AUTOMORPHISMS OF HYPERBOLIC SURFACES

Notes made for thesis project

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1 Torus and it's Automorphisms

Definition 1.1. Define the torus, T^2 as the quotient space $\mathbb{R}^2/\mathbb{Z}^2$ where two points are equivalent if their difference is in $\mathbb{Z}^2$.

Note that this is homeomorphic to the usual definition $S^1 \times S^1$. The universal covering space of T^2 is $\mathbb{R}^2$ and it's fundamental group is $\mathbb{Z}^2$. We identify the equivalance class of the closed curve $\gamma(t)$ based at $[(0,0)]$ with $\tilde{\gamma}(1) \in \mathbb{Z}^2$, where $\tilde{\gamma}$ is the lift of γ based at $(0,0)$ in $\mathbb{R}^2$. This map, from $\pi_1(T^2) \to \mathbb{Z}^2$ is a bijection since $\mathbb{R}^2$ is simply connected (see theorem 54.4 on pg 345 in [1]).

Automorphisms on the torus correspond to elements in $GL_2(\mathbb{Z})$. Suppose that $\phi : T^2 \to T^2$ is an automorphism then it induces a map $\phi_* : \pi_1(T^2) \to \pi_1(T^2)$ which is an isomorphism. Since isomorphisms of $\mathbb{Z}^2$ are just invertible integer matrices. These are just $GL_2(\mathbb{Z})$ which is the same as matrices with determinant ± 1. On the other hand any $A \in GL_2(\mathbb{Z})$ will induce an automorphism ϕ_A on T^2 where the mapping is just $[(x,y)] \mapsto [(x,y)A^t]$. The automorphism is orientation preserving if and only if the corresponding matrix A has positive determinant, i.e. 1.

Proposition 1.2. The correspondence $\mathrm{Aut}(T^2) \to GL_2(\mathbb{Z})$ is a homomorphism. Moreover if A is in $GL_2(\mathbb{Z})$ then $(\phi_A)_* = A$; i.e. the correspondence is surjective.

Proof. Since $(\phi \circ \psi)_* [\gamma] = [\phi \circ \psi \circ \gamma] = \phi_* \circ \psi_*([\gamma])$ it follows that the map $\phi \mapsto \phi_*$ is a homomorphism. Let A be $GL_2(\mathbb{Z})$, then ϕ_A is well defined automorphism of the torus. Now ϕ_{A*} acts on $(m,n) \in \mathbb{Z}^2$ in the following way: (m,n) corresponds to the unique class $[\gamma]$ where $\tilde{\gamma}(1) = (m,n)$, so the action is given by $\phi_{A*}(m,n) = \widetilde{\phi_A \circ \gamma}(1)$. Since $\phi_A \circ \gamma(t) = \gamma(t)A^t$, the lifting of this at $(0,0)$ will be just $\tilde{\gamma}(t)A^t$ (since liftings are unique and this is a lift). Thus $\phi_{A*}(m,n) = (m,n)A^t$. Thus ϕ_{A*} just corresponds to the matrix A in $GL_2(\mathbb{Z})$. $\square$

Let A be a matrix in $SL_2(\mathbb{Z})$ and ϕ_A be the corresponding orientation preserving automorphism, then we can classify ϕ_A by looking at the properties of the matrix A. The characteristic equation of such a matrix is given by $x^2 - \tau x + 1$, where τ is the trace. We break this into three possibilities:

1. $\tau = 0, \pm 1$. In this case the characteristic equation is $x^2 + 1$, $x^2 - x + 1$, or $x^2 + x + 1$. Thus the eigenvalue are complex in this case. Using Cayley-Hamilton theorem A solves its characteristic equation. In each case we have $A^4 = I$, $A^6 = I$, or $A^3 = I$ resp.; hence $A^{12} = I$ in each case. Thus the map ϕ_A is also a finite order map. In this case ϕ_A is said to be periodic.

2. $\tau = \pm 2$. In this case the characteristic is $(x \pm 1)^2$. Both eigenvalues are either 1 or both are -1 respectively. Eigenvector of A is integeral and thus correspond to (class of) closed curves on T^2. The map preserves the (equivalance class of the) curve (reverses the direction when $\tau = -2$) represented by the eigenvector. No other curve is preserved under the map. These are powers of the Dehn Twists in C.

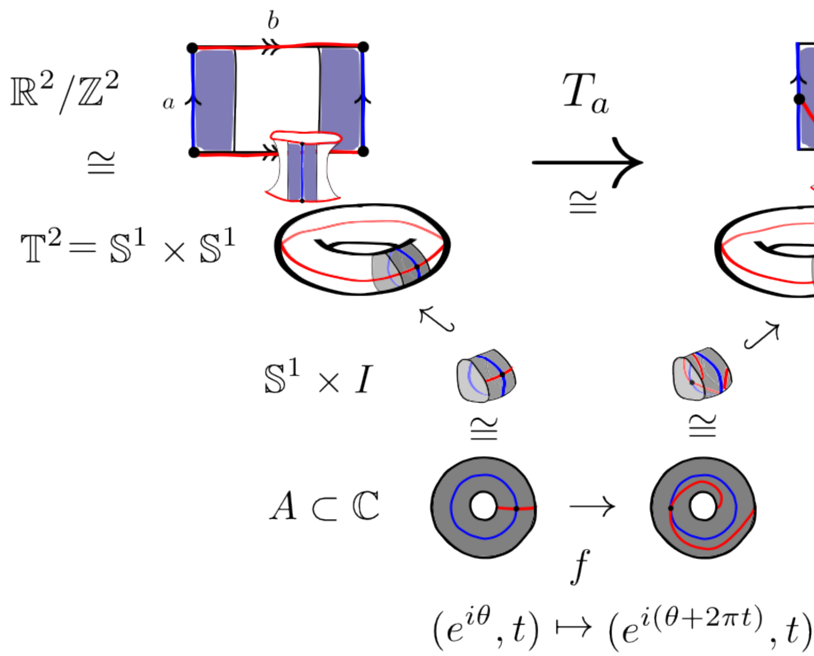

Figure 1: Dehn twist along the class of curves represented by $(1, 0)$ (blue). The red curve is in the class represented by $(0, 1)$. Source of image is wikipedia.

Definition 1.3. A Dehn Twist along a curve γ is defined in the following way: Let A be a regular neighborhood containing C such that A is homeomorphic to an annulus parametrized as (r, θ). The the extension of the homeomorphism $\phi(r, \theta) = (r, \theta + 2\pi r)$ to the whole of the torus (via characteristic function on A), is called the Dehn twist.

Matrices corresponding to Dehn twists which preserve the curves corresponding to $(1, 0)$ and $(0, 1)$ are

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \ \& \ S = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

respectively. *These matrices generate $SL_2(\mathbb{Z})$ as a group.*

3. $|\tau| > 2$. In this case the eigenvalues are distinct reals. The eigenvalues satisfy the relation $\lambda_1\lambda_2 = 1$. Let $|\lambda_1| > 1$. Thus the eigenvalues are of the form $\lambda, 1/\lambda$. Let v_1, v_2 be the corresponding eigenvectors. Think of these as elements of $T_p(\mathbb{R}^2/\mathbb{Z}^2)$ (tangent space). Consider the curves $\gamma_i = v_i t$. The action of the map ϕ_A on γ_1 expands it by a factor of λ and on γ_2 it contracts it by λ^{-1}. Such maps are called *Anosov Maps*.

2 Hyperbolic Plane

Definition 2.1. The upper half plane model of the hyperbolic plane is the set of all points in $\mathbb{C}$ with positive imaginary part, $\text{Im}(z) > 0$ with the metric given by

$$\mathrm{d}s^2 = \frac{(\mathrm{d}x^2 + \mathrm{d}y^2)}{y^2}$$

We denote this by $\mathbb{H}$.

Definition 2.2. The hyperbolic length of a piecewise differentiable curve $\gamma : [0,1] \to \mathbb{H}$ is given by

$$h(\gamma) = \int_0^1 \frac{|\dot{\gamma}(t)|}{\text{Im}(\gamma(t))} \mathrm{d}t$$

The distance between two points z_1, z_2 is given by,

$$\rho(z_1, z_2) = \inf_{\gamma} h(\gamma), \ \gamma(0) = z_1 \quad \text{and} \quad \gamma(1) = z_2$$

this is well defined since $\mathbb{C}$ is path connected.

Proposition 2.3. ρ is a metric.

Proof. Suppose that $\rho(z_1, z_2) = \ell_1$ and $\rho(z_2, z_1) = \ell_2$. Now let γ be any curve from z_2 to z_1. Then $\gamma(1-t)$ is a curve from z_1 to z_2 and since $h(\gamma) = h(\gamma(1-t)) \geq \ell_1$ it follows that ℓ_1 is also a lower bound and we must have $\ell_1 \leq \ell_2$. Repeating the same in the opposite direction we have $\ell_2 \leq \ell_1$. Hence ρ is symmetric.

Let z_1, z_2, z_3 be three points. Represent curves from $z_1 \to z_2$ by γ, $z_2 \to z_3$ by σ and $z_1 \to z_3 by \tau$. Given any γ and σ we can construct a curve from z_1 to z_3 in following way

$$\tau(t) = \gamma * \sigma(t) = \begin{cases} \gamma(2t), \ 0 \leq t \leq 1/2 \\ \sigma(2t-1), \ 1/2 \leq t \leq 1. \end{cases}$$

The length $h(\gamma * \sigma) = h(\gamma) + h(\sigma)$ (linearity and change of variables). It follows

$$\begin{aligned} &\inf_{\tau} h(\tau) \leq \inf_{\gamma,\sigma} h(\gamma * \sigma) = \inf_{\gamma} h(\gamma) + \inf_{\sigma} h(\sigma) \\ \implies &\rho(z_1, z_3) \leq \rho(z_1, z_2) + \rho(z_2, z_3). \end{aligned}$$

Hence ρ satisfies triangle inequality.

Clearly $\rho(z,z)=0$. Let z_1, z_2 be two points and γ be a curve between them. Then

$$\begin{aligned} h(\gamma) &= \int_0^1 \frac{|\dot{\gamma}|}{\mathrm{Im}(\gamma)}\mathrm{d}t \geq \left|\int_0^1 \frac{\dot{\gamma}}{\mathrm{Im}(\gamma)}\mathrm{d}t\right| \\ &> \frac{1}{M}\left|\int_0^1 \dot{\gamma}\mathrm{d}t\right| = \frac{1}{M}|z_2 - z_1| \end{aligned} \tag{1}$$

Where $M = \sup_{t\in[0,1]}(Im(\gamma(t)))$ (this is well defined since its a continuous function on a compact interval). This completes the proof. Suppose $\rho(z_1, z_2) = 0$ then $h(\gamma) < 0$ and hence $|z_1 - z_2| < 0$. Hence $z_1 = z_2$. ☐

Proposition 2.4. The set of all Mobius transforms from $\mathbb{C} \to \mathbb{C}$ of the form

$$z \mapsto \frac{az+b}{cz+d},\ a,b,c,d \in \mathbb{R}\ ad-bc=1$$

form a group, under composition. This group is isomorphic to $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{\pm I\}$.

Proof. The first part is trivial. For the second consider the map from $SL_2(\mathbb{R})$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(z \mapsto \frac{az+b}{cz+d}\right)$$

This is clearly surjective. Suppose that

$$\frac{az+b}{cz+d} = z \implies -cz^2 + (a-d)z + b = 0 \implies c=0,\ a=d,\ b=0.$$

Since $ad - bc = 1$ it follows that $a = \pm 1$. Hence ker of the map is just $\pm I$. ☐

Note that any Mobius transformation in $PSL_2(\mathbb{R})$ can be written by composing the functions $z \mapsto az$, $z \mapsto z+b$, $z \mapsto -1/z$.

Proposition 2.5. The metric topology of $\mathbb{H}^2$ is equivalent to the subspace topology induced from $\mathbb{C}^2$.

Proof. Consider the two points $z, w \in \mathbb{H}^2$. Consider the curve $\gamma(t) = (z-w)t + w$. By definition

$$\rho(z,w) \leq \int_0^1 \frac{|\dot{\gamma}|}{\mathrm{Im}(\gamma)}\mathrm{d}t = |z-w|\int_0^1 \frac{\mathrm{d}t}{(y-v)t+v} \leq |z-w|\int_0^1 \frac{\mathrm{d}t}{\min\{y,v\}} = \frac{|z-w|}{\min\{y,v\}}$$

Where $\mathrm{Im}(z) = y$ and $\mathrm{Im}(w) = v$. Hence $\rho(z,w) \leq K_w|z-w|$ for some $K_w > 0$. From (1) we already know that

$$|z-w| \leq M\rho(z,w)$$

Hence the metrics are equivalent. ☐

This is an important result since we can just check the continuity of functions under the regular metric. This helps reduce calculations.

Theorem 2.6. $PSL_2(\mathbb{R})$ acts on $\mathbb{H}$ by homeomorphisms.

Proof. Suppose $z \in \mathbb{H}^2$ and

$$w = \frac{az+b}{cz+d}$$

then

$$\begin{aligned} w - \bar{w} &= \frac{az+b}{cz+d} - \frac{a\bar{z}+b}{c\bar{z}+d} \\ &= \frac{z-\bar{z}}{|cz+d|^2} > 0 \end{aligned}$$

Thus points on $\mathbb{H}^2$ are mapped to $\mathbb{H}^2$ itself. Since mobius transformations are bijective, continuous, and the inverse again is a mobius transformation it follows that it is a homeomorphism on $\mathbb{H}^2$. $\square$

Theorem 2.7. $PSL_2(\mathbb{R})$ is isomorphic to a subgroup of the isometry group of $\mathbb{H}^2$.

Proof. Let $z, w \in \mathbb{H}^2$ and $T \in PSL_2(\mathbb{R})$ be a mobius transformation (a, b, c, d). Let γ be a curve from z to w then $\sigma(t) = T(\gamma(t))$ is a curve from Tz to Tw. Since

$$\dot{\sigma}(t) = \frac{\dot{\gamma}}{(c\gamma+d)^2},$$

and

$$\text{Im}\{\sigma(t)\} = \frac{\text{Im}\{\gamma\}}{|c\gamma(t)+d|^2}$$

it follows that

$$h(\sigma) = h(\gamma).$$

Since T is bijective there is a bijective correspondence between curves from z to w and curves from Tz to Tw. Thus they have the same infimum, meaning that $\rho(w, z) = \rho(Tw, T, z)$. $\square$

Definition 2.8. Geodesics are curves with the shortest length between any two points in a metric space.

Theorem 2.9. The geodesics in $\mathbb{H}^2$ are straight lines and semi circles perpendicular to the real axis. Moreover, between any two points in $\mathbb{H}^2$ there exists a unique geodesic.

Proof. Consider the two points $z = a_0 + ia$ and $w = a_0 + ib$ in $\mathbb{H}^2$. For any curve $\gamma(t) = x(t) + iy(t)$ between these points,

$$\begin{aligned} h(\gamma) &= \int_0^1 \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{y(t)} \mathrm{d}t \\ &\geq \int_0^1 \frac{|\dot{y}|}{y(t)} \mathrm{d}t \\ &\geq \left| \int_0^1 \frac{\dot{y}}{y} \mathrm{d}t \right| \\ &= \left| \log\left(\frac{b}{a}\right) \right| \end{aligned} \tag{2}$$

But since the curve $\gamma_0(t) = i(b-a)t + ia + a_0$ also has the length $|\log(b/a)|$ it follows that $\rho(z, w) = |\log(b/a)|$. The geodesic, γ_0, in this case is a straight line perpendicular to $\mathbb{R}$.

Now consider any two points $z_1, z_2 \in \mathbb{H}^2$. Then there is a unique circle which passes through z_1, z_2 and is perpendicular to the real line: $|z - a| = r$ where

$$a = \frac{|z_1|^2 - |z_2|^2}{2(\mathrm{Re}(z_1 - z_2))} \quad \& \quad r = |z_1 - a|.$$

There also exists a $T \in PSL_2(\mathbb{R})$ which maps the above semi-circle in $\mathbb{H}^2$ to the positive imaginary line. Explicitly this is:

$$T = \frac{1}{\sqrt{2r}} \begin{pmatrix} 1 & -a - r \\ 1 & -a + r \end{pmatrix}$$

Suppose that z lies on the semi-circle, then $z - a = re^{i\theta}$. Thus under the transformation

$$T(z) = \frac{e^{i\theta} - 1}{e^{i\theta} + 1} \in i\mathbb{R}^+$$

Hence the semi circle gets mapped to the imaginary axis, which is a geodesic. Since T is an isometry it follows that the semicircle is also a geodesic. The uniqueness follows from the fact that for any curve other than the straight line the inequality in 2 is strict, and hence only the straight line achives the minimum length. By isometry argument it generalizes to the aribtrary case. □

Definition 2.10. The set of points on the unique geodesic connecting w and z is represented by $[w, z]$.

Corollary 2.11. Let $z, w, \xi \in \mathbb{H}^2$ then,

$$\rho(z, w) = \rho(z, \xi) + \rho(\xi, w)$$

if and only if $\xi \in [z, w]$.

Proof. Suppose that $\xi \in [z, w]$. Then the geodesic γ from z to w on restriction gives a geodesic between z and ξ, because if not then there is some other σ between z, ξ such that $h(\sigma) < h(\gamma|_{[z,\xi]})$, and then the curve

$$\tilde{\gamma}(t) = \begin{cases} \sigma(2t), \; 0 \leq t \leq 1/2 \\ \gamma\big|_{[\xi,w]}(2t-1), \; 1/2 \leq t \leq 1 \end{cases}$$

has smaller length, in contradiction to the fact that γ is the geodesic. Hence

$$\rho(z,w) = h(\gamma) = h(\gamma\big|_{[z,\xi]}) + h(\gamma\big|_{[\xi,w]}) = \rho(z,\xi) + \rho(\xi,w).$$

Conversly, suppose that the equality holds. Let γ_1, γ_2 be the geodesics between z, ξ and ξ, w respectively. The concatenation as above defines a curve between z and w, and

$$\rho(z,w) = \rho(z,\xi) + \rho(\xi,w) = h(\gamma_1) + h(\gamma_2) = h(\gamma_1 * \gamma_2)$$

Hence the concatenation is a geodesic. Thus $\xi \in [z,w]$. ☐

Theorem 2.12. $PSL_2(\mathbb{R})$ maps geodesics to geodesics.

Proof. Since, as seen already in theorem 2.7, $h(T\gamma) = h(\gamma)$ for all $T \in PSL_2(\mathbb{R})$ and curves γ between z, w. Thus it follows trivially that if γ is a geodesic then so is $T\gamma$ ☐

Definition 2.13. A cross ratio, denoted $(z_1, z_2; , z_3, z_4)$ is defined as

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

Proposition 2.14. Cross ratios are preserved under Mobius transformations.

Proof. The transformation T, given by

$$T(z) = \frac{z - z_2}{z - z_4}\frac{z_3 - z_4}{z_3 - z_2},$$

maps $z_2 \mapsto 0$, $z_4 \mapsto \infty$, $z_3 \mapsto 1$. $T(z_1)$ is exactly the cross ratio. Let S be any mobius transformation then TS^{-1} is the transformation which maps $Sz_2 \mapsto 0$, $Sz_4 \mapsto \infty$, and $Sz_3 \mapsto 1$. Hence $TS^{-1}(z) = (z, Sz_2; Sz_3, Sz_4)$. Hence $(Sz_1, Sz_2; Sz_3, Sz_4) = TS^{-1}(Sz) = T(z) = (z_1, z_2; z_3, z_4)$. ☐

Theorem 2.15. Let $w, z \in \mathbb{H}^2$ and γ be the geodesic between them. Extend γ in both directions and let $w^*, z^* \in \mathbb{R} \cup \{\infty\}$ be the end points of the extended curve (semi-circle or straight line perpendicular to real axis) such that $z \in [z^*, w]$. Then

$$\rho(w,z) = \log(w, z^*; z, w^*)$$

Proof. As seen before there exists a $T \in PSL_2(\mathbb{R})$ such that T maps the extended γ to the imaginary axis. Explicitly such a T is

$$T(\xi) = i\frac{\xi - z^*}{\xi - w^*} \cdot \frac{z - w^*}{z - z^*}$$

this maps z^* to 0, w^* to ∞, and z to i. Note that the coefficient of T are indeed all real, and it can be made determinant 1 by multiplying and dividing by a real constant. And moreover,

$$T(w) = i \underbrace{\frac{w - z^*}{w - w^*} \cdot \frac{z - w^*}{z - z^*}}_{r}.$$

r must be greater than 1 since $|z - w^*| > |w - w^*|$ and $|w - z^*| > |z - z^*|$ (since we choose w to be closer to w^* and z^* is closer to z). The hyperbolic distance between $T(z)$ and $T(w)$ is $\log(r)$. Since $r = (ir, 0; i, \infty) = (T(w), T(z^*); T(z), T(w^*)) = (w, z^*; z, w^*)$. Hence the statement of the theorem follows. $\square$

Now we describe the Poincare Disk model of the Hyperbolic plane. Consider the unit disk $\mathbb{D}$, and the map $\phi : \mathbb{H}^2 \to \mathbb{D}$

$$\phi(z) = \frac{iz + 1}{z + i}$$

Clearly $|\phi(z)| = |z - i|/|z + i| < 1$ if and only if $z \in \mathbb{H}^2$. Also ϕ maps the real line to the boundary of $\mathbb{D}$. This map induces a distance ρ^* on $\mathbb{D}$ given by

$$\rho^*(w, z) = \rho(\phi^{-1}(w), \phi^{-1}(z))$$

It follows that,

$$\begin{aligned}
\rho^*(w, z) &= \inf_\gamma \int_0^1 \frac{|\frac{\mathrm{d}\phi^{-1} \circ \gamma}{\mathrm{d}t}|}{\mathrm{Im}(\phi^{-1} \circ \gamma)} \mathrm{d}t \\
&= \inf_\gamma \int_0^1 \frac{|\frac{\mathrm{d}\phi^{-1}}{\mathrm{d}z}\big|_\gamma \dot{\gamma}|}{\mathrm{Im}(\phi^{-1} \circ \gamma)} \mathrm{d}t \\
&= \inf_\gamma \int_0^1 \frac{2|\dot{\gamma}(t)|}{1 - |\gamma(t)|^2} \mathrm{d}t
\end{aligned}$$

This gives the metric on $\mathbb{D}$ to be

$$\mathrm{d}s = \frac{2|\mathrm{d}z|}{1 - |z|^2}.$$

This model of the hyperbolic plane is called the Poincare Disk. The geodesics here are circles perpendicular to $\mathbb{D}$ and diametric lines in $\mathbb{D}$.

Definition 2.16. Let the group of all 2×2 matrices in $\mathbb{R}$ with determinant ± 1 be denoted $S^*L_2(\mathbb{R})$. Let $PS^*L_2(\mathbb{R})$ be the group $S^*L_2(\mathbb{R})/\{\pm I\}$.

Proposition 2.17. Let $z, w \in \mathbb{H}^2$. Then

$$\sinh\left(\frac{1}{2}\rho(z,w)\right) = \frac{|z-w|}{2\sqrt{\operatorname{Im}(z)\operatorname{Im}(w)}}$$

Proof. Let T be in $PSL_2(\mathbb{R})$ then T leaves the LHS invariant. It is straight forward to check that the RHS is also invariant. Suppose $z = ia$ and $w = ib$ $(b > a)$ then we know that $\rho(ia, ib) = \log(b/a)$ and thus

$$\sinh\left(\frac{1}{2}\rho(ia,ib)\right) = \frac{b-a}{2\sqrt{ab}} = \frac{|ib|-|ia|}{2\sqrt{\operatorname{Im}(ia)\operatorname{Im}(ib)}}$$

Using the fact that there exists a T which maps the geodesic between arbitrary z, w to a geodesic between ia, ib; the result follows. $\square$

Theorem 2.18. The isometry group of $\mathbb{H}^2$ is isomorphic to $PS^*L_2(\mathbb{R})$.

Proof. Let ϕ be any isometry. If $\xi \in [z, w]$ then

$$\rho(\phi(z),\phi(z)) = \rho(z,w) = \rho(z,\xi) + \rho(\xi,w) = \rho(\phi(z),\phi(\xi)) + \rho(\phi(\xi),\phi(w))$$

Thus $\phi(\xi) \in [\phi(z), \phi(w)]$. This means that isometries map geodesics to geodesics. Consider the positive imaginary line I which is a geodesic. Then $\phi(I)$ is also some geodesic. There exists a $g \in PSL_2(\mathbb{R})$ such that g maps $\phi(I)$ to I. Without loss of generality we can assume $g(\phi(i)) = i$ (since $g(\phi(i)) = ai$, and dividing by a we get another element of $PSL_2(\mathbb{R})$) and that it maps $(0, i)$ and (i, ∞) onto themselves (like in the previous theorem). Suppose that $g \circ \phi(yi) = vi$ then

$$|\log(y)| = \rho(yi, i) = \rho(g \circ \phi(yi), i) = |\log(v)|$$

Either $v = y$ or $v = 1/y$, but since the intervals $(0, i)$ and (i, ∞) are fixed it follows that $v = y$. Hence $g \circ \phi$ fixes I. Let $\gamma \circ \phi(x + iy) = u + iv$ then using the previous proposition on the points $x + iy$ and it we get

$$\frac{x^2 + (y-t)^2}{2y} = \frac{u^2 + (v-t)^2}{2v}$$

dividing by t^2 and taking $t \to \infty$ we get $v = y$ and $x^2 = u^2$. Thus

$$g \circ \phi(z) = z \quad \text{or,} \quad -\bar{z}$$

In the first case ϕ is in $PSL_2(\mathbb{R})$ and in the second case it is of the form

$$\phi(z) = g^{-1}(-\bar{z}) = \frac{a\bar{z}+b}{c\bar{z}+d},\ ad - bc = -1$$

Thus ϕ can be naturally mapped to an element of $S^*L_2(\mathbb{R})$. The homomorphism part of the mapping follows easily, and the kernel of the map is $\{\pm I\}$. Thus the isometry group is isomorphic to $PS^*L_2(\mathbb{R})$. $\square$

Note that $PSL_2(\mathbb{R})$ along with the map $h : z \mapsto -\bar{z}$ generates the isomoetry group. This means that the quotient space $PS^*L_2(\mathbb{R})/PSL_2(\mathbb{R})$ is just $\{PSL_2(\mathbb{R}), h \cdot PSL_2(\mathbb{R})\}$ and thus has index 2. This means that $PSL_2(\mathbb{R})$ is normal in the isometry group.

The Riemannian metric of the Hyperbolic plane is induced by the inner product $\langle \cdot, \cdot \rangle : T_z\mathbb{H}^2 \times T_z\mathbb{H}^2 \to \mathbb{R}$ given by

$$\langle \zeta_1, \zeta_2 \rangle = \frac{1}{\operatorname{Im}(z)^2} \operatorname{Re}(\zeta_1 \bar{\zeta}_2)$$

This is an inner product on $T_z\mathbb{H}^2$ over $\mathbb{R}$. This induces a norm $\|\cdot\|$ on $T_z\mathbb{H}^2$ defined as

$$\|\zeta\| = \sqrt{\langle \zeta, \zeta \rangle} = \frac{|\zeta|}{\operatorname{Im}(z)}$$

Since all isometries of $\mathbb{H}^2$ are (real) differentiable, their pushforward gives a the map $\mathrm{d}\phi_z : T_z\mathbb{H}^2 \to T_{\phi(z)}\mathbb{H}^2$

$$\mathrm{d}\phi_z(\zeta) = \frac{\pm\zeta}{(cz+d)^2}, \text{ where, } \phi(z) = \frac{az+b}{cz+d}, \ \& \ ad - bc = \pm 1.$$

The pushforward is norm preserving since

$$\|\mathrm{d}\phi_z(\zeta)\| = \frac{|\zeta|}{|cz+d|^2 \operatorname{Im}(\phi(z))} = \frac{|\zeta|}{\operatorname{Im}(z)} = \|\zeta\|$$

Using the polarization identity,

$$\langle \zeta_1, \zeta_2 \rangle = \frac{1}{2}(\|\zeta_1\| + \|\zeta_2\| - \|\zeta_1 - \zeta_2\|)$$

we can conclude that the pushforward of isometries preserve the absolute value of angles between vectors.

Definition 2.19. Angle between geodesics in $\mathbb{H}^2$ is defined as the angle between the tangent vectors at the point of intersection.

Definition 2.20. A map on $\mathbb{H}^2$ is said to be confromal if it preserves angles, and anti-conformal if it preserves absolute value of the angle but reverses direction.

Theorem 2.21. Transformations in $PSL_2(\mathbb{R})$ are conformal and the other isometries are anti-conformal.

Proof. We saw already that the pushforward preserves the absolute value of angles. But since the pushforward at each point is of the form

$$\mathrm{d}\phi_z(\zeta) = \frac{\pm\zeta}{(cz+d)^2}, \text{ where, } \phi(z) = \frac{az+b}{cz+d}, \ \& \ ad - bc = \pm 1.$$

it follows that $PSL_2(\mathbb{R})$ preserves direction while $z \mapsto -\bar{z}$ reverses orientation. $\square$

Definition 2.22. Hyberbolic area of a subset A of $\mathbb{H}^2$ is defined as

$$\mu(A) = \int_A \frac{\mathrm{d}x\mathrm{d}y}{y^2}$$

Theorem 2.23. If $A \subset \mathbb{H}^2$ and $\mu(A)$ exists then the hyperbolic area is invariant under transformations of $PSL_2(\mathbb{R})$.

Proof. Suppose that $z = x + iy$ and $Tz = u + iv$. Then using the Cauchy-Riemann equations the determinant of the Jacobian $\partial(u,v)/\partial(x,y)$ is given by

$$\begin{aligned} \left|\frac{\partial(u,v)}{\partial(x,y)}\right| &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \\ &= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \\ &= \left|\frac{\mathrm{d}T}{\mathrm{d}z}\right|^2 \\ &= \frac{1}{|cz+d|^4} \end{aligned}$$

Thus by change of variables

$$\mu(T(A)) = \int_{T(A)} \frac{\mathrm{d}u\mathrm{d}v}{v^2} = \int_A \frac{|cz+d|^4}{|cz+d|^4 y^2} \mathrm{d}x\mathrm{d}y = \mu(A)$$

Thus the area is invariant under $PSL_2(\mathbb{R})$. $\square$

Definition 2.24. An $n-$sided polygon in $\mathbb{H}^2 \cup \mathbb{R} \cup \{\infty\}$ is defined by the area enclosed by n distinct geodesics. The vertices can lie on the boundary.

Theorem 2.25 (Gauss-Bonnet Theorem)**.** A hyperbolic triangle Δ with angles α, β, γ has area $\mu(\Delta) = \pi - \alpha - \beta - \gamma$.

Proof. Case 1. Consider a triangle with one point on $\mathbb{R} \cup \{\infty\}$ then there exists a transformation in $PSL_2(\mathbb{R})$ which takes the vertex on $\mathbb{R} \cup \{\infty\}$ to ∞. Thus w.l.o.g. we consider a triangle with two sides being lines perpendicular to the imaginary axis. Again w.l.o.g. (by a $PSL_2(\mathbb{R})$ transformation) we can translate and scale the triangle such that the center of the semi-circle (the third side) is at 0 with radius 1. All these transformations preserve the area and the angles. The area of this triangle can be calculated

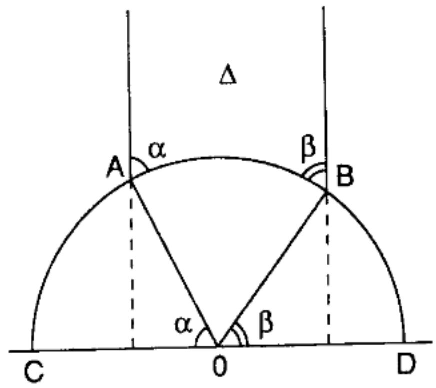

Figure 2: Case 1.

$$
\begin{aligned}
\mu(\Delta) &= \int_{\Delta} \frac{\mathrm{d}x\mathrm{d}y}{y^2} \\
&= \int_a^b \mathrm{d}x \int_{\sqrt{1-x^2}}^{\infty} \frac{\mathrm{d}y}{y^2} \\
&= \int_a^b \frac{\mathrm{d}x}{\sqrt{1-x^2}} \\
&= \int_{\pi-\alpha}^{\beta} \frac{-\sin\theta \mathrm{d}\theta}{\sin\theta} \\
&= \pi - \alpha - \beta.
\end{aligned}
$$

Case 2. Suppose that none of the vertices are on $\mathbb{R} \cup \{\infty\}$. There exists a transforamtion such that no two vertices lies on a vertical geodesic. Extend the side AB, of the triangle ABC, to a point $D \in \mathbb{R}$. Let $\Delta_1 = ACD$ and $\Delta_2 = CBD$. Then

$$\mu(\Delta) = \mu(\Delta_1) - \mu(\Delta_2) = \pi - \alpha - (\gamma + \theta) - \pi + \theta + (\pi - \beta) = \pi - \alpha - \beta - \gamma$$

This proves the theorem. □

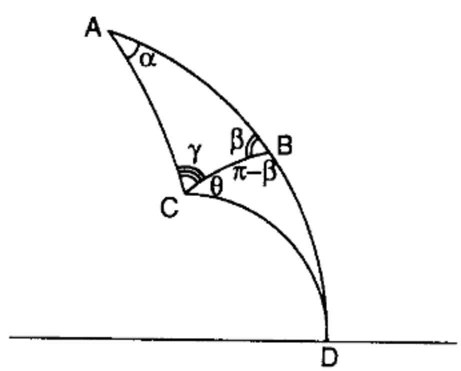

Figure 3: Case 2.

Corollary 2.26. The area of an $n-$gon with angles $\alpha_1 \cdots, \alpha_n$ is $(n-2)\pi - \alpha_1 - \cdots - \alpha_n$.

Proof. This follows from induction. It is true for a trianlge. Suppose it is true for an $n-1$-gon. An $n-$gon can be divided into a triangle and an $n-1$-gon by drawing an appropriate geodesic curve. Adding the areas of the two we get the result. ☐

Theorem 2.27 (Brouwer's Fixed Point Theorem)**.** Let $f: \mathbb{D} \to \mathbb{D}$ be a continous bijection on the disk. There are is atleast one fixed point.

Amazing proof using Functors. Suppose f fixes no point. Define the function $r: \mathbb{D} \to S^1$ in the following way: draw extend the line from $f(x)$ to x to the boundary of $\mathbb{D}$ where it intersects with the circle at $r(x)$. Note that r fixes each point on S^1. We have the short exact sequence

$$0 \longrightarrow S^1 \xrightarrow{i} \mathbb{D} \xrightarrow{r} S^1 \longrightarrow 0$$

Since there is a functor $\pi_1 : \mathrm{Top}_* \to \mathrm{Grp}$ which maps a based topological space (X, x_0) to it's fundametal group at x_0. Applying this functor to the above exact sequence at any point, gives the short exact sequence

$$0 \longrightarrow \pi_1(S^1) \xrightarrow{\pi_1(i)} \pi_1(\mathbb{D}) \xrightarrow{\pi_1(r)} \pi_1(S^1) \longrightarrow 0$$

Since $\pi_1(S^1) = \mathbb{Z}$ and $\pi_1(\mathbb{D}) = 0$ we get that $\mathrm{id}_{S^1} = \pi_1(r \circ i) = \pi_1(r) \circ \pi_1(i) = 0$, a contradiction. ☐

Brouwer's theorem tells us that isometries of $\mathbb{H}^2$ must fix atleast one point, since $\mathbb{H}^2$ is isometric to the disk. The following is a classification of

Theorem 2.28. Let T be an orientation preserving isometry of $\mathbb{H}^2$. Then one of the following happens:

1. T fixes only one point in $\mathbb{H}^2$.
2. T fixes only one point on the boundary of $\mathbb{H}^2$.
3. T fixes two points on the boundary of $\mathbb{H}^2$.

Proof. Since $T \in PSL_2(\mathbb{R})$, z is a fixed point if

$$z = \frac{az+b}{cz+d} \implies cz^2 + (d-a)z - b = 0$$

Thus

$$z = \frac{(a-d) \pm \sqrt{(a+d)^2 - 4}}{2c}$$

Thus case 1 corresponds to $a+d < 2$, 2 corresponds to $a+d = 2$, and 3 corresponds to $a+d > 2$. ☐

3 Hyperbolic Structures on Surfaces

For the sake of consistency with the texts, I use the Poincare model of hyperbolic plane here.

Definition 3.1. Let X be a surface. A hyperbolic structure on X is an atlas of charts such that each chart $\phi : U \to \phi(U) \subset \mathbb{H}^2$ is a homeomorphism and the transition maps in the atlas are restrictions of orientation preserving isometries of $\mathbb{H}^2$.

Let $p \in X$ and (U, ϕ) be a chart around p. We can define an inner product on T_pX as

$$\langle u, v \rangle_{T_pX} = \langle \phi_* u, \phi_* v \rangle$$

This is well defined since if (V, ψ) is another chart containing p then

$$\langle \psi_* u, \psi_* v \rangle = \langle \psi_* \circ \phi_*^{-1} \circ \phi_* u, \psi_* \circ \phi_* \circ \phi_* v \rangle = \langle \phi_* u, \phi_* v \rangle.$$

Since $\langle \cdot, \cdot \rangle_{T_pX}$ is a smooth bilinear map it serves as a Riemann metric on X. From here on I drop the subscript T_pX on the metric. The norm induced by the metric will just be written $\| \cdot \|$. From here on we can think of Hyperbolic surface to be a topological surface with a hyperbolic Atlas and a Riemannian metric which is isometric to the hyperbolic metric locally. The metric induces a distance on X which is given by the infimum of the length of the curves.

Proposition 3.2. Let $\gamma : [0, 1] \to U \subset X$ be a curve in X between x_0 and x_1. If γ is a geodesic then there exist charts (U_α, ϕ_α) such that they cover γ and $\phi_\alpha \circ \gamma\big|_{U_\alpha}$ is a segment of a geodesic in $\mathbb{H}^2$.

Proof. Suppose that γ is a geodesic. Let x be a point on γ. There exists a neighborhood U of this point which is isometric to $\mathbb{H}^2$ by the ϕ. Let σ be some curve between the end-points of the curve $\phi \circ \gamma$ (which always exist since $\mathbb{H}^2$ is complete). Then,

$$L(\sigma) = L(\phi^{-1} \circ \sigma) \geq L(\gamma) = L(\phi \circ \gamma)$$

since γ is a geodesic. Thus it follows that $\phi \circ \gamma$ is a geodesic. □

Theorem 3.3 (Half of Hopf-Rinow theorem)**.** In a complete hyperbolic surface, all geodesics can be extended indefnitely.

Proof. Suppose that $\gamma : (-\epsilon, \epsilon) \to X$ is a bounded geodesic in X. Then consider a sequence of points $\gamma(t_n)$ where $t_n \to \epsilon$. Since X is complete the Cauchy sequence $\gamma(t_n)$ converges to a unique point, say x_1. Let U, ϕ be some chart centered around x_1. The image $\phi \circ \gamma$ is a geodesic by previous proposition. Extend this in $\mathbb{H}^2$ indefnitely. Then the pull back of this extension extends γ at x_1 till a new end point x_2. Repeating this process one can indefnitely extend γ. □

Theorem 3.4. Any complete, connected, simply-connected hyperbolic surface is isometric to $\mathbb{H}^2$.

Proof. Suppose X is a space with the mentioned properties. Consider the maps $E : \mathbb{H}^2 \to X, D : X \to \mathbb{H}^2$ defined as follows:

- *The exponential.* Choose a point $a \in X$ and a chart (U, ϕ) such that $\phi(a) = 0$. For $x \in \mathbb{H}^2$ let γ be the geodesic between 0 and x and then extend the geodesic $\phi^{-1}(\gamma)$. Define $E(x)$ as the point on the extended geodesic such that $\text{dist}(a, E(x)) = \rho(0, x)$.

- *The developing.* Fix a point $a \in X$ and a chart (U, ϕ) around it. There exists a map $D : X \to \mathbb{H}^2$ such that D is a local isometry and $D\big|_U = \phi$. This claim is proven below:

  proof of existence of D. Choose a path γ between points $a, b \in X$. The path can be cover by finitely many convex coordinate charts (due to compactness), say (U_i, ϕ_i) with $(U_0, \phi_0) = (U, \phi)$. Refine the covering such that it is minimal (so that U_i only intersects with $U_{i\pm1}$ and no U_i is contained in U_j). Choose points $x_0 = a, \cdots, x_i, \cdots, x_n = b$ on γ such that $[x_i, x_{i+1}] \subset U_i$. If the maps ϕ_i and ϕ_{i+1} do not agree on $U_i \cap U_{i+1}$, which contains x_{i+1},then there exists an isometry g (unique extension of $\phi_{i+1} \circ \phi_i^{-1}$) such that $g \circ \phi_i = \phi_{i+1}$ on their intersection. Thus without loss of generality we can assume that all the charts agree on the intersection (by replacing ϕ_{i+1} with $g \circ \phi_1$). Now define $D(b) = \phi_n(b)$.

  (Well defined-ness). Clearly D is not dependent on the choice x_i. Now suppose $(U'_i, \phi'_i)_{i=0}^m$ is a different set of charts which minimally cover γ with $U_0, \phi_0 = U, \phi$ and such that the coordinate charts agree on the intersection. We show by induction that whenever $U_i \cap U'_j \neq \emptyset$ then $\phi'_j = \phi_i$ in the intersection. By construction $U_0 = U'_0$ and $\phi'_0 = \phi_0$. Since $U'_0 \cap U_1 = U_0 \cap U_1$ it follows that in this intersection $\phi'_0 = \phi_0 = \phi_1$. This is the base case of the induction. Suppose now that for all $s < j$ if $U'_s \cap U_i \neq \emptyset$ then $\phi'_s = \phi_i$ in the intersection for all i. Consider U'_j and suppose that it intersects with some U_i. There are two cases:

  1. $U'_{j-1} \cap U_i \neq \emptyset$. In this case consider the intersection $U_i \cap U'_j \cap U'_{j-1}$. In this region $\phi'_j = \phi'_{j-1} = \phi_i$. Restricted to $\phi'_j(U_i \cap U'_j)$ the map $g = \phi_i \circ (\phi'_j)^{-1}$ is in $PSL_2(\mathbb{R})$. Since on $\phi'_j(U_i \cap U'_j \cap U_i)$ the map g is identity it follows that g is identity everywhere in $\phi'_j(U_i \cap U'_j)$ (since Mobius maps are fixed by 3 points).
  2. $U'_{j-1} \cap U_i = \emptyset$. Then U'_{j-1} intersects U_{i-1} (by construction). In the region $U_{i-1} \cap U_j \cap U_{j-1}$ we have $\phi'_j = \phi'_{j-1} = \phi_{i-1}$ and thus $\phi' \circ \phi_{i-1}^{-1}$ is identity on infinitely many points. Thus they are the same on $U'_j \cap U_{i-1}$. In the region $U'_j \cap U_i \cap U_{i-1}$ we have $\phi'_j = \phi_{i-1} = \phi_i$. Using the same argument as before we have that $\phi_i = \phi'_j$ everywhere on $U_i \cap U'_j$.

  Hence the induction step is complete. Since $b \in U'_m \cap U_n$ it follows that $\phi'_m(b) = \phi_n(b)$. Hence D is not dependent on the covering of γ. Now we need to show that D does not depend on γ. If γ' is some other curve. Since X is simply connected it follows that there is a Homotopy H between γ and γ'. Using continuity of H there exists an

ϵ so that the curves $H(s,t)$ and $H(s,t+\epsilon)$ can be covered by the same charts. Hence $\phi_n(b)$ is the same for both. Thus it follows that D is well defined. $\square$

Now that we have these two functions, note that $D \circ E = 1_{\mathbb{H}^2}$: let $x \in \mathbb{H}^2$ then $E(x)$ lies on a geodesic γ from a to $E(x)$ such that $\phi \circ \gamma$ is part of the geodesic connecting 0 and x. Let U_i be any minial cover of the geodesic from a to $E(x)$. Then $\phi_n(E(x))$ lies on the extension of the geodesic $\phi \circ \gamma$ and $\rho(0, D(E(x))) = \rho(0, x)$ since D and E are local isometries, but there is only one such point on the geodesic: x. Hence $D \circ E(x) = x$.

Note that on the image of E in X the map $E \circ D$ is identity. $E(\mathbb{H}^2)$ is closed and open (since E is local injection it follows by Invariance of domain theorem). Since X is connected the only non-trivial clopen subset is X itself. Thus $E(\mathbb{H}^2) = X$. Hence $E \circ D = 1_X$. $\square$

As a consequence of the above theorem it follows that the universal cover of any closed hyperbolic surface X is isometric to $\mathbb{H}^2$. Thus for any hyperbolic surface X, we can write it as $\mathbb{H}^2/\Gamma$ where Γ is the fundamental group of the space X (since deck transformations is isomorphic to fundamental group for universal covers). Note that since Γ acts on $\mathbb{H}^2$ by automorphisms, it is a subgroup of $PSL_2(\mathbb{R})$. Now suppose that $x \in X$, then the pre-image of x under the covering map is a discrete set, which is equivalent to saying that that the orbit of each point is discrete which in turn is equivalent to saying that Γ is discrete. This means Γ is a Fuchsian group which acts properly discontinuously on $\mathbb{H}^2$ (deck transformations act properly discontinuously). As a result Γ has no elliptic elements.

If $\mathbb{H}^2/\Gamma$ is compact then the Dirichlet domain D of Γ in $\mathbb{H}^2$ is compact, let ϵ be the maximum distance between the points in the dirichlet domain. Then $\rho(z, \gamma z) \geq \epsilon$. But for parabolic elements, which are essentially translations, say $z \mapsto z+1$, the distance between $z, z+1$ is smaller than ϵ for large enough $\text{Im}(z)$. Thus Γ can only have hyperbolic elements.

Proposition 3.5. The torus is not a hyperbolic surface.

Proof. Suppose it is, then it has $\mathbb{H}^2$ as it's universal cover. But that would mean that the fundamental group $\mathbb{Z} \oplus \mathbb{Z}$ is a Fuchsian group. But since all abelian Fuchsian groups are cyclic it follows that $Z \oplus \mathbb{Z}$ is not cyclic. $\square$

Definition 3.6. A closed curve is essential if it is not null homotopic.

Proposition 3.7. Every essential closed curve in a compact hyperbolic surface F is freely homotopic to a unique closed geodesic.

Proof. Let C be a closed curve in $F = \mathbb{H}^2/\Gamma$ and let $\tilde{C}$ be it's unique lift at fixed point x in $\mathbb{H}^2$. Then the other end point of the lift is some gx, for some $g \in \Gamma$ since it's a loop when pushed down. Let $\tilde{\gamma}$ be the unique geodesic, which is the fixed axis of the hyperboic element g. $\square$

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