Computational Methods for Detection,
Estimation and Identification 2022/2023

Assignment 2

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1 Overview

This assignment, as the previous one, revolves around a robotic platform equipped with a sonar system that measures the distance to a wall. The goal of the experiment is to evaluate the performance of the vehicle's breaking system.

Differently from the last assignment, an old sonar with overheating problems is being used. As time goes by the device warms up and the measuring accuracy decreases.

For the purpose of the experiment it can be assumed that the noise standard deviation for the i^{th} reading is $\sigma_i = \text{i}.\sigma$ where σ is a constant. Note that the true value of the standard deviation is not known.

2 Motion parameters estimation

2.1 Mathematical Model

As of the last assignment, it's possible to derive the structure of matrices G and m, knowing that the system has with n = 3 parameters and m = 100 observations.

$$G = \begin{bmatrix} 1 & t_1 & \frac{t_1^2}{2} \\ 1 & t_2 & \frac{t_2^2}{2} \\ & \vdots & \\ 1 & t_{100} & \frac{t_{100}^2}{2} \end{bmatrix}_{100\times3} \tag{1}$$

$$m = \begin{bmatrix} y_0 \\ v_0 \\ a_0 \end{bmatrix}_{3\times1} \tag{2}$$

$$d = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{100} \end{bmatrix}_{100\times1} \tag{3}$$

In accordance with the description of the problem, it is also possible to obtain the t matrix, containing all the time stamps of the collection of the readings.

$$t = \begin{bmatrix} 0.1 & 0.2 & 0.3 & \dots & 9.9 & 10 \end{bmatrix}_{1\times100} \tag{4}$$

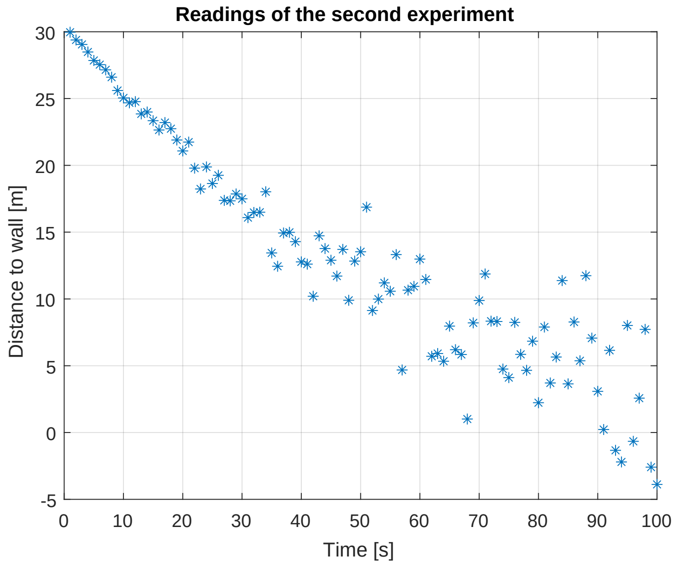

Figure 1: Visualization of measurements in d.

2.2 Visualization of Data

In order to ease the comprehension of the obtained data, and once the readings matrix was obtained, the first step was to visualize the data being worked, as shown on figure (1).

It is possible to notice that distance to the wall decreases as time passes in what appears to be a quadratic behaviour, in agreement with the obtained mathematical model. It is also important to notice that as the time progresses, the readings show a lot of dispersion.

3 Least Squares solution

3.1 Estimation

In order to obtain an initial estimation of the motion parameters, the Least Squares solution was calculated.

Assuming that the model and observations are full rank, ie., rank(G) = n (or, equally, det(G^TG) $\neq$ 0) the solution of the least squares solution is given by the Pseudo-Inverse of Moore-Penrose.

$$m_{L_2} = (G^TG)^{-1}G^T.d \tag{5}$$

Knowing this information, and given the data in d, m_{L_2} was calculated using equation (5).

$$m_{L2} = \begin{bmatrix} y_0 = 29.578 \\ v_0 = -4.519 \\ a_0 = 0.367 \end{bmatrix}_{3\times1} \tag{6}$$

From the results of (6) and knowing that d_{est} = G.m_{L2}, the results of d_{est} were calculated. These values can be seen in figure (2).

3.2 Residuals interpretation

At first glance, the estimation seems to correspond to the data. In order to further analyse the obtained estimation re-

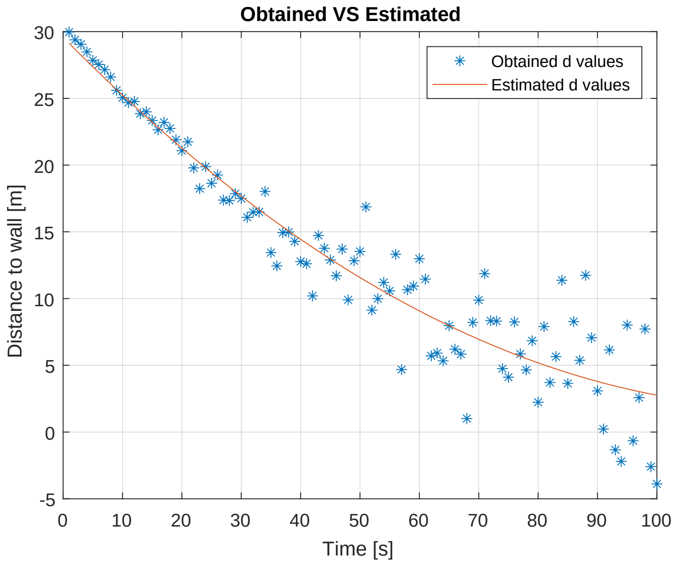

Figure 2: Estimated reading values.

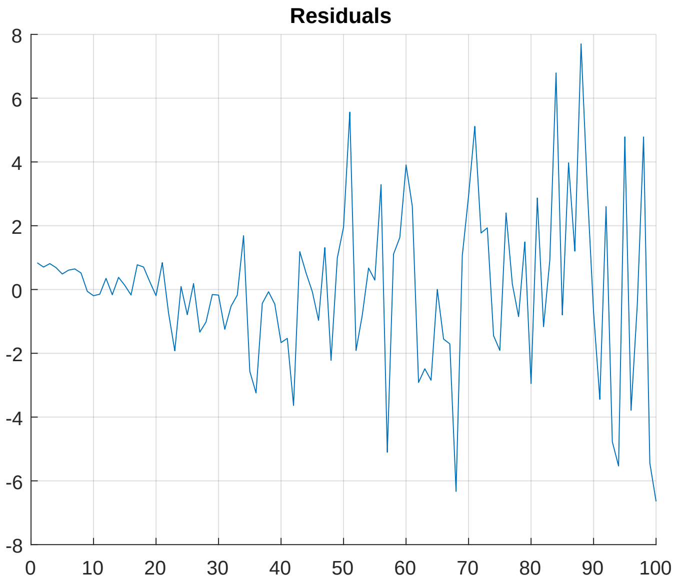

Figure 3: Calculated residuals.

sults, the residuals were calculated using expression (7).

$$r = d - d_{est} = d - G.m_{L2} \tag{7}$$

The calculated residuals can be seen in figure (3). By visual analysis, although is clear to see that they appear to be random, the standard deviation seems to increase as the time increases, as previously established by the problem statement. This phenomenon is called a proportional effect, and must be corrected.

In order to analyse if the residuals follow a normal distribution, they were subjected to a quantile-quantile test, as seen on figure (4).

By visual inference of the quantile-quantile test is possible to affirm that the residuals do not follow a normal distribution, as the sample data dos not follow a linear relation that resembles that of a standard normal distribution.

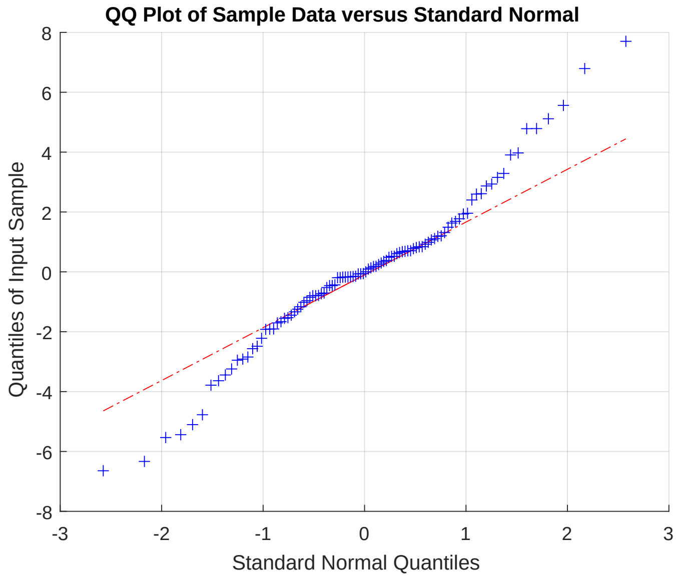

Figure 4: Quantile-quantile test of calculated residuals.

4 Maximum Likelihood

4.1 Principles

The maximum likelihood estimation is a general method in which the essential problem is to find the most likely model, as characterized by the elements of the parameter vector m, for the set of observations contained in the vector d. After some algebraic manipulation, the maximum likelihood solution to the problem can be simplified and be given by (8).

$$min \sum_{i=1}^{m} \frac{(d_i - (G.m)i)^2}{{\sigma_i}^2} \tag{8}$$

Aside from the distinct $\frac{1}{{\sigma_i}^2}$ factors in each term of the sum, this is identical to the least squares problem for Gm = d. (8) is commonly referred to as the weighted least squares problem. In order to incorporate the data standard deviations into the weighted least square solution, a matrix (9) of weight is created.

$$W = diag(\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \ldots, \frac{1}{\sigma_m}) \tag{9}$$

Using (9) to weight the system of equations is possible to obtain (10) and (11).

$$G_w = W.G \tag{10}$$

$$d_w = W.d \tag{11}$$

Using (10) and (11), the weighted system of equations is given by (12).

$$G_w.m = d_w \tag{12}$$

And thus the maximum likelihood solution can be given by the least square solution to (12), as described by (13).

$$m_{L_{2_w}} = (G_w^T G_w)^{-1} G_w^T . d_w \tag{13}$$

4.2 Unknown Standard Deviation

Since in this case the standard deviation is unknown, we need to calculate an approximation to this value. Using the residuals vector r calculated from the Least Squares solution, it's possible to obtain an approximated value (14).

$$s = \frac{||r||_2}{\sqrt{\nu}} \tag{14}$$

Using this expression and the vector of residuals r, it was possible to calculate a value of s = 2.5915.

4.3 Proportional Effect

As said before, the data is under the influence of the proportional effect, and as so it's necessary to build a vector s_i (15), which contains the approximated standard deviation for each measurement.

$$s_i = t.s \tag{15}$$

This vector s_i will be used to calculate the weight matrix W (9), in order to obtain the maximum likelihood solution.

4.4 Estimation

Now that all the necessary components and knowledge to calculate the maximum likelihood estimation is gathered, let's calculate it and analyse the results.

First, a matrix W was calculated using (9) and (15) with s = 2.5915. Using W and the expressions (10) and (11), the matrices G_w and d_w were obtained.

Next, $m_{L_{2_w}}$ was calculated using expression (13). It produced the results in (16).

$$m_{L2_w} = \begin{bmatrix} y_0 = 30.4676 \\ v_0 = -5.1501 \\ a_0 = 0.5032 \end{bmatrix}_{3\times1} \tag{16}$$

To obtain the estimated weighted d values, d_w, the following expression was used (17).

$$d_w = G.m_{L2_w} \tag{17}$$

Figure (5) shows the results of d_w against the original data and the first estimation. Note that G and not G_w was used, as that would describe a different model. In order to further analyse the results, the individual weighted residual vector was calculated. It is given by expression (18).

$$r_{w,i} = \frac{(d - G.m)_i}{\sigma_i} = (d_w - G_w.m)_i \tag{18}$$

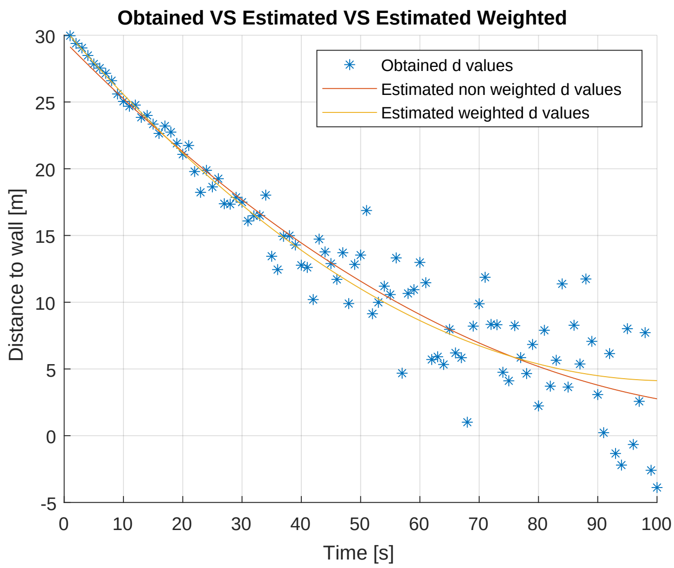

Figure 5: Data Vs LS Estimation Vs ML Estimation.

Figure 6: Residuals comparison.

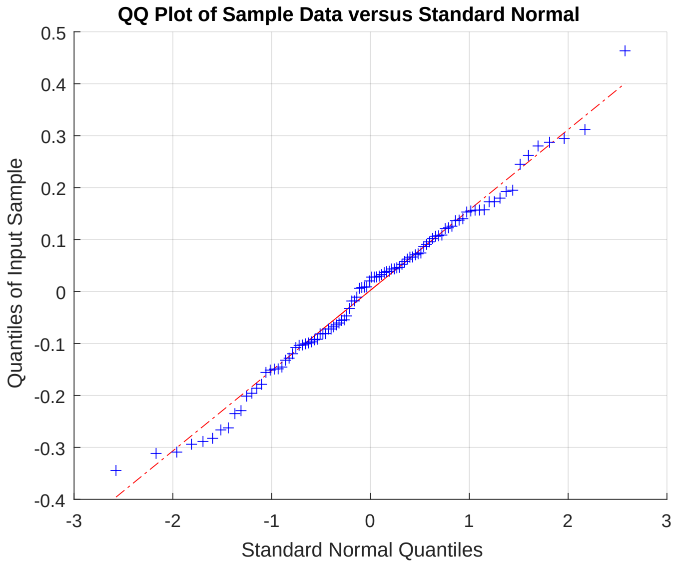

Figure 7: Quantile-quantile test of calculated weighted residuals.

As a way to understand the impact of weighting the system, the weighted residuals were plotted against the original residuals, as seen on figure (6).

By analysing the weighted residuals, it's possible to affirm that they don't follow a specific trend, differently from the previous calculated residuals. In order to show that the weighted residuals follow a normal distribution, they were subjected to a quantile-quantile test. The result of this test can be seen in figure (7).

By analysing the plot, it's possible to affirm that the weighted residuals follows a normal distribution, unlike the original residuals.

5 P-test

The estimation obtained via the Maximum Likelihood estimation cannot be checked using p-test because the true value of σ is unknown. To prove this let's try to calculate χ^2_{obs} using equation (19) with $\sigma = s$ (14).

$$\chi^2_{obs} = \sum_{i=1}^{m} \frac{(d_i - (Gm_{L2})_i)^2}{\sigma_i^2} \tag{19}$$

$$\chi^2_{obs} = \sum_{i=1}^{m} \frac{(d_i - (Gm_{L2})_i)^2}{(\frac{||r||_2}{\sqrt{\nu}})^2} = \sum_{i=1}^{m} (\sqrt{\nu})^2 = \nu \tag{20}$$

As can be seen, $\chi^2_{obs} = \nu$, and as so the model would always pass the χ^2 test.

6 Confidence Intervals

The confidence intervals for the model parameters are given by (21).

$$m_{L_2} \pm 1.96 diag(Cov(m_{L_2}))^{\frac{1}{2}} \tag{21}$$

Since the value of σ is unknown and only an estimation s is know, the covariance matrix is also not know and as so an estimation is needed (22).

$$Cov(m_{L_2}) \approx \tilde{C} = s^2(G^TG)^{-1} \tag{22}$$

In this specific case, since the desired results are the confidence intervals of the Maximum Likelihood estimation the correct expression is (23).

$$\tilde{C} = s^2(G_w^TG_w)^{-1} \tag{23}$$

Solving this equation gives the results (24).

$$m_{L2_w} = \begin{bmatrix} y_0 = 30.4676 \pm 1.2839 \\ v_0 = -5.1501 \pm 3.2780 \\ a_0 = 0.5032 \pm 1.0396 \end{bmatrix}_{3\times1} \tag{24}$$

References

[1] R. C. Aster, B. Borchers, and C. H. Thurber, Parameter Estimation and Inverse Problems. Elsevier, 2018.

[2] Nuno Gonçalves, sebenta_TCDEI_chapter1_linear_inverse_problems, University of Coimbra, 2023