Computational Methods for Detection, Estimation and Identification 2022/2023

Assignment 1

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1 Overview

This assignment revolves around a robotic platform equipped with a sonar system that measures the distance to a wall. The goal of the experiment is to evaluate the performance of the vehicle's breaking system.

To do so, the robot was set on motion towards the wall with a constant velocity v_0. At a distance d_0 from the wall, the break is actuated, in order to induce a constant acceleration a_0. During this, the sonar system measures the decreasing distance to the wall with a frequency of 10 Hz. The readings are collected during 10 seconds and stacked in a data vector d with dimension 100 $\times$ 1.

After 50 repetitions of the experiment, a 100x50 matrix is produced, in which each column is a data vector d with 100 readings.

It is also stated that the noise affecting the readings is independent and follows a standard normal distribution. This means that the mean μ = 0 and the standard deviation σ^2 = 1.

2 Motion parameters estimation

2.1 Mathematical Model

Since we are presented with a matrix d of dimensions 100x50 (50 runs with 100 measures each) readings and want to derive the vector m of dimensions 3x50 (50 runs with 3 motion parameters each) of the motion parameters for each repetition, we are facing an inverse estimation problem.

As so, our system is of the form

$$G.m = d \tag{1}$$

with

- G = operator/math model
- m = model/parameters
- d = data/measures

This experiment has been addressed during the theoretical class. As so, the mathematical model describing the position of the robot is available in the theoretical slides, and is given by equation (2).

$$G(m,t) = y_0 + v_0.t + \frac{t^2}{2}.a_0 \tag{2}$$

From (1) and (2) is possible to derive the structure of matrices G and m, knowing that the system has with n = 3 parameters and m = 100 observations in 50 repetitions.

$$G = \begin{bmatrix} 1 & t_1 & \frac{t_1^2}{2} \\ 1 & t_2 & \frac{t_2^2}{2} \\ & \vdots & \\ 1 & t_{100} & \frac{t_{100}^2}{2} \end{bmatrix}_{100\times3} \tag{3}$$

$$m = \begin{bmatrix} y_{0_1} & y_{0_2} & & y_{0_{50}} \\ v_{0_1} & v_{0_2} & \dots & v_{0_{50}} \\ a_{0_1} & a_{0_2} & & a_{0_{50}} \end{bmatrix}_{3\times50} \tag{4}$$

$$d = \begin{bmatrix} d_{1_1} & d_{2_1} & \dots & d_{50_1} \\ d_{1_2} & d_{2_2} & \dots & d_{50_2} \\ \vdots & \vdots & \ddots & \vdots \\ d_{1_{100}} & d_{2_{100}} & \dots & d_{50_{100}} \end{bmatrix}_{100\times50} \tag{5}$$

In accordance with the description of the problem, it is also possible to obtain the t matrix, containing all the time stamps of the collection of the readings.

$$t = \begin{bmatrix} 0.1 & 0.2 & 0.3 & \dots & 9.9 & 10 \end{bmatrix}_{1\times100}$$

2.2 Visualization of Data

In order to ease the comprehension of the obtained data, and once the readings matrix was obtained, the first step was to visualize the data being worked.

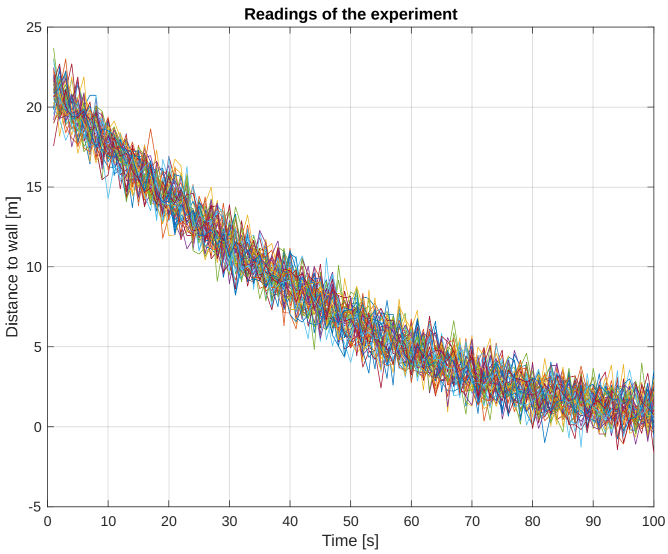

Figure 1: Visualization of measurements in d.

It is possible to notice that distance to the wall decreases as time passes in what appears to be a quadratic behaviour, in agreement with the obtained mathematical model.

2.3 Least Squares solution

As stated before, the readings in this experiment (as well as all real readings ever obtained) are affected by noise. Therefore there is no solution to the problem that satisfies (1) exactly. In this case, an approximate solution that minimizes the residuals vector (6) between the actual data and G.m may still be found.

$$r = d - G.m \tag{6}$$

One of the most common methods to obtain this approximation is called the Least Squares solution. It is defined so that the minimisation objective is the minimisation of the sum of the residuals squared (7), using the L_2 norm.

$$min \sum (d_i - (G.m)i)^2 \tag{7}$$

Assuming that the model and observations are full rank, ie., rank(G) = n (or, equally, det$(G^T G) \neq 0$) the solution of the least squares solution is given by the Pseudo-Inverse of Moore-Penrose.

$$m_{L_2} = (G^T G)^{-1} G^T.d \tag{8}$$

Knowing this information, and given the data in d, m_{L_2} was calculated using the (8).

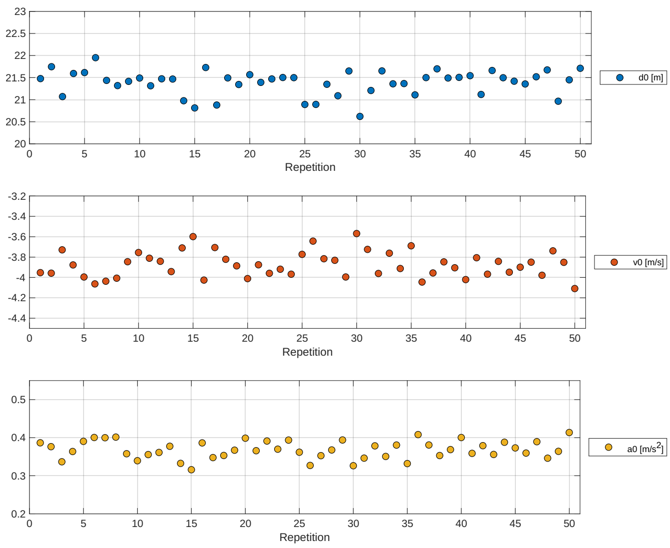

Figure 2: Estimated motion parameters.

In Fig2 it is possible to observe the estimated values of d_0, v_0 and a_0 for every repetition od the experiment.

Using the estimated motion parameters, the estimated data for the measurements was obtained by the application of (2) with m = m_{L_2}. A comparison of both data can be seen in Fig3.

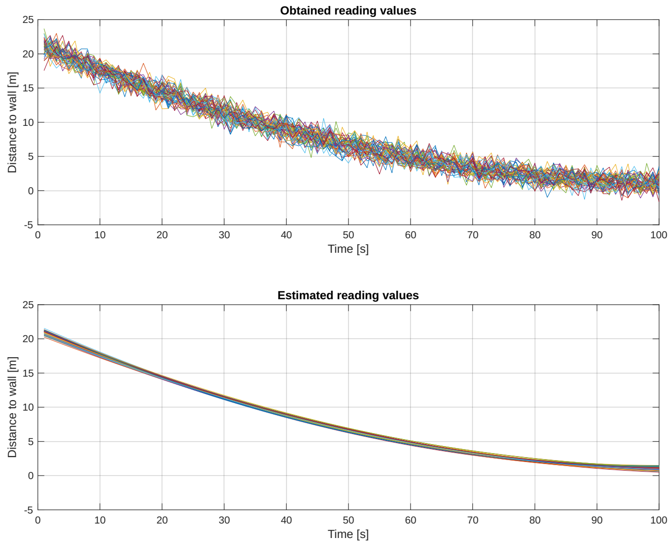

Figure 3: Observed VS Estimated motion parameters.

2.4 Discussion of results

The LS solution is, however, insufficient in this form. The complete analysis of the estimation problem involves the following three steps:

1. parameters estimation;
2. estimate and characterise the errors in the parameters and consequently compute a confidence interval for the estimations;
3. evaluating statistically the estimation quality.

3 P-test

3.1 The χ^2 statistic

The sum of the squares of the residuals also provides useful statistical information about the quality of the model estimated obtained with least squares. The chi-square statistic is given by

$$\chi^2_{obs} = \sum_{i=1}^{m} \frac{d_i - (Gm_{L2})i)^2}{\sigma_i^2}$$

But since in our case the standard deviation $\sigma^2 = 1$, this expression can be simplified and reduced to

$$\chi^2_{obs} = \sum_{i=1}^{m} \frac{d_i - (Gm_{L2})i)^2}{1} = \sum_{i=1}^{m} (d_i - (Gm_{L2})i)^2 = \sum_{i=1}^{m} r_i^2 \tag{9}$$

with r = $[r_1 \ r_2 \ ... \ r_i]$ being the residuals vector.

It can be shown that under our assumptions χ^2_{obs} has a χ^2 distribution with k = m n = 100 - 3 = 97 degrees of freedom (dof).

3.2 The χ^2 test

The χ^2 test provides a statistical assessment of the assumptions that were used in finding the least squares solution. In this test, we compute χ^2_{obs} and compare it to the theoretical χ^2 distribution with k dof. The probability of obtaining a χ^2 value as large or larger than the observed value (and hence a worse misfit between data and model data predictions than that obtained) is called the p-value of the test, and is given by

$$p = \int_{\chi^2_{obs}}^{\infty} f_{\chi^2}(x)\, dx \tag{10}$$

3.3 P-values calculation

Using the Matlab cdf() (cumulative distribution function) it is possible to calculate

$$cdf = \int_{-\infty}^{\chi^2_{obs}} f_{\chi^2}(x)\, dx$$

As so, using this function is possible to calculate the p-value associated with repetition

$$p = 1 - cdf = 1 - \int_{-\infty}^{\chi^2_{obs}} f_{\chi^2}(x)\, dx = \int_{\chi^2_{obs}}^{\infty} f_{\chi^2}(x)\, dx \tag{11}$$

In Fig4 it is possible to see the calculated p-values distribution.

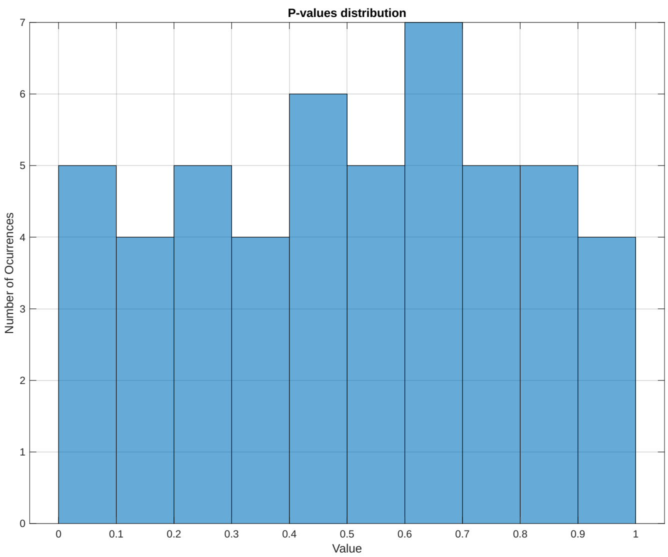

Figure 4: Histogram of p values.

3.4 P-values distribution

Observing Fig4, there's a temptation to say that the p-values obtained from the p-test follow an uniform distribution.

Let's prove it using the χ^2 test of goodness of fit.

In this case, the null hypothesis states that the obtained p-values follow an uniform distribution. The next step is to construct an uniformly distributed random variable. Since our variable p = [5 4 5 4 6 5 7 5 5 4] has a mean value of 5, our uniformly distributed random variable p_{unif} = [5 5 5 5 5 5 5 5 5 5].

Using Matlab chi2gof() function with p and p_{unif} as inputs for O(obtained distribution) and E(expected distribution) with and $\alpha = 0.05$ (95% confidence), an h = 0 is obtained, which confirms the established null hypothesis, and so confirms that the obtained p-values follow an uniform distribution.

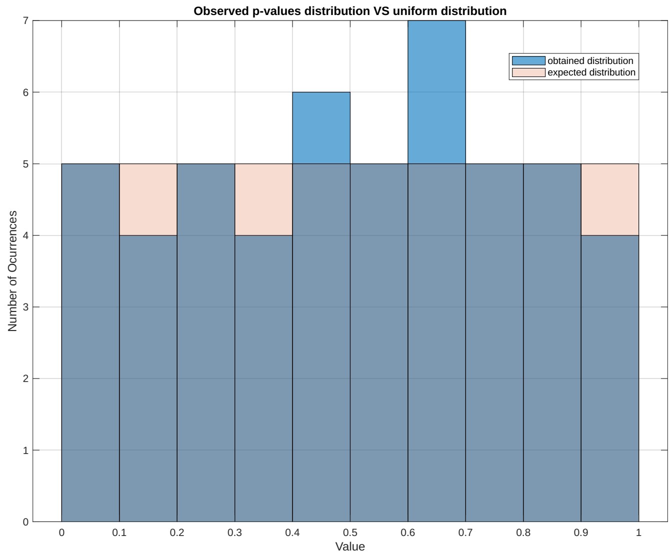

Figure 5: Histogram of obtained VS expected p values.

4 Confidence Intervals

The confidence intervals correspond to dispersion metrics in which the central value of a certain data point corresponds to the value which minimizes the residues (Least Square Solution) - has the maximum likelihood.

We can compute 95% confidence intervals for individual model parameters using the fact that each model parameter m_i has a normal distribution with a mean given by the corresponding element of m_{true} and variance $Cov(m_{L_2})_{i,i}$. The 95% confidence intervals are given by

$$m_{L_2} \pm 1.96 diag(Cov(m_{L_2}))^{\frac{1}{2}} \tag{12}$$

with

$$(Cov(m_{L2=\sigma^2(G^TG)^{-1}} \tag{13}$$

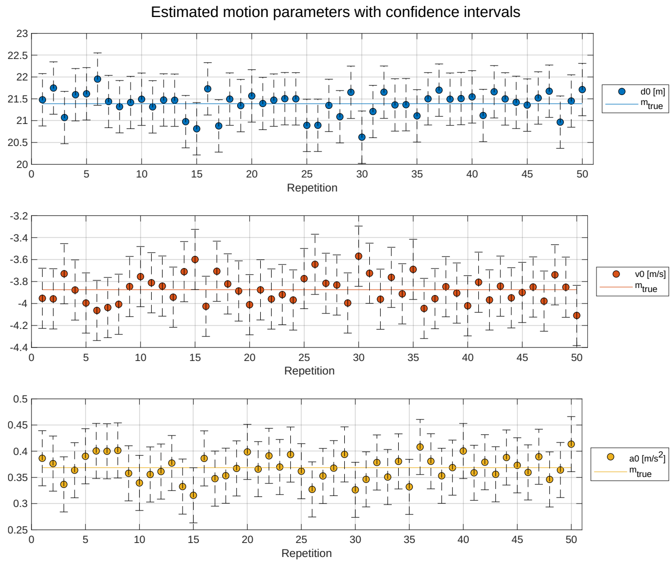

Figure 6: Confidence intervals.

Since more then 95% of the calculated intervals fall on the m_{true} value, the intervals match the results.