1 The Backus-Gilbert and Tikhonov regularisation methods

Let us briefly review the essence of the Backus-Gilbert method. We are aimed at the solution of the integral equation

$$C(\tau) = \int \frac{d\omega}{\pi}\rho(\omega)\frac{\cosh(\omega(\beta/2-\tau))}{\sinh(\beta/2\omega)} = \int \frac{d\omega}{\pi}\rho(\omega)K(\tau,\omega), \quad \beta = \frac{1}{T} \tag{1}$$

where $\rho(\omega)$ is the spectral function to be determined, $K(\tau,\omega)$ is the thermal kernel. The correlation function $C(\tau)$ is defined at the lattice points $\tau_i = a \cdot i, \quad i = 0, 1, .., N_t - 1$.

The reconstruct the spectral function $\rho(\omega)$ we introduce the estimator $\hat{\rho}(\omega)$

$$\hat{\rho}(\bar{\omega}) = \int d\omega\rho(\omega)\delta(\omega,\bar{\omega}), \tag{2}$$

with the resolution function $\delta(\omega,\bar{\omega})$ which is built as an expansion over the thermal kernel

$$\delta(\omega,\bar{\omega}) = \sum_i g_i(\bar{\omega})K(\tau_i,\omega). \tag{3}$$

The resolution function $\delta(\omega,\bar{\omega})$ has a peak at the $\omega \sim \bar{\omega}$ and the width of this function is expected to be sufficiently small. In this case the estimator reproduces the spectral function. For the resolution function (3) the estimator is given by

$$\hat{\rho}(\bar{\omega}) = \sum_i g_i(\bar{\omega})C(\tau_i). \tag{4}$$

The coefficients $g_i(\bar{\omega})$ can be found from the requirement to make the resolution function as narrow as possible. This requirement is reduced to the minimisation of the width functional

$$A(g_i) = \int d\omega(\omega-\bar{\omega})^2(\delta(\omega,\bar{\omega}))^2 \tag{5}$$

Unfortunately this approach leads to large uncertainties and requires regularization. In particular, instead of the minimisation of the (5) one minimises the function

$$W(g_i) = (1-\lambda)A(g_i) + \lambda B(g_i), \tag{6}$$

where B gives the statistical uncertainty in the values of the correlation function $C(\tau_i)$: $B(g_i) = g^T \cdot \mathrm{Cov} \cdot g$, where Cov is the covariance matrix of the correlator. Thus formula (6) is the compromise between narrow resolution function and statistical uncertainty in the correlator. The procedure how we calculate the g_i, the λ-parameter ($\lambda \in (0,1)$) and estimate the uncertainties is described in our previous paper.

Tikhonov regularisation method is very similar to the Backus-Gilbert with minor modifications in the regularisation procedure. In our study we employed Tikhonov regularisation method, which gives the results shown in Fig. 1.

2 Improved Backus-Gilbert method

One of the problem with the Tikhonov regularization procedure as well as Backus-Gilbert method is that the resolution function depends on lattice data. This dependence is very week. However, because of this dependence it is not strictly correct to compare the spectral functions obtained at different lattice spacings. In addition, the resolution function is not simple function, which complicates the comparison of the obtained results with different models. In order to solve these problems the authors of paper [1] proposed a new approach to the solution of the equation (1). This approach can be briefly formulated as follows

1. The authors of [1] get rid of the requirement to find as narrow resolutions function as possible. Instead they introduced the target function

$$\delta_0(\omega,\bar{\omega}) = \gamma\exp\left(-\frac{(\omega-\bar{\omega})^2}{2\sigma^2}\right), \tag{7}$$

where the constant γ can be determined from the normalization $\int d\omega\delta_0(\omega,\bar{\omega}) = 1$. The parameter σ controls the width of the target function and influences on systematic uncertainties of the approach(see below).

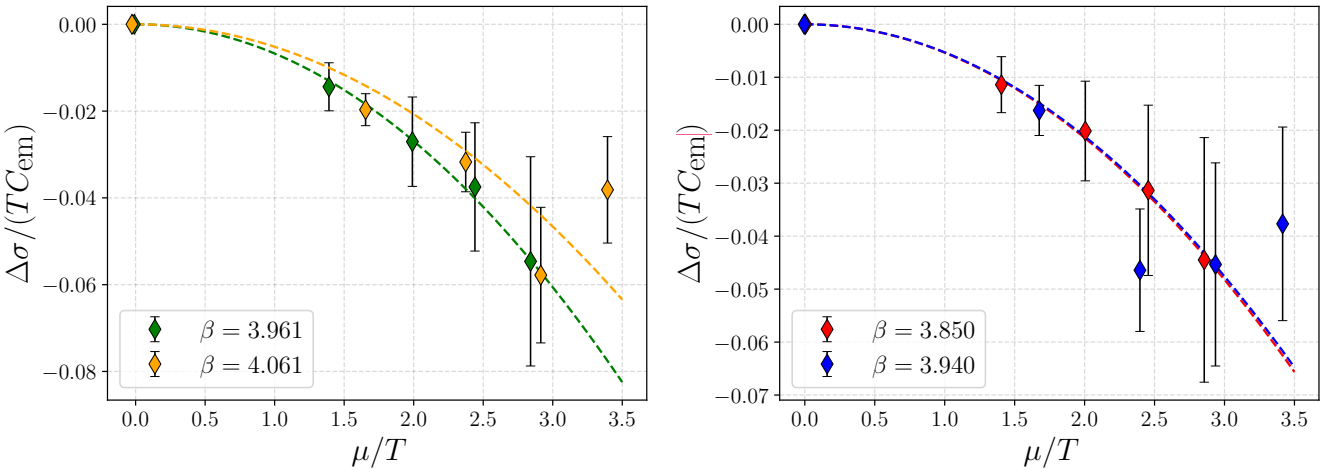

Figure 1: The results obtained within the Tikhonov regularization procedure.

2. The estimator and the resolution function are given by the formulas from the Backus-Gilbert approach (2), (3), (4). However, now we don't need as narrow resolution function as possible. In the new approach the requirement is that the resolution function should be as close to the $\delta_0(\omega, \bar{\omega})$ as possible. To fulfill this requirement it is proposed to minimise the functional

$$A(g_i) = \int d\omega (\delta(\omega, \bar{\omega}) - \delta_0(\omega, \bar{\omega}))^2 \tag{8}$$

3. Similarly to the original Backus-Gilbert method the new approach requires regularisation, which is introduced through the functional

$$W(g_i) = (1-\lambda)A(g_i) + \frac{\lambda}{(C(0))^2} B(g_i), \tag{9}$$

where $B(g_i) = g^T \cdot \mathrm{Cov} \cdot g$ with the covariance matrix Cov and it accounts the statistical uncertainty in the correlator. Notice the difference in last formula as compared to formula (6). The term proportional to the $B(g_i)$ has been divided by the value of the correlation function at zero Euclidean time.

4. An important disadvantage of the Backus-Gilbert method is that the λ parameter is not fixed by the predure. The authors of [1] proposed to fix the λ by the value at which the functional (9) has extremum. Notice that in this case we first find the values of the coefficients g_i which depend on the λ then put them back to functional (9) and find how the W depend on λ.

5. New approach does not have uncertainty due to the finite width of the resolution function. However it acquires a new uncertainty which results from the deviation of the resolution function (3) from the target resolution function. It was proposed to estimate this systematic uncertainty Δ_{syst} though the following formula

$$\Delta_{syst} = |r|\hat{\rho}(\bar{\omega}), \tag{10}$$

where r is the relative deviation of the resolution from the target function at the peak position $\omega = \bar{\omega}$

$$r = 1 - \frac{\delta(\bar{\omega}, \bar{\omega})}{\delta_0(\bar{\omega}, \bar{\omega})} \tag{11}$$

6. Finally the width parameter σ has to be determined. In order to do this one can use the fact that the systematic uncertainty Δ_{syst} is very sensitive to the value of the σ. Thus the value of the σ can be found from the requirement to have acceptable systematic uncertainties of the calculation.

Now let us proceed to application of the new approach to our lattice data. Before presenting our results we would like to discuss some modifications of the approach proposed in [1]. First since we are interesting in the conductivity which is related to the behaviour of the spectral function at $\omega = 0$, we employ all above formulas for

$\omega = 0$. Notice also that in the case of staggered fermions one has two branches of lattice data: even and odd. So we apply the approach to both branches and then take average of the even and odd results as has been done in our previous paper.

We tried to build the target function (7) as expansion (3). Unfortunately it is not possible to do this since we have only three independent terms in expansion (3) (for the lattice with $N_t = 12$) and this not sufficient to reproduce (3). Instead we decided to change the target function form (7) to

$$\delta_0(\omega) = \frac{1}{(\sigma\pi/2)^2} \frac{\omega}{\sinh(\omega/\sigma)}. \tag{12}$$

The advantage of the usage of this function stems from the fact that at $\sigma = 2T$ this function coincides with the thermal kernel at $\tau = \beta/2$: $\delta_0(\omega) = K(\beta/2, \omega)$ (see formula (1)). This implies that at $\sigma = 2T$ only one point is needed to reproduce the $\delta_0(\omega)$ with 100% accuracy. At $\sigma \neq 2T$ we need more points but not as many as for the target function (7).

Now let us consider the question: what width of the target function σ can be well approximated with our lattice data. We have already noticed that $\sigma = 2\,T$ can be well approximated since in this case the target function coincides with the thermal kernel at $\tau = 2\,T$. To study the target function with smaller width in Fig. 2 we plot the relative deviation of the resolution from the target function r as a function of λ for various values of $\sigma < 2\,T$. From this figure one sees that the smaller the σ the larger the deviation of the resolution function from the target function. This deviation is also larger for larger values of the λ due to the fact that regularisation leads to larger deviation from the target function. From Fig. 2 one can see that for the uncertainty 5%(dashed line) it is reasonable to take $\sigma \sim 1.7T$[1].

Further let us discuss formula (9) and the way how the authors of paper [1] propose to find the optimal value of the λ parameter. First, notice that formula (9) and the position of extremum strongly depend on the value $C(0)$. Instead of the $C(0)$ we divided the term $B(g_i)$ by arbitrary constant and studied how the extremum position depends on this factor. We found that for sufficiently large or small values of this factor extremum disappears. Notice also that for plenty of QCD correlators the value of the $C(0)$ represents ultraviolet physics whereas it is interesting to study infrared properties. Based on these facts and on our numerical experiments we have drawn a conclusion that the way how it was propose to fix the λ is not convincing. We should invent something better.

However, in order to obtain some results we divided the $B(g_i)$ by the $(C(0))^2$. In this case one can find the extremum of the $W(\lambda)$ functional. We have shown the $W(\lambda)$ in Fig. 3 for $\mu/T = 3$ and $\beta = 3.96$. Similar plots can be drawn for all lattice parameters. Using this position of the extremum we calculated the conductivity for all lattice data. The results are shown in Fig. 4. In order to study the λ dependence of our results in Fig. 5 we plot the conductivity for various values of the λ parameter. Comparing plots on Fig. 4 and Fig. 5 one sees that they agree within the uncertainty. In addition one can see that the plots obtained at the peak value of the λ parameter overestimates the uncertainty. So to get better results(smaller uncertainties) one can take the value of the λ parameter smaller than the peak position.

References

[1] Martin Hansen, Alessandro Lupo, and Nazario Tantalo. Extraction of spectral densities from lattice correlators. *Physical Review D*, 99(9), May 2019.

[1] Notice that the parameter σ paramerises the width of the target function, but this is not the width. Real width of the resolution function is given by the expression $\sqrt{\langle\omega^2\rangle}$. For $\sigma = 2\,T$ $\sqrt{\langle\omega^2\rangle} = 4.4\,T$, whereas for $\sigma = 1.7\,T$ $\sqrt{\langle\omega^2\rangle} = 3.8\,T$

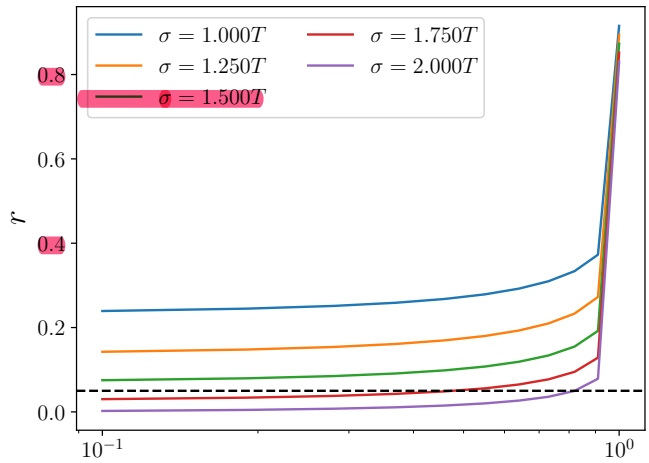

Figure 2: The relative deviation of the resolution from the target function r (11) as a function of the λ for various value of the σ parameter. The dashed line corresponds to the uncertainty 5% in the approximation of the target function.

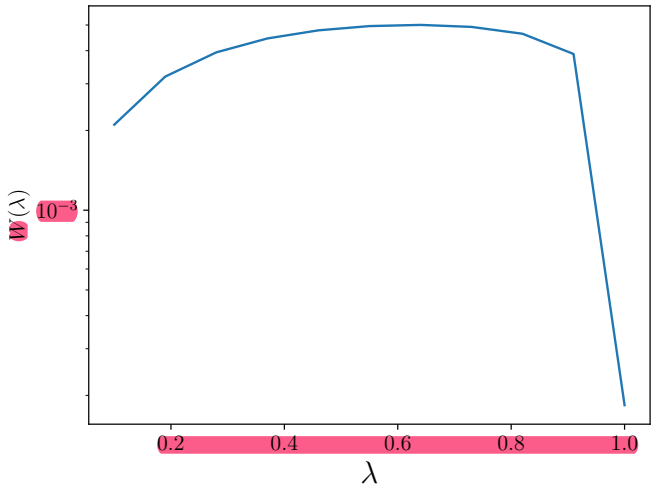

Figure 3: Functional $W(\lambda)$ evaluated with division at $C(0)^2$, $\mu/T = 3.0$, $\sigma = 1.7\,T$ and $\beta = 3.96$.

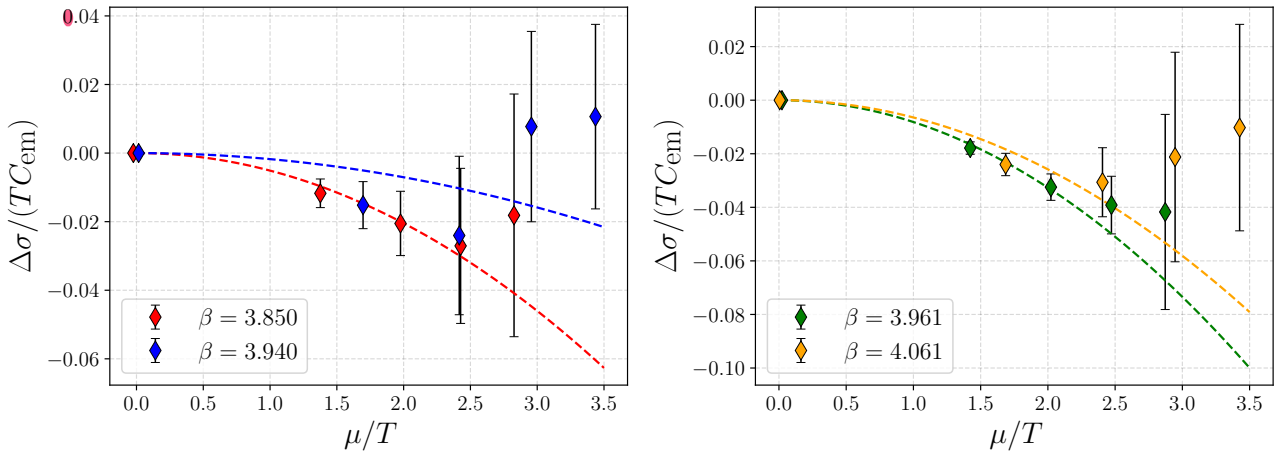

Figure 4: Conductivities as the function of μ/T evaluated at $\lambda = 0.6$, corresponding to extremum value of $W(\lambda)$.

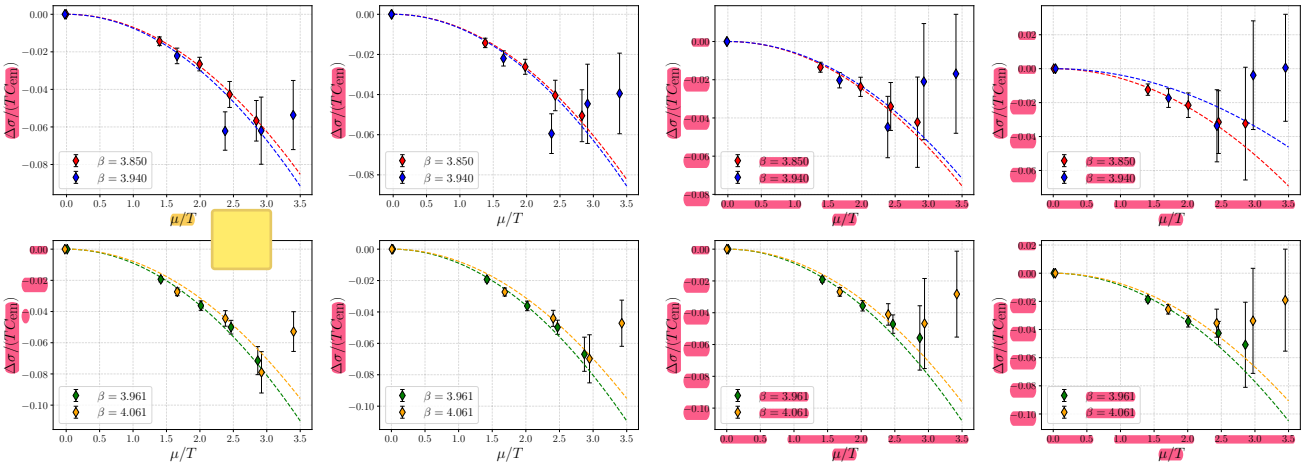

Figure 5: Conductivity as the function of μ/T with $\sigma = 1.7T$. The values of λ are 10^{-3}, 10^{-2}, 10^{-1}, 0.3 (from left to right).