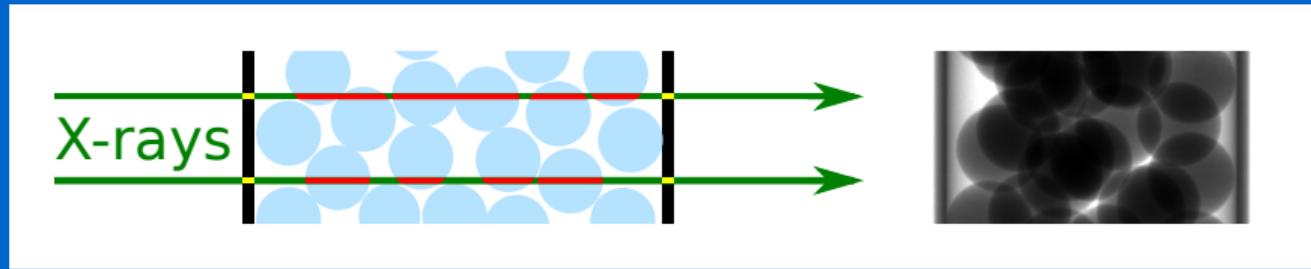


# Measuring the volume fraction of **dynamic** granular systems

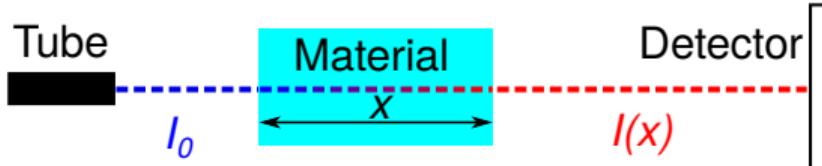


## Correction of beam hardening in X-ray radiograms

Baur *et al*, *Rev. Sci. Instrum.* (2019)

In collaboration with Norman Uhlmann, Fraunhofer EZRT

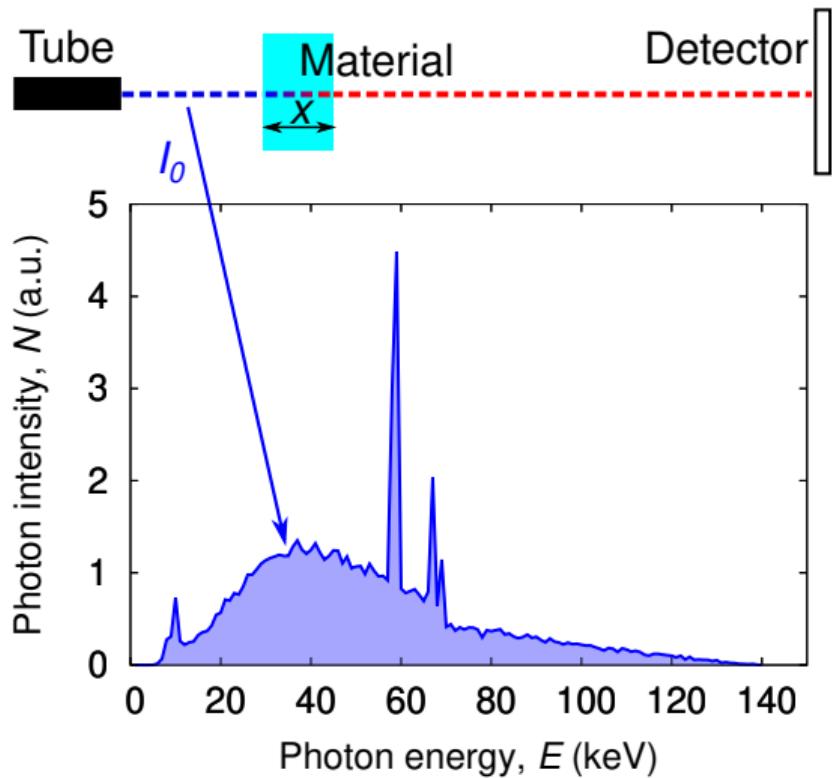
# Attenuation of X-rays



Beer-Lambert's law

$$I(x) = I_0 \exp(-\mu x)$$

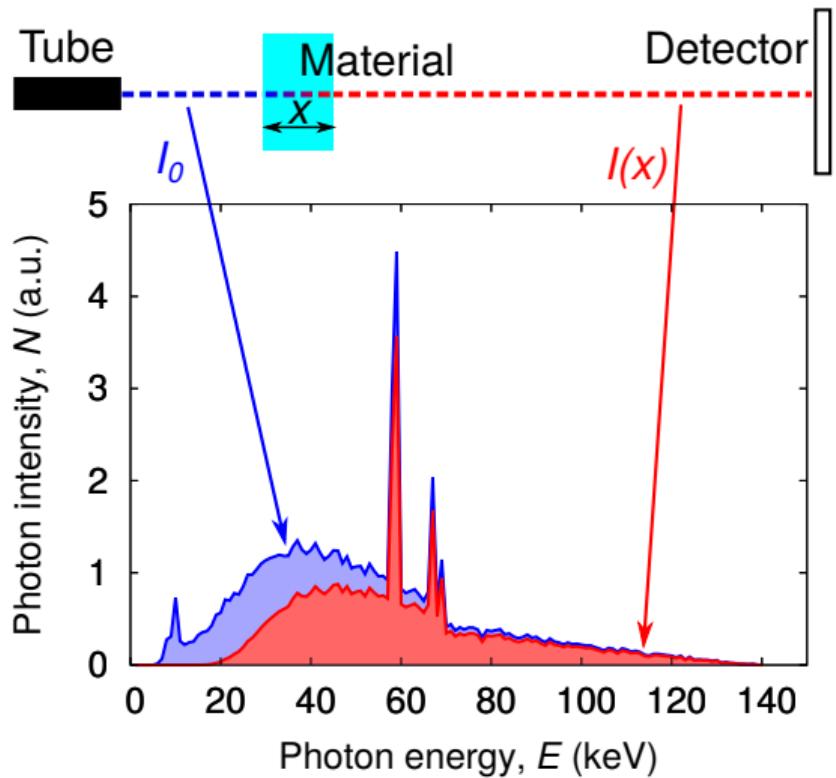
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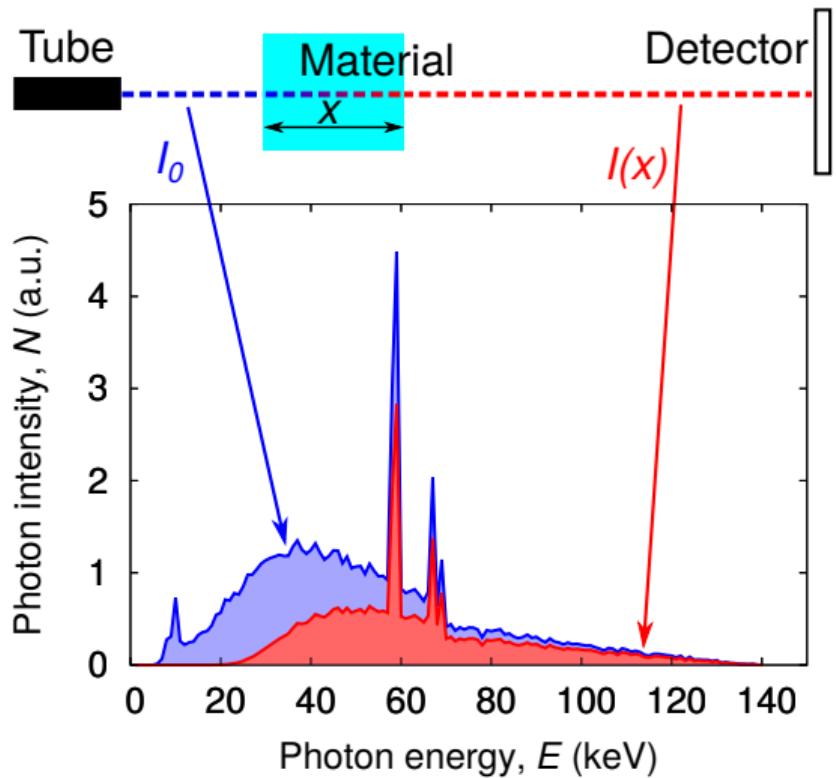
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$$I(x) = I_0 \exp(-\mu x)$$

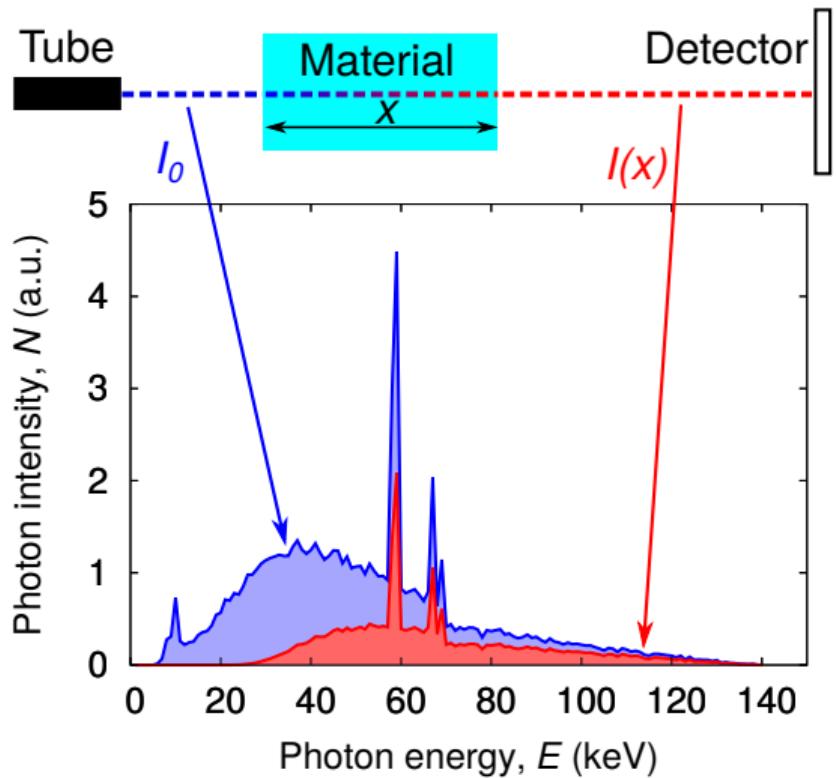
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Beer-Lambert's law

$$I(x) = I_0 \exp(-\mu x)$$

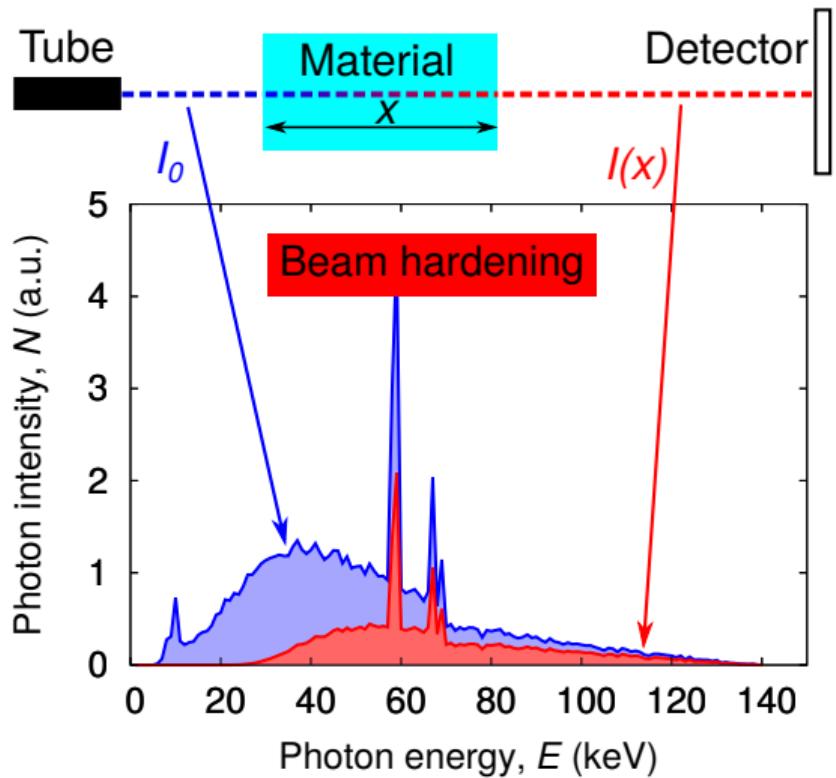
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$$I(x) = I_0 \exp(-\mu x)$$

# Attenuation of X-rays

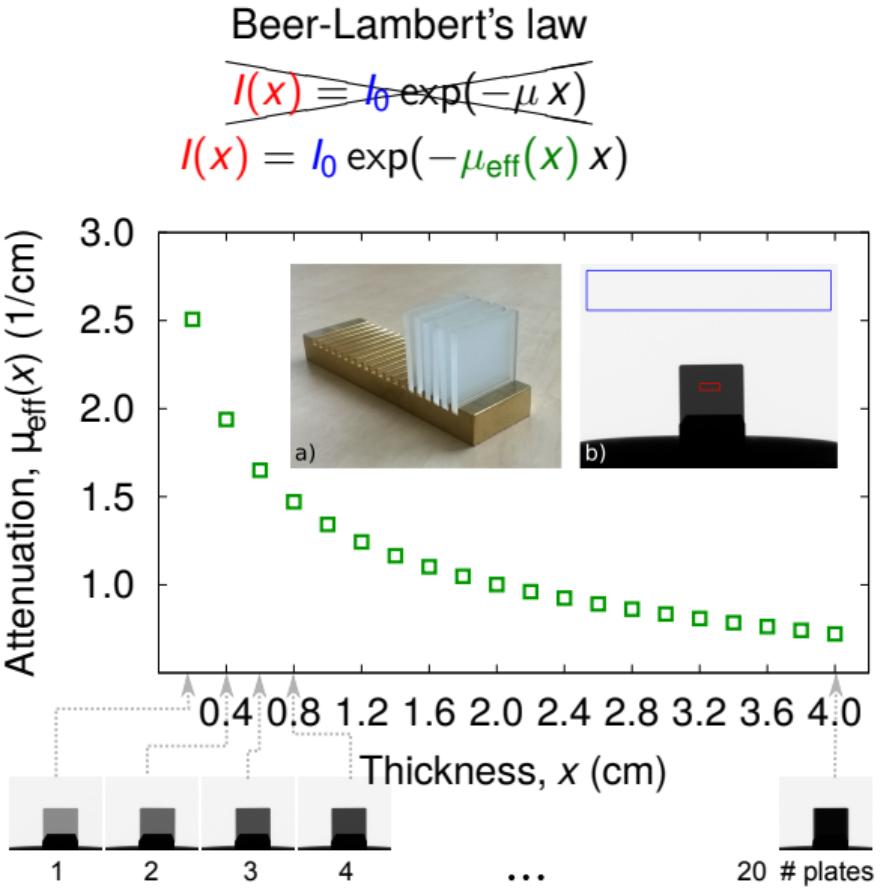
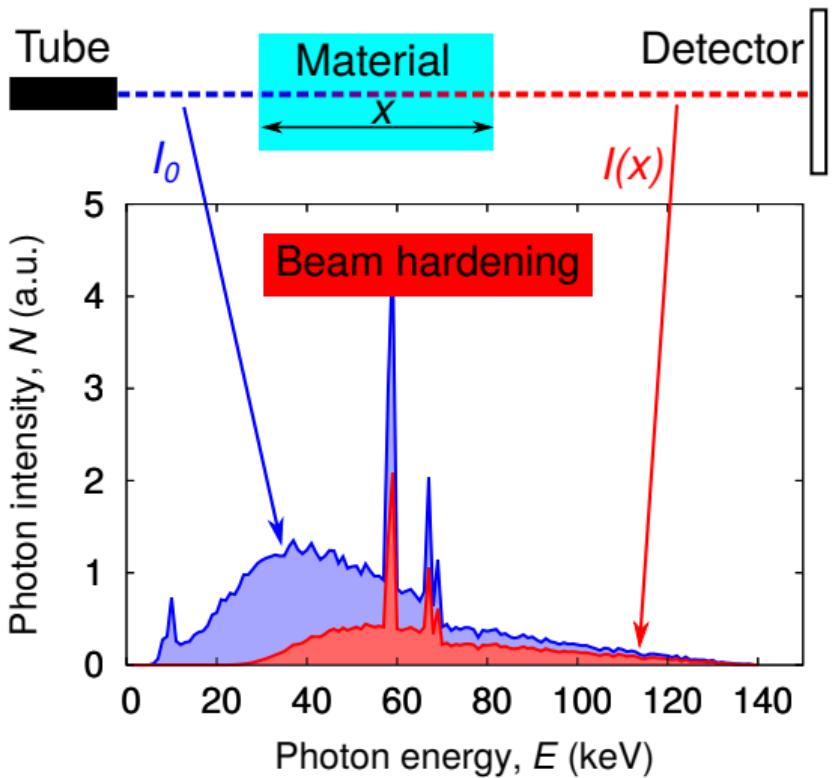


Beer-Lambert's law

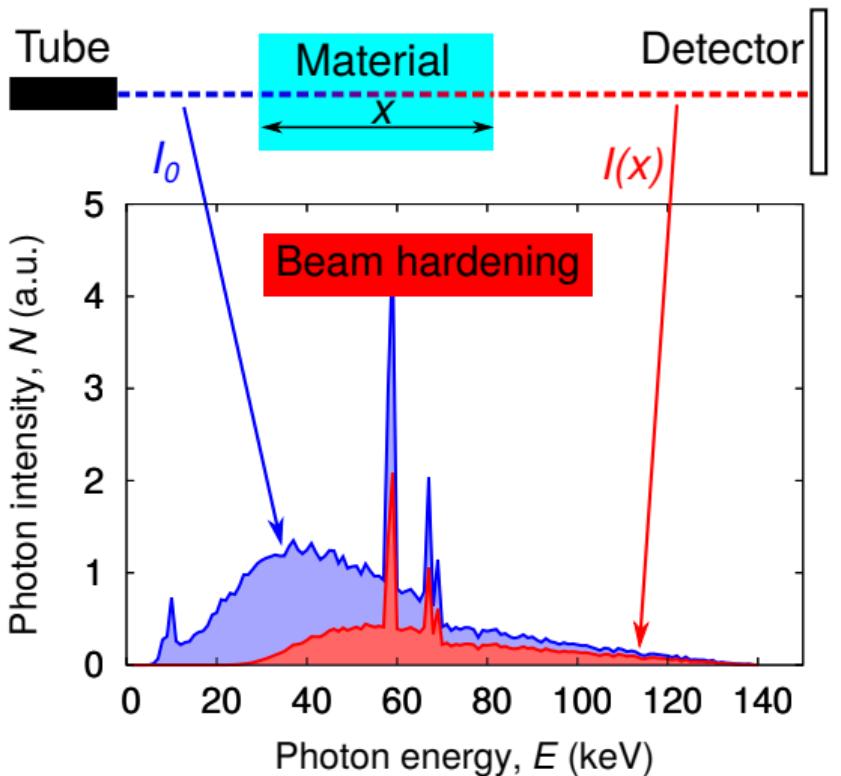
$$I(x) \equiv I_0 \exp(-\mu x)$$

$$I(x) = I_0 \exp(-\mu_{\text{eff}}(x) x)$$

# Attenuation of X-rays

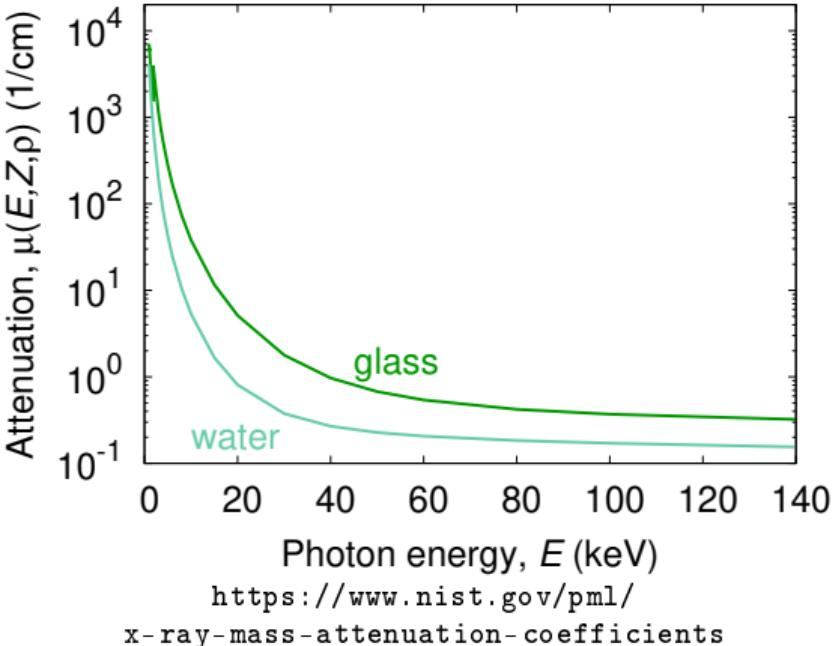


# Attenuation of X-rays



Beer-Lambert's law

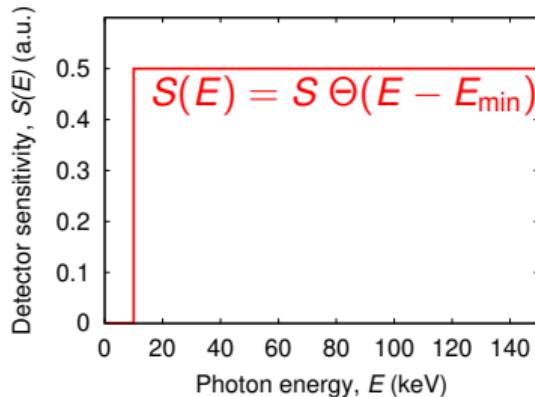
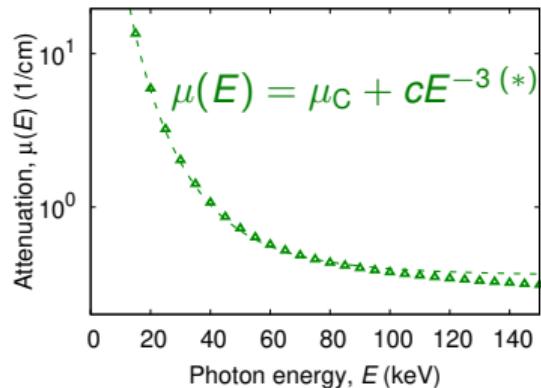
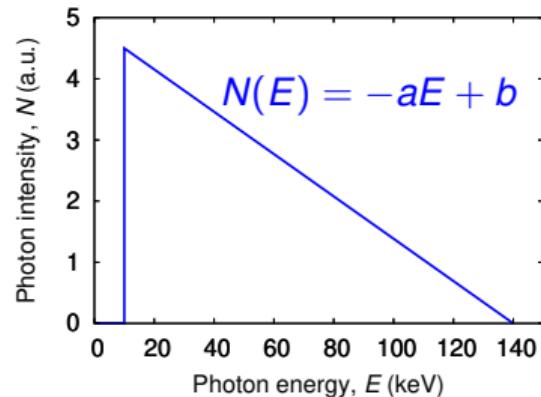
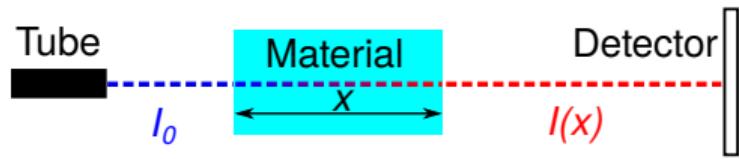
$$I(x) = I_0 \exp(-\mu(E, Z, \rho) x)$$



<https://www.nist.gov/pml/>

x-ray-mass-attenuation-coefficients

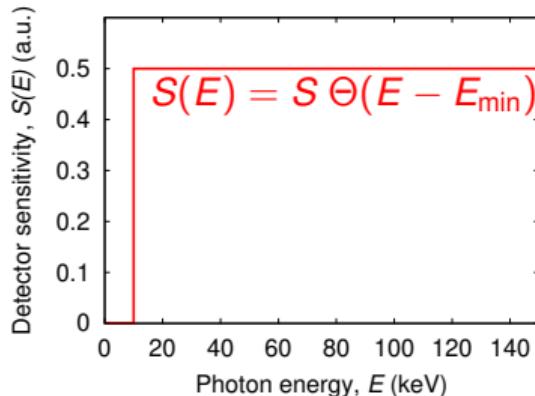
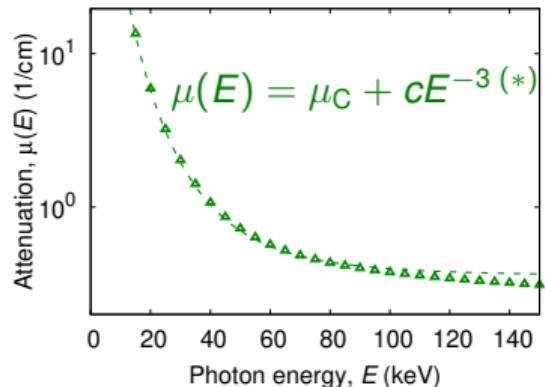
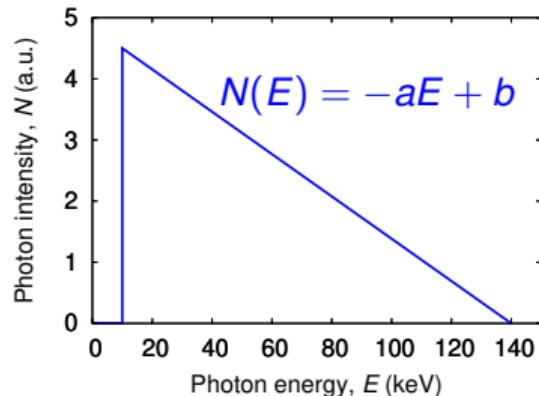
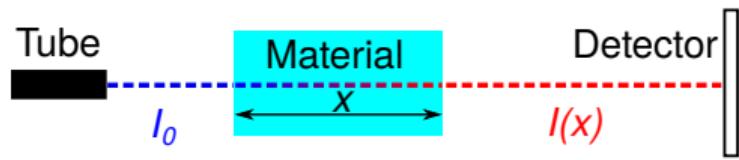
# Modeling of $\mu_{\text{eff}}(x)$



$$I(x) \propto \int N(E) \exp\{-\mu(E)x\} S(E) dE$$

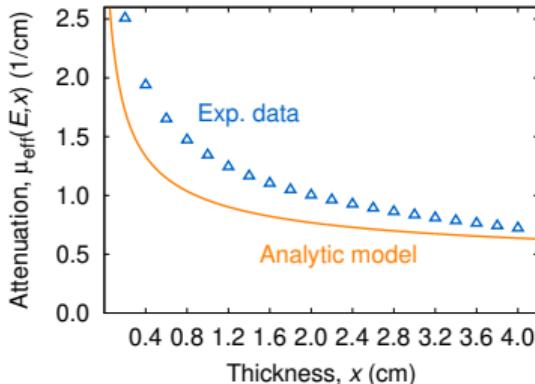
(\*) XCOM supplied by NIST

# Modeling of $\mu_{\text{eff}}(x)$



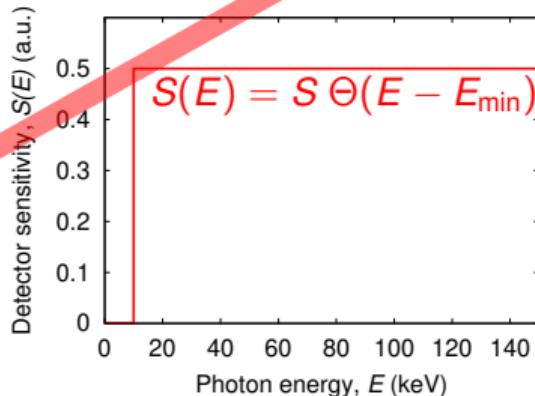
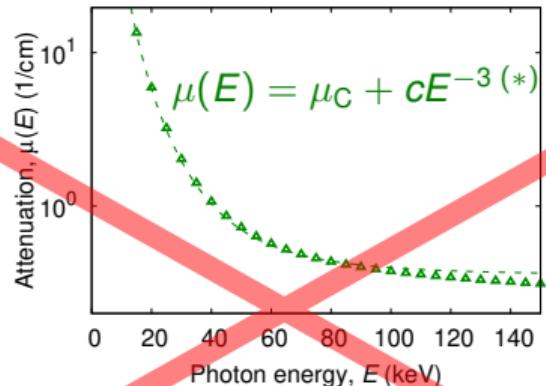
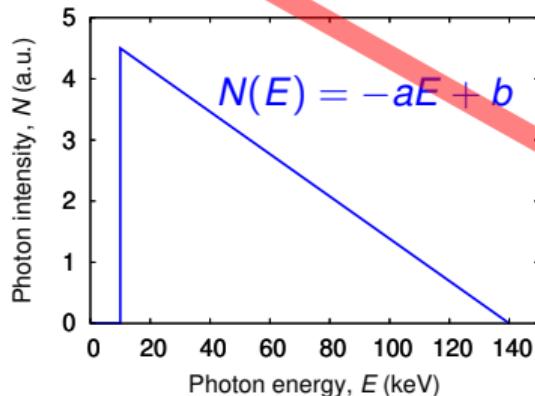
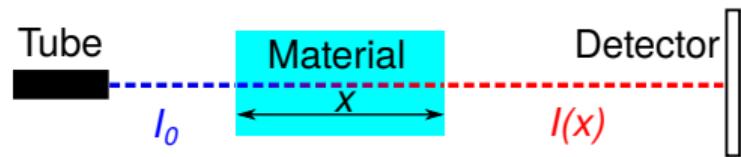
$$I(x) \propto \int N(E) \exp\{-\mu(E)x\} S(E) dE$$

$$\propto S \int_{E_{\min}}^{E_{\max}} (-aE + b) \exp\{-(\mu_C + cE^{-3})x\} dE$$



(\*) XCOM supplied by NIST

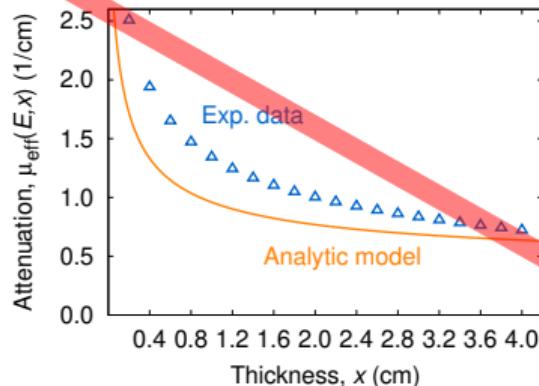
# Modeling of $\mu_{\text{eff}}(x)$



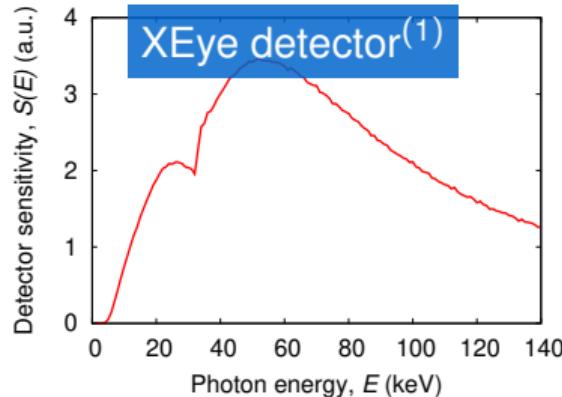
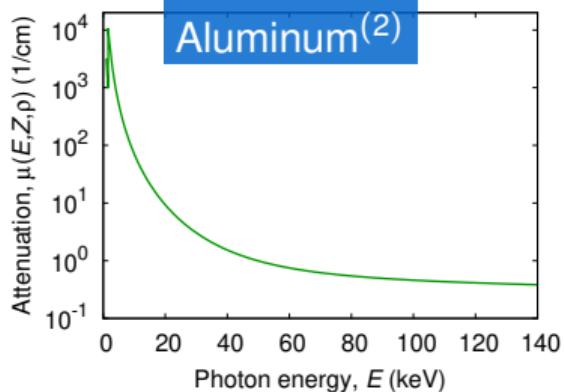
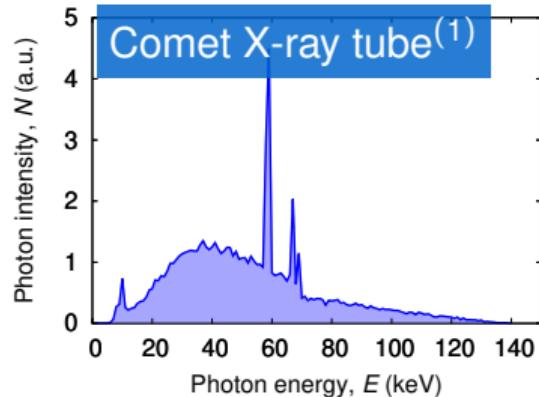
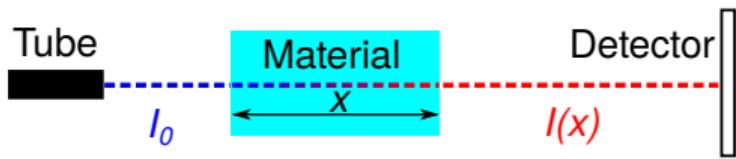
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(\*) XCOM supplied by NIST



# Numerical approx. of $\mu_{\text{eff}}(x)$



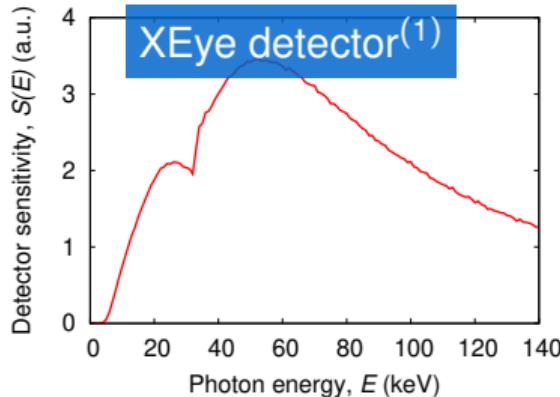
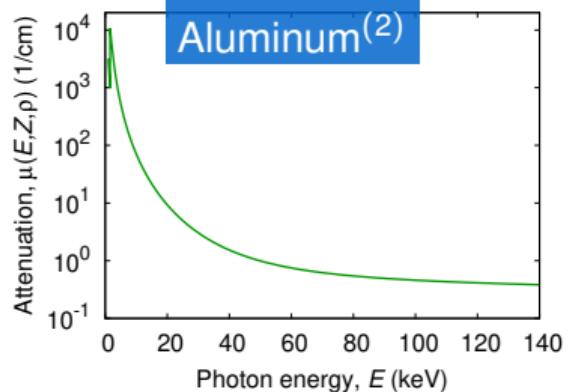
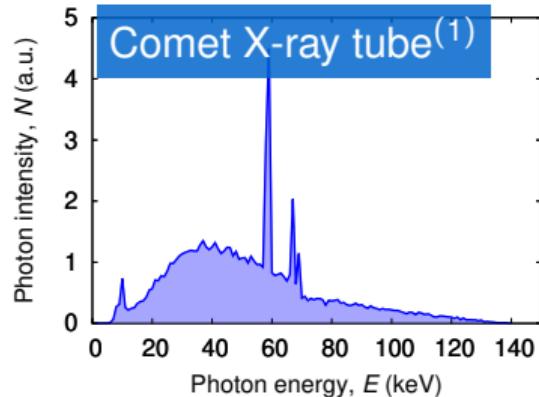
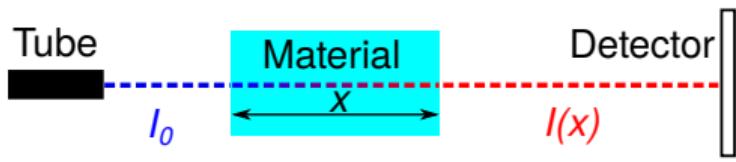
$$I(x) \propto \int \mathbf{N}(E) \exp\{-\mu(E)x\} \mathbf{S}(E) dE$$

$$\int \rightarrow \sum$$

(1) Supplied by Norman Uhlmann, Fraunhofer EZRT

(2) XCOM supplied by NIST

# Numerical approx. of $\mu_{\text{eff}}(x)$

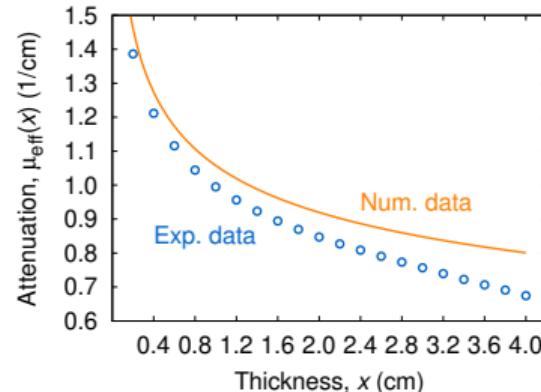


$$I(x) \propto \int \mathbf{N}(E) \exp\{-\mu(E)x\} \mathbf{S}(E) dE$$

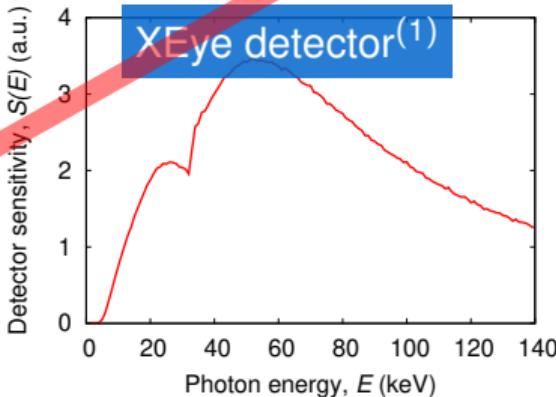
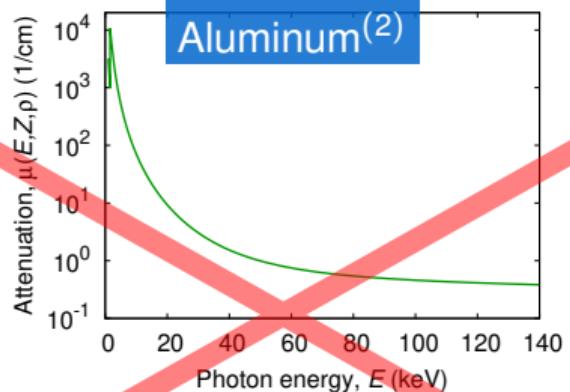
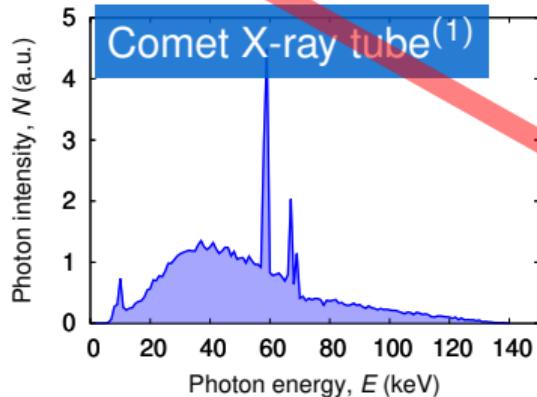
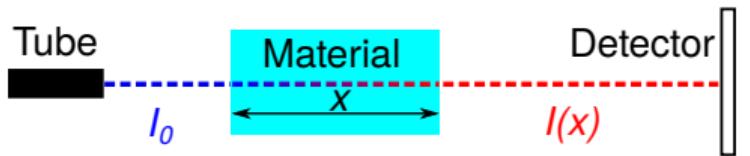
$$\int \rightarrow \sum$$

(1) Supplied by Norman Uhlmann, Fraunhofer EZRT

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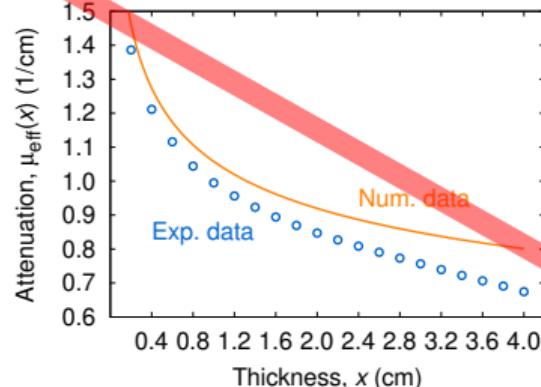


$$I(x) \propto \int N(E) \exp\{-\mu(E)x\} S(E) dE$$

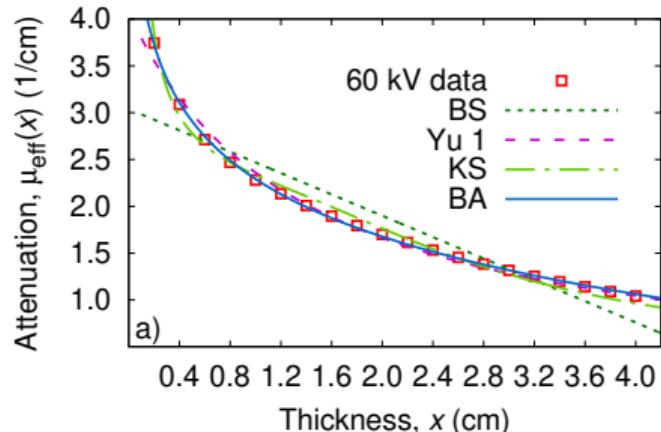
$\int \rightarrow \sum$

(1) Supplied by Norman Uhlmann, Fraunhofer EZRT

(2) XCOM supplied by NIST



# Heuristic model functions for $\mu_{\text{eff}}(x)$



$$\mu_{\text{eff}}(x) = \mu_0 - \lambda x$$

Bjärngard & Shackford  
(1994)

$$\mu_{\text{eff}}(x) = \frac{\mu_0}{1 + \lambda x}$$

Yu *et al.* (1997)

$$\mu_{\text{eff}}(x) = \mu(E_{\text{max}}) + \frac{2\mu_1}{x\sqrt{-\lambda_1^2+4\lambda_2}} \times$$

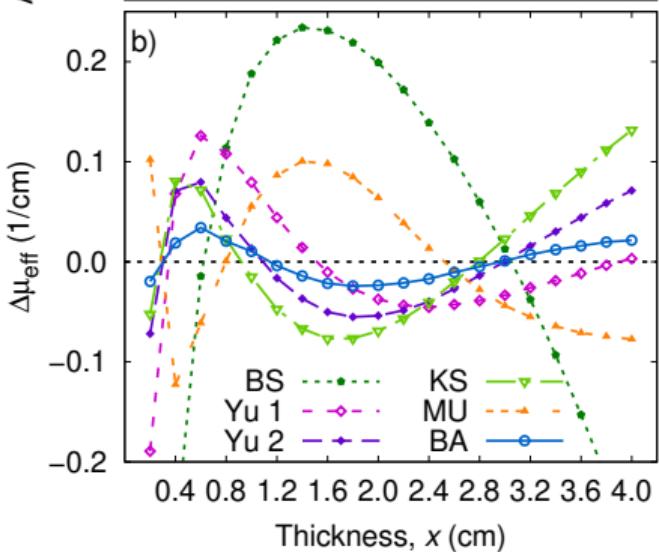
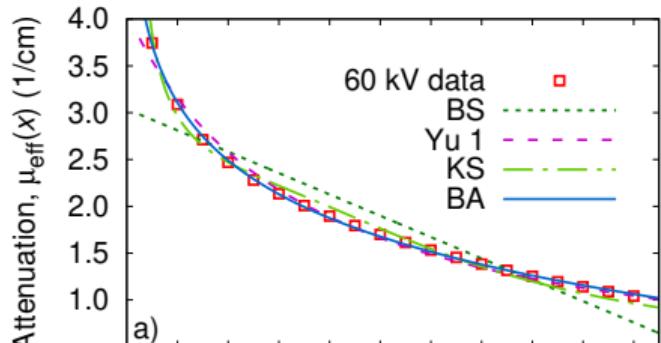
$$\left[ \arctan\left(\frac{\lambda_1+2\lambda_2x}{\sqrt{-\lambda_1^2+4\lambda_2}}\right) - \arctan\left(\frac{\lambda_1}{\sqrt{-\lambda_1^2+4\lambda_2}}\right) \right]$$

Kleinschmidt (1999)

$$\mu_{\text{eff}}(x) = a + \frac{b}{x^\alpha}$$

Baur *et al.* (2019)  
(this work)

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$$\mu_{\text{eff}}(x) = \frac{\mu_0}{(1 + \lambda x)^\beta}$$

Yu *et al.* (1997)

$$\mu_{\text{eff}}(x) = \mu(E_{\text{max}}) + \frac{2\mu_1}{x\sqrt{-\lambda_1^2 + 4\lambda_2}} \times \left[ \arctan\left(\frac{\lambda_1 + 2\lambda_2 x}{\sqrt{-\lambda_1^2 + 4\lambda_2}}\right) - \arctan\left(\frac{\lambda_1}{\sqrt{-\lambda_1^2 + 4\lambda_2}}\right) \right]$$

Kleinschmidt (1999)

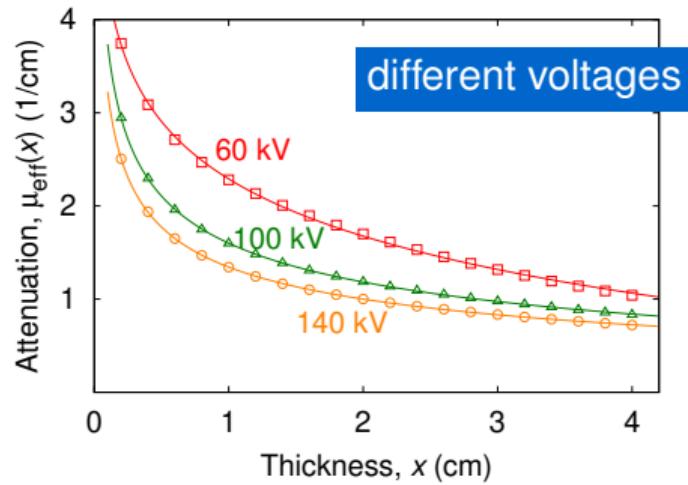
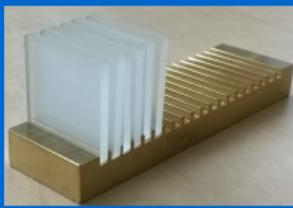
$$\mu_{\text{eff}}(x) = -\frac{1}{x} \ln [A + B \exp(-x/C)]$$

Mudde *et al.* (2008)

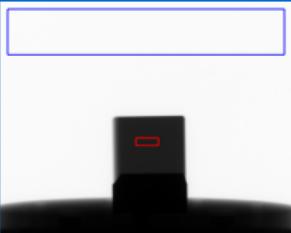
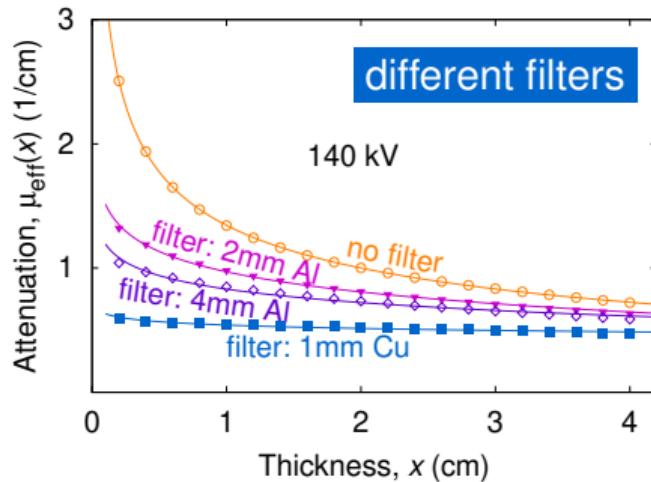
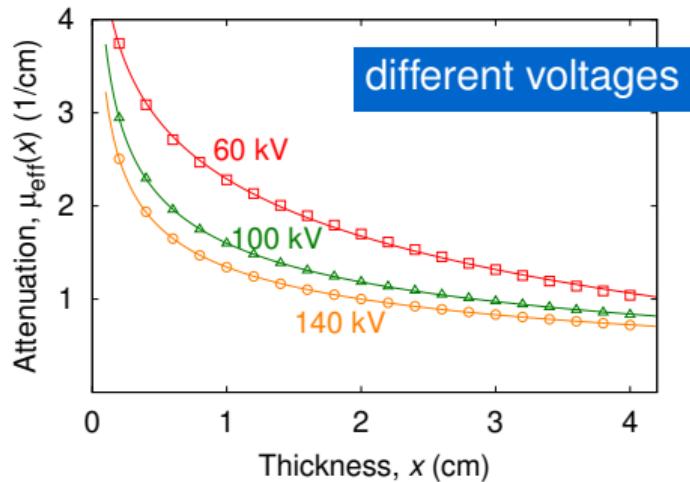
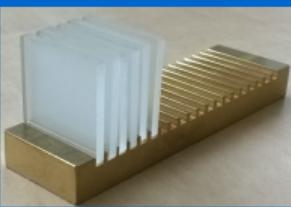
$$\mu_{\text{eff}}(x) = a + \frac{b}{x^\alpha}$$

Baur *et al.* (2019)  
(this work)

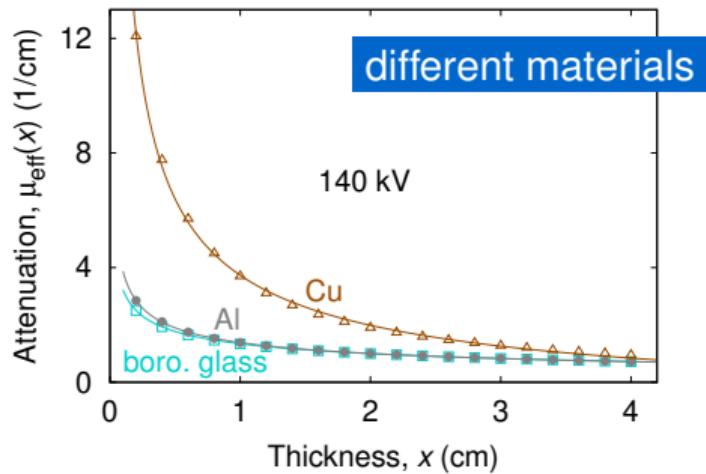
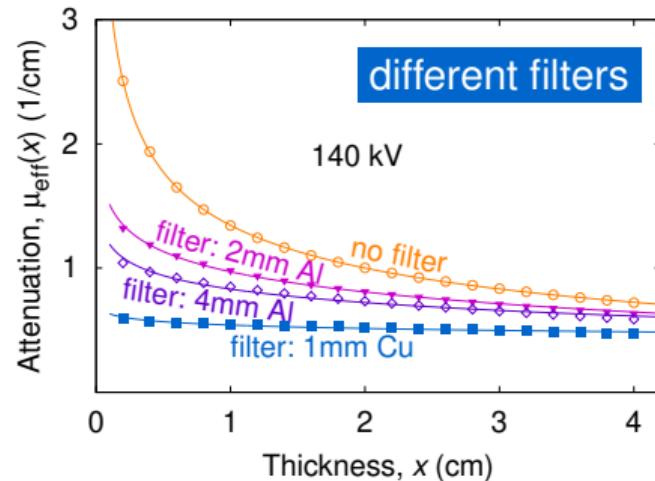
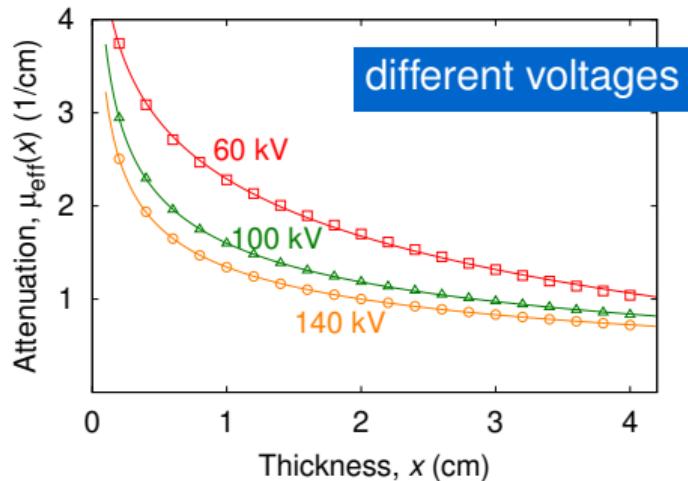
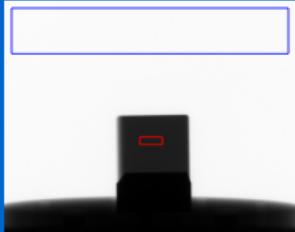
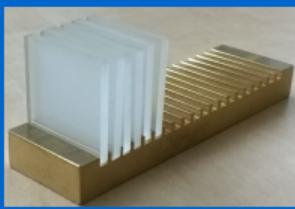
Applicability of  
 $\mu_{\text{eff}}(x) = a + \frac{b}{x^\alpha}$



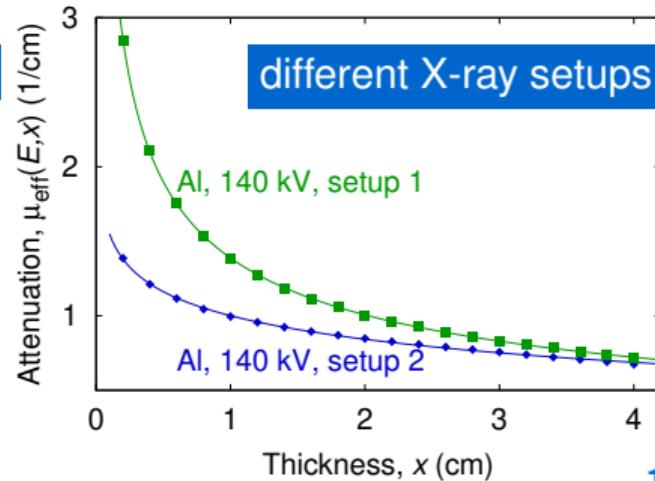
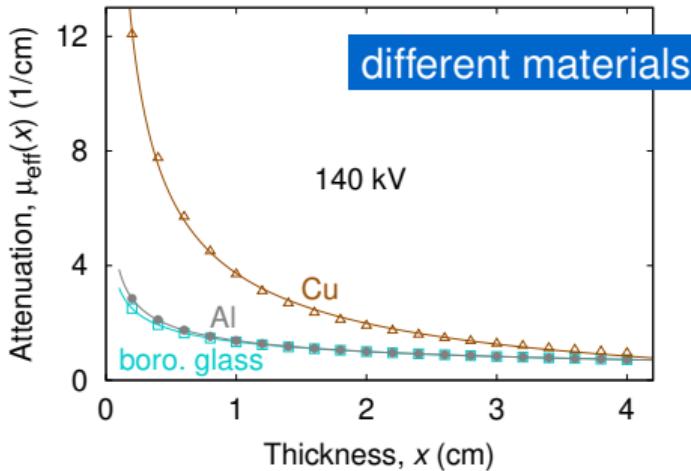
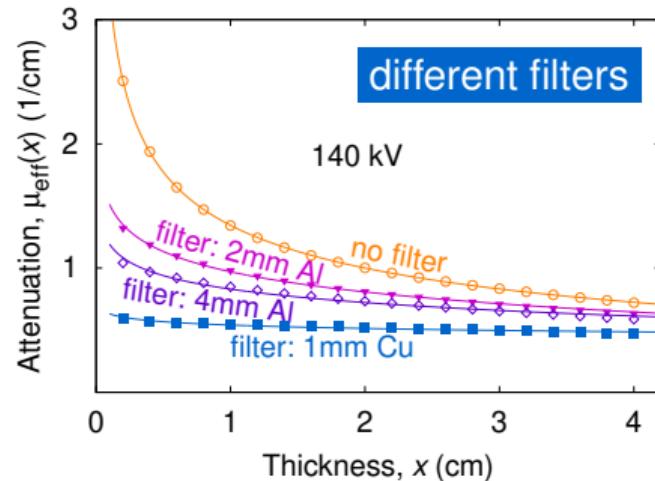
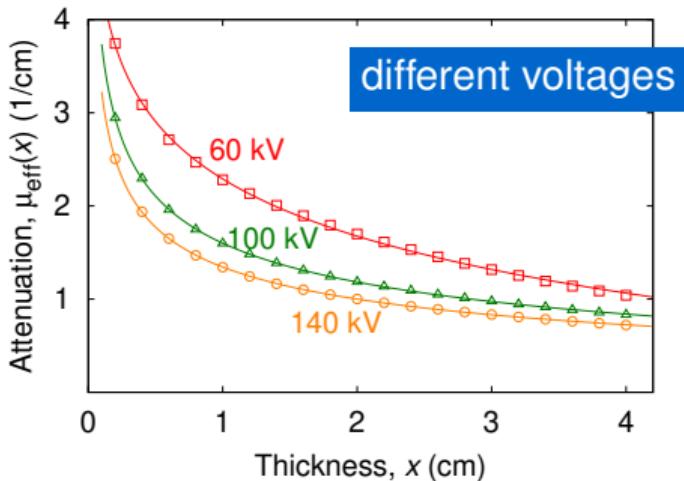
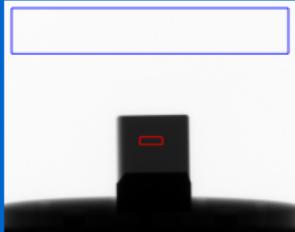
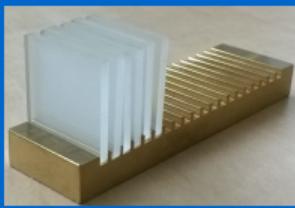
Applicability of  
 $\mu_{\text{eff}}(x) = a + \frac{b}{x^\alpha}$



Applicability of  
 $\mu_{\text{eff}}(x) = a + \frac{b}{x^\alpha}$



# Applicability of $\mu_{\text{eff}}(x) = a + \frac{b}{x^\alpha}$



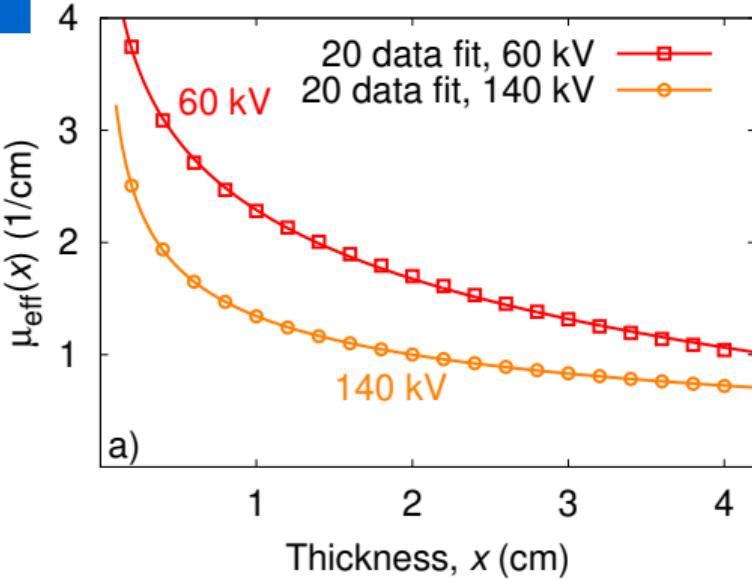
# Determining the material thickness $x$

Generalized Beer-Lambert

$$I(x) = I_0 \exp(-\mu_{\text{eff}}(x) x)$$

Model function

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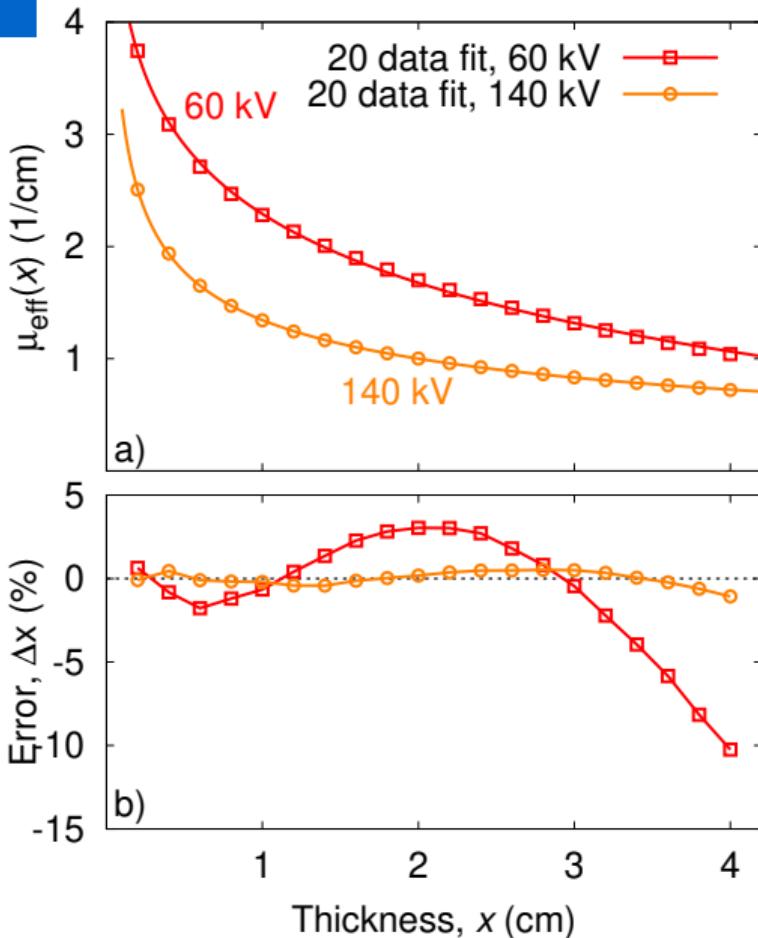
Model function

$$\mu_{\text{eff}}(x) = a + \frac{b}{x^\alpha}$$

Solve

$$ax + bx^{1-\alpha} + \ln\left(\frac{I(x)}{I_0}\right) = 0$$

e.g. Newton's method or look-up table



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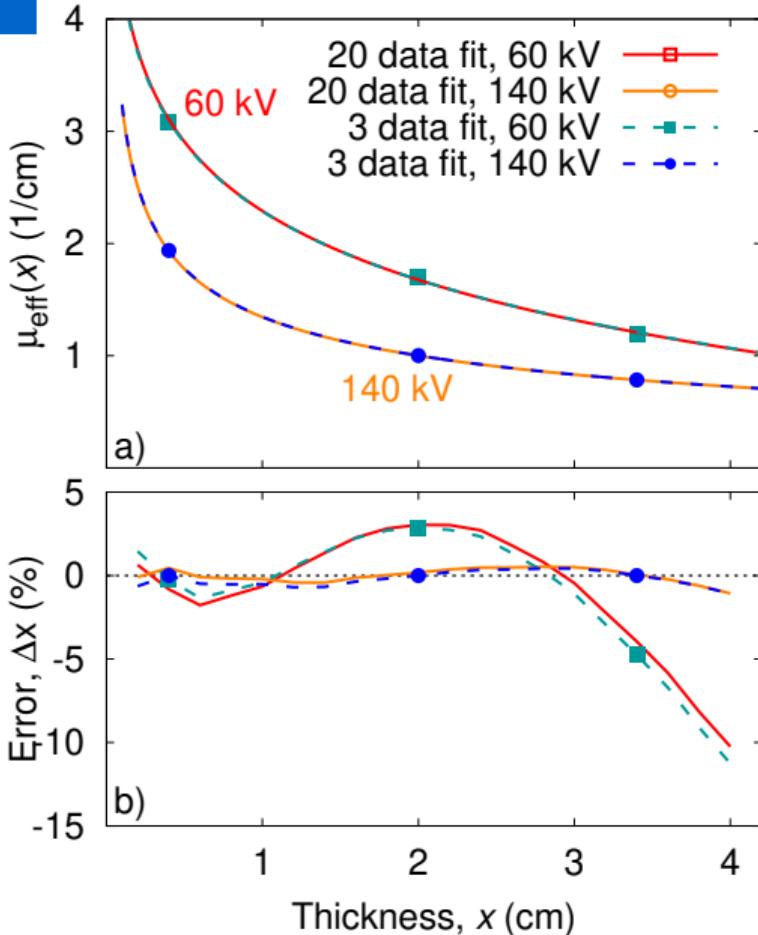
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# Migrating shear bands in shaken granular matter, Kollmer *et al* (2020)

