On the Gromov–Hausdorff distance for metric pairs and tuples

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Introduction

The Gromov–Hausdorff distance and convergence have become fundamental tools for quantitatively comparing and studying metric spaces. Gromov introduced this concept in several well-known papers as a generalization of the classical Hausdorff distance, which compares closed subsets of the same metric space. Beyond its significant theoretical contributions in geometric group theory and differential geometry—such as Gromov's compactness theorem—the Gromov–Hausdorff distance has found numerous applications. In particular, Topological Data Analysis (TDA) uses it as a foundational tool in proving key stability theorems for objects like persistence diagrams. Moreover, it continues to serve as a reference for recent developments in stability and computational results.

The foundations of the **Gromov–Hausdorff distance for metric pairs and tuples** were established in [1]. Ahumada-Gómez and Che proved results such as embedding, completeness, and compactness theorems. Additionally, they presented a relative version of Fukaya's theorem on quotient spaces under Gromov–Hausdorff equivariant convergence, as well as a version of the Grove–Petersen–Wu finiteness theorem for stratified spaces.

We continue building the theory behind the Gromov–Hausdorff convergence and distance for metric pairs and tuples as well as presenting some applications that broaden the scope of this tool [4].

Definition of GH for metric pairs and tuples

A *metric pair* (X, A) is a metric space X with a closed subset $A \subset X$. More generally, a *metric tuple* (or k-tuple, to emphasize its length) $(X, X_k, \ldots, X_1)$ is formed by a metric space X and a nested sequence of closed subsets

$$X \supseteq X_k \supseteq X_{k-1} \supseteq \cdots \supseteq X_1.$$

Observe that a metric 1-tuple is the same thing as a metric pair.

The **Gromov–Hausdorff distance between two compact metric pairs** (X, A) and (Y, B) is defined as

$$d_{GH}((X,A),(Y,B)) := \inf\{d_{\mathrm{H}}^{\delta}((X,A),(Y,B)) : \delta \text{ admissible on } X \sqcup Y\}.$$

More generally, the **Gromov–Hausdorff distance between two metric tuples** $(X, X_k, \ldots, X_1)$ and $(Y, Y_k, \ldots, Y_1)$ is defined as

$$d_{GH}((X, X_k, \ldots, X_1), (Y, Y_k, \ldots, Y_1)) := \inf\{d_{\mathrm{H}}^{\delta}((X, X_k, \ldots, X_1), (Y, Y_k, \ldots, Y_1))\},$$

where the infimum is taken over all admissible metrics δ in $X \sqcup Y$. We denote by (GH_k, d_{GH}) the metric space of compact metric k-tuples. In both definitions, δ denotes an admissible metric on the disjoint union of the elements of the k-tuple.

Separability and geodecisity

Another important properties of metric spaces are:

- *Separability:* A metric space is separable if it contains a countable dense subset.

Theorem [Ahumada-Gómez, Che & C., 2025, [4]]
The metric space (GH_1, d_{GH}) is separable.

- *Geodecisity:* A length space is geodesic if every two points can be joined by a curve of minimal length.

For geodecisity, we followed the same spirit as Chowdhury and Mémoli in [3] for the Gromov–Hausdorff space. *First*, we present an **equivalent formulation** for the Gromov–Hausdorff distance for metric tuples.

We say that $R \subset X \times Y$ is a *pair correspondence* between (X, A) and (Y, B) if the following conditions hold:

- for every point $x \in X$ there exists a point $y \in Y$ such that $(x, y) \in R$ and, in particular, for every point $a \in A$ there exists $b \in B$ such that $(a, b) \in R$,
- for every point $y \in Y$ there exists a point $x \in X$ such that $(x, y) \in R$ and, in particular, for every point $b \in B$ there exists $a \in A$ such that $(a, b) \in R$.

In other words, R is a correspondence between X and Y such that the restriction $R|_{A\times B} := R \cap A \times B$ is a correspondence between A and B, in the usual sense.

The *distortion* of R is defined by

$$\operatorname{dis}(R) = \frac{1}{2}\left(\sup\{|d_X(x,x') - d_Y(y,y')| : (x,y),(x',y') \in R\} + \sup\{|d_X(a,a') - d_Y(b,b')| : (a,b),(a',b') \in R|_{A\times B}\}\right).$$

Now, we can state the *reformulation Theorem*:

Theorem [Ahumada-Gómez, Che & C., 2025, [4]]
For any two metric pairs (X, A) and (Y, B),

$$d_{\mathrm{GH}}((X,A),(Y,B)) = \frac{1}{2}\inf_R \operatorname{dis}(R).$$

In other words, $d_{\mathrm{GH}}((X,A),(Y,B))$ is equal to the infimum of $r > 0$ for which there exists a pair correspondence between (X, A) and (Y, B) with $\operatorname{dis}(R) < 2r$.

And now, the *Geodesicity Theorem*:

Theorem [Ahumada-Gómez, Che & C., 2025, [4]]
The metric space $(\mathrm{GH}_1, d_{\mathrm{GH}})$ is geodesic. Moreover, let $(X,A),(Y,B) \in \mathrm{GH}_1$, $R \in \mathfrak{R}^{\mathrm{opt}}((X,A),(Y,B))$, and a curve $\gamma_R : [0,1] \to \mathrm{GH}_1$ between (X, A) and (Y, B) such that

$$\gamma_R(0) := (X,A), \quad \gamma_R(1) := (Y,B) \quad \text{and}$$
$$\gamma_R(t) := (R, R|_{A\times B}, d_{\gamma_R(t)}) \quad \text{for} \quad t \in (0,1),$$

where for each $(x,y),(x',y') \in R$ and $t \in (0,1)$, the metric $d_{\gamma_R(t)}$ is given by

$$d_{\gamma_R(t)}((x,y),(x',y')) := (1-t)\, d_X(x,x') + t\, d_Y(y,y').$$

Then, γ_R is a geodesic.

Arzelà–Ascoli Theorem

We prove a version of the classical Arzelà–Ascoli theorem for metric pairs, generalizing the one presented in [5] for maps between pointed metric spaces.

Theorem [Ahumada-Gómez, Che & C., 2025, [4]]
Consider metric pairs $(X_i, A_i) \xrightarrow{\mathrm{GH}} (X_\infty, A_\infty)$ and $(Y_i, B_i) \xrightarrow{\mathrm{GH}} (Y_\infty, B_\infty)$, where A_i is compact for all $i \in \mathbb{N} \cup \{\infty\}$, and relatively equicontinuous, relatively uniformly bounded maps $f_i \colon (X_i, A_i) \to (Y_i, B_i)$, for all $i \in \mathbb{N}$.

Then there exists a subsequence of $\{f_i\}_{i\in\mathbb{N}}$ converging relatively uniformly to a continuous relative map $f_\infty \colon (X_\infty, A_\infty) \to (Y_\infty, B_\infty)$. Moreover, if $\operatorname{Lip}(f_i) \le K$ for all $i \in \mathbb{N}$ then $\operatorname{Lip}(f_\infty) \le K$, and if $\operatorname{Lip}(f_i|_{A_i}) \le L$ for all $i \in \mathbb{N}$ then $\operatorname{Lip}(f_\infty|_{A_\infty}) \le L$.

Approximation by surfaces

In [2], Cassorla proved that any compact **length space can be approximated by Riemannian surfaces.** We prove a generalization of that result but for metric pairs.

Theorem [Cassorla '92, [2]]
If $\mathcal{M}$ is the set of compact inner metric spaces (a metric space where distances are computed as the infimum of lenght curves) and if $\mathcal{S}$ is the subset of $\mathcal{M}$ consisting of 2–dimensional Riemannian manifolds which can be isometrically embedded in $\mathbb{R}^3$, then $\overline{\mathcal{S}} = \mathcal{M}$ in $\mathrm{d_{GH}}$.

Quoting Cassorla: *"This result is philosophically interesting because it shows one can visualize in $\mathbb{R}^3$ a manifold that is GH close to a compact inner metric space of arbitrarty dimension".*

Theorem [Ahumada-Gómez, Che & C., 2025, [4]]
If $\mathcal{M}$ is the set of compact length metric pairs and if $\mathcal{S}$ is the subset of $\mathcal{M}$ consisting of 2-dimensional Riemannian manifolds which can be isometrically embedded in $\mathbb{R}^3$ together with a 2-dimensional submanifold with boundary, then $\overline{\mathcal{S}} = \mathcal{M}$ in d_{GH_1}.

References

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