June 17, 2022

FEEDBACK HW3 :

① Aim: Approximate $\mathbb{E}[Q(u)]$ via MLMC

- First, make the approx:

$$\mathbb{E}[Q(u)] \approx \mathbb{E}[Q(u^N)]$$

where u^N is the solution of the PDE with

$$a^N(x,\omega) = \exp(k^N(x,\omega))$$

and

$$k^N(x,\omega) = \sum_{n=1}^{N} \sqrt{\lambda_n}\, Y_n(\omega) e_n$$

- Let $h_0 > h_1 > \cdots > h_L$

$$h_\ell = h_0 2^{-\ell}$$

Denote $g_\ell(Y_1,\dots,Y_N) = Q(u^N_{h_\ell})$

$$, \ \ell = 0, \dots, L$$

- Telescoping Sum

$$\mathbb{E}[g_L] = \mathbb{E}[g_0] + \sum_{\ell=1}^{L} \mathbb{E}[g_\ell - g_{\ell-1}]$$

- MLMC estimator:

$$A_{MLMC} = \frac{1}{M_0} \sum_{j=1}^{M_0} g_0\left(Y_1^{(0,j)}, \ldots, Y_N^{(0,j)}\right)$$
$$+ \sum_{\ell=1}^{L} \frac{1}{M_\ell} \sum_{j=1}^{M_\ell} \underbrace{g_\ell\left(Y_1^{(\ell,j)}, \ldots, Y_N^{(\ell,j)}\right) - g_{\ell-1}\left(Y_1^{(\ell,j)}, \ldots, Y_N^{(\ell,j)}\right)}_{\text{Use the same } Y_1, \ldots, Y_N}$$

- Mean Square Error (MSE)

$$\mathbb{E}[(g - A_{MLMC})^2]$$

$$= \text{Bias}^2 + \text{Variance}$$

$$= \underbrace{\left(\mathbb{E}[g - g_L]\right)^2}_{TOL^2/2} + \underbrace{\text{Var}(A_{MLMC})}_{TOL^2/2}$$

Choosing Optimal Parameters

- L is chosen such that

$$\text{Bias}^2 = \frac{TOL^2}{2}$$

$$\text{Bias} = |\mathbb{E}[g - g_L]| = c h_L^{\alpha} = \frac{TOL}{\sqrt{2}}$$

⚠ - Find α, c using pilot run

- Consider relative error and not absolute error

- Optimal Sample Sizes $\{M_\ell\}_{\ell=0}^{L}$

Run a pilot to estimate

$$C_\ell : \quad \ell = 0, \ldots, L$$

$$V_\ell = \begin{cases} \mathrm{Var}[g_0] , & \ell = 0 \\ \mathrm{Var}[g_\ell - g_{\ell-1}] , & \ell > 0 \end{cases}$$

Optimal $$M_\ell^* = \left\lceil \mathrm{TOL}^{-2} \sqrt{\frac{V_\ell}{C_\ell}} \sum_{j=0}^{L} \sqrt{V_j C_j} \right\rceil$$

Choice of Level 0

$$\mathrm{Var}[g_1 - g_0] \ll \mathrm{Var}[g_0]$$

or

$$\mathrm{Cost}[A_{MLMC}(0, L)] < \mathrm{Cost}[A_{MLMC}(1, L)]$$

② Approximate $\mathbb{P}[Q(u) < K]$

- Denote $g_\ell(Y_1, \dots, Y_N) = \mathbb{1}_{\{Q(u^N_{n_\ell}) < K\}}$

- $\mathbb{E}[g_L] = \mathbb{E}[g_0] + \sum_{\ell=1}^{L} \mathbb{E}[g_\ell - g_{\ell-1}]$

For all levels, use the same IS shifts as you did in HW1.

③ MLMC + QMC

- Same choice of L and level 0 as in question 1.
- Optimal number of QMC points

$$
\begin{cases}
\min\limits_{m, \{M_\ell\}_{\ell=0}^{L}} m \left\{ M_0 + 2M_1 + \cdots + 2^L M_L \right\} \\
\text{s.t} \quad \dfrac{1}{m} \left\{ \sum\limits_{\ell=0}^{L} \dfrac{C_\ell}{M_\ell^{-1-\delta_\ell}} \right\} \leq TOL^2
\end{cases}
$$

m: number of random shifts for RQMC

C_ℓ and $0 < \delta_\ell < 1$ determined via a pilot