Automated tight Lyapunov analysis for first-order methods

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Adrien Taylor

Pontus Giselsson

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This talk

- Based on:
  - U.M., Banert, S., Taylor, A.B., Giselsson, P. *Automated tight Lyapunov analysis for first-order methods*. Mathematical Programming (2024)
    - Code: `https://github.com/ManuUpadhyaya/TightLyapunovAnalysis`
  - Banert, S., U.M., Giselsson, P. *The Chambolle–Pock method converges weakly with* $\theta > 1/2$ *and* $\tau\sigma\|L\|^2 < 4/(1+2\theta)$ [arXiv:2309.03998] (2023)
- Content:
  - Methodology for proving algorithm convergence
  - Focus on first-order (splitting) methods for convex optimization problems

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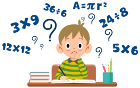
Proving convergence

- Pages of inequalities



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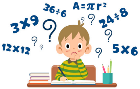

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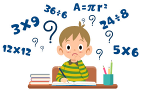

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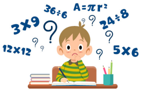

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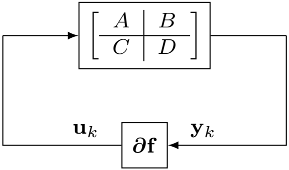
- Our approach



and $\qquad V_{k+1} \leq \rho V_k - R_k$

One example of what we can show with our methodology

- Problem:

$$\underset{y \in \mathcal{H}}{\text{minimize}} \quad f(y) + g(y)$$

  where $f, g \in \mathcal{F}_{0,\infty}$, i.e. lower semicontinuous, proper and convex.

- Method (Chambolle and Pock, 2011, Algorithm 1):

$$x_{k+1} = \text{prox}_{\tau f}(x_k - \tau y_k),$$
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  where $\tau, \sigma > 0$, $\theta \in \mathbb{R}$, prox is the proximal operator and g^* is the convex conjugate of g (linear operator set to identity mapping)

- Parameter choices that give ($\mathcal{O}(1/k)$ ergodic primal-dual gap) convergence:

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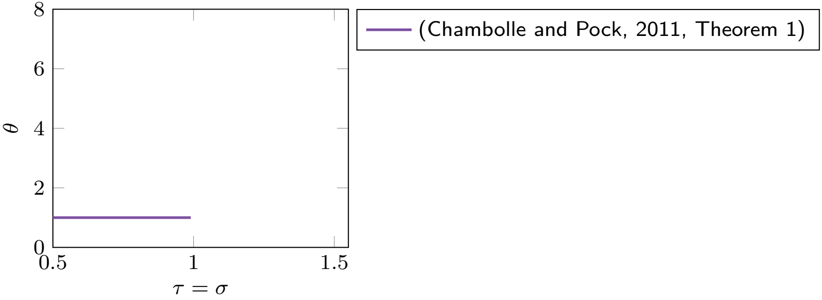

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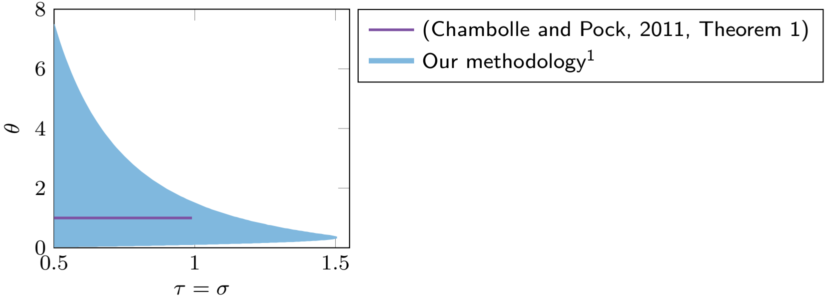

[1]Parameters evaluated on a square grid of size 0.01×0.01 with the restriction that $\tau = \sigma \geq 0.5$

An extended convergence analysis of the Chambolle–Pock method

- The numerical result above was the inspiration for the analytical result in (Banert et al., 2023)
- Problem:

$$\underset{y \in \mathcal{H}}{\text{minimize}} \quad f(y) + g(Ly)$$

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- The problem has the equivalent primal–dual formulation

$$\underset{x\in\mathcal{H}}{\text{minimize}}\ \underset{y\in\mathcal{G}}{\text{maximize}}\ \ f(x) + \langle Lx, y\rangle - g^*(y)$$

- We assume the problem has at least one solution $(x_\star, y_\star) \in \mathcal{H} \times \mathcal{G}$ that satisfies the Karush–Kuhn–Tucker (KKT) condition

$$\begin{aligned} -L^* y_\star &\in \partial f(x_\star), \\ Lx_\star &\in \partial g^*(y_\star) \end{aligned}$$

- We define the *primal–dual gap function* (our convergence measure)

$$\mathcal{D}_{x_\star,y_\star}(x, y) = \mathcal{L}(x, y_\star) - \mathcal{L}(x_\star, y) \geq 0$$

where

$$\mathcal{L}(x, y) = f(x) + \langle Lx, y\rangle - g^*(y)$$

is the *Lagrangian function*

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$$x_{k+1} = \mathrm{prox}_{\tau f}(x_k - \tau L^* y_k),$$
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$$\theta \geq 1/2 \quad \text{and} \quad \tau\sigma\|L\|^2 \leq 4/(1+2\theta)$$

where at least one is strict. Then

$$\mathcal{D}_{x_\star, y_\star}\left(\frac{1}{K}\sum_{k=1}^{K} x_k, \frac{1}{K}\sum_{k=1}^{K} y_k\right) = \mathcal{O}\left(\frac{1}{K}\right)$$

- **Result 2 [Weak convergence]:** Suppose that

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Then

$$(x_k, y_k) \rightharpoonup (x_\star, y_\star)$$

- **Result 3 [Tightness]:** Suppose that

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Then there exists an example such that the Chambolle–Pock method does not converge

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Comparison to results in the literature

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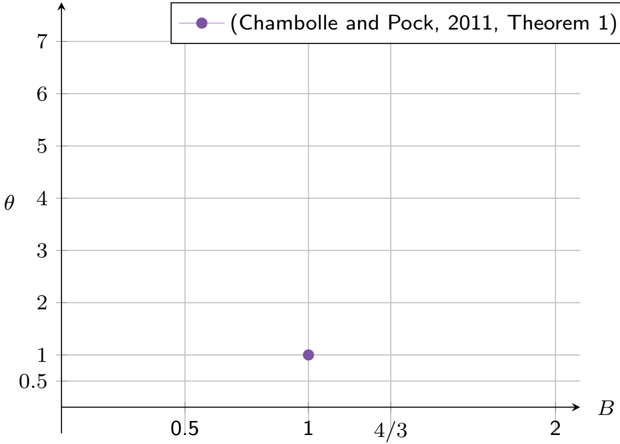
Convergence if $\tau\sigma \|L\|^2 < B$ where[2]



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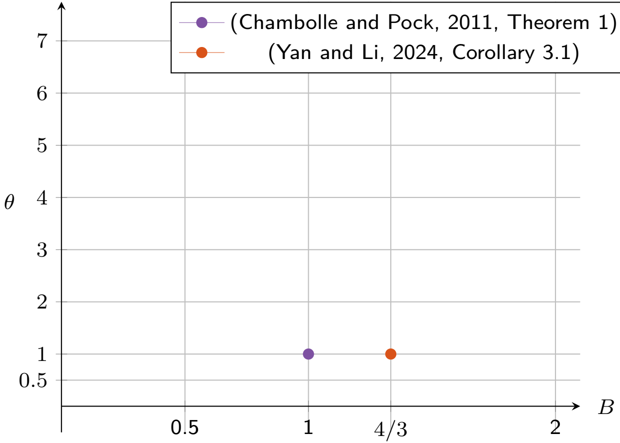

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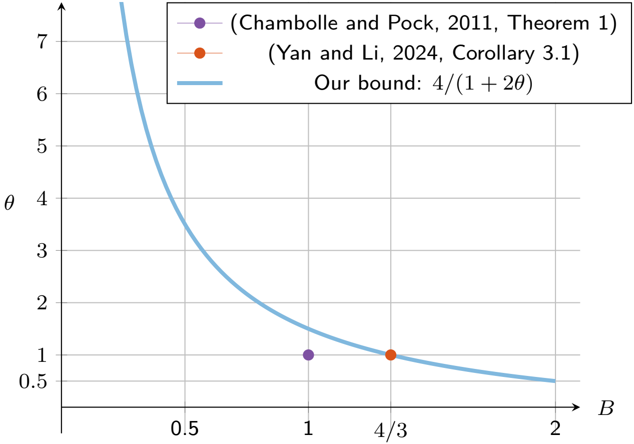

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Outline

1. Problem class
2. Algorithm representation
3. Lyapunov inequalities
4. Main result - A necessary and sufficient condition
5. Numerical results
6. Outlook

Problem class - Preliminaries

- $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ real Hilbert space. Associated norm $\|\cdot\|^2 = \langle\cdot,\cdot\rangle$
- Let $0 \leq \sigma < +\infty$ and $0 \leq \beta \leq +\infty$.

  $\mathcal{F}_{\sigma,\beta}$ class of all functions $f : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ that are proper, lower semicontinuous, σ-strongly convex and β-smooth (if $\beta < +\infty$)

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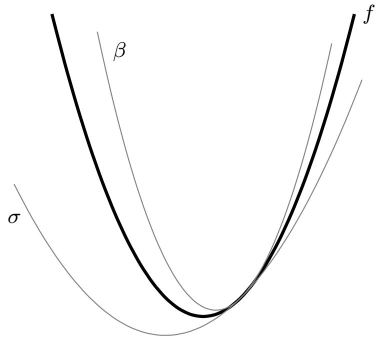

Problem class

- Convex optimization problem

$$\underset{y \in \mathcal{H}}{\text{minimize}} \quad \sum_{i=1}^{m} f_i(y)$$

where $f_i \in \mathcal{F}_{\sigma_i,\beta_i}$ and $0 \leq \sigma_i < \beta_i \leq +\infty$, for each $i \in [\![1, m]\!]$ [3]

- Associated inclusion problem

$$\text{find } y \in \mathcal{H} \ \text{ such that } \ 0 \in \sum_{i=1}^{m} \partial f_i(y)$$

where ∂f_i are subdifferential operators

- Problem class $\mathcal{F}_{\boldsymbol{\sigma},\boldsymbol{\beta}}$ is all $(f_1, \ldots, f_m) \in \prod_{i=1}^{m} \mathcal{F}_{\sigma_i,\beta_i}$ such that inclusion is solvable

[3] $[\![a, b]\!] = \{j \in \mathbb{Z} \mid a \leq j \leq b\}$

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1 Problem class

2 **Algorithm representation**

3 Lyapunov inequalities

4 Main result - A necessary and sufficient condition

5 Numerical results

6 Outlook

Algorithm representation

Algorithms on state-space form[4] [5]:

$$\begin{aligned}\mathbf{x}_{k+1} &= (A \otimes \mathrm{Id})\mathbf{x}_k + (B \otimes \mathrm{Id})\mathbf{u}_k \\ \mathbf{y}_k &= (C \otimes \mathrm{Id})\mathbf{x}_k + (D \otimes \mathrm{Id})\mathbf{u}_k \\ \mathbf{u}_k &\in \boldsymbol{\partial}\mathbf{f}(\mathbf{y}_k) \\ \mathbf{F}_k &= \mathbf{f}(\mathbf{y}_k)\end{aligned}$$

where

$$A \in \mathbb{R}^{n \times n} \qquad B \in \mathbb{R}^{n \times m} \qquad C \in \mathbb{R}^{m \times n} \qquad D \in \mathbb{R}^{m \times m}$$

$$\mathbf{x}_k = \left(x_k^{(1)}, \ldots, x_k^{(n)}\right) \in \mathcal{H}^n \quad \mathbf{y}_k = \left(y_k^{(1)}, \ldots, y_k^{(m)}\right) \in \mathcal{H}^m \quad \mathbf{u}_k = \left(u_k^{(1)}, \ldots, u_k^{(m)}\right) \in \mathcal{H}^m$$

and

$$\begin{aligned}&\mathbf{f} : \mathcal{H}^m \to (\mathbb{R} \cup \{+\infty\})^m : \left(y^{(1)}, \ldots, y^{(m)}\right) \mapsto \left(f_1\left(y^{(1)}\right), \ldots, f_m\left(y^{(m)}\right)\right) \\ &\boldsymbol{\partial}\mathbf{f} : \mathcal{H}^m \rightrightarrows \mathcal{H}^m : \left(y^{(1)}, \ldots, y^{(m)}\right) \mapsto \prod_{i=1}^{m} \partial f_i\left(y^{(i)}\right).\end{aligned}$$

[4] Model used in control literature, (Lessard et al., 2016), and similar to the model in (Morin et al., 2024).

[5] Let $M \in \mathbb{R}^{m \times n}$ and $\mathbf{z} = (z^{(1)}, \ldots, z^{(n)}) \in \mathcal{H}^n$. Then $(M \otimes \mathrm{Id})\mathbf{z} = \left(\sum_{j=1}^{n} [M]_{1,j} z^{(j)}, \ldots, \sum_{j=1}^{n} [M]_{m,j} z^{(j)}\right)$.

Algorithm representation

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$$\begin{aligned}\mathbf{x}_{k+1} &= ({\color{red}A} \otimes \mathrm{Id})\mathbf{x}_k + ({\color{red}B} \otimes \mathrm{Id})\mathbf{u}_k \\ \mathbf{y}_k &= ({\color{red}C} \otimes \mathrm{Id})\mathbf{x}_k + ({\color{red}D} \otimes \mathrm{Id})\mathbf{u}_k \\ \mathbf{u}_k &\in \boldsymbol{\partial}\mathbf{f}(\mathbf{y}_k) \\ \mathbf{F}_k &= \mathbf{f}(\mathbf{y}_k)\end{aligned}$$

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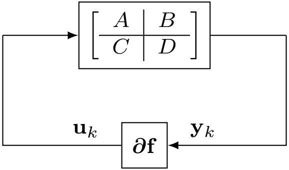

Examples[6]

- gradient method
- proximal point method
- proximal gradient method
- Nesterov accelerated gradient method
- gradient method with heavy-ball momentum
- triple momentum method
- FISTA
- Davis–Yin three-operator splitting method
- Chambolle–Pock method
- etc.

[6] constant parameter versions

Algorithm representation - Chambolle–Pock method

- The problem:

$$\underset{y \in \mathcal{H}}{\text{minimize}} \quad f_1(y) + f_2(y)$$

- Method (Chambolle and Pock, 2011, Algorithm 1):

$$\begin{aligned} x_{k+1} &= \operatorname{prox}_{\tau_1 f_1}(x_k - \tau_1 y_k), \\ y_{k+1} &= \operatorname{prox}_{\tau_2 f_2^*}(y_k + \tau_2 \left(x_{k+1} + \theta(x_{k+1} - x_k)\right)) \end{aligned}$$

  where $\tau_1, \tau_2 > 0$, $\theta \in \mathbb{R}$ (linear operator set to identity mapping)

- On state-space form:

$$\begin{aligned} \mathbf{x}_{k+1} &= \left(\begin{bmatrix} 1 & -\tau_1 \\ 0 & 0 \end{bmatrix} \otimes \mathrm{Id} \right) \mathbf{x}_k + \left(\begin{bmatrix} -\tau_1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \mathrm{Id} \right) \mathbf{u}_k \\ \mathbf{y}_k &= \left(\begin{bmatrix} 1 & -\tau_1 \\ 1 & \frac{1}{\tau_2} - \tau_1(1+\theta) \end{bmatrix} \otimes \mathrm{Id} \right) \mathbf{x}_k + \left(\begin{bmatrix} -\tau_1 & 0 \\ -\tau_1(1+\theta) & -\frac{1}{\tau_2} \end{bmatrix} \otimes \mathrm{Id} \right) \mathbf{u}_k \\ \mathbf{u}_k &\in \boldsymbol{\partial} \mathbf{f}(\mathbf{y}_k) \end{aligned}$$

Algorithm representation - Fixed points

- Algorithm *fixed points* $\boldsymbol{\xi}_\star = (\mathbf{x}_\star, \mathbf{u}_\star, \mathbf{y}_\star, \mathbf{F}_\star)$ satisfy

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Algorithm representation - Fixed-point encoding property

- We are only interested in algorithms such that

$$\text{"finding a fixed point} \quad \Longleftrightarrow \quad \text{solving inclusion problem"}$$

- More specifically[7]:
  - from each solution, it should be possible to construct a fixed point
  - from each fixed point, it should be possible to extract a solution
- Such algorithms have the *fixed-point encoding property* (FPEP)

[7] For the precise way to construct fixed points and extract solutions, see (Upadhyaya et al., 2024). This has been omitted from the presentation for clarity and simplicity

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- Let

$$N = \begin{bmatrix} I \\ -\mathbf{1}^\top \end{bmatrix} \in \mathbb{R}^{m \times (m-1)}$$

where $\mathbf{1}$ denotes the column vector of all ones of comfortable size

- Result:

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$$\iff$$

The matrices (A, B, C, D) satisfy

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- Recall: $f_i \in \mathcal{F}_{\sigma_i,\beta_i}$ for each $i \in [\![1, m]\!]$ and

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- **Well-posedness**: Can we find at least one $\mathbf{x}_{k+1}$ for each $\mathbf{x}_k$?
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D *lower triangular with nonpositive diagonal and*

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- Under the sufficient condition above, the algorithm can be implemented as

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\text{for } k = 0, 1, \ldots \\
\quad \text{for } i = 1, \ldots, m \\
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Outline

1 Problem class

2 Algorithm representation

3 **Lyapunov inequalities**

4 Main result - A necessary and sufficient condition

5 Numerical results

6 Outlook

Lyapunov inequalities

- Let $\boldsymbol{\xi}_k = (\mathbf{x}_k, \mathbf{u}_k, \mathbf{y}_k, \mathbf{F}_k)$ and $\boldsymbol{\xi}_\star = (\mathbf{x}_\star, \mathbf{u}_\star, \mathbf{y}_\star, \mathbf{F}_\star)$
- Many first-order methods analyzed using *Lyapunov inequalities*

$$V(\boldsymbol{\xi}_{k+1}, \boldsymbol{\xi}_\star) \leq \rho V(\boldsymbol{\xi}_k, \boldsymbol{\xi}_\star) - R(\boldsymbol{\xi}_k, \boldsymbol{\xi}_\star)$$

  where $\rho \in [0, 1]$,
  - $V : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ is a *Lyapunov function*
  - $R : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ is a *residual function*

  and $\mathcal{S} = \mathcal{H}^n \times \mathcal{H}^m \times \mathcal{H}^m \times \mathbb{R}^m$
- Traditional way to find Lyapunov inequalities:
  - Use inequalities for the function classes involved (e.g. $\mathcal{F}_{\sigma_i, \beta_i}$)
  - Combine with algorithm updates
  - Manipulate to arrive at a Lyapunov inequality
- We want to automatically find such Lyapunov inequalities!

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- Many first-order methods analyzed using *Lyapunov inequalities*

$$V(\boldsymbol{\xi}_{k+1}, \boldsymbol{\xi}_\star) \leq \rho V(\boldsymbol{\xi}_k, \boldsymbol{\xi}_\star) - R(\boldsymbol{\xi}_k, \boldsymbol{\xi}_\star)$$

  where $\rho \in [0, 1]$,
    - $V : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ is a *Lyapunov function*
    - $R : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ is a *residual function*

  and $\mathcal{S} = \mathcal{H}^n \times \mathcal{H}^m \times \mathcal{H}^m \times \mathbb{R}^m$
- Traditional way to find Lyapunov inequalities:
    - Use inequalities for the function classes involved (e.g. $\mathcal{F}_{\sigma_i, \beta_i}$)
    - Combine with algorithm updates
    - Manipulate to arrive at a Lyapunov inequality
- We want to automatically find such Lyapunov inequalities!

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- We want to automatically find such Lyapunov inequalities!

Lyapunov inequalities - Quadratic ansatzes

- We consider quadratic ansatzes of V and R:

$$V(\boldsymbol{\xi}, \boldsymbol{\xi}_\star) = \langle (\mathbf{x} - \mathbf{x}_\star, \mathbf{u}, \mathbf{u}_\star), (Q \otimes \mathrm{Id})(\mathbf{x} - \mathbf{x}_\star, \mathbf{u}, \mathbf{u}_\star) \rangle + q^\top (\mathbf{F} - \mathbf{F}_\star)$$
$$R(\boldsymbol{\xi}, \boldsymbol{\xi}_\star) = \langle (\mathbf{x} - \mathbf{x}_\star, \mathbf{u}, \mathbf{u}_\star), (S \otimes \mathrm{Id})(\mathbf{x} - \mathbf{x}_\star, \mathbf{u}, \mathbf{u}_\star) \rangle + s^\top (\mathbf{F} - \mathbf{F}_\star)$$

  where $Q, S \in \mathbb{S}^{n+2m}$, $q, s \in \mathbb{R}^m$ parameterize the functions[8]
- Our methodology searches for/provides (Q, q, S, s) that gives a valid Lyapunov inequality

[8] Inner-product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}^d$ is given by

$$\langle \mathbf{z}_1, \mathbf{z}_2 \rangle = \sum_{i=1}^{d} \left\langle z_1^{(i)}, z_2^{(i)} \right\rangle$$

for each $\mathbf{z}_i = (z_i^{(1)}, \ldots, z_i^{(d)}) \in \mathcal{H}^d$ and $i \in [\![1, 2]\!]$

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Lyapunov inequalities - Lower bounds

- However, we do not know (Q, q, S, s) that parameterize V and R in advance $\implies$ can not control convergence conclusions
- Solution: enforce nonnegative quadratic lower bounds on V and R

$$V(\boldsymbol{\xi}_k, \boldsymbol{\xi}_\star) \geq \langle (\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star), (P \otimes \mathrm{Id})(\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star) \rangle + p^\top (\mathbf{F}_k - \mathbf{F}_\star) \geq 0$$
$$R(\boldsymbol{\xi}_k, \boldsymbol{\xi}_\star) \geq \langle (\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star), (T \otimes \mathrm{Id})(\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star) \rangle + t^\top (\mathbf{F}_k - \mathbf{F}_\star) \geq 0$$

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Lyapunov inequalities - Lower bounds - Convergence conclusions - Linear case

- Recall:
  C1 $V(\boldsymbol{\xi}_{k+1}, \boldsymbol{\xi}_\star) \leq \rho V(\boldsymbol{\xi}_k, \boldsymbol{\xi}_\star) - R(\boldsymbol{\xi}_k, \boldsymbol{\xi}_\star)$
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  $$0 \leq \langle (\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star), (P \otimes \mathrm{Id})(\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star) \rangle + p^\top (\mathbf{F}_k - \mathbf{F}_\star) \leq \rho^k V(\boldsymbol{\xi}_0, \boldsymbol{\xi}_\star) \to 0$$

  i.e., lower bound

  $$\left\{ \langle (\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star), (P \otimes \mathrm{Id})(\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star) \rangle + p^\top (\mathbf{F}_k - \mathbf{F}_\star) \right\}_{k \in \mathbb{N}_0}$$

  converges ρ-linearly to 0

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Lyapunov inequalities - Lower bounds - Convergence conclusions - Sublinear case

- Recall:

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  C3 $R(\boldsymbol{\xi}_k, \boldsymbol{\xi}_\star) \geq \langle (\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star), (T \otimes \mathrm{Id})(\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star) \rangle + t^\top (\mathbf{F}_k - \mathbf{F}_\star) \geq 0$

- For $\rho = 1$, a telescoping summation gives

$$0 \leq \sum_{k=0}^{\infty} \Big(\langle (\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star), (T \otimes \mathrm{Id})(\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star) \rangle + t^\top (\mathbf{F}_k - \mathbf{F}_\star) \Big) \leq V(\boldsymbol{\xi}_0, \boldsymbol{\xi}_\star)$$

  i.e., lower bound

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  is summable (and converges to zero)

- If the lower bound is convex in its arguments, Jensen's inequality gives that

$$\langle (\overline{\mathbf{x}}_k - \mathbf{x}_\star, \overline{\mathbf{u}}_k, \mathbf{u}_\star), (T \otimes \mathrm{Id})(\overline{\mathbf{x}}_k - \mathbf{x}_\star, \overline{\mathbf{u}}_k, \mathbf{u}_\star) \rangle + t^\top \Big(\overline{\mathbf{F}}_k - \mathbf{F}_\star \Big) = \mathcal{O}\left(\frac{1}{K} \right)$$

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Lyapunov inequalities - Lower bounds - Convergence conclusions - Sublinear case

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  - **C2** $V(\boldsymbol{\xi}_k, \boldsymbol{\xi}_\star) \geq \langle (\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star), (P \otimes \mathrm{Id})(\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star)\rangle + p^\top (\mathbf{F}_k - \mathbf{F}_\star) \geq 0$
  - **C3** $R(\boldsymbol{\xi}_k, \boldsymbol{\xi}_\star) \geq \langle (\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star), (T \otimes \mathrm{Id})(\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star)\rangle + t^\top (\mathbf{F}_k - \mathbf{F}_\star) \geq 0$
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Lyapunov inequalities - Full definition

- (P, p, T, t, ρ)-*quadratic Lyapunov inequality* for algorithm and $\boldsymbol{\mathcal{F}_{\sigma,\beta}}$:
  C1 $V(\boldsymbol{\xi}_{k+1}, \boldsymbol{\xi}_\star) \leq \rho V(\boldsymbol{\xi}_k, \boldsymbol{\xi}_\star) - R(\boldsymbol{\xi}_k, \boldsymbol{\xi}_\star)$
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- Technical difficulty: Given (A, B, C, D), we only want this to hold for algorithm-consistent points ξ_k, fixed points $\xi_\star$, and $\mathrm{f} \in \mathcal{F}_{\boldsymbol{\sigma},\boldsymbol{\beta}}$

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Outline

1. Problem class
2. Algorithm representation
3. Lyapunov inequalities
4. **Main result - A necessary and sufficient condition**
5. Numerical results
6. Outlook

Main result - Necessary and sufficient condition

There exists a (P, p, T, t, ρ)-quadratic Lyapunov inequality

if and only if[9]

a particular SDP involving (Q, q, S, s) is feasible

[9] Assuming dimension independence and Slater condition

Main result - Necessary and sufficient condition

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a particular SDP involving (Q, q, S, s) is feasible

$$\text{C1}\begin{cases}\lambda^{\text{C1}}_{(l,i,j)} \geq 0 \text{ for each } l \in [\![1, m]\!] \text{ and distinct } i, j \in \{\varnothing, +, \star\},\\ \Sigma_\varnothing^\top(\rho Q - S)\Sigma_\varnothing - \Sigma_+^\top Q \Sigma_+ + \sum_{l=1}^{m} \sum_{\substack{i,j\in\{\varnothing,+,\star\}\\ i\neq j}} \lambda^{\text{C1}}_{(l,i,j)} \mathbf{M}_{(l,i,j)} \succeq 0,\\ \begin{bmatrix}\rho q - s\\ -q\end{bmatrix} + \sum_{l=1}^{m} \sum_{\substack{i,j\in\{\varnothing,+,\star\}\\ i\neq j}} \lambda^{\text{C1}}_{(l,i,j)} \mathbf{a}_{(l,i,j)} = 0,\end{cases}$$

$$\text{C2}\begin{cases}\lambda^{\text{C2}}_{(l,i,j)} \geq 0 \text{ for each } l \in [\![1, m]\!] \text{ and distinct } i, j \in \{\varnothing, \star\},\\ \Sigma_\varnothing^\top(Q - P)\Sigma_\varnothing + \sum_{l=1}^{m} \sum_{\substack{i,j\in\{\varnothing,\star\}\\ i\neq j}} \lambda^{\text{C2}}_{(l,i,j)} \mathbf{M}_{(l,i,j)} \succeq 0,\\ \begin{bmatrix}q - p\\ 0\end{bmatrix} + \sum_{l=1}^{m} \sum_{\substack{i,j\in\{\varnothing,\star\}\\ i\neq j}} \lambda^{\text{C2}}_{(l,i,j)} \mathbf{a}_{(l,i,j)} = 0,\end{cases}$$

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[9] Assuming dimension independence and Slater condition

Main result - How did we find this condition?

- Let us look at **C1**:[10] $V(\boldsymbol{\xi}_+, \boldsymbol{\xi}_\star) \leq \rho V(\boldsymbol{\xi}, \boldsymbol{\xi}_\star) - R(\boldsymbol{\xi}, \boldsymbol{\xi}_\star)$
- **C1** equivalent to that optimal value of

$$\begin{array}{ll} \text{maximize} & V(\xi_+, \xi_\star) - \rho V(\xi, \xi_\star) + R(\xi, \xi_\star) \\[1em] \text{subject to} & \xi \text{ is algorithm consistent for } \mathbf{f}, \\ & \xi_+ \text{ is a successor of } \xi \text{ for } \mathbf{f}, \\ & \xi_\star \text{ is a fixed point for } \mathbf{f}, \\ & \mathbf{f} \in \mathcal{F}_{\boldsymbol{\sigma}, \boldsymbol{\beta}}, \end{array} \qquad \text{(PEP)}$$

  is nonpositive!
- Arrived at the condition using:
  - Convex interpolation conditions (Taylor et al., 2017*b*)
  - Performance estimation problem (PEP) reformulations (Drori and Teboulle, 2014)

[10] We use the same trick for **C2** and **C3**

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Numerical results - Douglas–Rachford method

- The problem:

$$\underset{y\in\mathcal{H}}{\text{minimize}} \quad f_1(y) + f_2(y)$$

where $f_1 \in \mathcal{F}_{1,2}$ and $f_2 \in \mathcal{F}_{0,\infty}$

- Douglas–Rachford method:

$$\begin{aligned} y_k^{(1)} &= \text{prox}_{\gamma f_1}(x_k) \\ y_k^{(2)} &= \text{prox}_{\gamma f_2}\left(2y_k^{(1)} - x_k\right) \\ x_{k+1} &= x_k + \lambda\left(y_k^{(2)} - y_k^{(1)}\right) \end{aligned}$$

where $\gamma \in \mathbb{R}_{++}$ and $\lambda \in \mathbb{R} \setminus \{0\}$ ($\lambda = 1$ in the plot below)

- $(P, p, T, t, \rho) \implies$ squared distance to the solution convergence ρ-linearly to zero[11]

[11] Smallest ρ via bisection search

Numerical results - Douglas–Rachford method

- The problem:

$$\underset{y \in \mathcal{H}}{\text{minimize}} \quad f_1(y) + f_2(y)$$

  where $f_1 \in \mathcal{F}_{1,2}$ and $f_2 \in \mathcal{F}_{0,\infty}$
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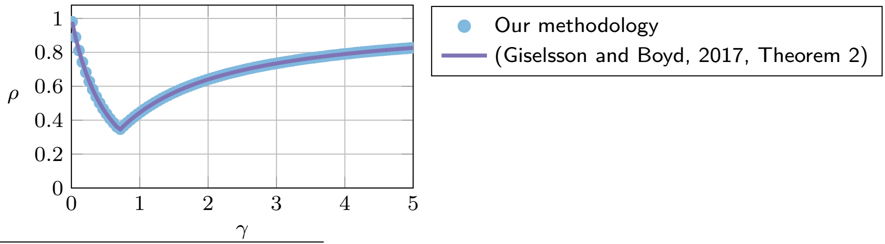

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- The problem:

$$\underset{y \in \mathcal{H}}{\text{minimize}} \quad f_1(y)$$

  where $f_1 \in \mathcal{F}_{0,1}$
- Gradient method with heavy-ball momentum:

$$x_{k+1} = x_k - \gamma \nabla f_1(x_k) + \delta(x_k - x_{k-1})$$

- $(P, p, T, t, \rho) \implies \lim_{k\to\infty}(f_1(x_k) - f_1(x_\star)) = 0$ and $f_1\left(\frac{1}{K}\sum_{k=1}^{K} x_k\right) - f_1(x_\star) = \mathcal{O}\left(\frac{1}{K}\right)$

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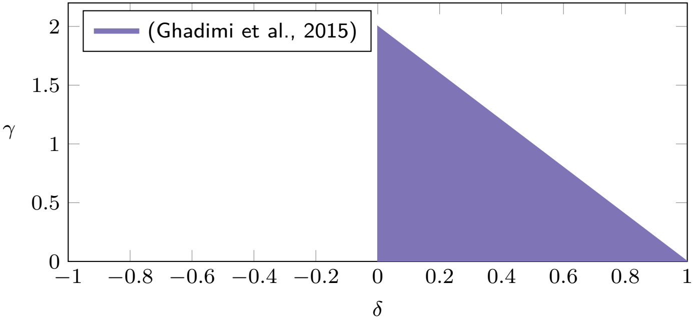

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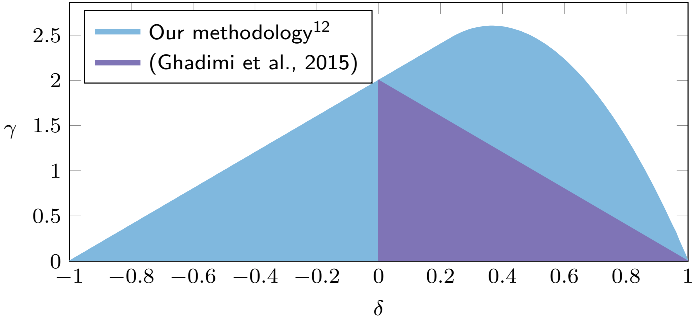

[9] Parameters evaluated on a square grid of size 0.01×0.01

Outline

1. Problem class
2. Algorithm representation
3. Lyapunov inequalities
4. Main result - A necessary and sufficient condition
5. Numerical results
6. **Outlook**

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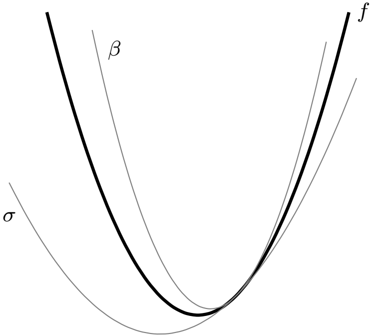
Appendix - Preliminaries

- $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ real Hilbert space. Associated norm $\|\cdot\|^2 = \langle \cdot, \cdot \rangle$
- Let $f : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$. Then:
  (i) *effective domain* of f is the set $\operatorname{dom} f = \{x \in \mathcal{H} \mid f(x) < +\infty\}$
  (ii) f *proper* if $\operatorname{dom} f \neq \emptyset$
  (iii) *subdifferential* of a proper function f is the set-valued operator $\partial f : \mathcal{H} \to 2^{\mathcal{H}}$ such that $x \mapsto \{u \in \mathcal{H} \mid \forall y \in \mathcal{H},\ f(y) \geq f(x) + \langle u, y - x \rangle\}$
- Let $f \colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ and $\sigma, \beta \in \mathbb{R}_+$. The function f is:
  (i) *convex* if $f((1-\lambda)x + \lambda y) \leq (1-\lambda) f(x) + \lambda f(y)$ for each $x, y \in \mathcal{H}$ and $0 \leq \lambda \leq 1$
  (ii) σ*-strongly convex* if f is proper and $f - (\sigma/2) \|\cdot\|^2$ is convex
  (iii) β*-smooth* if f is differentiable and $\|\nabla f(x) - \nabla f(y)\| \leq \beta \|x - y\|$ for each $x, y \in \mathcal{H}$

Appendix - More preliminaries

- Let $0 \leq \sigma < +\infty$ and $0 \leq \beta \leq +\infty$. $\mathcal{F}_{\sigma,\beta}$ class of all functions $f : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ that are proper, lower semicontinuous, σ-strongly convex and β-smooth (if $\beta < +\infty$)



- Let $f \in \mathcal{F}_{0,\infty}$ and $\gamma > 0$. Then the *proximal operator* $\mathrm{prox}_{\gamma f} : \mathcal{H} \to \mathcal{H}$ is defined as the single-valued operator given by

$$\mathrm{prox}_{\gamma f}(x) = \underset{z \in \mathcal{H}}{\mathrm{argmin}} \left(f(z) + \frac{1}{2\gamma} \|x - z\|^2 \right)$$

  for each $x \in \mathcal{H}$

- The *convex conjugate* of f, denoted $f^* : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$, is the proper, lower semicontinuous and convex function given by $f^*(u) = \sup_{x \in \mathcal{H}} (\langle u, x \rangle - f(x))$ for each $u \in \mathcal{H}$

Appendix - Some choices of (P, p, T, t, ρ)

Suppose $\rho \in [0, 1[$, let e_i be ith standard basis vector and

$$(P, p, T, t) = \left(\begin{bmatrix} C & D & -D \end{bmatrix}^\top e_i e_i^\top \begin{bmatrix} C & D & -D \end{bmatrix}, 0, 0, 0 \right).$$

Then

$$\langle (\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star), (P \otimes \mathrm{Id})(\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star) \rangle + p^\top (\mathbf{F}_k - \mathbf{F}_\star) = \left\| y_k^{(i)} - y_\star \right\|^2 \geq 0$$

and the distance to the solution squared converges ρ-linear to zero.

Appendix - Some choices of (P, p, T, t, ρ)

Suppose $\rho = 1$, $m = 1$ and let

$$(P, p, T, t) = (0, 0, 0, 1).$$

Then

$$\langle (\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star), (T \otimes \mathrm{Id})(\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star) \rangle + t^\top (\mathbf{F}_k - \mathbf{F}_\star) = f_1\left(y_k^{(1)}\right) - f_1(y_\star) \geq 0$$

which gives

- function value suboptimality converges to zero
- $\mathcal{O}(1/k)$ ergodic function value suboptimality convergence (via Jensen's inequality)

Appendix - Some choices of (P, p, T, t, ρ)

Suppose $\rho = 1$ and let

$$(P, p, T, t) = \left(0, 0, \begin{bmatrix} C & D & -D \\ 0 & 0 & I \end{bmatrix}^\top \begin{bmatrix} 0 & -\frac{1}{2}I \\ -\frac{1}{2}I & 0 \end{bmatrix} \begin{bmatrix} C & D & -D \\ 0 & 0 & I \end{bmatrix}, \mathbf{1}\right).$$

Then

$$\begin{aligned} &\langle (\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star), (T \otimes \mathrm{Id})(\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star) \rangle + t^\top (\mathbf{F}_k - \mathbf{F}_\star) \\ &= \sum_{i=1}^{m} \left(f_i\left(y_k^{(i)}\right) - f_i\left(y_\star^{(i)}\right) - \left\langle u_\star^{(i)}, y_k^{(i)} - y_\star^{(i)} \right\rangle \right) \\ &= \mathcal{L}(\mathbf{y}_k, \mathbf{u}_\star) - \mathcal{L}(\mathbf{y}_\star, \mathbf{u}_k) \geq 0 \end{aligned}$$

where $\mathcal{L} : \mathcal{H}^m \times \mathcal{H}^m \to \mathbb{R}$ is a *Lagrangian function* giving

- primal-dual gap converges to zero,
- $\mathcal{O}(1/k)$ ergodic primal-dual gap convergence (via Jensen's inequality).

Reduces to function value suboptimality when $m = 1$.

Appendix - Chambolle–Pock - Lyapunov inequality

- Lyapunov function:

$$
\begin{aligned}
V_k = \theta(f(x_{k+1}) - f(x_\star) + \langle L^* y_\star, x_{k+1} - x_\star \rangle) + \frac{1}{2\tau} \|x_{k+1} - x_\star\|^2 \\
+ \frac{1}{2\sigma} \|y_k - y_\star + \sigma\theta(Lx_{k+1} - Lx_k)\|^2 + \frac{\theta}{2\tau} \|x_{k+1} - x_k\|^2 \\
- \frac{\sigma\left(4\theta^2 + 1\right)}{16} \|Lx_{k+1} - Lx_k\|^2 \geq 0
\end{aligned}
$$

- Lyapunov inequality:

$$
\begin{aligned}
& V_{k+1} - V_k + \mathcal{D}_{x_\star, y_\star}(x_{k+1}, y_{k+1}) \leq \\
& - \frac{1}{2\sigma} \left\| y_{k+1} - y_k - \sigma\left(\frac{1}{2}(Lx_{k+1} - Lx_{k+2}) - \theta(Lx_k - Lx_{k+1}) \right) \right\|^2 \\
& - \frac{8\theta - \tau\sigma \|L\|^2 \left(4\theta^2 + 1\right)}{16\tau} \left\| x_{k+2} - x_{k+1} - \frac{4\left(1 - \tau_1 \sigma\theta \|L\|^2\right)}{8\theta - \tau\sigma \|L\|^2 (4\theta^2 + 1)} (x_{k+1} - x_k) \right\|^2 \\
& - \frac{\left(4\theta^2 - 1\right)\left(4 - \tau\sigma \|L\|^2 (2\theta + 1)\right)\left(4 - \tau\sigma \|L\|^2 (2\theta - 1)\right)}{16\tau\left(8\theta - \tau\sigma \|L\|^2 (4\theta^2 + 1)\right)} \|x_{k+1} - x_k\|^2 \leq 0
\end{aligned}
$$

Appendix - Chambolle–Pock - Counterexample

- For the problem instance ($\mathcal{H} = \mathcal{G} = \mathbb{R}$, $f = g^* = 0$, and $L = 1$)

$$\underset{x\in\mathbb{R}}{\text{minimize}}\,\underset{y\in\mathbb{R}}{\text{maximize}}\ xy$$

  the Chambolle–Pock method becomes

$$\begin{aligned} x_{k+1} &= x_k - \tau y_k, \\ y_{k+1} &= y_k + \sigma(x_{k+1} + \theta(x_{k+1} - x_k)), \end{aligned}$$

  or equivalently

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & -\tau \\ \sigma & 1 - \tau\sigma(1+\theta) \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix},$$

  where we assume that $\tau, \sigma, \theta > 0$
- If

$$\tau\sigma \geq 4/(1+2\theta),$$

  the matrix above has an eigenvalue with a magnitude greater or equal to 1
- The Chambolle–Pock method no longer converges for all initial points