Lecture 4

1 What is a statistical model?

Let X be a random variable (or a collection of them). X can represent cross-sectional, time series, or panel data. A *statistical model for* X is collection of different probability distributions indexed by θ:

$$\{P_\theta\}_{\theta \in \Theta}.$$

A time series model is a *statistical model* for time series data.

A time series model for time series data

$$\{X_t\}_{t \in T}$$

is said to be *strongly stationary* if the distribution of any collection

$$(X_t, X_{t+h_1}, \ldots, X_{t+h_k})$$

does not depend on t. A strongly stationary time series model is *weakly stationary*, but not the other way around.

2 A non-stationary model

1. Consider the model

$$X_t = \sum_{j=1}^{t} \epsilon_j$$

where ϵ_t are mean 0, variance σ, uncorrelated random variables.[1] Is this

[1] Time series with these properties are usually referred to as white noise

model stationary? Answer: No. This is why:

$$\begin{aligned} X_1 &= \epsilon_1 \\ X_2 &= \epsilon_1 + \epsilon_2 \\ X_3 &= \epsilon_1 + \epsilon_2 + \epsilon_3 \end{aligned}$$

Consquently

$$Cov(X_1, X_2) = Cov(\epsilon_1, \epsilon_1) = \sigma^2,$$

but,

$$Cov(X_2, X_3) = Cov(\epsilon_1 + \epsilon_2, \epsilon_1 + \epsilon_2) = 2\sigma^2.$$

3 Example of weakly stationary models: Gaussian MA(1)

For $t = 1, 2, \ldots, T$ consider:

$$X_t = \theta_0 \epsilon_t + \theta \epsilon_{t-1},$$

where $\epsilon_t \sim \mathcal{N}(0, \sigma^2)$. The mean function is

$$E[X_t] = 0 \quad \text{for all } t \tag{1}$$

And the first order autocovariance is:

$$Cov(X_t, X_{t+1}) = Cov(\theta_0 \epsilon_t + \theta_1 \epsilon_{t-1}, \theta_0 \epsilon_{t+1} + \theta_1 \epsilon_t),$$

which equals $\theta_0 \theta_1 \sigma^2$ and does not depend on t.

How about $\gamma_X(h)$ for $h > 1$? Algebra shows that

$$Cov(X_t, X_{t+h}) = 0.$$

Therefore, the covariance function is given by:

$$\gamma_X(h) = \begin{cases} (\theta_0^2 + \theta_1^2)\sigma^2 & \text{if} \quad h = 0 \\ \theta_0\theta_1\sigma^2 & \text{if} \quad h = 1 \\ 0 & \text{if} \quad h > 1. \end{cases}$$

Check the jupyter notebook that we used in class and see if the simulated autocovariance function matches our derivation.