Homework 4
Numerical methods for random partial differential equations: hierarchical approximation and machine learning approaches

June 25, 2023

IMPORTANT: Please refer to the course syllabus and the course outline for what an acceptable level on a handed in homework is.

Discrete L^2 Regression methods

Objective

Hands-on experience with Discrete L^2 Regression methods. Tasks include numerical verification. The idea is to connect theory and practical aspects and experiment with the methods.

1 L^2 discrete regression under exact sampling (continuation of Ex. 1)

1.1 Objective

This problem offers hands-on experience on random discrete L^2 projection under the idealized assumptions of exact sampling.

1.2 Problem

We are interested in approximating

$$g = \underset{v \in P_\Lambda}{\operatorname{argmin}} \|f - v\|^2_{L^2_{([0,1]^N)}} \tag{1}$$

for some given function f. We try different cases, which mainly differ by the regularity degree of the function f.

1.3 Model problems

We look at different examples of f for some real constants $\{c_n, w_n\}_{n=1}^N$ taken from

1. Oscillatory: $f(\mathbf{x}) = \cos\left(2\pi w_1 + \sum_{n=1}^N c_n x_n\right)$, with $c_n = 9/N$, $w_1 = \frac{1}{2}$.

2. Product peak: $f(\mathbf{x}) = \prod_{n=1}^N \left(c_n^{-2} + (x_n - w_n)^2\right)^{-1}$, with $c_n = 7.25/N$ and $w_n = \frac{1}{2}$.

3. Corner peak: $f(\mathbf{x}) = \left(1 + \sum_{n=1}^N c_n x_n\right)^{(-N+1)}$
with $c_n = 1.85/N$.

4. Gaussian: $f(\mathbf{x}) = \exp\left(-\sum_{n=1}^N c_n^2 (x_n - w_n)^2\right)$, with $c_n = 7.03/N$ and $w_n = \frac{1}{2}$.

$$\int f(x)\ \mathrm{d}x = \prod_{i=1}^N \frac{-\sqrt{\pi}}{2} c_i^{-1} \mathrm{erf}\left(c_i(w_i - x_i)\right)$$

5. Continuous: $f(\mathbf{x}) = \exp\left(-\sum_{n=1}^N c_n |x_n - w_n|\right)$,with $c_i = 2.04/N$ and $w_i = \frac{1}{2}$.

6. Discontinuous: $f(\mathbf{x}) = \begin{cases} 0 & \text{if } x_1 > w_1 \text{ or } x_2 > w_2 \\ \exp\left(\sum_{n=1}^N c_n x_n\right) & \text{otherwise} \end{cases}$, with $c_i = 4.3/N$, $w_1 = \frac{\pi}{4}$ and $w_2 = \frac{\pi}{5}$.

1.4 Numerical tasks

1. Compute a polynomial approximation for f based on a weighted discrete L^2 projection for (1), namely

$$\hat{f}_{\Lambda,M} = \underset{v \in \mathbb{P}_\Lambda([0,1]^N)}{\operatorname{argmin}} \frac{1}{M} \sum_{k=1}^M \frac{\rho(\mathbf{y}^{(k)})}{\hat{\rho}(\mathbf{y}^{(k)})} |f - v|^2(\mathbf{y}^{(k)}).$$

   Use the following isotropic polynomial spaces to carry out the projection:

   a) Total degree

   b) Hyperbolic cross

Remember to choose the number of samples to preserve the stability of the random discrete L^2 projection. Compute the bound of $K(\Lambda)$, plot it with respect to the dimension of the polynomial subspace to conclude an appropriate choice of the number of samples using the sampling inequality. Try both a linear and a quadratic relation between the number of samples and the dimension of the polynomial subspace. Sample both from a uniform and a Chebyshev distribution and expand the function in Legendre polynomials. For all the methods, check that $\| \frac{1}{M} D^T D - I_d \| \to 0$ as M increases (where D is the design matrix). Estimate the numerical error using cross-validation. Plot the error estimates versus the number of samples used. Estimate the convergence rate on each case. Do your computations at least for $N = 2$ and $N = 5$. For the product peak and the discontinuous functions, try to work with the scaled functions or use the relative error in all examples to have the same scale for the error.

2. Consider the function $f_1 : \mathbb{R}^N \to \mathbb{R}$ defined as

$$f_1(\mathbf{x}) = \mathrm{Si}\left(\frac{x_1}{a}\right) e^{-4|\mathbf{x}|^2} \tag{2}$$

where $a = 1$ and

$$\mathrm{Si}(v) = \int_0^v \frac{\sin(t)}{t} \mathrm{d}t$$

is the so called *Sine integral*. The function (2) is illustrated for $a = 1$ and $a = 10^{-2}$ in Figure 1 and Figure 2 respectively when $N = 1$. Compute a trigonometric approximation for the $[0, 2\pi]^N$-periodic function

$$f(\mathbf{x}) = f_1(((\mathbf{x} + \pi\mathbf{1}) \bmod 2\pi) - \pi\mathbf{1}),$$

where mod is taken componentwise, based on a discrete L^2 projection for (1),

Implement an adaptive procedure to select the optimal trigonometric space for the L^2 approximation of the given function. Do your computations at least for $N = 1$ and $N = 2$. To simplfity your search for the set of indices, try to impose a constraint to have a real valued output. Also, in case you have an odd or even function to approximate, try to impose that constraint too.

Hint: In `MATLAB` the Sine integral function has the name `sinint`. Type `help sinint` in the command window for more information.

3. Repeat the previous task using instead the value $a = 10^{-2}$ in (2).

Implement an adaptive procedure to select the optimal trigonometric space for the L^2 approximation of the given function. Do your computations at least for $N = 2$ and $N = 5$.

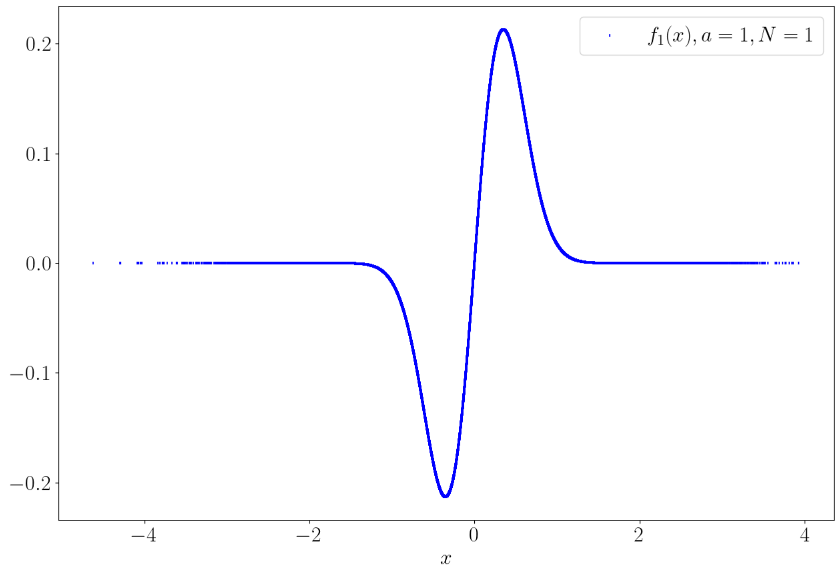

Figure 1: The figure presents the function (2), for $a = 1$ and $N = 1$ evaluated at 10^5 points sampled from the standard normal distribution.

2 Random discrete L^2 projection

2.1 Objective

Deepening the understanding of random discrete L^2 projection by using it in the presence of discretization errors, that is, approximate sampling.

2.2 Problem

Consider the random discrete L^2 projection, improving the stability of the scheme by sampling from a Chebyshev distribution and correcting the least square weights accordingly, i.e.

$$Q(\hat{u}_h)_{\Lambda,M} = \underset{v \in \mathbb{P}_\Lambda(\Gamma)}{\operatorname{argmin}} \frac{1}{M} \sum_{k=1}^{M} \frac{\rho(\mathbf{y}^{(k)})}{\hat{\rho}(\mathbf{y}^{(k)})} (Q(u_h) - v)^2(\mathbf{y}^{(k)}),$$

where the sample $\{\mathbf{y}^k\}_{k=1}^M$ is drawn from the Chebyshev distribution $\hat{\rho}(\mathbf{y})d\mathbf{y}$ and $\rho(\mathbf{y}^{(k)})$ is the uniform probability density function (see course slides). The probability density function of a Chebyshev random variable is $\hat{\rho}(y) = \frac{1}{\pi}\frac{1}{\sqrt{1-y^2}}$, and you can generate samples of a Chebyshev random variable as $Y = 2\sin^2(\frac{\pi}{2}X) - 1$, where $X \sim \mathcal{U}([-1, 1])$.

1. Implement an adaptive procedure to select the optimal polynomial space for the L^2 approximation of the Quantity of Interest of the linear elliptic PDE test-case

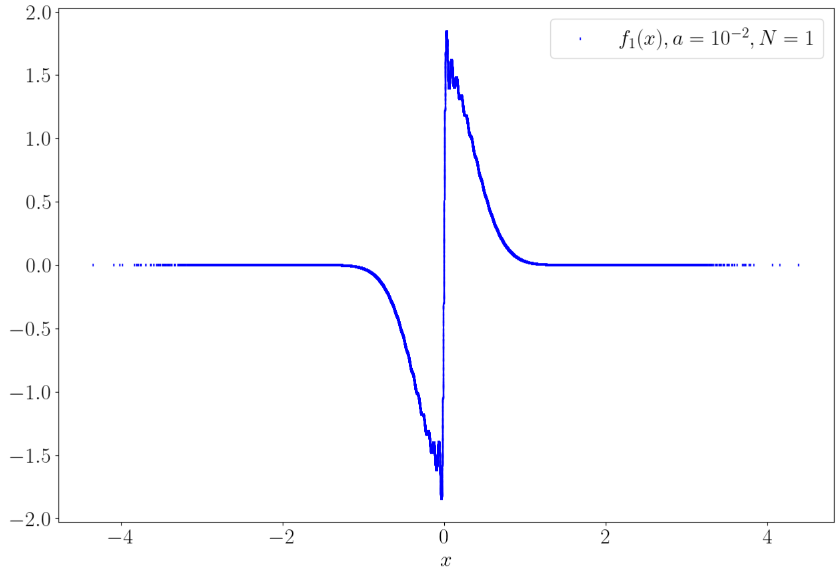

Figure 2: The figure presents the function (2), for $a = 10^{-2}$ and $N = 1$ evaluated at 10^5 points sampled from the standard normal distribution.

considered in Homework 1 with 0.5% L^2 error. To this end, consider Model 1 (Piecewise constant coefficient) in Homework 1 for the random coeeficient a, and only a fixed number of random variables $N = 10$. Choose the spatial discretization parameter h to meet the accuracy requirement.

2. Redo the last part using a bilevel discrete L^2-regression approximation.