that $(E \cap A) \cup (E \cup A^c \cup B) = E \cap A \cup B$. Hence $A \cup B \in \mathscr{M}$. Now extending this to the countable case, let $A_j \in \mathscr{M}$, $A = \bigcup_{j \geq 1} A_j$, and $B_n = \bigcup_{j=1}^{n} A_j$. Using closure under finite unions we can say that

$$\pi^*(E) = \pi^*(E \cap B_n) + \pi^*(E - B_n)$$

Since $B_n \subset A \implies E - B_n \supset E - A$. Hence,

$$\pi^*(E) \geq \pi^*(E \cap B_n) + \pi^*(E - A)$$

Define the sets $F_1 = A_1, ..., F_j = A_j - B_{j-1}, ...$; and observe that $F_j \in \mathscr{M}$, $A = \bigcup_{j \geq 1} F_j$, and that these sets are pairwise disjoint. If we define $G_n = \bigcup_{j=1}^{n} F_j$, then using a similar logic as B_n

$$\pi^*(E) \geq \pi^*(E \cap G_n) + \pi^*(E - A).$$

Using induction one can show that

$$\pi^*(E \cap \bigcup_{j=1}^{n} F_j) = \sum_{j=1}^{n} \pi^*(E \cap F_j).$$

For $n = 1$ it is obviously true. Assuming it to be true for some n,

$$\begin{aligned}
\pi^*(E \cap \bigcup_{j=1}^{n+1} F_j) &= \pi^*(E \cap \bigcup_{j=1}^{n+1} F_j \cap F_{n+1}) + \pi^*(E \cap \bigcup_{j=1}^{n+1} F_j \cap F_{n+1}^c) \\
&= \pi^*(E \cap F_{n+1}) + \pi^*(E \cap \bigcup_{j=1}^{n} F_j) \\
&= \pi^*(E \cap F_{n+1}) + \sum_{j=1}^{n} \pi^*(E \cap F_j) \\
&= \sum_{j=1}^{n+1} \pi^*(E \cap F_j).
\end{aligned}$$

Hence using this property,

$$\pi^*(E) \geq \pi^*(E \cap G_n) + \pi^*(E - A) = \sum_{j=1}^{n} \pi^*(E \cap F_j) + \pi^*(E - A).$$

Taking the limit $n \to \infty$,

$$\pi^*(E) \geq \sum_{j \geq 1} \pi^*(E \cap F_j) + \pi^*(E - A) \geq \pi^*(E \cap A) + \pi^*(E - A).$$

The final inequality comes from sub-additivity of π^*. This completes the proof that $\mathscr{M}$ is a $\sigma-$algebra. ■

Since the algebra $\mathscr{A}$ is a subset of $\mathscr{M}$ and $\mathscr{M}$ is a $\sigma-$algebra it follows that $\mathscr{M}$ contains the $\sigma-$algebra generated by $\mathscr{A}$.

PROPOSITION 3.10 $\pi^*(A) = \nu(A) \; \forall \; A \in \mathscr{A}$.

Proof | Let $A \in \mathscr{A}$. Consider the covering $\{A_1 = A,\ A_j = \emptyset\ j \geq 2\}$. Then $\pi^*(A) \leq \nu(A)$ by definition. The opposite inequality can be proved by constructing sets $F_1 = E_1$, $F_n = E_n - \bigcup_{j=1}^{n-1} E_j$, where $\{E_n \in \mathscr{A}\}$ is some covering of A. As discussed in the previous proof F_j are pairwise disjoint. Since,

$$\begin{aligned} & A \subset \bigcup_{j\geq 1} F_j \\ \implies & A = \bigcup_{j\geq 1} F_j \cap A \\ \implies & \nu(A) = \nu(\bigcup_{j\geq 1} F_j \cap A) \\ \implies & \nu(A) = \sum_{j\geq 1} \nu(F_j \cap A) \leq \sum_{j\geq 1} \nu(E_j) \end{aligned}$$

The last inequality comes from the fact that $F_j \cap A \subset E_j$. This inequality shows that $\nu(A)$ is infact the infimum of the sum over all coverings of A. Hence $\pi^*(A) = \nu(A)$. ∎

Proposition 3.11 $\pi^*|_{\mathscr{M}}$ is $\sigma-$additive.

Proof | It is clear that $\pi(\emptyset) = 0$. Let $A_1, A_2, ... \in \mathscr{M}$ be pairwise disjoint sets and let their union be A. Since $\mathscr{M}$ is a $\sigma-$algebra $A \in \mathscr{M}$. Since we have already shown that for any pairwise disjoint sets $F_1, F_2, ... \in \mathscr{M}$ and any $E \subset \Omega$

$$\pi^*(E \cap \bigcup_{j=1}^{n} F_j) = \sum_{j=1}^{n} \pi^*(E \cap F_j).$$

Letting $E = A$ and $F_j = A_j$,

$$\pi^*(\bigcup_{j=1}^{n} A_j) = \pi^*(A \cap \bigcup_{j=1}^{n} A_j) = \sum_{j=1}^{n} \pi^*(A \cap A_j) = \sum_{j=1}^{n} \pi^*(A_j).$$

Since $\bigcup_{j=1}^{n} A_j \subset \bigcup_{j\geq 1} A_j$, using monotonicity of π^*

$$\pi^*(A) \geq \sum_{j=1}^{n} \pi^*(A_j).$$

Taking the limit

$$\pi^*(A) \geq \sum_{j\geq 1} \pi^*(A_j).$$

Since using sub-additivity we already know that

$$\pi^*(A) \leq \sum_{j\geq 1} \pi^*(A_j)$$

it follows that π^* acting on $\mathscr{M}$ is $\sigma-$additive. ∎

Definition 3.12 A set Ω is said to be $\sigma-$finite with respect to a function μ if there exists a sequence $E_1, E_2, ... \subset \Omega$, such that $E_j \uparrow \Omega \implies \mu(E_j) < \infty$.

Definition 3.13 A class $\mathscr{G} \subset \mathfrak{P}(\Omega)$ is said to be a monotone class if all monotonic sequences of sets converge in $\mathscr{G}$.

Proposition 3.14 If $\mathscr{G}_\alpha$ where $\alpha \in I \subset \mathbb{R}$ are monotone classes then the intersection $\bigcap_{\alpha \in I} \mathscr{G}_\alpha$ is also a monotone class.

Proof | If $A_1, A_2 ...$ is any monotone sequence in $\bigcap_{\alpha \in I} \mathscr{G}_\alpha$ then it is in all $\mathscr{G}_\alpha$ and hence converge in all $\mathscr{G}_\alpha$. ■

Using this proposition one can define the smallest monotone class generated by some class $\mathscr{C}$ as the intersection of all the monotone classes containing $\mathscr{C}$.

Lemma 3.16 Let $\mathscr{A}$ be any algebra of subsets of Ω, $\mathscr{G}$ be the monotone class generated by $\mathscr{A}$, and let $\mathscr{F}$ be the $\sigma-$algebra generated by $\mathscr{A}$. Then $\mathscr{G} = \mathscr{F}$.

Proof | Let $A_j \in \mathscr{F}$ monotonically increase (decrease) to A; since the countable union (intersection) of A_j is in $\mathscr{A}$ so A must also be in $\mathscr{F}$. Since $\mathscr{A} \subset \mathscr{F}$ it follows that $\mathscr{G} \subset \mathscr{F}$.

All that remains to be shown is $\mathscr{F} \subset \mathscr{G}$. If we can show that $\mathscr{G}$ is an algebra, then for any $A_1, A_2, ... \in \mathscr{G}$ let $B_i = \bigcup_{j=1}^{i} A_i \in \mathscr{G}$ (since if $\mathscr{G}$ is an algebra it will be closed under finite union), it follows that $\bigcup_{j \geq 1} B_j = \bigcup_{j \geq 1} A_j \in \mathscr{G}$ (using the fact that $\mathscr{G}$ is monotone class). To prove that $\mathscr{G}$ is an algebra define $A \in \mathscr{G}$ the class:

$$\mathscr{M}(A) = \{M \in \mathscr{G} \mid A - M,\ M - A,\ A \cap M \in \mathscr{G}\}.$$

Clearly $\mathscr{M}(A) \subset \mathscr{G}$. Let $E_1 \subset E_2 \subset ... \in \mathscr{M}(A)$ converge to some E. Then $E - A = \bigcup_{i \geq 1}(E_i - A)$, but since by definition $E_i - A \in \mathscr{G}$ and $E_i - A \subset E_{i+1} - A$ it follows that $E - A \in \mathscr{G}$ (since $\mathscr{G}$ is monotone class). Similarly it can be shown that $A - E$ and $A \cap E$ are in $\mathscr{G}$, and thus $E \in \mathscr{M}(A)$. The same argument can be used to show that if $E_1 \supset E_2 \supset ... \in \mathscr{M}(A)$ converges to E then $E \in \mathscr{M}(A)$, hence concluding that $\mathscr{M}(A)$ is a monotone class. $\mathscr{M}(A)$ is also symmetric in the sense that if $A \in \mathscr{M}(B) \iff B \in \mathscr{M}(A)$, because if $A - B, B - A, A \cap B \in \mathscr{G}$ then both $A \in \mathscr{M}(B)$ and $B \in \mathscr{M}(B)$.

Let $A, B \in \mathscr{A}$ then we know that $A - B, B - A, A \cap B \in \mathscr{A}$. Hence $B \in \mathscr{M}(A)$ forall $A, B \in \mathscr{A}$, implying that $\mathscr{A} \subset \mathscr{M}(A)\ \forall\ A \in \mathscr{A}$. Since $\mathscr{G}$ is the smallest monotone class containing $\mathscr{A}$, $\mathscr{G} \in \mathscr{M}(A)\ \forall\ A \in \mathscr{A}$. Since $M \in \mathscr{A}$ forall $M \in \mathscr{G}$ using the symmetry it implies that $A \in \mathscr{M}(M)$. Hence $\mathscr{A} \in \mathscr{M}(M)$. Therefore $\mathscr{G} = \mathscr{M}(M)$ forall $M \in \mathscr{G}$. This means that $\mathscr{G}$ is closed under finite difference and intersection, proving that it is an algebra. This completes the proof. ■

Theorem 3.17 (Uniqueness of Extension) Let $\mu_1, \mu_2 : \mathscr{F} \to \mathbb{R}^+$ be $\sigma-$additive functions, where $\mathscr{F}$ is the $\sigma-$algebra generated by algebra $\mathscr{A}$ of a set Ω which is $\sigma-$finite with respect to μ_1 and μ_2 (with the additional condition that the finite sequence exists in $\mathscr{A}$), be such that $\mu_1|_{\mathscr{A}} = \mu_2|_{\mathscr{A}}$ then $\mu_1 = \mu_2$.

Proof | Let $E_1, E_2, ... \in \mathscr{A}$ be the sequence such that $\mu_1(E_n) < \infty$ and $\mu_2(E_n) < \infty$ for all n and $E_n \uparrow \Omega$. This sequence is gauranteed by the $\sigma-$finiteness of Ω. Define $\mathscr{B}_n = \{E \in \mathscr{F} \mid \mu_1(E \cap E_n) =$

$\mu_2(E \cap E_n)\}$. Clearly $\mathscr{B}_n \subset \mathscr{F}$. If $E \in \mathscr{A}$ then $E \cap E_n \in \mathscr{A}$ and since $\mu_1|_{\mathscr{A}} = \mu_2|_{\mathscr{A}}$ it follows that $\mathscr{A} \subset \mathscr{B}_n$. Let $A_1, A_1, ... \in \mathscr{B}$ be a sequence monotonically converging to some A. Since

$$\begin{aligned} \mu_1(A_j \cap E_n) &= \mu_2(A_j \cap E_n) \\ \implies \mu_1(A \cap E_n) &= \mu_2(A \cap E_n) \end{aligned}$$

where we have used lemma 2.17 and the finiteness of E_n in case of continuity from above. It follows that $\mathscr{B}_n$ is a monotone class. Since it contains the algebra $\mathscr{A}$ as well it must contain the monotone class generated by $\mathscr{A}$. Using lemma 3.16 we can conclude that $\mathscr{F} \subset \mathscr{B}_n$ and hence $\mathscr{B}_n = \mathscr{F}$. Let $A \in \mathscr{F}$ then

$$\begin{aligned} \lim_{n \to \infty} \mu_1(A \cap E_n) &= \lim_{n \to \infty} \mu_2(A \cap E_n) \\ \implies \mu_1(A) &= \mu_2(A) \end{aligned}$$

Therefore the extension is unique. ∎

As a result of this theorem we can conclude that the function $\pi^* : \mathscr{F} \to \mathbb{R}^+$ is a $\sigma-$additive extension of the $\sigma-$additive measure $\nu : \mathscr{A} \to \mathbb{R}^+$ on the $\sigma-$algebra $\mathscr{F}$ generated by the algebra $\mathscr{A}$ and is uniquely determined. This is known as Caratheodory theorem. The formal statement of this theorem is:

Theorem 3.18 (Caratheodory Theorem) Let $\mathscr{A}$ be an algebra, $\nu : \mathscr{A} \to \mathbb{R}^+$ be a $\sigma-$additive measure, and $\mathscr{F}$ be the $\sigma-$algebra generated by $\mathscr{A}$. Then there exists a unique $\sigma-$additive measure $\pi : \mathscr{M} \to \mathbb{R}^*$ such that $\pi|_{\mathscr{A}} = \nu$. Explicity this measure is given by restricting the outer measure $\pi^* : \mathfrak{P}(\Omega) \to \mathbb{R}^+$,

$$\pi^*(A) = \inf_{\{E_j \in \mathscr{A}\}} \sum_{j \geq 1} \nu(E_j), \text{ where } \{E_j\} \text{ is a covering of } A$$

on $\mathscr{M}$; i.e. $\pi = \pi^*|_{\mathscr{M}}$.

4 Lebersgue Measure

In this section we define a $\sigma-$additive measure on a class of subsets $\mathbb{R}$ and formalise the notion of length of subsets of $\mathbb{R}$. The procedure to do this is as follows:

1) Construct a semi-algebra $\mathscr{S}$ and a $\sigma-$additive measure $\mu : \mathscr{S} \to \mathbb{R}^+$.
2) Use theorem theorem 2.18 to construct a σ-additive measure ν on the algebra $\mathscr{A}$ generated by $\mathscr{S}$.
3) Use the caratheodory theorem to determine the $\sigma-$measure on $\mathscr{F}$, the $\sigma-$algebra generated by $\mathscr{A}$.

Definition 4.1 Let $\mathscr{S} = \{\emptyset, \mathbb{R}, (a,b], (a,\infty), (-\infty,b]\}$. It is easy to check that $\mathscr{S}$ is a semi-algebra of subsets of $\mathbb{R}$. Let $F : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function. Then define $\mu_F : \mathscr{S} \to \mathbb{R}^+$ as:

$$\begin{aligned} \mu_F(\emptyset) = 0, \quad \mu_F(\mathbb{R}) &= F(\infty), \quad \mu_F((a,b]) = F(b) - F(a), \\ \mu_F((a,\infty)) &= F(\infty) - F(a), \quad \mu_F((-\infty,b]) = F(b) - F(-\infty). \end{aligned}$$

Observe that if we construct a function $G : \mathbb{R} \to \mathbb{R}$ given by $G(x) = \lim_{n\to\infty} F(x_n)$ when $-\infty < x < \infty$ and $G(\pm\infty) = F(\pm\infty)$, where $x_n \downarrow x$. It is easy to verify that G is non-decreasing, right continuous and $\mu_G(A) = \mu_F(A)$, $\forall\ A \in \mathscr{S}$. Hence without loss of generalization it is fair to assume that F is always right continuous. Also observe that μ_F is monotone.

PROPOSITION 4.3 μ_F is a σ-additive measure.

Proof By definition we have that $\mu_F(\emptyset) = 0$. Consider the interval $(a,b] = \bigcup_{j=1}^{n}(a_j, b_j]$, where $(a_j, b_j]$ are pairwise disjoint. Then it is always possible to reindex the intervals such that $b_j = a_{j+1}$ when $j < n$, $a_{n+1} \equiv b_n = b$, and $a_1 = a$ (this uses both the fact that the intervals are disjoint and their union is $(a,b]$). Since,

$$
\begin{aligned}
F(b) - F(a) &= F(a_{n+1}) - F(a_1) \\
&= F(a_{n+1}) - F(a_2) + F(a_2) - F(a_1) \\
&= (F(a_{n+1}) - F(a_n)) + ... + (F(a_2) - F(a_1)) \\
&= \sum_{j=1}^{n} F(a_{j+1}) - F(a_j)
\end{aligned}
$$

This sum can again be reindexed such that

$$F(b) - F(a) = \sum_{j=1}^{n} F(b_j) - F(a_j)$$

This implies that

$$\mu_F((a,b]) = \sum_{j=1}^{n} \mu_F((a_j, b_j])$$

Hence μ_F is additive. Now consider the case when $(a,b] = \bigcup_{j\geq 1}(a_j, b_j]$ where $(a_j, b_j]$ are pairwise disjoint. Using montonicity and additivity of μ_F

$$\mu_F((a,b]) \geq \mu_F(\bigcup_{j=1}^{k}(a_j, b_j]) = \sum_{j=1}^{k} \mu_F((a_j, b_j]).$$

Taking the limit $k \to \infty$,

$$\mu((a,b]) \geq \sum_{j\geq 1} \mu_F((a_j, b_j]).$$

All that remains to be proven is that the $\leq$ ineqality. To prove this fix an $\epsilon > 0$. Choose a $c > a$ such that $F(c) - F(a) < \epsilon$, choose $d_j > b_j$ such that $F(d_j) - F(b_j) < \epsilon/2^j$ and $[c,b] \subset \bigcup_{j\geq 1}(a_j, d_j)$ (such choices are possible since F is continuous from the right). Using Heine-Borel theorem, since $[c,b]$ is closed and bounded and $\{(a_j, d_j)\}$ forms an open cover of $[c,b]$, there exists a finite subcover $\{(a_j, d_j) \mid j \leq k\}$ of $[c,b]$. Without loss of generality we can assume that $c \in (a_1, d_1)$ and $b \in (a_k, d_k)$. Since,

$$\mu_F((a,b]) = F(b) - F(a) < F(b) - F(c) + \epsilon = \mu_F((b,c]) + \epsilon$$

Then using the montonicity of μ_F,

$$
\begin{aligned}
\mu_F((c,b]) &\leq \mu_F(\bigcup_{j=1}^{k}(a_j, d_j]) \\
&\leq \sum_{j=1}^{k} \mu_F((a_j, d_j]) \\
&\leq \sum_{j=1}^{k} F(d_j) - F(a_j)
\end{aligned}
$$

$$< \sum_{j=1}^{k} F(b_j) - F(a_j) + \epsilon$$
$$< \sum_{j\geq 1} F(b_j) - F(a_j) + \epsilon.$$

Thus for any $\epsilon > 0$

$$\sum_{j\geq 1} \mu_F((a_j, b_j]) \leq \mu_F((a,b]) < \sum_{j\geq 1} \mu_F((a_j, b_j]) + 2\epsilon$$

It is easy to show that this is true also for intervals $(-\infty, b]$ and (a, ∞). This proves $\sigma-$additivity. ■

Using the extension theorem and then caratheodory theorem the function $\mu_F^* : \mathscr{F} \to \mathbb{R}^+$

$$\mu_F^*(A) = \inf \left\{ \sum_j \mu_F(A_j) \mid A_j \in \mathscr{A} \ \ \& \ \ A \subset \bigcup_{j\geq 1} A_j \right\}$$

is a unique $\sigma-$additive extension of μ on the $\sigma-$algebra $\mathscr{M}_{\mu^*}$ (which is the set of measurable functions w.r.t. μ^*). The measure space $(\mathbb{R}, \mathscr{M}_{\mu^*}, \mu_F^*)$ is called Lebesgue-Stieltjes measure space. In the case when $F(a) = a$ and hence $\mu((a,b]) \equiv \mu_F((a,b]) = b - a$, $(\mathbb{R}, \mathscr{M}_{\mu^*}, \mu^*)$ is called the Lebesgue measure space.

Convention From now on we refer to $(\Omega, \mathscr{F}, \mu)$ a measure space if Ω is some set, $\mathscr{F}$ is some $\sigma-$algebra containing Ω, and μ is a $\sigma-$additive measure. From now we also adopt the convention of calling $\sigma-$additive measures as just measures.

5 Complete Measures

Definition 5.1 A measure space $(\Omega, \mathscr{F}, \mu)$ is said to be *complete* if $A \in \mathscr{F}$, $\mu(A) = 0$ and $E \subset A$ imply that $E \in \mathscr{F}$.

Definition 5.2 Let $(\Omega, \mathscr{F}, \mu)$ measure space and $A \in \mathscr{F}$ such that $\mu(A) = 0$. Then subsets of A are said to be *negligible sets*.

Proposition 5.3 If $(\Omega, \mathscr{F}, \mu)$ is a measure space and $\mathscr{F}'$ is defined as

$$\mathscr{F}' = \{A \cup N \mid A \in \mathscr{F} \ \ \& \ \ N \subset E \in \mathscr{F} \text{ where } \mu(E) = 0\}.$$

Then $\mathscr{F}'$ is a $\sigma-$algebra.

Proof | Let $A \in \mathscr{F}$ and $E = \emptyset$ (implying that $\mu(E) = 0$) then $A \cup N = A$ where $N \subset E$, hence $A \in \mathscr{F}'$. It further follows that $\mathscr{F} \subset \mathscr{F}'$. This means that $\Omega \in \mathscr{F}'$.

Let $A \in \mathscr{F}'$. Then $A = E \cup N$ where $E \in \mathscr{F}$ and $N \subset H$ such that $\mu(H) = 0$. One can then write $A^c = E^c \cap N^c = (E^c \cap H^c) \cup (E^c \cap (H - N))$. Clearly $E^c \cap H^c \in \mathscr{F}$ and $E^c \cap (H - N) \subset H - N \subset H$. Hence $A^c \in \mathscr{F}'$.

Let $A_1, A_2, ... \in \mathscr{F}'$. Let $A_j = E_j \cup N_j$ where $E_j \in \mathscr{F}$, $N_j \subset H_j$ and $H_j \in \mathscr{F}$ such that $\mu(H_j) = 0$. Then

$$\bigcup_{j\geq 1} A_j = \left(\bigcup_{j\geq 1} E_j \right) \cup \left(\bigcup_{j\geq 1} N_j \right)$$

Since $\bigcup_{j\geq 1} E_j \in \mathscr{F}$, $\bigcup_{j\geq 1} N_j \subset \bigcup_{j\geq 1} H_j$ and $\mu(\bigcup_{j\geq 1} H_j) = \sum_{j\geq 1} \mu(H_j) = 0$, it follows that $\bigcup_{j\geq 1} A_j \in \mathscr{F}'$. ■

Definition 5.4 Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and $\mathscr{F}' \supset \mathscr{F}$ be a $\sigma-$algebra as defined in proposition 5.3. Then define $\mu' : \mathscr{F}' \to \mathbb{R}^+$ as follows. If $A \in \mathscr{F}'$ and $A = E \cup N$ where $A \in \mathscr{F}$ and N is a negligible set then

$$\mu'(A) = \mu(E)$$

Proposition 5.5 μ' is a unique, $\sigma-$additive extension of μ.

Proof | Let $E \cup N = F \cup M$ where $E, F \in \mathscr{F}$ and $N \subset H, M \subset H'$ where $H, H' \subset \mathscr{F}$ and $\mu(H) = \mu(H') = 0$. Clearly $E \subset E \cup N = F \cup M \subset F \cup H'$. Using monotonicity we have $\mu(E) \leq \mu(F)$. Similarly it can be shown that $\mu(F) \leq \mu(E)$, implying that $\mu(E) = \mu(F)$. This shows that $\mu'(E \cup N) = \mu(E) = \mu(F) = \mu'(F \cup M)$. Hence μ' is well defined.

Clearly $\mu'(\emptyset) = 0$. Let $A_1, A_2, ... \in \mathscr{F}'$ be pairwise disjoint, and let their representation be $A_j = E_j \cup N_j$ where $E_j \in \mathscr{F}$ and N_j are negligible. By definition $\mu'(A_j) = \mu(E_j)$. Also since A_j are paorwise disjoint so will be E_j. Thus:

$$\begin{aligned}
\mu'(\bigcup_{j\geq 1} A_j) &= \mu'(\bigcup_{j\geq 1} E_j \cup \bigcup_{j\geq 1} N_j) \\
&= \mu(\bigcup_{j\geq 1} E_j) \\
&= \sum_{j\geq 1} \mu(E_j) \\
&= \sum_{j\geq 1} \mu'(A_j)
\end{aligned}$$

Hence μ' is $\sigma-$additive.

Let $A \in \mathscr{F}$. Then clearly $\mu'(A) = \mu'(A \cup \emptyset) = \mu(A)$. Thus μ' is an extension of μ. To prove that this is a unique extension let $\mu_1, \mu_2 : \mathscr{F}' \to \mathbb{R}^+$ be $\sigma-$additive functions such that $\mu_1(A) = \mu_2(A) = \mu(A)\ \forall\ A \in \mathscr{F}$. Then for some $E \cup N$, where $E, H \in \mathscr{F}$, $N \subset H$, and $\mu(H) = 0$:

$$\mu_1(E \cup N) \leq \mu_2(E \cup H) = \mu_2(E) \leq \mu_2(E \cup N)$$

Similarly it can be shown that $\mu_1(E \cup N) \geq \mu_2(E \cup N)$. Hence μ' is also unique. ■

Proposition 5.6 The measure space $(\Omega, \mathscr{F}', \mu')$ is complete.

Proof | If $A \in \mathscr{F}', \mu'(A) = 0, A = F \cup M$ where $F \in \mathscr{F}$ and M is negligible. Then $\mu(F) = \mu'(A) = 0$. Thus we could simply make the choice $M \subset F$ and hence $A = F$. Therefore shown that if $\mu'(A) = 0$ then $A \in \mathscr{F}$. Let $E \subset F$. Then we can simply represent E as $\emptyset \cup E$. Since $\emptyset \in \mathscr{F}$ and $E \subset A \in \mathscr{F}$ where $\mu(A) = 0$, it follows that $E \in \mathscr{F}'$. ■

Proposition 5.7 Let $(\Omega, \mathscr{M}, \pi^*|_{\mathscr{M}})$ be the measure space as defined in theorem 3.18. This is a complete measure space.

Proof | Let $B \in \mathscr{M}$, $\pi^*(B) = 0$, and $A \subset B$. For any $F \subset \Omega$,

$$F \cap A \subset A \subset B,$$

hence $\pi^*(F \cap A) \leq \pi^*(B) = 0$. Since $F \cup A^c \subset F \implies \pi^*(F \cup E^c) \leq \pi^*(F)$. Adding these two inequalities we get:

$$\pi^*(F) \geq \pi^*(F \cap A) + \pi^*(F \cap A^c).$$

Thus $A \in \mathscr{M}$. ■

6 Integration

In this section we would like to formulate the concept of integral in the context of measure spaces. In the Riemann integral we first partition the domain and then approximate then integral to be:

$$\int f \approx \sum_{k \geq 1} y_k (x_k - x_{k-1})$$

Using a similar concept we would later define an integral, called the Lebesgue integral, where we partition the y-axis, take the inverse of that interval to get a set in the domain, and then use a measure to find the "length" of this interval. Then we approximate the integral as follows:

$$\int f \approx \sum_{k \geq 1} y_k \mu(f^{-1}(A_k))$$

See fig. 1 to understand this better. But in order to define this integral the function must have the property that its inverse belongs to the $\sigma-$algebra on which the measure μ is defined. For this a new class of functions known as measurable functions is defined which hold this property.

Definition 6.1 Let $(X, \mathscr{T})$ be a topological space. Then the $\sigma-$algebra generated by the open sets of this space is called the *Borel $\sigma-$algebra*, and represented $\mathscr{B}(X, \mathscr{T})$.

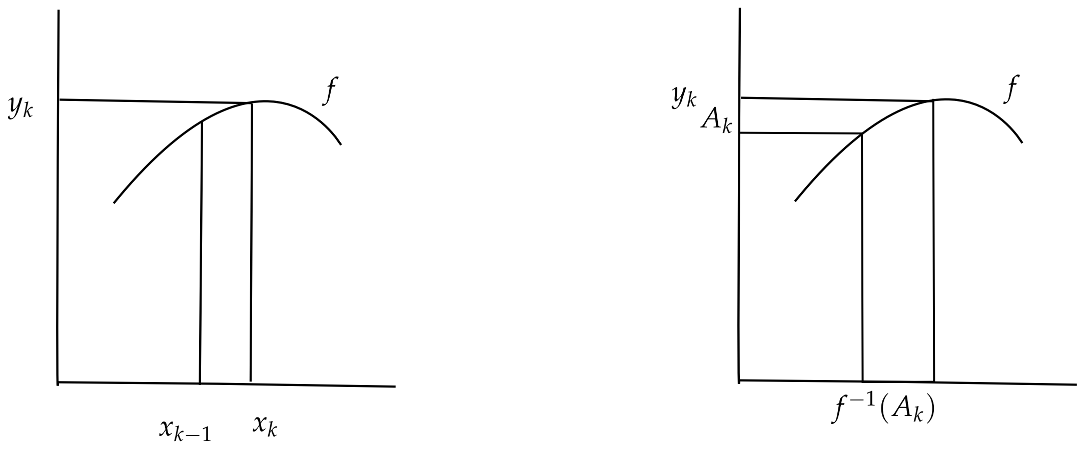
In the case when $X = \mathbb{R}^n$ and $\mathscr{T}$ is the usual topology on $\mathbb{R}$, the Borel $\sigma-$algebra is simply denoted $\mathscr{B}(\mathbb{R}^n)$.

Definition 6.2 Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, and $f : \Omega \to \mathbb{R}$ be a function Ω then f is said to be *$\mathscr{F}-$measurable*, or simply measurable, if $B \in \mathscr{B}(\mathbb{R}) \implies f^{-1}(B) \in \mathscr{F}$.

In general if $(\Omega_1, \mathscr{F}_1, \mu_1)$ and $(\Omega_2, \mathscr{F}_2, \mu_2)$ are measure spaces then $f : \Omega_1 \to \Omega_2$ is said to be $\langle \mathscr{F}_1, \mathscr{F}_2 \rangle$-measurable if $A \in \mathscr{F}_2 \implies f^{-1}(A) \in \mathscr{F}_1$. Hence $\mathscr{F}-$measurable functions are just $\langle \mathscr{F}, \mathscr{B}(\mathbb{R}) \rangle$-measurable.

Lemma 6.3 Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and $f : \Omega \to \mathbb{R}$. Then f is measurable $\iff f^{-1}((-\infty, x]) \in \mathscr{F}$.

Proof | Before we begin proving this lemma, note that $\mathscr{B}(\mathbb{R})$ is the $\sigma-$algebra generated by the usual topology $\mathscr{T}$ on $\mathbb{R}$. Let $\mathscr{C} = \{(-\infty, x] \mid x < \infty\}$ and $\mathscr{G}$ be the $\sigma-$algebra generated by $\mathscr{C}$. Since for any $(-\infty, x]$ the sequence of elements $(-\infty, x_n) \in \mathscr{B}(\mathbb{R})$ where $x_n \downarrow x$ satisfies $(-\infty, x] = \bigcap_{j \geq 1} (-\infty, x_j)$. This shows that $(-\infty, x] \in \mathscr{B}(\mathbb{R})$ (using closure under countable intersection), furhter implying



(a) Riemann integral

(b) Lebesgue integral

Figure 1: Visualization of the integral

that $\mathscr{G} \subset \mathscr{B}(\mathbb{R})$. Similarly it can be shown that any open set (x, y) can be expressed as countable unions and intersections of elements of $\mathscr{G}$, implying that $\mathscr{B}(\mathbb{R}) \subset \mathscr{G}$. Hence $\mathscr{B}(\mathbb{R}) = \mathscr{G}$.

$(\implies)$ If f is measurable then $A \in \mathscr{B}(\mathbb{R}) \implies f^{-1}(A) \in \mathscr{F}$, and since $(-\infty, x] \in \mathscr{B}(\mathbb{R})$ this implies that $f^{-1}((-\infty, x]) \in \mathscr{F}$.

$(\impliedby)$ Let $\mathscr{M} = \{A \in \mathscr{B}(\mathbb{R}) \mid f^{-1}(A) \in \mathscr{F}\}$. Since $\mathbb{R} \in \mathscr{B}(\mathbb{R})$ and $f^{-1}(\mathbb{R}) = \{\omega \mid f(\omega) \in \mathbb{R}\} = \Omega \implies \Omega \in \mathscr{M}$. Also observing that f^{-1} preserves countable unions and compliments, if $A, A_1, ... \in \mathscr{M}$ then $f^{-1}(A^c) = (f^{-1}(A))^c \in \mathscr{F}$ and $f^{-1}(\bigcup_{j \geq 1} A_j) = \bigcup_{j \geq 1} f^{-1}(A_j)$. This shows that $\mathscr{M}$ is closed under compliments and countable unions. Hence $\mathscr{M}$ is a $\sigma-$algebra. Since we are showing the backward implication we assume that $\mathscr{C} \subset \mathscr{M}$. Since $\mathscr{M}$ is a $\sigma-$algebra containing $\mathscr{C}$ it follows that $\mathscr{B}(\mathbb{R}) \subset \mathscr{M}$. Since by definition of $\mathscr{M}$ it is a subset of $\mathscr{B}(\mathbb{R})$ further follows that $\mathscr{M} = \mathscr{B}(\mathbb{R})$. Thus proving the lemma. ∎

Note that the key to proving the lemma really was showing that $\mathscr{B}(\mathbb{R}) \subset \mathscr{M}$, which required the fact that $\mathscr{B}(\mathbb{R})$ was the $\sigma-$algebra generated by $\mathscr{C}$. Hence this theorem can be easily extended to any class $\mathscr{C}$ which generates the $\sigma-$algebra $\mathscr{B}(\mathbb{R})$, in the following sense:

Lemma 6.5 Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, $\mathscr{C}$ be a class of subsets of $\mathbb{R}$ which generates the $\sigma-$algebra $\mathscr{B}(\mathbb{R})$, and $f : \Omega \to \mathbb{R}$. Then f is measurable $\iff$ $f^{-1}(A) \in \mathscr{F}$ where $A \in \mathscr{C}$.

Hence lemma 6.3 also holds if $(-\infty, x]$ is replaced with one of $(-\infty, x)$, (x, ∞), or $[x, \infty)$.

Definition 6.6 Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, $E_1, ..., E_n \in \mathscr{F}$ be pairwise disjoint sets such that $\Omega = \bigcup_{j=1}^{n} E_j$, and 1_{E_j} be the indicator function of E_j. Then a *simple function* $f : \Omega \to \mathbb{R}$ is a function which can be written as:

$$f(\omega) = \sum_{j=1}^{n} c_j 1_{E_j}(\omega)$$

where $c_j \in \mathbb{R}$.

PROPOSITION 6.7 Simple functions are measurable.

Proof | Let $f : \Omega \to \mathbb{R}$ be a simple function expressed as:

$$f(\omega) = \sum_{j=1}^{n} c_j 1_{E_j}(\omega)$$

The set $f^{-1}((-\infty, x]) = \{\omega \mid f(\omega) \leq x\}$ should belong to $\mathscr{F}$ for f to be measurable by lemma 6.3. Notice that $f(\omega) \leq x$ only when $\omega \in E_j$ such that the corresponding $c_j \leq x$. Hence $f^{-1}((-\infty, x]) = \bigcup_{j \mid c_j \leq x} E_j$. Since each $E_j \in \mathscr{F}$ so will be any finite union. Hence $f^{-1}((-\infty, x]) \in \mathscr{F}$. ■

DEFINITION 6.8 (Integral of Non-negative Simple functions) Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and let f be a non-negative simple function of the form

$$f(\omega) = \sum_{j=1}^{n} c_j 1_{E_j}(\omega),\ c_j \leq 0$$

Then we define:

$$\int f = \sum_{j=1}^{n} c_j \mu(E_j)$$

The non-negative condition was applied to avoid cases like the following: $\mu(E_1) = \mu(E_2) = \infty$ and $c_1 = -c_2$. Then the sum on the RHS would have $\infty - \infty$, which is not well defined.

PROPOSITION 6.9 The integral of non-negative simple function is well defined.

Proof | Let $f : \Omega \to \mathbb{R}$, $\{E_1, ..., E_n\}, \{F_1, ..., F_2\} \in \mathscr{F}$ be two partitions of Ω, and f be represented as:

$$f(\omega) = \sum_{j=1}^{n} c_k 1_{E_j} = \sum_{j=1}^{n} d_k 1_{F_j}$$

where $c_j, d_j \geq 0$. Consider the case when $E_{j_0} \cap F_{k_0} \neq \emptyset$. If $\omega \in E_{j_0} \cap F_{k_0}$ then $f(\omega) = c_{j_0} = d_{k_0}$. Since

$$\begin{aligned} \mu(E_j) &= \mu(E_j \cap \Omega) \\ &= \mu(E_j \cap \bigcup_{k=1}^{n} F_k) \\ &= \sum_{k=1}^{n} \mu(E_j \cap F_k), \end{aligned}$$

it follows that

$$\int f = \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \mu(E_j \cap F_k).$$

Similarly in case of F_j,

$$\int f = \sum_{j=1}^{n} \sum_{k=1}^{n} d_k \mu(E_j \cap F_k).$$

When $E_j \cap F_k = \emptyset$ the corresponding term in the sum is 0, and when $E_j \cap F_k \neq \emptyset$ then $d_j = c_j$. Thus both the sums are equal. ∎

Lemma 6.10 Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, $f, g : \Omega \to \mathbb{R}$ be measurable functions, and α be some constant; then

1) αf,
2) $f + \alpha$,
3) $f + g$,
4) f^2,
5) $1/f$,
6) $f^{\pm}$, where $f^{\pm}(\omega) = \max(\pm f(\omega), 0)$,
7) $|f|$,
8) fg

are measurable functions.

Proof | From lemma 6.3, in each case we only have to show that the inverse map of $(-\infty, x]$ belongs to $\mathscr{F}$, given that $A \in \mathscr{B}(\mathbb{R}) \implies f^{-1}(A), g^{-1}(A) \in \mathscr{F}$.

1) In this case we need to show that $\{\omega \mid \alpha f(\omega) \leq x\} \in \mathscr{F}$. When $\alpha = 0$, forall $x \geq 0$ the set in question is Ω and when $x < 0$ it is $\emptyset$. Both of these are in $\mathscr{F}$. When $\alpha > 0$, the set $\{\omega \mid \alpha f(\omega) \leq x\} = \{\omega \mid f(\omega) \leq x/\alpha\} \in \mathscr{F}$. Similarly for $\alpha < 0$, $\{\omega \mid \alpha f(\omega) \leq x\} = \{\omega \mid f(\omega) \geq x/\alpha\} \in \mathscr{F}$.

2) Using similar logic as above $\{\omega \mid -\infty < f(\omega) + \alpha \leq x\} = \{\omega \mid -\infty < f(\omega) \leq x - \alpha\} \in \mathscr{F}$.

3) Consider the set $\{\omega \mid f(\omega) + g(\omega) \leq x\}$. Using density of $\mathbb{Q}$ in $\mathbb{R}$ we know that it is always possible to find $r \in \mathbb{Q}$ such that $f(\omega) \leq r$ and hence $g(\omega) \leq x - r$. Thus $\{\omega \mid f(\omega) + g(\omega) \leq x\} = \bigcup_{r \in \mathbb{Q}} \{\omega \mid f(\omega) \leq r\} \cap \{\omega \mid g(\omega) \leq x - r\}$. Since each $\{\omega \mid f(\omega) \leq r\}$ and $\{\omega \mid g(\omega) \leq x - r\}$ is in $\mathscr{F}$, by closure under countable unions and intersections $\{\omega \mid f(\omega) + g(\omega) \leq x\} \in \mathscr{F}$.

4) Consider the set $\{\omega \mid f^2(\omega) \leq x\}$. In the case when $x < 0$, the set $\{\omega \mid f^2(\omega) \leq x\} = \emptyset \in \mathscr{F}$. In the case when $x \geq 0$, $\{\omega \mid f^2(\omega) \leq x\} = \{\omega \mid -\sqrt{x} \leq f(\omega) \leq \sqrt{x}\} \in \mathscr{F}$.

5) Consider the set $\{\omega \mid 1/f(\omega) < x\}$. In the case when $x > 0$,

$$\begin{aligned}\{\omega \mid 1/f(\omega) < x\} &= \{\omega \mid 1/f(\omega) < 0\} \cup \{\omega \mid 0 < 1/f(\omega) < x\} \\ &= \{\omega \mid f(\omega) \leq 0\} \cup \{\omega \mid 0 < f(\omega) \leq 1/x\}\end{aligned}$$

Since each set in the RHS is in $\mathscr{F}$ it follows that $\{\omega \mid 1/f(\omega) < x\} \in \mathscr{F}$. When $x = 0$, $\{\omega \mid 1/f(\omega) < 0\} = \{\omega \mid f(\omega) < 0\} \in \mathscr{F}$. When $x < 0$ then

$$\{\omega \mid 1/f(\omega) < x\} = \{\omega \mid 0 > f(\omega) > 1/x\}$$

which is clearly in $\mathscr{F}$.

6) The set $\{\omega \mid f^+(\omega) \leq x\} = \{\omega \mid f(\omega) \leq x\}$ when $x \geq 0$, and $\{\omega \mid f^+(\omega) \leq x\} = \emptyset$ when $x < 0$. Thus $\{\omega \mid f^+(\omega) \leq x\} \in \mathscr{F}$. Similarly since $\{\omega \mid f^-(\omega) \leq x\} = \{\omega \mid f(\omega) \geq x\}$ when $x \geq 0$ (and $\emptyset$ otherwise). Hence $\{\omega \mid f^-(\omega) \leq x\} \in \mathscr{F}$.

7) Since $|f| = f^+ + f^-$, it is obvious that $|f|$ is also measurable.

8) In the case of product of measurable functions, we simply use the identity:

$$fg = \frac{1}{2}((f+g)^2 - f^2 - g^2)$$

Since $f+g$, f^2, and g^2 are measurable (by points 3,4) it follows that fg is also measurable (again by point 3).

∎

Proposition 6.11 Let $(\Omega, \mathscr{F}, \mu)$ be measure space, $f : \Omega \to \mathbb{R}$ be some function, and $f_j : \Omega \to \mathbb{R}$ are a sequence of measurable functions. Then

1) $\sup f_n$,
2) $\inf f_n$,
3) $\limsup f_n$,
4) $\liminf f_n$, and
5) $\lim f_n$

are measurable functions.

Proof | 1) Consider the set $\{\omega \mid \sup f_n < x\}$. Since $f_n \leq \sup f_n$ forall n, hence $f_n < x$. Thus

$$\{\omega \mid \sup f_n < x\} = \bigcup_{n \geq 1} \{\omega \mid f_n < x\} \in \mathscr{F}$$

2) Since $\inf f_n = -\sup\{-f_n\}$, it is clear that $\inf f$ is also measurable.
3) Since

$$\limsup f_n = \inf_n \sup_{m \geq n} f_n$$

and both the infimum and supremum of sequence of measurable functions is measurable, it follows that $\limsup f_n$ is also measurable.
4) Similar argument as above for $\liminf f_n$.
5) For converging sequences $\limsup f_n = \liminf f_n = \lim f_n$. Hence the limit of a converging sequence of functions is converging.

∎

Definition 6.12 A property P is said to be true almost everywhere w.r.t. μ, also written as a.e. (μ), if $\mu(\{\omega \mid P \text{ is false }\}) = 0$. In other words P is true everywhere except in a set with zero measure.

Proposition 6.13 Let f and g be simple functions, then:

1) Integral is linear, i.e.

$$\int af = a \int f \quad \& \quad \int f + g = \int f + \int g.$$

2) Integral is monotonic, i.e.

$$f \leq g \implies \int f \leq \int g.$$

3) $\int f = 0 \iff f = 0$, a.e. (μ).
4) If $f = g$ a.e. (μ), then $\int f = \int g$.

Proof | Let $\{E_j\}$, $\{F_j\}$ be a partition of Ω and let the representation of f, g be:

$$f(\omega) = \sum_{j\geq 1} c_j 1_{E_j}(\omega)$$
$$g(\omega) = \sum_{j\geq 1} d_j 1_{F_j}(\omega)$$

then:

1) The representation of af would be:

$$f(\omega) = \sum_{j\geq 1} (ac_j) 1_{E_j}(\omega))$$

Thus its integration would be:

$$\begin{aligned} \int af &= \sum_{j\geq 1} (ac_j)\mu(E_j) \\ &= a \sum_{j\geq 1} c_j \mu(E_j) \\ &= a \int f. \end{aligned}$$

The representation of $f + g$ would be:

$$\begin{aligned} (f+g)(\omega) &= \sum_{j\geq 1} c_j 1_{E_j}(\omega) + \sum_{j\geq 1} d_j 1_{F_j}(\omega) \\ &= \sum_{j\geq 1}\sum_{k\geq 1} (c_j + d_k) 1_{E_j \cap F_k}(\omega) \end{aligned}$$

The final inequality is due to the fact that if $\omega \in E_j \cap F_k \neq \emptyset$ in which case we have c_j from the first sum and d_k from the second sum in the LHS and a $c_j + d_k$ in the RHS. Since both the summations are countable it is possible to write it out as a single sum (using the fact that product of countable sets is countable). Thus

$$\begin{aligned} \int (f+g) &= \sum_{j\geq 1}\sum_{k\geq 1} (c_j + d_k)\mu(E_j \cup F_k) \\ &= \sum_{j\geq 1}\sum_{k\geq 1} c_j \mu(E_j \cup F_k) + \sum_{j\geq 1}\sum_{k\geq 1} d_j \mu(E_j \cup F_k) \\ &= \int f + \int g \end{aligned}$$

2) If $\omega \in E_i \cap F_j \neq \emptyset$ then

$$c_i = f(\omega) \leq g(\omega) = d_j,$$

and since

$$\int f = \sum_{i\geq 1}\sum_{j\geq 1} c_i \mu(E_i \cap F_j)$$

Either $E_i \cap F_j = \varnothing$, or $E_i \cap F_j \neq \varnothing$ in which case $c_i \leq d_j$. Thus

$$\int f \leq \sum_{i\geq 1}\sum_{j\geq 1} d_j \mu(E_i \cap F_j) = \int g$$

3) Its clear that when $f = 0$, then $\int f = 0$. For the forward implication let $D = \{\omega \mid f > 0\}$ and $D_n = \{\omega \mid f > 1/n\}$. Clearly $D_n \uparrow D$. Since

$$\begin{aligned} f &\geq f 1_{D_n} \geq \frac{1}{n} 1_{D_n} \\ 0 = \int f &\geq \int \frac{1}{n} 1_{D_n} = \frac{1}{n}\mu(D_n) \\ \implies \mu(D_n) &\leq 0 \implies \mu(D_n) = 0 \end{aligned}$$

Using monotone continuity from below, we get that $\mu(D) = 0$. Thus $f = 0$ a.e. (μ).

4) Let $h = f - g$. Thus $h = 0$, a.e. (μ). From the previous property we know that $\int h = 0$. Then using linearity $\int f = \int g$.

Thus completing the proof. ■

The second property can be further generalised to $f \leq g$ a.e. (μ) implies that $\int f \leq \int g$ using the fourth property.

Lemma 6.15 Let $f : \Omega \to \mathbb{R}$ be a non-negative function, then there exists a sequence of non-negative simple functions $f_n : \Omega \to \mathbb{R}$ such that $f_n \uparrow f$.

Proof | Consider the sequence of simple functions $(f_n)_{n\geq 1}$ given by:

$$f_n(\omega) = \begin{cases} n, \text{ if } f(\omega) > n \\ \frac{k}{2^n}, \text{ if } \frac{k}{2^n} \leq f(\omega) \leq \frac{k+1}{2^n},\ 0 \leq k \leq n2^n - 1 \end{cases}$$

If $f(\omega) = \infty$, then $f_n(\omega) = n$ implying that $f_n(\omega) \to f(\omega)$. When $f(\omega) < \infty$, then $\exists\, n_0$ such that $f(\omega) < n_0$. If $n > n_0$ then

$$f_n(\omega) = \frac{[2^n f(\omega)]}{2^n} \leq f(\omega).$$

Since $f \geq 0$ it follows that:

$$\begin{aligned} \frac{2^n f(\omega) - 1}{2^n} \leq f_n(\omega) &= \frac{[2^n f(\omega)]}{2^n} \leq f(\omega) \\ \implies f(\omega) &\leq \lim_{n\to\infty} f_n(\omega) \leq f(\omega) \end{aligned}$$

Thus as $n \to \infty$ we get $f_n \to f$.

Now it is requred to show that $f_{n+1} > f_n$. In the case when $f(\omega) = \infty$ clearly $f_n(\omega) = n < f_{n+1}(\omega) = n + 1$. When $f(\omega) > n + 1$, then $f_n(\omega) < f_{n+1}(\omega)$. When $n < f(\omega) < n + 1$, we know that

$$f_{n+1}(\omega) = \frac{[2^{n+1} f(\omega)]}{2^{n+1}} \geq \frac{[n2^{n+1}]}{2^{n+1}} \geq n = f_n(\omega)$$

And finally when $f_n(\omega) < n < n+1$ then:

$$f_{n+1}(\omega) = \frac{[2^{n+1} f(\omega)]}{2^{n+1}}$$

Since for any $x > 0$ we have:

$$[2x] = \begin{cases} 2[x]+1, \text{ if } \{x\} \geq 0.5 \\ 2[x], \text{ if } \{x\} < 0.5 \end{cases}$$

Thus:

$$\frac{[2^{n+1} f(\omega)]}{2^{n+1}} = \begin{cases} \frac{2[2^n f(\omega)]+1}{2^{n+1}}, \\ \frac{[2^n f(\omega)]}{2^n} \end{cases}$$
$$\implies f_{n+1}(\omega) \geq f_n(\omega)$$

Hence we have proved that $f_n \uparrow f$. ■

Definition 6.16 (Definition of integral of non-negative functions) Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and $f : \Omega \to \mathbb{R}$ be a non-negative function, and f_n be a sequence of non-negative simple functions such that $f_n \uparrow f$. Then

$$\int f := \lim_{n \to \infty} \int f_n$$

Proposition 6.17 Integral in definition 6.16 is well defined.

Proof | Let f_n, g_n be sequences of non-negative simple functions such that $f_n, g_n \uparrow f$. Let the representation of f_n, g_n be as follows,

$$f_n = \sum_{i \geq 1} c_{ni} 1_{E_{ni}}$$
$$g_n = \sum_{i \geq 1} d_{ni} 1_{F_{ni}},$$

let

$$I_n = \int f_n = \sum_{i \geq 1} c_{ni} \mu(E_{ni}) = \sum_{i \geq 1} \sum_{j \geq 1} c_{ni} \mu(E_{ni} \cap F_{nj})$$
$$J_n = \int g_n = \sum_{j \geq 1} d_{nj} \mu(F_{nj}) = \sum_{j \geq 1} \sum_{i \geq 1} d_{nj} \mu(F_{nj} \cap E_{ni}),$$

and let $\epsilon > 0$. Since we know that f_n and g_n converge to the same function, f, assuming that $\omega \in E_{nk} \cap F_{nl} \neq \emptyset$ it follows that $\exists\, N$ such that $n > N$ implies

$$|f_n(\omega) - g_n(\omega)| < \frac{\epsilon}{2^{k+l} \mu(E_{nk} \cap F_{nl})}$$
$$\implies |\sum_{i \geq 1} c_{ni} 1_{E_{ni}}(\omega) - \sum_{i \geq 1} d_{ni} 1_{F_{ni}}(\omega)| < \frac{\epsilon}{2^{k+l} \mu(E_{nk} \cap F_{nl})}$$
$$\implies |c_{nk} - d_{nl}| < \frac{\epsilon}{2^{k+l} \mu(E_{nk} \cap F_{nl})}.$$

Also,

$$|I_n - J_n| = \left| \sum_{j\geq 1} \sum_{i \geq 1} (c_{ni} - d_{nj}) \mu(F_{nj} \cap E_{ni}) \right|$$

In this sum we either have that $E_{ni} \cap F_{nj} = \emptyset$, or $E_{ni} \cap F_{nj} \neq \emptyset$ in which case we know that $|c_{ni} - d_{nj}|$ is arbitrarily close to zero. Hence we get

$$\begin{aligned} |I_n - J_n| &\leq \sum_{j\geq 1} \sum_{i \geq 1} \left|c_{ni} - d_{nj}\right| \mu(F_{nj} \cap E_{ni}) \\ &< \sum_{j\geq 1} \sum_{i \geq 1} \frac{\epsilon}{2^{i+j} \mu(E_{ni} \cap F_{nj})} \mu(F_{nj} \cap E_{ni}) = \epsilon \end{aligned}$$

Thus $n > N \implies |I_n - J_n| < \epsilon$. Hence

$$\lim I_n = \lim J_n = \int f$$

proving that the integral is well defined. ■

> **PROPOSITION 6.18** The properties in proposition 6.13 extend to non-negative measurable functions.

Proof | Let f, g be non-negative measurable functions, let $f_n \uparrow f$, and $g_n \uparrow g$ where f_n, g_n are non-negative simple functions. Then:

1) Since $f_n \uparrow f \implies af_n \uparrow af$. Thus

$$\begin{aligned} \int af &= \lim_{n\to\infty} \int af_n \\ &= a \lim_{n\to\infty} \int f_n \\ &= a \int f. \end{aligned}$$

2) If $f \leq g$ then there exists N such that $n \geq N \implies f_n \leq g_n$. Thus

$$\begin{aligned} &\int f_n \leq \int g_n \\ \implies &\lim_{n\to\infty} \int f_n \leq \lim_{n\to\infty} \int g_n \\ \implies &\int f \leq \int g. \end{aligned}$$

3) The proof for the third property did not assume anything about the function f thus it is true for non-negative functions too.
4) Since $f = g$ a.e. (μ), let $\epsilon > 0$, then there exists N such that $n > N$ implies that

$$\begin{aligned} &g_n - \epsilon < f_n < g_n + \epsilon, \text{ a.e. } (\mu) \\ \implies &\int g_n - \epsilon\mu(\Omega) < \int f_n < \int g_n + \epsilon\mu(\Omega), \text{ everywhere} \\ \implies &\lim_{n\to\infty} \int g_n - \epsilon\mu(\Omega) < \lim_{n\to\infty} \int f_n < \lim_{n\to\infty} \int g_n + \epsilon\mu(\Omega) \\ \implies &\int g \leq \int f \leq \int g \end{aligned}$$

$$\implies \int f = \int g$$

Thus completing the proof.

■

THEOREM 6.19 (Monotone Convergence Theorem) If $(f_n)_{n\geq 1}$, f be non-negative measurable functions such that $f_n \uparrow f$ a.e. (μ). Then,

$$\int f = \lim_{n\to\infty} \int f_n.$$

Proof ■