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Distributed Node Coloring in the SINR Model

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***Abstract*—Given a palette P of at most $\mathcal{V}$ colors, and a parameter d, a $(d, \mathcal{V})$-coloring of a graph is an assignment of a color from the palette P to every node in the graph such that any two nodes at distance at most d have different colors. We prove that for every n-node unit disk graph with maximum degree Δ, there exists a distributed algorithm computing a $(1, O(\Delta))$-coloring under the SINR (Signal-to-Interference-plus-Noise Ratio) physical model in at most $O(\Delta \log n)$ time slots, which is optimal up to a logarithmic factor. Our result is based on revisiting a previous coloring algorithm, due to T. Moscibroda and R. Wattenhofer, described in the so called graph-based model [1]. We also prove that, for a well defined constant d, a $(d, O(\Delta))$-coloring allows us to schedule an interference free TDMA-like MAC protocol under the physical SINR constraints. As a corollary, any uniform interference-free message passing algorithm with running time τ can be simulated in the SINR model in $O(\Delta(\log n + \tau))$ time slots. The latter generic result provides new insights into the distributed scheduling of radio network tasks under the harsh SINR constraints.**

I. Introduction

Context and motivation: Finding distributed scheduling schemes allowing nodes to communicate efficiently while avoiding message loss due to simultaneous transmissions is an important research topic in multi-hop radio networks, such as sensor and wireless ad-hoc networks. In fact, by essence wireless radio communications are subject to interference, thus without using efficient interference avoidance mechanisms one cannot hope to solve a distributed task efficiently. Coping with wireless interferences turns out to be more challenging when the nodes of the network act in a fully distributed environment, that is when no global knowledge of the network topology is available. In such an environment, the nodes themselves must coordinate their actions locally and decide which local strategy to use in order to optimize the network global performances. In other words, designing efficient distributed algorithms allowing nodes to self-organize their communications plays a fundamental role when dealing with interferences and message collisions in fully distributed radio networks.

There exist many distributed solutions aiming at minimizing interferences in several wireless settings. For instance, an access protocol in a MAC layer (Medium Access Control) allows nodes to schedule their transmissions within the same physical radio spectrum in a distributed way while avoiding collisions. Setting up common access protocols for wireless networks such as TDMA (Time Division Multiple Access) is tightly related to constructing coloring-like network structures. For instance, assigning different colors to wireless nodes that could interfere with each others and associating colors with different time slots can be viewed as a scheduled MAC protocol without direct interferences. Generally speaking, computing node colorings with their various variants are of great practical importance. On the theoretical side, coloring problems have attracted a lot of attention from the distributed computing research community, where it is considered as one the most fundamental symmetry breaking problems. Although the impressive amount of existing work enables a good understanding of the locality of computing network colorings, the existing algorithms still suffer from some weaknesses. May be the most relevant open questions come from the abstractions made when defining the distributed model. In fact, assuming simplified models of radio communication does enable a good understanding of the algorithmic issues but the obtained solutions still need to be adapted and improved to fit more sophisticated network constraints.

For the problem of wireless network coloring, many past studies considered rather simplified models of interferences. The best time-efficient coloring algorithms by Wattenhofer et al [2], [1] assume the so-called simple graph based model of interferences. In that model, interferences are modeled as a localized function where a node can hear a message iff exactly one neighbor is transmitting. In other words, only a neighbor transmitting at the same time slot can prevent the correct transmission of a message. In practice however, interferences are caused not only by neighboring nodes but also by other nodes being farther in the network. A more realistic model capturing these issues is the Signal-to-Interference-plus-Noise-Ratio (SINR) model, also known as the physical model [3], [4]. In the SINR physical model, a node experiences an interference if the ratio of the received signal strength and the sum of the interference caused by all nodes sending simultaneously in the network, plus ambient

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noise is less then a hardware-defined threshold β, where the signal fades with the distance to the power of some path-loss exponent α.

Contributions: In this paper, we build on the coloring algorithm of T. Moscibroda and R. Wattenhofer [1] (called the MW algorithm throughout the rest of the paper), and we show that computing a coloring under the harsh SINR constraints is still feasible in an essentially optimal way. More specifically, by carefully tuning the parameters of the MW algorithm, we analytically prove that with high probability a $(1, O(\Delta))$-coloring of any n-node unit disk graph (UDG) of maximum degree Δ can be computed in $O(\Delta \log n)$ time under the SINR physical constraints[1]. One should note that the number of colors produced by the algorithm is optimal up to a constant factor, and the time complexity is optimal up to a $\log n$ factor. To the extent of our knowledge, this is the first time where an almost optimal coloring algorithm is given in the SINR model.

As stated previously, one important application of coloring-based structures is to design MAC protocols. However, it is well known that a $(1, O(\Delta))$-coloring is not sufficient to design interference free protocols even under the simple graph based model. In this paper we show that under the SINR model a TDMA-like protocol without direct interference can be scheduled with $O(\Delta)$ time frames using a $(d, O(\Delta))$-coloring for a well chosen constant d. As a direct application, we obtain upper bounds on the simulation of any point-to-point message passing algorithm in the SINR model which enables a better understanding of the complexity of scheduling distributed tasks in a wireless environment [5]. In fact, the latter generic result tightens the gap between our theoretical understanding of the wireless nature of some distributed models and the well known point-to-point message passing model.

Related work: There is an impressive amount of work concerning network coloring with several variants and applications. In the following, we only focus on the locality results concerning *node* coloring and recall the best *time efficient* algorithms. For arbitrary graphs, the best deterministic (resp. randomized) algorithm runs in $2^{O(\sqrt{\log n})}$ time [6] (resp. $O(\log n)$ [7], [8]) and produces a $(1, \Delta+1)$-coloring. For unit disk graphs, the same coloring can be computed in $O(\log^* n)$ time [9] matching Linial's well-known lower bound [8] (The reader is referred to [10], [11], [12], [13], [14] for more related results).

The above mentioned algorithms were all designed for classical message passing models not taking into account interferences and message collisions. Moscibroda and Wattenhofer [15] came up with a nice algorithm for properly coloring the node of a unit disk graphs using $O(\Delta)$ colors in $O(\Delta \log n)$ time. That algorithm was then generalized in [1] to run on bounded independence graphs, that is graphs such that the size of the maximum independent set in the 2-hop neighborhood of any node is bounded by a constant, e.g., this includes unit ball graphs with constant doubling dimension. As stated previously, the MW algorithm assumes the simple graph based model of interferences. The algorithm does not need any collision detection mechanism and it has the nice property to work on an asynchronous wake-up model. Very recently, Schneider and Wattenhofer [2] improved the number of colors used by the MW algorithm by a constant factor while reducing the running time by a log factor. Specifically, the number of colors is $\Delta+1$ and the running time is $O(\Delta + \log \Delta \log n)$. Generally speaking, computing a network coloring is tightly related to many other structures (e.g., independent/dominated sets) dealing with the distributed initialization of a wireless network (See for instance [1], [2] for an overview).

Due to the inaccuracy of localized interference models in realistic settings, the focus of the algorithmic community has recently shifted to more realistic models such as the SINR physical model. In particular, one can find some recent results on computing coloring-related structures dealing with the SINR constraints. In particular, a lot of work has been done on scheduling a set of transmitter/receiver requests under the SINR constraints, e.g., [16], [17], [18], [19]. Distributively computing an initial network structure based on constant density dominated sets was investigated analytically in [20]. The basic distributed local broadcasting problem is studied in [21] where almost optimal algorithms are given. Distributed edge coloring was investigated recently in [22] where the authors described lower/upper bounds on the number of colors needed to guarantee connectivity for some particular topologies, e.g., 1 and 2 dimensional grids. Wang et al addressed in [23] the problem of distributively (and sequentially) computing an edge coloring allowing to design a TDMA scheme under a variant of the so-called protocol model[2]. Nevertheless, as pointed in their conclusion the authors left open the challenging question of finding *time-efficient* distributed algorithms with optimal guarantees. In [24], it is shown that for uniformly distributed nodes, protocols designed for the UDG collision free model can be emulated in the SINR model with a polylogarithmic multiplicative overhead. Only few other results concerning the impact of the SINR constraints in comparison to classical graph-based models exist in the literature [25], [26], [27], [28].

II. Model and Definitions

In this paper, we assume that nodes are placed arbitrarily in the one dimensional Euclidean space, i.e., the plane. Given two nodes u and v, we denote by $\delta(u, v)$ the euclidian distance between nodes u and v. We assume time to be divided

[1]See Section II for a more formal definition of a $(d, \mathcal{V})$-coloring in a unit disk graph and more details about the distributed model.

[2]Roughly speaking, in this interference model a node can only interfere with other nodes within a fixed distance.

into discrete time-slots that are synchronized between all nodes. We assume that nodes may wake up asynchronously at any time. Furthermore, we assume that nodes wake up spontaneously, i.e., sleeping nodes are not necessarily woken up by incoming messages.

Under the SINR constraints [3], [4], a node u successfully receives a message from a sender v if we have:

$$\frac{\frac{P}{\delta(u,v)^\alpha}}{N \ + \ \sum_{w\in V\setminus\{v\}} \frac{P}{\delta(u,w)^\alpha}} \quad \geq \quad \beta$$

where P is the power level of the transmission of nodes[3], $\alpha > 2$ is the so-called path-loss exponent, which depends on external conditions of the medium, $\beta \geq 1$ denotes the minimum signal to interference ratio required for a message to be successfully received, N is the ambient noise, and $\sum_{w\in V\setminus\{v\}} P_w/\delta(u,w)^\alpha$ is the total amount of interference experienced by receiver u and caused by all simultaneously transmitting nodes in the network.

In absence of any other simultaneous transmission, the SINR condition tells us that for a node u to hear a node v it must verify $\delta(u,v) \leq R_{\max} = (P/N\beta)^{1/\alpha}$. In this paper, we additionally assume that to hear a message the receiver u must be at distance at most $R_T < R_{\max}$ from the sender v. Equivalently, this assumption means that the signal power at the receiver is required to be enough strong compared to noise in order to be able to decode the message. In this paper, R_T is called the transmission range and it is defined by[4]: $R_T = (P/(2N\beta))^{1/\alpha}$. Having these assumptions, the network can now be clearly modeled by a 'unit' disc graph $G = (V, E, R_T)$ where there exists an edge $(u,v) \in E$ between two nodes u and v if they are at distance at most R_T, i.e., in absence of simultaneous transmissions node u can hear node v at distance $\delta(u,v) \leq R_T$.

Given a positive constant $d \geq 1$ and an integer $\mathcal{V}$, a $(d,\mathcal{V})$-coloring of G is a coloring of the nodes of G using at most $\mathcal{V}$ colors such that if u and v are two nodes verifying $\delta(u,v) \leq d\ R_T$, then u and v have different colors. An independent set in $G = (V, E, R_T)$ is a set of nodes $\mathcal{I} \subseteq V$ such that for every two nodes u and v in $\mathcal{I}$, $\delta(u,v) > R_T$. We denote by $\phi(R)$ the size of the largest independent set in any disc of radius $R > 0$ around any node of the graph[5].

For every node v, the transmission region B_v of node v is the disc of radius R_T around v. We also denote (the interference disk) I_v the disc of radius R_I around v where R_I is defined by: $R_I = 2R_T\left(96\ \rho\ \beta \cdot \frac{\alpha-1}{\alpha-2}\right)^{1/(\alpha-2)}$. Note that $R_I \geq 2R_T$ for a well chosen constant $\rho > 1$. For any constant $c \geq 5$, we define the following constants:

$$\begin{aligned}
\lambda &= \frac{1-1/\rho}{e^{\phi(R_I)/\phi(R_I+R_T)}} \cdot \left(1 - \frac{\phi(R_I)}{\phi(R_I+R_T)^2\cdot\Delta}\right) \\
&\qquad \cdot \left(1 - \frac{1}{\phi(R_I+R_T)}\right)^{\phi(R_I)} \\
\lambda' &= \frac{1-1/\rho}{e\cdot\phi(R_I+R_T)} \cdot \left(1 - \frac{1}{\phi(R_I+R_T)\Delta}\right) \\
&\qquad \cdot \left(1 - \frac{1}{\phi(R_I+R_T)}\right)^{\phi(R_I+R_T)}
\end{aligned}$$

$$\begin{aligned}
\sigma &= \frac{2c}{\lambda'} \quad , \quad \gamma = \frac{c\cdot\phi(R_I+R_T)}{\lambda} \\
q_\ell &= \frac{1}{\phi(R_I+R_T)} \quad , \quad q_s = \frac{1}{\phi(R_I+R_T)\Delta}
\end{aligned}$$

By a routine computation, one can easily verify that $\sigma > 2\gamma$. Finally, we shall use any constant variables η and μ verifying: $\eta \geq 2\gamma\phi(2R_T) + \sigma + 1$ and $\mu \geq \gamma$. In addition, it is not difficult to see that the following inequality holds:

Fact 1: $\forall x \geq 1$, $|t| \leq x$, $e^t\left(1 - \frac{t^2}{x}\right) \leq \left(1 + \frac{t}{x}\right)^x \leq e^t$.

The algorithm analyzed in the following section ensures that all nodes decide on a correct color after wake up. We term time complexity of the algorithm, the maximum number of time slots a node spends before deciding on its color.

III. THE MW COLORING ALGORITHM [1], [15]

$P_v := \varnothing$; $\zeta_i := \begin{cases} 1 & \text{if} \quad i = 0 \\ \Delta & \text{if} \quad i > 0 \end{cases}$;
$\mathcal{A}_{suc} := \begin{cases} \mathcal{R} & \text{if} \quad i = 0 \\ \mathcal{A}_{i+1} & \text{if} \quad i > 0 \end{cases}$;
for $\lceil \eta\Delta\ln n \rceil$ *time slots* **do**
  for *each* $w \in P_v$ **do** $d_v(w) := d_v(w) + 1$;
  if $M^i_{\mathcal{A}}(w, c_w)$ *received* **then** $P_v := P_v \cup \{w\}$; $d_v(w) := c_w$;
  if $M^i_{\mathcal{C}}(w)$ *received* **then** $state := \mathcal{A}_{suc}$; $L(v) := w$;
$c_v := \chi(P_v)$, where $\chi(P_v)$ is the maximum value s.t., $\chi(P_v) \notin \{d_v(w) - \lceil\gamma\zeta_i\ln n\rceil, \cdots, d_v(w) + \lceil\gamma\zeta_i\ln n\rceil\}$ for each $w \in P_v$, and $\chi(P_v) \leq 0$;
while $state = \mathcal{A}_i$ **do**
  $c_v := c_v + 1$;
  for *each* $w \in P_v$ **do** $d_v(w) := d_v(w) + 1$;
  if $c_v \geq \lceil\sigma\Delta\ln n\rceil$ **then** $state := \mathcal{C}_i$;
  transmit $M^i_{\mathcal{A}}(v, c_v)$ with probability q_s;
  if $M^i_{\mathcal{C}}(w)$ *received* **then** $state := \mathcal{A}_{suc}$; $L(v) := w$;
  if $M^i_{\mathcal{A}}(w, c_w)$ *received* **then**
    $P_v := P_v \cup \{w\}$; $d_v(w) := c_w$;
    if $|c_v - c_w| \leq \lceil\gamma\zeta_i\ln n\rceil$ **then** $c_v := \chi(P_v)$

Figure 1: Code for node v in state $\mathcal{A}_i$ [1], [15]

To make the paper self-contained, we recall the general ideas of the MW algorithm (see [15], [1] for more technical details[6]). Each node can be in three state classes $\mathcal{A}$, $\mathcal{R}$ or $\mathcal{C}$

[3]In the paper, we assume that all nodes have the same power level.

[4]This choice of R_T is made for simplicity but any other value verifying $R_T < R_{\max}$ will do well.

[5]Notice that $\phi(R)$ can be roughly bounded as following: $\phi(R) \leq \frac{\pi(R+R_T/2)^2}{\pi(R_T/2)^2} = \left(\frac{2R}{R_T}+1\right)^2$. Notice also that in our proofs, knowing only an upper bound on $\phi(R)$ affects our bound only by a constant, i.e., knowing the exact value of $\phi(R)$ is not required to prove our results.

[6]We have used the same symbols used in [15], [1] unless when they interfere with the usual and standard notations used for the SINR model.

1 color$_v$:= i;
2 **if** $i > 0$ **then**
3 **repeat** transmit $M^i_{\mathcal{C}}(v)$ with probability q_s **until** *protocol stopped;*
4 **else if** $i = 0$ **then**
5 $tc := 0$; $\mathcal{Q} := \varnothing$;
6 **repeat**
7 **if** $M_{\mathcal{R}}(w, v)$ *received and* $w \notin \mathcal{Q}$ **then** add w to $\mathcal{Q}$;
8 **if** $\mathcal{Q}$ *is empty* **then**
9 transmit $M^0_{\mathcal{C}}(v)$ with probability q_ℓ;
10 **else**
11 $tc := tc + 1$;
12 Let w be the first element in $\mathcal{Q}$;
13 **for** $\lceil \mu \ln n \rceil$ *time slots* **do** transmit $M^0_{\mathcal{C}}(v, w, tc)$ with probability q_ℓ;
14 Remove w from $\mathcal{Q}$;
15 **until** *protocol stopped*;

Figure 2: Code for node v in state $\mathcal{C}_i$ [1], [15]

1 **while** $state = \mathcal{R}$ **do**
2 transmit $M_{\mathcal{R}}(v, L(v))$ with probability q_s;
3 **if** $M^0_{\mathcal{C}}(L(v), v, tc_v)$ *received* **then**
4 $state := \mathcal{A}_{tc_v(\phi(2R_T)+1)}$;

Figure 3: Code for node v in state $\mathcal{R}$ [1], [15]

as depicted in figures 1, 2 and 3. It is important to remark that we do *not* use the same parameters and constants used in [15], [1]. In fact, the values of the parameters (namely, γ, σ, η, μ, the time intervals, and the sending probabilities) obtained from the previous section are carefully tuned to fit the SINR constraint and are crucial to prove our results.

Upon wake up, a node enters state $\mathcal{A}_0$ and executes the algorithm of Fig. 1. Whenever a node enters state $\mathcal{R}$ (resp. $\mathcal{C}_i$) it executes the algorithm of Fig. 3 (resp. Fig. 2). Generally speaking, the idea of the MW algorithm is quite simple. First, the algorithm attempts to compute an independent set of the graph: nodes in state $\mathcal{A}_0$ compete in order to be in the independent set. Once a node becomes in the independent set, it enters state $\mathcal{C}_0$. Then, a clustering of the graph is implicitly computed, i.e., each node is associated with one leader (variable $L(v)$ Line 5 and 12 of Fig. 1) and each node of the independent set is the leader of its cluster. Once a node has joined a cluster it requests a color from its leader (state $\mathcal{R}$). The leader assigns a unique cluster color $tc \leq \Delta$ to each node in its cluster (line 13 of Fig. 2 and line 3 of Fig. 3) but it cannot ensure that those assigned colors are different from colors of other nodes in the 2-neighborhood. In fact, two neighbors in two different clusters may be assigned the same cluster color tc. Therefore, once a node is assigned a color from its leader, it must compete with neighbors belonging to different clusters in order to decide on its final color. Since, there is only a constant number of neighbors that may have been assigned the same cluster color (namely, at most $\phi(2R_T)$), a node sequentially tries to pick color $tc \cdot (\phi(2R_T)+1)$, then color $tc \cdot (\phi(2R_T)+1)+1$, and so on. Nodes competing for color $i > 0$ are in state $\mathcal{A}_i$ (while loop of Fig. 1). Once a node decides on its final color, it enters state class $\mathcal{C}_i$ (line 3 of Fig. 2).

The major technical difficulty of the MW algorithm is first to ensure that two neighbors do not enter state $\mathcal{C}_0$ at the same time thus violating the independence of leaders, and second to ensure that two nodes competing for the same final color do not end picking the same color. Of course, nodes have to communicate to handle these issues and thus managing interferences is crucial. Roughly speaking, the MW algorithm makes use of two techniques to make nodes safely coordinate their actions and make correct decisions w.h.p.. First, to ensure that a given information is correctly delivered, a message is sent with a carefully defined probability or during a carefully defined period of time (Lines 2, 11 of Fig. 1, Lines 3, 9, 13 of Fig. 2, and Line 2 of Fig. 3). Second, a node picks its final color by trying to increment a counter up to a carefully defined threshold (Lines 8 and 10 of Fig. 1). The counter can be reseted to prevent neighboring nodes picking the same color (Line 15 of Fig. 1).

In the analysis made originally in [15], [1], the interference at a node depends by definition on its direct neighbors. Thus, the proof there goes in a very localized fashion and it did not care about nodes being farther away. In contrast, when switching to the additive SINR model, the interference experienced by a node depends on all other nodes in the network. The key idea of our proof is to show that actually the SINR constraint at a node u depends most importantly on the transmission probabilities of nodes in the disc I_u, i.e., in the t-neighborhood for a well defined constant $t > 1$. Then the algorithm is proven to be correct by carefully tuning the sending probabilities, the time periods, and the counter threshold as in [15], [1].

IV. Analysis in the SINR model

Given a time slot and two nodes u and v, we denote p_v the sending probability of node v at that time slot and $\Psi^v_u = p_v/\delta(u,v)^\alpha$ the probabilistic interference caused by v to node u. For every node u, we define the probabilistic interference induced by nodes outside some region R in a given time slot as following: $\Psi^{v \notin R}_u = P \cdot \sum_{v \notin R} \Psi^v_u$.

Our analysis follows the same scheme than the analysis made in [1], [15]. The correctness of the coloring is based on the key property that nodes in state $\mathcal{C}_i$ form an independent set at any time. This is stated in Theorem 1 below. In order to prove the theorem, we need to prove some intermediary properties expressed in Lemmas 1 and 2. These two lemmas provide a bound on the time needed for a node in some particular state to transmit without experiencing interferences. To prove Lemmas 1 and 2 in the SINR model, we need to bound the interference experienced by a node at any step of the algorithm execution. This is the aim of key Lemma 3 below.

In the remainder of the paper, we say that a node transmits successfully, if it transmits a message and the message is received by all its neighbors.

Theorem 1: For all $i \geq 0$, with probability at least $1 - O(n^{2-c})$ the color class $\mathcal{C}_i$ forms an independent set throughout the execution of the algorithm.

Lemma 1: Assume $\mathcal{C}_0$ forms an independent set. Consider nodes u, u', and v such that $\delta(u,v) \leq R_T$, $\delta(u',v) \leq R_T$, and $u \in V \setminus \mathcal{C}_0$ and $u' \in \mathcal{C}_0$. Let J (resp. J') be a time interval of length $\gamma\Delta \ln n$ (resp. $\gamma \ln n$). The probability $\mathbb{P}_{no}$ (resp. $\mathbb{P}'_{no}$) that v does not get a message from u (resp. u') during interval J (resp. J') is upper bounded by $1/n^c$.

Lemma 2: Assume $\mathcal{C}_0$ forms an independent set. Consider a node $v \in \mathcal{A}_i$ for an arbitrary i. Let J be a time interval of length $|J| = \dfrac{\sigma}{2}\Delta \ln n$. With probability at least $1 - 1/n^c$, there is a time slot $t \in J$ such that, at least one node $w \in B_v \cap \mathcal{A}_i$ transmits successfully.

Lemma 3: Assume $\mathcal{C}_0$ forms an independent set. Then, for every node u, the probabilistic interference $\Psi_u^{v \notin I_u}$ verifies: $\Psi_u^{v \notin I_u} \leq \frac{P}{2\rho\beta R_T^\alpha}$.

Proof of Lemma 3. Consider a fixed time slot and let p_w the sending probability of node w. From the algorithm, if $w \in \mathcal{C}_0$ then $p_w = q_\ell$ otherwise $p_w = q_s$. Thus, the sum of transmitting probabilities over all nodes in a given region B_v can be bounded as following:

$$\sum_{w \in B_v} p_w = \sum_{w \in B_v \cap \mathcal{C}_0} q_\ell + \sum_{w \in B_v \cap V \setminus \mathcal{C}_0} q_s$$

$$\leq \sum_{w \in B_v \cap \mathcal{C}_0} \frac{1}{\phi(R_I + R_T)} + \sum_{w \in B_v \cap V \setminus \mathcal{C}_0} \frac{1}{\phi(R_I + R_T)\Delta}$$

By definition, the number of independent nodes in B_v is at most $\phi(R_T) \leq \phi(R_I + R_T)$. Hence, since the maximum degree of the graph is Δ, we get:

$$\sum_{w \in B_v} p_w \quad \leq \quad 2 \tag{1}$$

Given a node u, let $R_\ell = \{v \in V \mid \ell R_I \leq \delta(u,v) \leq (\ell+1)R_I\}$. Consider an independent set $\mathcal{I}$ of maximum size in R_ℓ. It is clear that $\cup_{z \in \mathcal{I}} B_z$ (the union of the transmission regions of nodes in $\mathcal{I}$) covers entirely the ring R_ℓ (otherwise we can add a new node to $\mathcal{I}$ leading to a contradiction since set $\mathcal{I}$ has maximum size). Since nodes in $\mathcal{I}$ are independent, the discs of radius $R_T/2$ around nodes in $\mathcal{I}$ are mutually disjoint (otherwise the independence of $\mathcal{I}$ is violated). We remark that these discs are located inside the extended region R_ℓ^+ defined by $R_\ell^+ = \{v \in V \mid \ell R_I - R_T/2 \leq \delta(u,v) \leq (\ell+1)R_I + R_T/2\}$. Thus, $|\mathcal{I}| \leq Area\left(R_\ell^+\right)/Area\left(Disc(R_T/2)\right)$. Thus, the probabilistic interference caused by nodes inside R_ℓ verifies:

$$\begin{aligned}
\Psi_u^{R_\ell} &= P \cdot \sum_{v \in R_\ell} \Psi_u^v \\
&\leq \frac{Area\left(R_\ell^+\right)}{Area(Disc(R_T/2))} \cdot \max_{v \in \mathcal{I}} \Big\{ \sum_{w \in B_v \cap R_\ell} \frac{P \cdot p_w}{(\ell R_I)^\alpha} \Big\} \\
&\leq_{\text{Eq. 1}} \frac{Area\left(R_\ell^+\right)}{Area(Disc(R_T/2))} \cdot \frac{2P}{\ell^\alpha R_I^\alpha} \\
&= \frac{\pi\left(((\ell+1)R_I + R_T/2)^2 - (\ell R_I - R_T/2)^2\right)}{\pi (R_T/2)^2} \cdot \frac{2P}{\ell^\alpha R_I^\alpha} \\
&= \frac{4(2\ell+1)\left(R_I^2 + R_I R_T\right)}{R_T^2} \cdot \frac{2P}{\ell^\alpha R_I^\alpha} \\
&\leq \frac{1}{\ell^{\alpha-1}} \cdot \frac{48 P R_I^2}{R_T^2 \cdot R_I^\alpha}
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \Psi_u^{v \notin I_u} &= \sum_{\ell=1}^{\infty} \Psi_u^{R_\ell} \leq \frac{48 P R_I^2}{R_T^2 \cdot R_I^\alpha} \cdot \sum_{\ell=1}^{\infty} \frac{1}{\ell^{\alpha-1}} \\
&\leq P \cdot \frac{48 R_I^{2-\alpha}}{R_T^2} \cdot \frac{\alpha-1}{\alpha-2} \leq \frac{P}{2\rho\beta R_T^\alpha}
\end{aligned}$$

■

Proof of Lemma 1. Consider a given time slot of the algorithm. Suppose that node u is the only node inside I_v that transmits at that time slot. Then, we shall show that node v is likely to hear the message sent by u, i.e., the SINR condition is likely to be verified. In fact, from the Markov inequality and using Lemma 3, we have that the probability that the interferences caused by nodes outside I_v exceeds $\rho \cdot \Psi_v^{w \notin I_v}$ is at most $1/\rho$. Thus, given that u is the only sending node in I_v, with probability at least $1 - 1/\rho$, the SINR at node v can be bounded as following:

$$\frac{\frac{P}{\delta(u,v)^\alpha}}{\rho \cdot \Psi_v^{w \notin I_v} + N} \geq \frac{\frac{P}{\delta(u,v)^\alpha}}{\frac{P}{2\beta R_T^\alpha} + \frac{P}{2\beta R_T^\alpha}} \geq \beta \tag{2}$$

Thus the probability $\mathbb{P}((u \longrightarrow v))$ that node v hears a message sent by u at a given time slot is at most:

$$\begin{aligned}
& (1 - 1/\rho) \cdot p_u \cdot \prod_{w \in I_v \setminus \{u\}} (1 - p_w) \\
=\ & (1 - 1/\rho) \cdot p_u \cdot \prod_{w \in I_v \cap \mathcal{C}_0} (1 - q_\ell) \cdot \prod_{w \in I_v \cap V \setminus \mathcal{C}_0} (1 - q_s) \\
\geq\ & (1 - 1/\rho) \cdot p_u \cdot \left(1 - \frac{1}{\phi(R_I + R_T)}\right)^{\phi(R_I)} \\
& \qquad \cdot \left(1 - \frac{1}{\phi(R_I + R_T) \cdot \Delta}\right)^{\phi(R_I) \cdot \Delta} \\
\geq\ & (1 - 1/\rho) \cdot p_u \cdot \left(1 - \frac{1}{\phi(R_I + R_T)}\right)^{\phi(R_I)} \\
& \quad \cdot \left(\left(1 - \frac{\phi(R_I)}{\phi(R_I + R_T)^2 \cdot \Delta}\right) e^{-\phi(R_I)/\phi(R_I + R_T)}\right) \\
=\ & \lambda\, p_u
\end{aligned}$$

For $u \in I_v \cap V \setminus \mathcal{C}_0$, i.e., $p_u = \frac{1}{\phi(R_I + R_T)\Delta}$, we get:

$$\begin{aligned}
\mathbb{P}_{no} &\leq (1 - \mathbb{P}((u \longrightarrow v)))^{|J|} \\
&= \left(1 - \frac{\lambda}{\phi(R_I + R_T)\Delta}\right)^{\frac{c\phi(R_I + R_T)}{\lambda}\Delta \ln(n)} \leq n^{-c}
\end{aligned}$$

Similarly, for $u' \in I_v \cap \mathcal{C}_0$, i.e., $p_u = \frac{1}{\phi(R_I+R_T)}$, we get:

$$\begin{aligned}\mathbb{P}_{no} &\leq (1-\mathbb{P}((u' \longrightarrow v)))^{|J'|} \\ &= \left(1-\frac{\lambda}{\phi(R_I+R_T)}\right)^{\frac{c\phi(R_I+R_T)}{\lambda}\ln(n)} \leq n^{-c}\end{aligned}$$

■

Proof of Lemma 2 Let $v \in \mathcal{A}$ a node and $w \in B_v \cap \mathcal{A}$ a neighbor of v. Consider a node u such that u is a neighbor of w. We remark that if w is the only sending node in the region I_u around u, then u is likely to hear the message sent by w, i.e., the SINR condition is likely to be verified for u. Thus, if w is the only sending node in $\cup_{u\in B_w} I_u$, then the message sent by w is likely to be received by w's neighbors.

More precisely, let $(w \longrightarrow \star)$ be the event "all w's neighbors hear a message from w". Using Lemma 3, we have that $\Psi_w^{x\notin \cup_{u\in B_w} I_u} \leq P/(2\rho\beta R_T^{\alpha})$. Thus, provided that w is the only sending node in $\cup_{u\in B_w} I_u$, and using the Markov inequality (as for Equation 2), it is not difficult to see that the SINR condition holds for all w's neighbor with probability at least $1-1/\rho$, i.e., if w is the only sending node in $\cup_{u\in B_w} I_u$ then for every node $u \in B_w$ the SINR condition holds and u hears w's message with probability at least $1-1/\rho$. Thus, $\mathbb{P}((w \longrightarrow \star))$ is at most:

$$\begin{aligned}& (1-\frac{1}{\rho}) \cdot p_w \cdot \prod_{r\in\left(\bigcup_{u\in B_w} I_u\right)\setminus\{w\}} (1-p_r) \\ = \;& (1-\frac{1}{\rho}) \cdot p_w \cdot \prod_{r\in\left(\left(\bigcup_{u\in B_w} I_u\right)\setminus\{w\}\right)\cap\mathcal{C}_0} (1-q_\ell) \\ & \cdot \prod_{r\in\left(\left(\bigcup_{u\in B_w} I_u\right)\setminus\{w\}\right)\cap(V\setminus\mathcal{C}_0)} (1-q_s) \\ \geq \;& (1-\tfrac{1}{\rho}) \cdot \tfrac{1}{\phi(R_I+R_T)\Delta} \cdot \left(1-\tfrac{1}{\phi(R_I+R_T)}\right)^{\phi(R_I+R_T)} \\ & \cdot \left(1-\tfrac{1}{\phi(R_I+R_T)\Delta}\right)^{\phi(R_I+R_T)\cdot\Delta} \\ \geq \;& (1-\tfrac{1}{\rho}) \cdot \tfrac{1}{\phi(R_I+R_T)\Delta} \cdot \left(1-\tfrac{1}{\phi(R_I+R_T)}\right)^{\phi(R_I+R_T)} \\ & \cdot \left(1-\tfrac{1}{\phi(R_I+R_T)\Delta}\right) \cdot e^{-1} \\ \geq \;& \lambda'/\Delta\end{aligned}$$

Thus the probability that w fails sending successfully during the interval J is at most:

$$(1-\mathbb{P}((w \longrightarrow \star)))^{|J|} = \left(1-\frac{\lambda'}{\Delta}\right)^{\frac{c}{\lambda'}\Delta\ln(n)} \leq n^{-c}$$

■

Proof of Theorem 1 The proof follows the same arguments than those used originally in [15], [1]. At the beginning, when the first node wakes up, the claim certainly holds, because $\mathcal{C}_i = \varnothing$ for all i. In the rest of the proof, we prove that w.h.p., no node can violate the independence property of set $\mathcal{C}_i$. Consider an arbitrary node $v \in \mathcal{A}_i$ and assume that v is the first node that violates the independence of $\mathcal{C}_i$ for an arbitrary $i \geq 0$. Let t_v^* be the time slot in which v enters any state $\mathcal{C}_i$. Thus, we know that $\mathcal{C}_0$ is a correct independent set for all time slots $t < t_v^*$. Thus, we shall use Lemma 1 until time slot t_v^*-1. Let w be a neighbor of v that has joined $\mathcal{C}_i$ at time $t_w^* \leq t_v^*$ before v. We consider two cases:

Case 1: If $t_w^* < t_v^* - \gamma\zeta_i \ln n$, then w entered state $\mathcal{C}_i$ at least $\gamma\zeta_i\ln n$ time slots before v. By definition of ζ_i, Lemma 1 holds at time slot $t_v^*-\gamma\zeta_i\ln n$, and thus the probability that w successfully sends a message $M_{\mathcal{C}}^i$ to v during these $\gamma\zeta_i\ln n$ time slots is at least $1-n^{-c}$. During these time slots v must be in $\mathcal{A}_i$ (since it joins $\mathcal{C}_i$ later at time t_v^*). Thus, by line 5 of Fig. 1, v moves to state $\mathcal{A}_{suc}$ upon receiving $M_{\mathcal{C}}^i$. Hence, v does not enter $\mathcal{C}_i$ at time t_v^* w.h.p.

Case 2: If $t_w^* \geq t_v^* - \gamma\zeta_i\ln n$, we shall compute the probability that v joins $\mathcal{C}_i$ within $\gamma\zeta_i\ln n$ time slots after t_w^*. From the algorithm it holds that $c_w \geq \sigma\Delta\ln n$ at time t_w^* (Line 10, when w enters state $\mathcal{C}_i$). Consider the time interval J_w of length $\gamma\zeta_i\ln n$ before t_w^*. Because in each time slot, nodes in state $\mathcal{A}_i$ increment their counters by one or reset them to $\chi(P_v) \leq 0$ and because $\sigma\zeta_i\ln n > 2\gamma\zeta_i\ln n$ (by definition $\sigma > 2\gamma$), it follows that c_w was not reset during J_w (otherwise it would not have reached $\sigma\Delta\ln n$ by time t_w^*). Similarly, v's counter c_v could not be reseted during J_w. Hence, neither c_w nor c_v were reset during the interval J_w and it holds that at time t_w^*, $c_v \geq \sigma\Delta\ln n - \gamma\zeta_i\ln n$. More generally, it holds that $|c_w(t_w^*-h)-c_v(t_w^*-h)| \leq \gamma\zeta_i\ln n$ for each $h = 0,\cdots,\gamma\zeta_i\ln n - 1$. By Lemma 1, the probability that v receives at least one message $M_{\mathcal{A}}^i$ from w during time interval J_w is at least $1-n^{-c}$, which makes v reset its counter (By line 15). Thus, v does not enter $\mathcal{C}_i$ at time t_v^* with probability at least $1-n^{-c}$.

Thus, the probability that v is the first node to violate the independence of $\mathcal{C}_i$ is at most $1-2n^{-c}$. Therefore, over all nodes, each state $\mathcal{C}_i$ remains independent with probability at least $1-O(n^{1-c})$. By remarking that the number of state classes $\mathcal{C}_i$ in a worst case scenario is n, the theorem holds. ■

Having Theorem 1, Lemma 1, and Lemma 2 at hand, the correctness and the time complexity analysis of the coloring algorithm in the SINR model goes similar than the analysis made in [1]. In fact, technically speaking we just need that our constants do verify the following carefully checked conditions $\eta \geq 2\gamma\phi(2R_T)+\sigma+1$ and $\mu \geq \sigma$.

For completeness, and since the MW algorithm analysis is quite technical, we briefly revisit the proof in the SINR model. The next lemma enables to bound the number of times a node could be in state class $\mathcal{A}$.

Lemma 4: Assume set $\mathcal{C}_0$ forms an independent set. For any node v and any $i > 0$, the number of nodes in B_v that ever enter state $\mathcal{A}_i$ is at most $\phi(2R_T)$.

Proof: Consider a node $u \in B_v$ that enters a state $\mathcal{A}_j$ for the first time after being in request state $\mathcal{R}$ and receiving a message $M_{\mathcal{C}}^0(w,u,tc_u)$ from its leader w. From the algorithm, we have that $j = tc_u(\phi(2R_T)+1)$. Since w assigns different colors tc to each node having chosen it as a leader, a node $u' \in B_v \setminus \{u\}$ that enters state $\mathcal{A}_j$ for the first time after being in request must have a different leader $w' \neq w$. These leaders are in the disc of radius $2R_T$ around v, and thus there are at most $\phi(2R_T)$ such node u'.

Now, node u moves to state $\mathcal{A}_{j+1}$ whenever it receives a message $M_{\mathcal{C}}^{j}$ from a neighbor. Hence, if u moves to a state $\mathcal{A}_i$ with $i > j$, then at least one neighbor must have joined $\mathcal{C}_i$ (after being in state $\mathcal{A}_i$). Thus, in the worst case, if node u reached state $\mathcal{A}_{j+\phi(2R_T)}$ without deciding on its color, i.e., without entering state $\mathcal{C}$, then this means that the at most $\phi(2R_T)$ neighbors that were assigned the same color tc (by different leaders) already entered state $\mathcal{C}$. Thus, u cannot move to state $\mathcal{A}_{j+\phi(2R_T)+1}$ since it cannot receive any message $M_{\mathcal{C}}^{j+\phi(2R_T)}$ anymore. ■

Using Lemma 4, it is not difficult to see that whenever a node enters state $\mathcal{A}$, that is whenever a node receives a color tc from its leader, it can only be in the following $\phi(2R_T)$ states $\mathcal{A}_{tc\cdot(\phi(2R_T)+1)}, \cdots, \mathcal{A}_{tc\cdot(\phi(2R_T)+1)+\phi(2R_T)}$.

Lemma 5: Assume $\mathcal{C}_0$ forms an independent set. Let c_v be the counter of node $v \in \mathcal{A}_i$. It holds that $c_v \geq -2\gamma\Delta \ln n$ if $i = 0$, and $c_v \geq -2\gamma\phi(2R_T)\Delta \ln n - 1$ otherwise.

Proof: From line 6 and 15 of Fig. 1, the only time c_v is set to a negative value is when setting c_v to $\chi(P_v)$. By line 4 of Fig. 1, set P_v contains only nodes that entered state $\mathcal{A}_i$. Thus, by Lemma 4, $|P_v| \leq \phi(2R_T)$ for $i > 0$. For $i = 0$, it trivially holds that $|P_v| \leq \Delta$. From the definition of $\chi(P_v)$ (Line 6), the number of values that are prohibited for $\chi(P_v)$ is at most $\phi(2R_T)\cdot 2\gamma\Delta \ln n$ for $i > 0$ and $\Delta \cdot 2\gamma \ln n$ for $i = 0$, and the lemma holds. ■

We denote by $T_v^{\mathcal{Y}}$ the number of time slots a node v stays in state set $\mathcal{Y}$. In next lemmas, we bound $T_v^{\mathcal{A}_i}$ and $T_v^{\mathcal{R}}$.

Lemma 6: With probability at least $1-O(n^{2-c})$, it holds for all v and i that $T_v^{\mathcal{A}_i} = O(\phi(R_I+R_T)\cdot\phi(2R_T)^2\cdot\Delta \ln n)$.

Proof: By Theorem 1, nodes in state $\mathcal{C}_0$ form an independent set with probability $1 - O(n^{2-c})$. Thus, we shall focus on this case. Let t_v be the time slot in which node $v \in \mathcal{A}_i$ executes Line 6 of Fig. 1. Until t_v , v spends $\eta\Delta \ln n$ time slots in $\mathcal{A}_i$. By Lemma 2, at least one node $w \in B_v \cap \mathcal{A}_i$ is able to transmit successfully during the interval $J = [t_v, t_v + \frac{\sigma}{2}\Delta \ln n]$ with probability $1 - n^{-c}$ (unless v leaves state $\mathcal{A}_i$ during that interval in which case the lemma holds). Let t_w^s be the time node w transmits successfully. According to Line 4 of Fig. 1, all nodes $u \in B_w \cap \mathcal{A}_i$ store a local copy $d_u(w)$ of w's current counter c_w upon receiving w's message $M_{\mathcal{A}}^{i}$ in time slot t_w^s. In Lines 3 and 9 of Fig. 1, this local copy is incremented by one in each subsequent time slot. Thus, as long as w's real counter is not reset to $\chi(P_w)$, every node $u \in B_w \cap \mathcal{A}_i$ has a correct local copy $d_u(w)$ of w's current counter c_w. We now show that w's counter c_w cannot be reset by any node $u \in B_w \cap \mathcal{A}_i$ after t_w^s anymore. In fact, from Line 15 of Fig. 1, every node $u \in B_w \cap \mathcal{A}_i$ whose counter $c_u(t_w^s)$ at time t_w^s is in the range $[c_w(t_w^s) - \gamma\zeta_i \ln n, \cdots, c_w(t_w^s) + \gamma\zeta_i \ln n]$ resets its own counter to $\chi(P_u)$. By the definition of $\chi(P_v)$ (Line 6), and since w transmitted successfully, we have that $|c_u(t_w^s+1) - c_w(t_w^s+1)| > \gamma\zeta_i \ln n$. Clearly, the same inequality also holds for other nodes in $B_w \cap \mathcal{A}_i$ whose counters were not in the critical range $[c_w(t_w^s) - \gamma\zeta_i \ln n, \cdots, c_w(t_w^s) + \gamma\zeta_i \ln n]$.

In summary, we have that in time slot $t_w^s + 1$, every node $u \in B_w \cap \mathcal{A}_i$ has a correct local copy $d_u(w)$ of c_w , and $|c_u(t_w^s+1) - c_w(t_w^s+1)| > \gamma\zeta_i \ln n$. Thus none of these nodes can cause w to reset its counter. Node w can thus increment its counter in each time slot and hence, all nodes $u \in B_w \cap \mathcal{A}_i$ continue to have a correct local copy of c_w after t_w^s. Thus, even if a neighboring node u has to reset its counter to $\chi(P_u)$, this cannot cause c_u to come within $\gamma\zeta_i \ln n$ of c_w by definition of $\chi(P_u)$. Thus, by induction over the subsequent time slots, no node $u \in \mathcal{A}_i$ is able to reset w's counter after time t_w^s. By Lemma 5, for all i, $c_w \geq -2\gamma\phi(2R_T)\Delta \ln n - 1$ at time t_w^s. Hence, if w stays in $\mathcal{A}_i$, it requires at most $(2\gamma\phi(2R_T)+\sigma)\Delta \ln n + 1$ time slots to reach the threshold $\sigma\Delta \ln n$, and thus to enter state $\mathcal{C}_i$. Also, nodes that join $\mathcal{A}_i$ after t_w^s do not transmit for at least $\eta\Delta \ln n$ time slots, and because $\eta > 2\gamma\phi(2R_T) + \sigma + 1$, it follows that such nodes cannot interfere with w's incrementing its counter. Hence, after a successful transmission, there remains only one way to prevent w from incrementing its counter and entering $\mathcal{C}_i$: if w receives a message $M_{\mathcal{C}}^{i}$ before its counter reaches $\sigma\Delta \ln n$.

In summary, we have that after a successful transmission, either w enters $\mathcal{C}_i$ within $(2\gamma\phi(2R_T)+\sigma)\ln n + 1$ time slots or there must exist a neighbor x of w that joins $\mathcal{C}_i$ earlier. In the first case, v receives a message $M_{\mathcal{C}}$ from w within $\gamma\zeta_i \ln n$ after w enters $\mathcal{C}_i$ with probability at least $1 - n^{-c}$. In the other case, the node x must be in the disc of radius $2R_T$ around v. If v is not covered by x, i.e., x is not a direct neighbor of v, and v remains in state $\mathcal{A}_i$, then at least one node $w' \in B_v \cap \mathcal{A}_i$ transmits successfully within $\frac{\sigma}{2}\Delta \ln n$ time slots thereafter w.h.p., (By Lemma 2), and the argument repeats itself. That is, as long as v is active in $\mathcal{A}_i$, at least one node in the disc of radius $2R_T$ around v enters $\mathcal{C}_i$ every $\sigma\Delta \ln n + (2\gamma\phi(2R_T) + \sigma)\Delta \ln n + 1$ time slots with probability $1 - n^{-c}$. The number of such (independent) nodes that do not cover v is at most $\phi(2R_T)$ (by definition). Finally, once v becomes covered, an additional $\gamma\zeta_i \ln n$ time slots in $\mathcal{A}_i$ may be required before, with probability $1-n^{-c}$, the first neighbor in $\mathcal{C}_i$ sends a message $M_{\mathcal{C}}^{i}$ to v.

As stated at the beginning of the proof, the previous discussion holds under the condition that the set $\mathcal{C}_0$ forms an independent set which is true with probability $1 - O(n^{2-c})$ by Theorem 1. Therefore, node v spends at most $\eta\Delta \ln n + \phi(2R_T)\cdot(\sigma\Delta \ln n + (2\gamma\phi(2R_T)+\sigma)\Delta \ln n + 1) + \gamma\zeta_i \ln n = O(\phi(2R_T)^2\cdot\phi(R_I+R_T)\cdot\Delta \ln n)$ time slots in state $\mathcal{A}_i$, with probability at least $1 - (\phi(2R_T)n^{-c} + n^{-c} + O(n^{2-c})) >$

$1 - O(n^{2-c})$, for large enough n or c (recall that γ, $\sigma = O(\phi(R_I + R_T))$ and $\eta = O(\phi(R_I + R_T) \cdot \phi(2R_T))$). ■

Lemma 7: With probability at least $1 - O(n^{2-c})$ it holds for each $v \in V$ that $T_v^{\mathcal{R}} = O(\phi(2R_T) \cdot \phi(R_I + R_T) \cdot \Delta \ln n)$.

Proof: $T_v^{\mathcal{R}}$ is the time between v starting to request a color from its leader $w = L(v) \in \mathcal{C}_0$ to the time v receives the color tc_v from w. By Lemma 1, v is able to send its request to w within time $\gamma\Delta \ln n$ with probability $1 - n^{-c}$. Upon receipt, w queues v's request until it has served all its other previously received requests. In Line 13 of Fig. 2, w first transmits messages $M_{\mathcal{C}}^0$ to the currently considered requesting node for $\mu \ln n$ time slots, before moving on to the next request. Since by definition $\mu \geq \gamma$, Lemma 1 holds for each w's response with probability $1 - n^{-c}$. Since w can have at most Δ requesting nodes in its queue, $T_v^{\mathcal{R}}$ is at most $\gamma\Delta \ln n + \Delta \cdot \mu \ln n = (\gamma + \mu)\Delta \ln n$ for each node $v \in V$ with probability at least $1 - 2n^{-c}$. Combining this with the fact that $\mathcal{C}_0$ is independent with probability $1 - O(n^{2-c})$, the lemma holds from the definition of γ and μ, and for large enough n or c. ■

Now, we can state the main theorem of the paper.

Theorem 2: With high probability, the algorithm produces a $(1, (\phi(2R_T) + 1)\Delta)$-coloring within at most $O(\phi(2R_T)^3 \cdot \phi(R_T + R_I) \cdot \Delta \ln n)$ time slots after nodes wake up.

Proof: **(sketch)** First, the running time is a straightforward consequence of Lemmas 6, 7 and 4. Second, let us bound the number of different colors used by the algorithm. Let w be a leader and let ℓ_w be the number of nodes having w as their leader. As discussed earlier and since the set of leaders $\mathcal{C}_0$ is independent (w.h.p.,), leader w assigns a unique color tc from $\{1, 2, \cdots, \ell_w\}$ to each node in its cluster, and each such node ends up selecting a color from the range $tc(\phi(2R_T)+1), \cdots, tc(\phi(2R_T)+1)+\phi(2R_T)$. Since $\ell_w \leq \Delta$ and every node is assigned to a leader, the number of colors assigned by the algorithm is at most $\Delta(\phi(2R_T)+1)$ which terminates the proof. ■

V. MAC LAYER IN THE SINR MODEL

In this section, we show how to design a generic TDMA-like interference free MAC protocol in the SINR model. Among our goals is to simulate classical interference free message passing algorithms in the SINR model. In fact, following the same motivation than [5], since designing distributed algorithms from scratch under the physical constraints turns out to be a hard task, simulation-based techniques for standard message-passing systems can indeed help providing plausible solutions.

In the classical point-to-point message passing model neighboring nodes are connected by a private channel allowing them to communicate in a bi-directional manner without any interferences, i.e., no interference can prevent a message sent by a node to arrive to the second node. For simplicity we assume that in such a model, any algorithm proceeds into *rounds*. In each round, a node can receive messages, do some local computations and send messages. We consider two classes of algorithms: the *uniform* model and the *general* model. In the former, a node can only send the same message to its neighbors at a given round, e.g. broadcast-based algorithms. In the latter, a node can send a different message to each neighbor.

Let us focus on a *uniform* algorithm. Our goal is to transform the algorithm so we can run it in the SINR model and obtain the same output. A classical idea is to simulate each round of the algorithm and to attempt to deliver each original message to destination while avoiding interferences (this was termed as *Single Round Simulation* procedure in [5]). For that purpose, we shall use the previous coloring algorithm to implement a TDMA-like MAC layer by associating each color with a time slot where a node can transmit. However, given a node u having two neighbors v and w with the same color c, if v and w both transmit a message during the time slot corresponding to color c, then an interference occurs at node u, i.e., node u does not receive any message. This is a well known problem which is commonly solved by using a distance-2 coloring, that is a coloring such that each node has a different color than the nodes in its 2-neighbors. Unfortunately, under the SINR additive constraints such a coloring does not allow us to avoid interferences.

The following theorem shows how to schedule a coloring-based MAC layer without direct interferences in the SINR model. In fact, suppose we have constructed a $(d+1, \mathcal{V})$-coloring for some parameter d, that is a coloring using at most $\mathcal{V}$ colors such that each node has a different color than nodes at distance $(d+1) \cdot R_T$ in the Euclidean plane. We associate each color $c \in \{1, \cdots, \mathcal{V}\}$ with a time slot t_c where nodes colored c can transmit in time slot t_c. Using this simple TDMA-like protocol, we obtain:

Theorem 3: For $d = \left(32 \cdot \frac{\alpha-1}{\alpha-2} \cdot \beta\right)^{1/\alpha}$, a $(d+1, \mathcal{V})$-coloring defines a scheduled MAC layer protocol allowing every node to successfully send a message to all its neighbors within at most $\mathcal{V}$ time slots.

Proof: Consider a time slot t_c and a node u having a neighbor v with color $c \in \{1, \cdots, \mathcal{V}\}$. Let us compute the amount of interference $\Phi_{u \setminus v}$ experienced by u and caused by nodes $w \in V \setminus \{v\}$ during time slot t_c.

Let $H_{\ell,d} = \{w \in V \mid \ell d R_T \leq \delta(u,v) \leq (\ell+1) d R_T\}$. The d-neighborhood of any node $w \in H_\ell$ must be located in an extended region defined by: $H_{\ell,d}^+ = \{w \in V \mid (\ell - 1/2) d R_T \leq \delta(u,v) \leq (\ell + 3/2) d R_T\}$. Let $Disc(r)$ be any disc of radius r. Let $\phi'(\ell, d)$ be the maximum number of nodes with the same color in $H_{\ell,d}$. Since the $(d/2)$-neighborhoods of two nodes with the same color c are disjoint, we can bound $\phi'(\ell,d)$ as following:

$$\begin{aligned}\phi'(\ell,d) &\le Area(H^+_{\ell,d})\ /\ Area(Disc(dR_T/2))\\ &= \tfrac{\pi((\ell+3/2)dR_T)^2-\pi((\ell-1/2)dR_T)^2}{\pi d^2R_T^2/4}\\ &= 4\cdot\left((\ell+3/2)^2-(\ell-1/2)^2\right) \le 16\ell\end{aligned}$$

Thus, at time slot t_c corresponding to color c, we get:

$$\begin{aligned}\Phi_{u\setminus v} &= \sum_{w\in V\setminus\{v\}}\frac{P}{\delta(w,u)^\alpha} = \sum_{\substack{w\in V\setminus\{v\}\\ w\text{ has color } c}}\frac{P}{\delta(w,u)^\alpha}\\ &= P\cdot\sum_{\ell=1}^{\infty}\sum_{\substack{w\in H_{\ell,d}\\ w\text{ has color } c}}\frac{1}{(\ell\cdot d\cdot R_T)^\alpha}\\ &\le P\cdot\sum_{\ell=1}^{\infty}\frac{\phi'(\ell,d)}{(\ell\cdot d\cdot R_T)^\alpha} \le \frac{16P}{d^\alpha R_T^\alpha}\cdot\sum_{\ell=1}^{\infty}\frac{1}{\ell^{\alpha-1}}\\ &\le \tfrac{16P}{d^\alpha R_T^\alpha}\cdot\tfrac{\alpha-1}{\alpha-2} \le \tfrac{P}{2\beta R_T^\alpha}\end{aligned}$$

Thus the SINR at node u verifies:

$$SINR_u = \frac{\frac{P}{R_T^\alpha}}{\Phi_{u\setminus v}+N} \ge \frac{\frac{P}{R_T^\alpha}}{\frac{P}{2\beta R_T^\alpha}+\frac{P}{2\beta R_T^\alpha}} = \beta$$

Thus, node u receives the message sent by v. Therefore, a message sent by node v at time slot t_c is correctly received by all its neighbors and the theorem is proved. ■

Now, suppose that we have an algorithm computing a $(1,O(\Delta))$-coloring of a given (unit disc) graph G. Assuming that wireless nodes are able to tune their transmission power, it is not difficult to adapt the algorithm to compute a distance-d coloring of G. In fact, we remark that a distance-1 coloring of $G^d=(V,E',dR_T)$ is also a $(d,O(\Delta_{G^d}))$-coloring of G where Δ_{G^d} is the maximum degree of G^d. The maximum degree of G^d can be upper bounded as following $\Delta_{G^d}\le\phi(d\cdot R_T)\Delta\le(2d+1)^2\Delta$. Thus, when executed on G^d, the algorithm produces a $(d,O(d^2\cdot\Delta))$-coloring for G. To summarize, computing a distance-1 coloring of G^d allows us to obtain a $(d,O(\Delta))$-coloring of G. A simple idea to compute a coloring of G^d is to set the transmission power of every node to $O(d^\alpha\cdot P)$ before switching again to P once the network is initialized. Since $\Delta_{G^d}=O(\Delta)$ for d constant, the time needed to output the coloring is still $O(\Delta\log n)$. Of course, all the parameters used by the algorithm have to be tuned for $R'_T=d\cdot R_T$ and $\Delta'=\Delta_{G^d}$. From Theorem 3, we then get:

Corollary 1: Consider a uniform (resp. general) point-to-point message passing algorithm A (resp. A') running on an n-node unit disk graph in $\tau\le n^{O(1)}$ time and using messages of size at most s bits. With high probability,

- Algorithm A can be simulated in the SINR physical model in $O(\Delta\cdot(\log n+\tau))$ time using messages of size $O(s\log n)$ bits.
- Algorithm A' can be simulated in the SINR physical model in $O(\Delta\cdot(\log n+\tau))$ (resp. $O(\Delta\cdot\log n+\Delta^2\cdot\tau))$ time using messages of size $O(s\Delta\log n)$ (resp. $O(s\log n)$) bits.

It is interesting to remark that the previous corollary allows us to improve the number of colors used by the MW algorithm itself. In fact, using a standard palette-reduction procedure [11], it is easy to see that it is possible to compute a $(1,\Delta+1)$-coloring in the SINR model in $O(\Delta\log n)$ distributed time, thus getting rid of the constants hidden in the big O notation of the MW algorithm. More precisely, starting with a $(d,O(\Delta))$-coloring, each color c being associated with 2 time slots period $\{t_c,t_c+1\}$, every node with color c first chooses a new legitimate color from $\{1,\cdots,\Delta+1\}$, and then communicates its new color to its neighbors.

VI. Conclusion and Open Questions

In this paper, we have proved that the harsh SINR physical constraints do *not* significantly affect the complexity of distributively computing one of the most fundamental network structures, i.e., node coloring. We have also showed that a distance-d coloring is sufficient to schedule an interference free MAC protocol in the SINR model. We believe that these results will help finding efficient solution for other fundamental distributed problems in the SINR model. Many coloring-specific distributed questions are also left open. In particular, we wonder whether it is possible to get rid of the knowledge of Δ and n in our analysis, as it is proven in [29], [2] for the simple interference model.

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