Filling radius and reach of isometrically embedded metric spaces

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Definition of the Filling Radius

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The canonical inclusion

$$\Phi : M \to L^\infty(M), \qquad p \to \mathrm{d}_p(\cdot) = \mathrm{d}(p,\cdot),$$

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Given $r > 0$, we will denote by $U_r(M)$ the r–neighbourhood of M in $L^\infty(M)$, and by $\iota_r : M \hookrightarrow U_r(M)$ the inclusion.

For a given coefficient ring $\mathbb{F}$, consider the homomorphism induced in n-homology by the inclusion

$$\iota_{r,*} : \mathrm{H}_n(M,\mathbb{F}) \to \mathrm{H}_n(U_r(M),\mathbb{F}).$$

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$$\iota_{r,*} : \mathrm{H}_n(M, \mathbb{F}) \to \mathrm{H}_n(U_r(M), \mathbb{F}).$$

Definition (Filling Radius)

The filling radius of M, denoted by $\mathrm{FillRad}(M)$, is the infimum of those $r > 0$ for which $\iota_{r,*}([M]) = 0$, where $[M]$ is the fundamental class of M.

Most significant results

Theorem 1 (Gromov, 1983)

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Theorem 2 (Katz, 1983)

Let M be a closed connected manifold, then

$$\mathrm{FillRad}(M) \leq \frac{1}{3}\,\mathrm{diam}(M).$$

Theorem 3 (Wilhelm, 1992)

Let $\mathbb{S}^n$ *denote the unit sphere in* $\mathbb{R}^{n+1}$*, and let* $\mathscr{M}$ *denote the class of closed, Riemannian* $n-$*manifolds with sectional curvature* ≥ 1*. For all* $M \in \mathscr{M}$,

1. $\mathrm{FillRad}(M) \leq \mathrm{FillRad}(\mathbb{S}^n)$.

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3. *There is a* $\delta(n) > 0$ *so that if* $\mathrm{FillRad}(\mathbb{S}^n) - \delta(n) < \mathrm{FillRad}(M)$, *then* M *is diffeomorphic to* $\mathbb{S}^n$.

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Theorem 4 (Yokota, 2013)

For any $n-$dimensional Alexandrov space X of curvature ≥ 1 with $\partial X = \emptyset$, either $\mathrm{FillRad}(X) < \mathrm{FillRad}(\mathbb{S}^n)$ *or X isometric to the round sphere $\mathbb{S}^n$.*

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- $\mathrm{FillRad}(\mathbb{C}P^n) \geq \frac{1}{2}\arccos\left(-\frac{1}{3}\right)$.
- $\mathrm{FillRad}(CaP^2) \geq \frac{1}{2}\arccos\left(-\frac{1}{9}\right)$.
- $\mathrm{FillRad}(\mathbb{H}P^n) \geq \frac{1}{2}\arccos\left(-\frac{1}{5}\right)$.

Submersions

Definition (Riemannian submersion)

A differentiable map $\pi : M^{m+n} \to B^n$ is called **submersion** if π is surjective, and for all $p \in M$, $d\pi_p : T_pM \to T_{\pi(p)}B$ has rank n. If M and B have Riemannian metrics, the submersion π is said to be Riemannian if, for all $p \in M$, $d\pi_p : T_pM \to T_{\pi(p)}B$ preserves the lengths of vectors orthogonal to F_p, where $F_p := \pi^{-1}(p)$.

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Theorem 5 (C. & Guijarro, 2022)

Let $\pi : M \to B$ *be a Riemannian submersion with* $\dim M > \dim B$. *Then*

$$\mathrm{FillRad}(M) \leq \frac{1}{2} \max_{b \in B} \{\operatorname{diam} \pi^{-1}(b)\}, \tag{1}$$

where the diameter of each fiber is considered in the extrinsic metric.

Submetries

Recall that a submetry between metric spaces is a map $\pi : X \to B$ such that for every $p \in X$, any closed ball $B(p,r)$ of radius $r > 0$ centred at p maps onto the ball $B(\pi(p),r)$. It is a *purely metric version* of Riemannian submersions.

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Theorem (Berestovskii & Guijarro, 2000)

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Corolary 5.1 (C. & Guijarro, 2022)

Let $(X,\widehat{\mathrm{d}}_X)$ be a metric manifold (i.e, a closed manifold with a distance), (Y,d_Y) a metric space and $\pi : X \to Y$ a submetry between them. Thus

$$\mathrm{FillRad}(X) \leq \frac{1}{2}\max_{y\in Y}\{\mathrm{diam}\,\pi^{-1}(y)\}.$$

More corolaries

Corolary 5.2 (C. & Guijarro, 2022)

For B, F closed Riemannian manifolds, let $f : B \to (0, \infty)$ be a smooth function, and $M = B \times_f F$ the warped product over B with fiber F. Then

$$\operatorname{FillRad}(M) \leq \min\{\operatorname{FillRad}(B), \frac{1}{2} \max f \cdot \operatorname{diam} F\}.$$

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Corolary 5.3 (C. & Guijarro, 2022)

Suppose M is a Riemannian manifold admitting a singular Riemannian foliation $\mathscr{F}$ with closed leaves. Then

$$\mathrm{FillRad}(M) \leq \frac{1}{2} \max_{N \in \mathscr{F}} \{\operatorname{diam} N\}.$$

Lower bound

Definition (Injectivity radius)

Let (M^n, g) a Riemannian manifold, then

1. for $p \in M$, the **injectivity radius at** p, $\mathrm{inj}_M(p)$, is defined as follows

$$\mathrm{inj}_M(p) = \sup\{r > 0 : \exp_p|_{B_r(p)} \text{ is diffeo.}\}.$$

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2. the **injectivity radius of** M is defined as follows

$$\mathrm{inj}_M = \inf_{p \in M} \mathrm{inj}_M(p).$$

Theorem 6 (Greene & Petersen, 1992 // C. & Guijarro, 2022)

Let M *be a closed Riemannian manifold with injectivity radius* $\operatorname{inj} M$ *and such that* $\sec \le \Delta$, *where* $\Delta \ge 0$. *Then*

$$\operatorname{FillRad}(M) \ge \frac{1}{4} \min\left\{ \operatorname{inj} M, \frac{\pi}{\sqrt{\Delta}} \right\}, \tag{2}$$

where $\pi / \sqrt{\Delta}$ *is understood as* ∞ *whenever* $\Delta = 0$.

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Corolary 6.1

Let $\pi : M \to B$ a Riemannian submersion between closed manifolds. Then

$$\frac{1}{2} \min \left\{ \operatorname{inj} M, \frac{\pi}{\sqrt{\Delta}} \right\} \le \max_{b \in B} \{ \operatorname{diam} \pi^{-1}(b) \},$$

where Δ is an upper positive curvature bound for $\sec_M$.

Reach of the Kuratowski embedding

Definition

Let (X, d) be a metric space and a $A \subset X$ a subset. Then, we define the set of points having a unique metric projection in A as

$$\mathrm{Unp}(A) = \{x \in X \colon \text{there exits a unique } a \in A \text{ such that } \mathrm{d}(x, A) = \mathrm{d}(x, a)\}.$$

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For $a \in A$, we define the reach of A at a, denoted as $\mathrm{reach}(a, A)$, as

$$\mathrm{reach}(a, A) = \sup\{r \geq 0 \colon B_r(a) \subset \mathrm{Unp}(A)\}.$$

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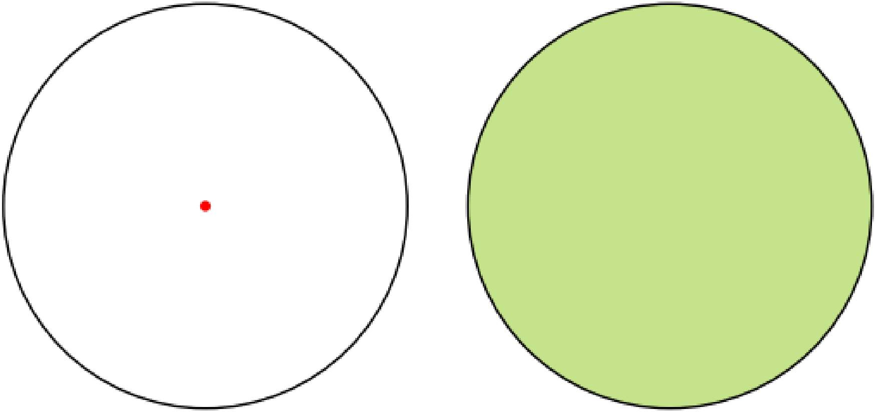
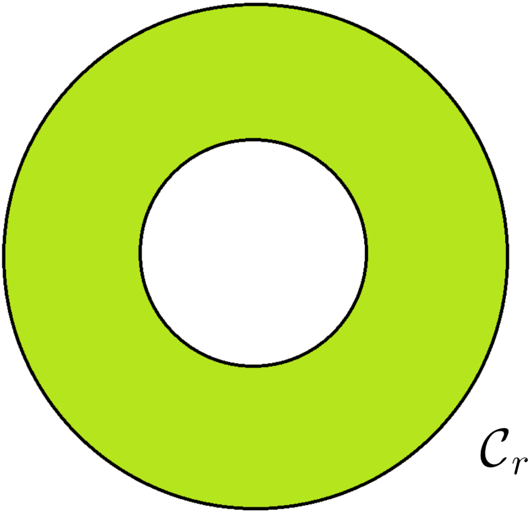
Finally, we define the global reach by

$$\mathrm{reach}(A \subset X) = \inf_{a \in A} \mathrm{reach}(a, A).$$



Example

Let $\mathscr{C}_r = \{\mathbb{D}^2 \backslash \mathbb{D}_r^2\}$ such that $r \to 0$, where $\mathbb{D}^2$ is considered with radius 1 and $\mathbb{D}_r^2$ has radius r


$\mathcal{C}_r$

Example

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$$\mathrm{reach}(\mathscr{C}_r \subset \mathbb{R}^2) = r, \text{ but } \mathrm{reach}(\mathbb{D}^2 \subset \mathbb{R}^2) = \infty.$$

Remark

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$$\operatorname{reach}(M) > 0.$$

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Theorem 7 (C. & Guijarro, 2022)

Let M^n *be a compact* n*-Riemannian manifold. For every* $p \in M$,

$$\operatorname{reach}(p, M \subset L^\infty(M)) = 0.$$

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Remark

- $\mathrm{Unp}(M \subset L^\infty(M)) = M$?
- *No!!* $f(x) = \mathrm{d}_p(x) + \delta \in \mathrm{Unp}(M)$.
- *Is* $\mathrm{Unp}(M \subset L^\infty(M))$ *dense?*

Proof

We are going to check that for every $\mathrm{d}_p \in M$ and all $\epsilon > 0$. There exists a function $f \in B_\epsilon(\mathrm{d}_p)$ such that

$$\mathrm{d}_\infty(f, M) = \|f - \mathrm{d}_p\|_\infty = \|f - \mathrm{d}_q\|_\infty .$$

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$$\mathrm{reach}(M) = 0.$$

We consider $\delta > 0$ such that $\delta << \epsilon$ and some $q \in M$ with $\mathrm{d}_M(p,q) = \delta$. Then, $\left\| \mathrm{d}_p - \mathrm{d}_q \right\|_\infty = \delta$. Now, our candidate is

$$f := \frac{1}{2}\left(\mathrm{d}_p + \mathrm{d}_q\right).$$

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3. $\nexists r \in M$ such that $\|f - \mathrm{d}_r\|_\infty < \|f - \mathrm{d}_p\|_\infty$.Suppose it does, then evaluating at r, we would obtain

$$|\mathrm{d}_r(r) - f(r)| = |f(r)| = \frac{1}{2}\left(\mathrm{d}_p(r) + \mathrm{d}_q(r)\right) < \frac{\delta}{2}. \tag{3}$$

On the other hand, by the triangular inequality,

$$\mathrm{d}_p(r) + \mathrm{d}_q(r) \geq \mathrm{d}_M(p, q) = \delta \text{ ¡!} \tag{4}$$

This part of the talk was based on:

- Manuel Cuerno and Luis Guijarro. "Upper and lower bounds on the filling radius". In: (2022). arXiv: 2206.08032 Accepted in *Indiana University Mathematics Journal*.

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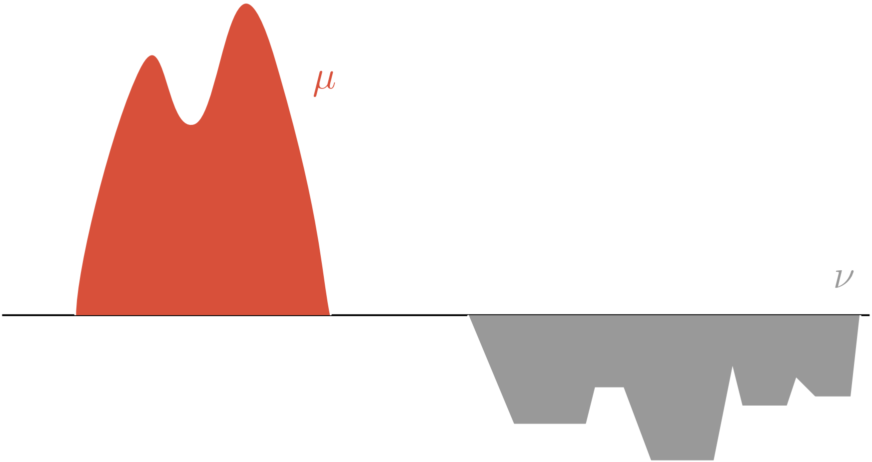
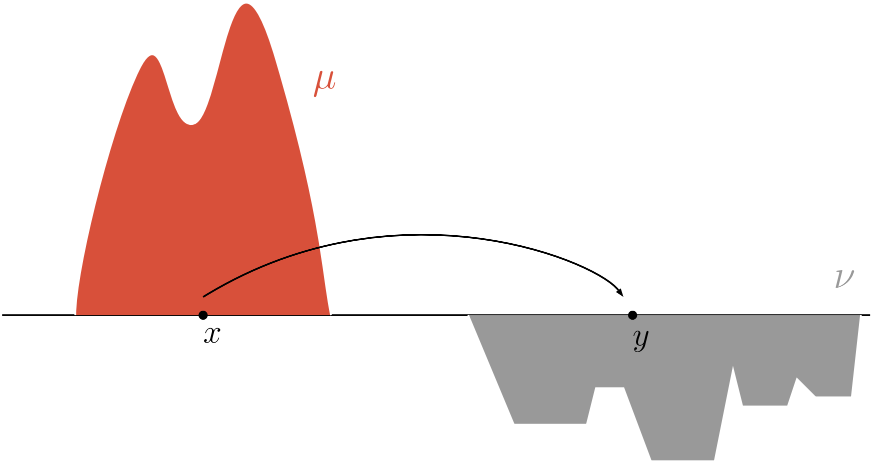
The next part is based on:

- Javier Casado, Manuel Cuerno, and Jaime Santos-Rodríguez. "On the reach of isometric embeddings into Wasserstein type spaces". In: (2023). arXiv: 2307.01051

Optimal Transport



Optimal Transport



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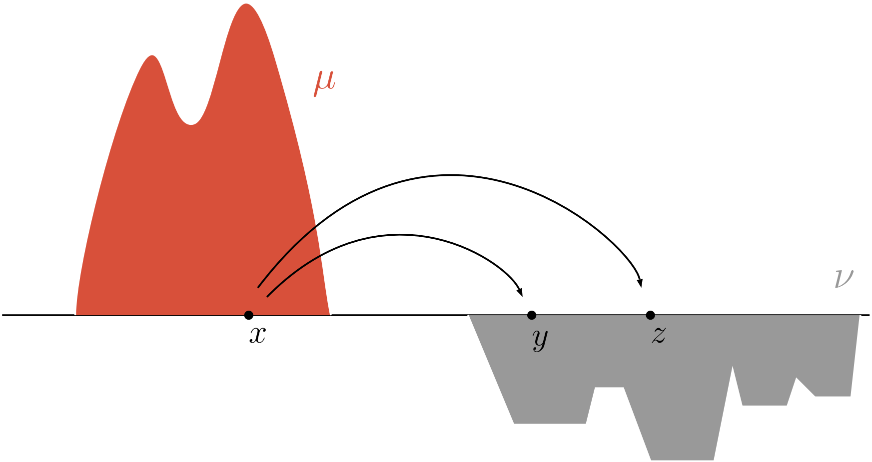

Let (X, d_X) be a complete and separable metric space. We denote by $\mathscr{P}(X)$ the set of probability measures of X and $\mathscr{P}_p(X)$ the ones with finite p–moment, i.e.

$$\mathscr{P}_p(X) := \{\sigma \in \mathscr{P}(X) : \int_X \mathrm{d}(x, x_0)^p \, d\sigma(x) < \infty \text{ for some } x_0 \in X\}.$$

Monge's formulation

Let μ, ν be two probability measures supported on X.

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$$\int \mathrm{d}(x, T(x))^2 \ d\mu(x), \tag{5}$$

among all measurable maps $T \colon X \to X$ such that $T_{\#}\mu = \nu$.

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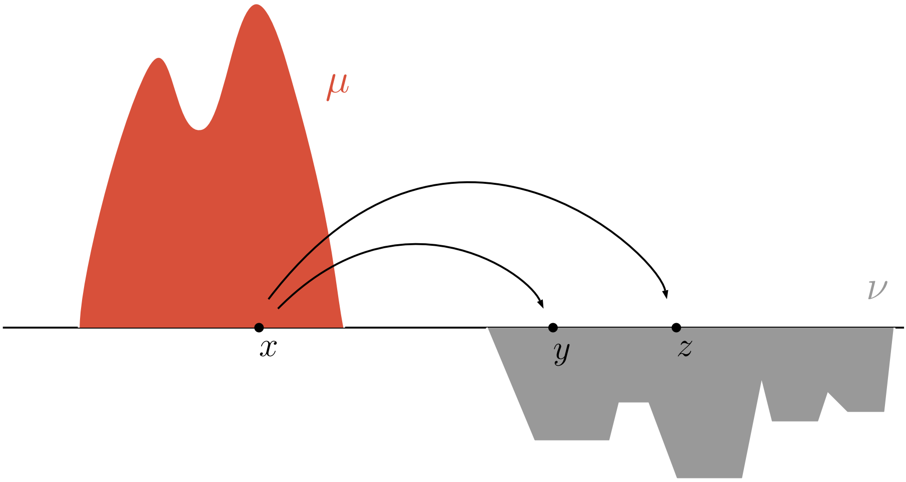

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Kantorovich presented a new formulation (1942) of the optimal transport problem, in some sense, weaker than Monge's one: instead of imposing the existence of some function T that sends one measure μ to another ν, he allows the possibility to split mass, i.e., he proposed to minimize the following functional:

$$\pi \mapsto \int \mathrm{d}_X(x,y)^2 d\pi,$$

among all *admissible measures* $\pi \in \Gamma(\mu,\nu)$.

Wasserstein space

Let (X, d, μ) a metric measure space. We denote by $\mathscr{P}(X)$ the set of probability measures of X and $\mathscr{P}_p(X)$ the ones with finite p–moment, i.e.

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Definition

A transference plan, *admissible plan* or *admissible measure* between two positive measures $\mu, \nu \in \mathscr{P}(X)$ is a finite positive measure $\pi \in \mathscr{P}(X \times X)$ which satisfies that, for all A, B Borel subsets of X,

$$\pi(A \times X) = \mu(A), \quad \text{and} \quad \pi(X \times B) = \nu(B).$$

We denote by $\Gamma(\mu,\nu)$ the set of transference plans between those two measures. Then, we define the p–Wasserstein distance for $p \geq 1$ and two probability measures by

$$W_p(\mu,\nu) := \left(\min_{\pi \in \Gamma(\mu,\nu)} \int_{X \times X} \mathrm{d}(x,y)^p \, d\pi(x,y) \right)^{\frac{1}{p}}.$$

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Thus, we define the p–**Wasserstein space** as the following metric space

$$W_p(X) := (\mathscr{P}_p(X), W_p).$$

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$$\Psi : X \to W_p(X)$$
$$x \to \delta_x$$

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Notation: We denote $x \in X$ as barycenter of μ if

$$W_p(\mu, X) = W_p(\mu, \delta_x).$$

Proposition (Casado, C., Santos-Rodríguez, 2023)

Let (X, d) *be a non-branching metric space and* $W_p(X)$ *with* $p > 1$ *its* p*–Wasserstein space. Then the set of unique points* $\mathrm{Unp}(X \subset W_p(X))$ *is dense in* $W_p(X)$.

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This implies that $W_p(\nu,\delta_z)\leq W_p(\nu,\delta_x)$ and with this we get

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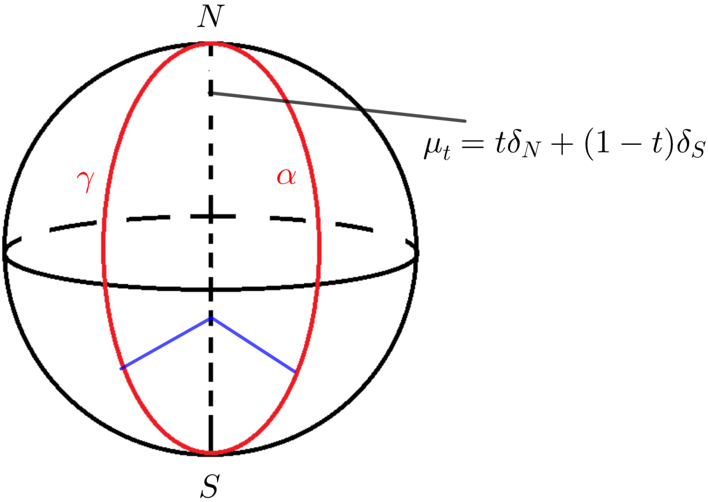
So then z is also a barycenter for μ.
Furthermore we notice that there is a branching geodesic joining μ with δ_z. This gives us the contradiction as $W_p(X)$ is non-branching. Then ν is a measure in $\mathrm{Unp}(X \subset W_p(X))$ which can be taken arbitrarily close to μ.

Proposition (Casado, C., Santos-Rodríguez, 2023)

Let X be a geodesic metric space, and $x, y \in X$ two points with $x \neq y$. Consider the probability measure

$$\mu_\lambda = \lambda\delta_x + (1-\lambda)\delta_y,$$

for $0 < \lambda < 1$. Then, μ_λ realizes its p–Wasserstein distance, $p > 1$, to X exactly once for every minimizing geodesic between x and y.



N
$\mu_t = t\delta_N + (1-t)\delta_S$
γ
α
S

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Let $\gamma(t) : [0,1] \to X$ be a minimizing geodesic from x to y. Then

$$W_p^p(\delta_{\gamma(t)}, \mu_\lambda) = \lambda d(\gamma(t), x)^p + (1-\lambda) d(\gamma(t), y)^p = (\lambda t^p + (1-\lambda)(1-t)^p) d(x,y)^p.$$

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The minimum will be achieved at the parameter t_0 which verifies

$$\frac{d}{dt}\Big|_{t=t_0} W_p^p(\delta_{\gamma(t)}, \mu_\lambda) = 0 \text{ which is } t_0 = \frac{(1-\lambda)^{p-1}}{\lambda^{p-1} + (1-\lambda)^{p-1}}.$$

$$W_p^p(\delta_{\gamma(t_0)}, \mu_\lambda) = \frac{\lambda(1-\lambda)^{(p-1)p} + (1-\lambda)\lambda^{(p-1)p}}{(\lambda^{p-1} + (1-\lambda)^{p-1})^p} \cdot d^p(x,y).$$

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Then, we only have to prove that the minimum can only be achieved inside a minimizing geodesic.
We will choose any $a \in X$, and we will construct another point a' inside a minimizing geodesic γ verifying $W_p^p(\delta_a, \mu_\lambda) \geq W_p^p(\delta_{a'}, \mu_\lambda)$.

The case $d(a,y) \geq d(x,y)$ is straightforward, as choosing $a' = x$ we have

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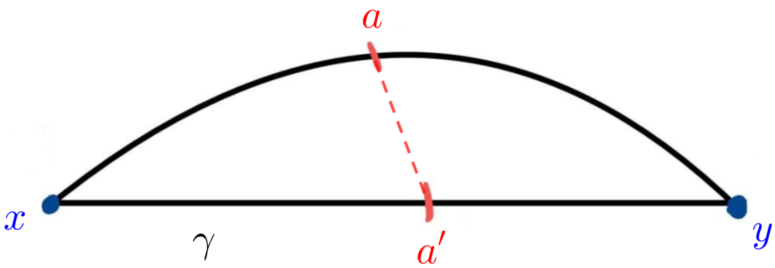
Now, if $d(a,y) < d(x,y)$, we can pick a' inside γ at distance $d(a,y)$ to y.


a
x
γ
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Now, if $d(a,y) < d(x,y)$, we can pick a' inside γ at distance $d(a,y)$ to y. Observe that $d(a,x) \geq d(a',x)$ or γ would not be minimizing. Then,

$$\begin{aligned} W_p^p(\delta_a, \mu_\lambda) &= \lambda d(a,x)^p + (1-\lambda) d(a,y)^p \\ &= \lambda d(a,x)^p + (1-\lambda) d(a',y)^p \\ &\geq \lambda d(a',x)^p + (1-\lambda) d(a',y)^p = W_p^p(\delta_{a'}, \mu_\lambda). \end{aligned}$$

Theorem 8 (Casado, C. & Santos-Rodríguez, 2023)

Let X be a geodesic metric space, and $x \in X$ a point such that there exists another $y \in X$ with the property that there exist at least two different minimising geodesics from x to y. Then, for every $p > 1$,

$$\mathrm{reach}(x, X \subset W_p(X)) = 0.$$

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Let (X,d) *be a metric space, and consider its* $1-$*Wasserstein space,* $W_1(X)$. *Then, for every accumulation point* $x \in X$, $\operatorname{reach}(x, X \subset W_1(X)) = 0$. *In particular, if* X *is not discrete,* $\operatorname{reach}(X \subset W_1(X)) = 0$.

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Future questions

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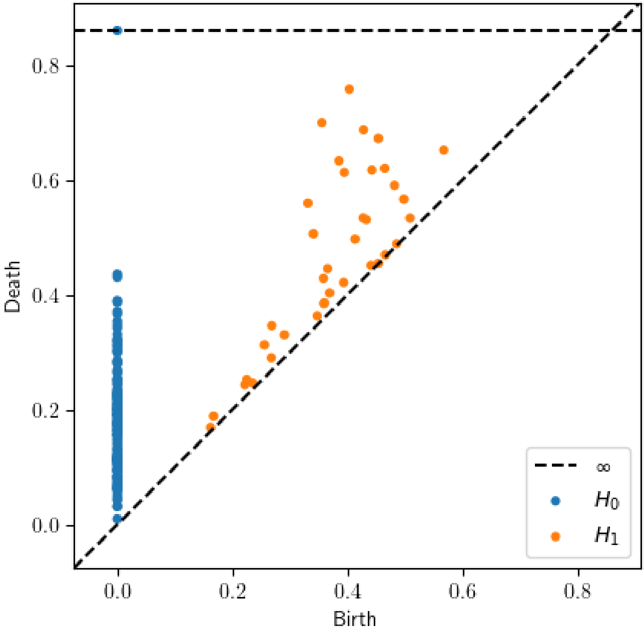
Definition (Persistence diagram)

A *persistence diagram* D is a multiset of points $(a, b) \in \mathbb{R}^2_< = \{(x, y) \in \mathbb{R}^2 : x < y\}$ and whose points represent the i–persistent homology of certain filtration $\mathscr{F}$.


0.8
0.6
0.4
0.2
0.0
Death
0.0
0.2
0.4
0.6
0.8
Birth
∞
H_0
H_1

Bottleneck Distance

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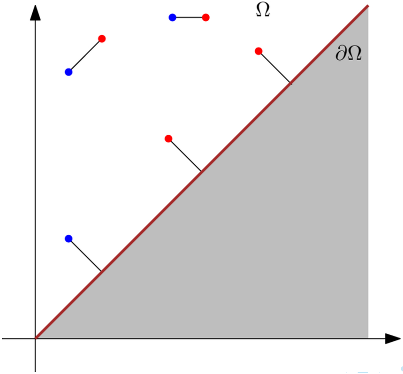

Definition (Cost of a partial matching)

Let $D_1\colon I_1 \to \mathbb{R}^2_<$ and $D_2\colon I_2 \to \mathbb{R}^2_<$ be persistence diagrams and (I_1', I_2', f) a partial matching between them. We endow $\mathbb{R}^2$ with the infinity metric $\mathrm{d}_\infty(a,b) = \|a-b\|_\infty = \max(|a_x - b_x|, |a_y - b_y|)$. We denote by $\mathrm{cost}_\infty(f)$ the ∞*–cost of* f, defined as follows:

$$\mathrm{cost}_\infty(f) = \max\Big\{\sup_{i\in I_1'} \mathrm{d}_\infty(D_1(i), D_2(f(i))), \sup_{j\in I_1\setminus I_1'} \mathrm{d}_\infty(D_1(j),\Delta), \sup_{k\in I_2\setminus I_2'} \mathrm{d}_\infty(D_2(k),\Delta)\Big\}.$$

If any of the terms in either expression is unbounded, we declare the cost to be infinity.

Definition (Bottleneck distance of persistence diagrams)

Let $p = \infty$ and D_1, D_2 persistence diagrams. Define

$$\tilde{w}_\infty(D_1, D_2) = \inf\{\mathrm{cost}_\infty(f) : f \text{ is a partial matching between } D_1 \text{ and } D_2\}.$$

Let $(\mathrm{Dgm}_\infty, w_\infty)$ denote the metric space of persistence diagrams D such that $\tilde{w}_\infty(D, \emptyset) < \infty$ with the relation $D_1 \sim D_2$ if $\tilde{w}_\infty(D_1, D_2) = 0$. The metric w_∞ is called the *bottleneck distance*.

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Remark

For $p<\infty$, we can use

$$\mathrm{cost}_p(f)=\Big(\sum_{i\in I_1'} \mathrm{d}_\infty(D_1(i),D_2(f(i)))^p+\sum_{j\in I_1\setminus I_1'} \mathrm{d}_\infty(D_1(j),\Delta)^p + \sum_{k\in I_2\setminus I_2'} \mathrm{d}_\infty(D_2(k),\Delta)^p\Big)^{1/p},$$

and then, define the p–Wasserstein distance for persistence diagrams.

Reach of the space of persistence diagrams

Bubenik and Wagner construct an explicit isometric embedding of bounded separable metric spaces into $(\mathrm{Dgm}_\infty, w_\infty)$.

$$\begin{aligned}\varphi : (X, \mathrm{d}) &\to (\mathrm{Dgm}_\infty, w_\infty)\\ x &\mapsto \{(2c(k-1), 2ck + \mathrm{d}(x, x_k))\}_{k=1}^{\infty},\end{aligned}$$

where $c > \mathrm{diam}(X)$ and $\{x_k\}_{k=1}^{\infty}$ is a countable, dense subset of (X, d).

Reach of the space of persistence diagrams

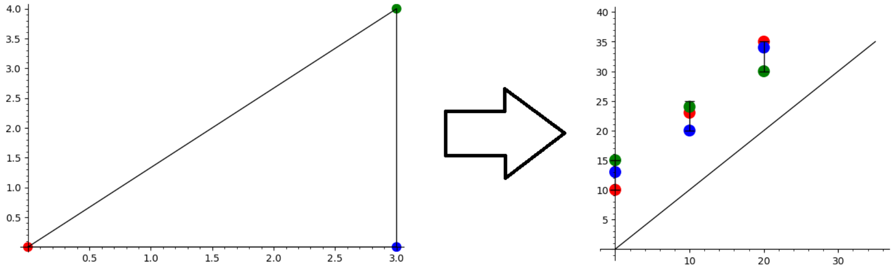

Figure: Bubenik's embedding of a triangle

Reach of the space of persistence diagrams

Bubenik and Wagner construct an explicit isometric embedding of bounded separable metric spaces into $(\mathrm{Dgm}_\infty, w_\infty)$.

$$\begin{aligned} \varphi : (X, \mathrm{d}) &\to (\mathrm{Dgm}_\infty, w_\infty) \\ x &\mapsto \{(2c(k-1), 2ck + \mathrm{d}(x, x_k))\}_{k=1}^{\infty}, \end{aligned}$$

where $c > \mathrm{diam}(X)$ and $\{x_k\}_{k=1}^{\infty}$ is a countable, dense subset of (X, d).

Theorem 11 (Casado, C. & Santos-Rodríguez, 2023)

Let (X, d) *be a separable, bounded metric space and* $(\mathrm{Dgm}_\infty, w_\infty)$ *the space of persistence diagrams with the bottleneck distance. If* $x \in X$ *is an accumulation point, then*

$$\mathrm{reach}(x, X \subset \mathrm{Dgm}_\infty) = 0.$$

In particular, if X *is not discrete,* $\mathrm{reach}(X \subset \mathrm{Dgm}_\infty) = 0$.

Thank you for your attention!

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