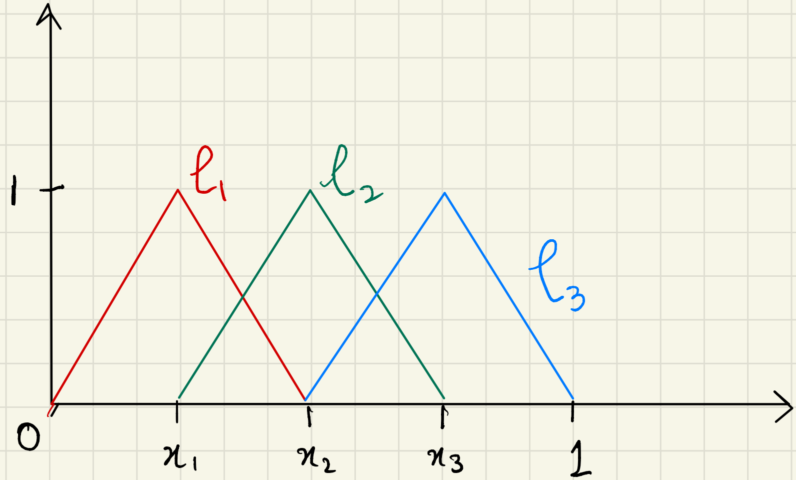
25-04-2022

Homework 1 - Part II (Feedback)

1. Piecewise Linear FEM Approximation

The basis functions ℓ_i are given by

$$
\ell_i(x) = \begin{cases} 0, & x < x_{i-1} \\ \dfrac{x - x_{i-1}}{\underbrace{x_i - x_{i-1}}_{=h}}, & x_{i-1} \le x < x_i \\ 1 - \dfrac{x - x_i}{\underbrace{x_{i+1} - x_i}_{=h}}, & x_i \le x < x_{i+1} \\ 0, & x \geq x_{i+1} \end{cases}
$$



Using the variational formulation,

$$\boxed{A(\omega)\, u_h(\omega) = F}$$

↳ Linear System

where

$$\begin{cases} u_h = [u_{h,1}, u_{h,2}, \dots, u_{h,I}] \\ A_{ij} = A(\phi_j, \phi_i) \\ F_i = F(\phi_i) \end{cases}$$

2. Models for Random Coefficient a

Model 1 : Piecewise Constant

$$a(x,\omega) = 1 + \sigma \sum_{n=1}^{N} Y_n(\omega)\, \mathbb{1}_{[\hat{x}_{n-1}, \hat{x}_n]}(x)$$

$$I = N \cdot 2^{\ell}, \quad \ell \geq 0$$

I → mesh for numerical discretization of PDE

N → mesh to define random field

! Choose σ to ensure coercivity

$$\boxed{1 - \sigma\sqrt{3} > 0}$$

Model 2 : KL Expansion

Observe that

$\nu = 0.5 \implies$ Exponential Covariance

$\nu \to \infty \implies$ Squared Exponential Cov.

$\nu \to$ controls pathwise regularity of random field

Small $\nu \implies$ Poor Regularity

$$K(x,\omega) \approx \sum_{n=1}^{N} \sqrt{\lambda_n}\, Y_n(\omega)\, e_n(x)$$

$\{Y_n\}_{n=1,\dots,N}$: iid standard Gaussian

$\{\lambda_n, e_n\}_{n=1,\dots,N}$ from

$$(*) \rightarrow \boxed{\int_0^1 C(x,y)\, e_n(y)\, dy = \lambda_n e_n(x)}$$

Discretize $(*)$ on grid

$$\{x_{i+1/2}\}_{i=0,\dots,I-1} \times \{x_{i+1/2}\}_{i=0,\dots,I-1}$$

$$\Rightarrow \boxed{C e_n = \lambda_n e_n}$$

where $C_{ij} = C(x_i, x_j)$

Use any eigensolver to solve

⚠ NORMALIZE $\{e_n\}_{n=1,\dots,N}$

Model 3: Fourier Expansion

Fourier coefficients

$$K_n^2 = \frac{2}{L_p} \int_0^{L_p} Cov_K^{\#}(x-y) \cos\left(\frac{n\pi(x-y)}{L_p}\right)$$

You can use

1) Numerical integration Scheme
2) FFT (Fast Fourier Transform)

to compute K_n

TASK 2.5 (i)

Let bias be

$$Q(u_h) - Q(u) = a_1 h^p + O(h^r)$$

for some $r > p$.

Using Richardson extrapolation, we can obtain an approximation of the bias that is r^{th} order accurate.

For example,

$$\begin{cases} ① - & Q(u_h) - Q(u) = a_1 h^p + a_2 h^r + \text{h.o.t} \\ ② - & Q(u_{h/2}) - Q(u) = \dfrac{a_1}{2^p} h^p + \dfrac{a_2}{2^r} h^r + \text{h.o.t} \end{cases}$$

① - ② $\Longrightarrow$

$$Q(u_n) - Q(u_{n/2}) = \left(1 - \frac{1}{2^p}\right) a_1 h^p + \mathcal{O}(h^r)$$

Approx. bias

$$\Rightarrow \boxed{\underset{(Q(u_n)-Q(u))}{\text{Bias}} \approx \frac{1}{\left(1 - 1/2^{p}\right)} \left[Q(u_n) - Q(u_{n/2})\right]}$$

Run pilot to first estimate p

$\Rightarrow$ Choose h to satisfy relative bias

$$\frac{\left|\mathbb{E}[Q(u_n)] - \mathbb{E}[Q(u)]\right|}{\mathbb{E}[Q(u)]}$$

Relative Statistical error = $\dfrac{C_\alpha \sqrt{\mathrm{Var}[Q(u_h)]}}{\mathbb{E}[Q(u)] \sqrt{M}}$

$\Rightarrow$ Choose M, h to satisfy TOL

Idea: One approach is to adaptively update M, h to handle relative errors.

Task 2.5 (2)

Generalized Control Variate

$$\frac{1}{M_1}\sum_{i=1}^{M_1} Q^{(i)}(u_{2h}) + \frac{1}{M_2}\sum_{j=1}^{M_2}\left(Q(u_h) - Q(u_{2h})\right)^{(j)}$$

$\Downarrow$

Same randomness

$Y_1, \dots, Y_n$

Choice of h

Choose h as in Task 2.5 (1)

Choice of M_1, M_2

For given h,

Variance of estimator,

$$V = \frac{Var[Q(u_{2h}(\omega))]}{M_1} + \frac{Var[Q(u_h(\omega)) - Q(u_{2h}(\omega))]}{M_2}$$

Work of estimator,

$$Work = M_1 C_1 + M_2 C_2$$

where C_1 = cost of one realization of $Q(u_{2h}(\omega))$

C_2 = cost of one realization of $[Q(u_h(\omega)) - Q(u_{2h}(\omega))]$

Solve

$$\begin{cases} \min\limits_{\{M_1, M_2\}} \left(M_1 C_1 + M_2 C_2\right) \\ \text{s.t} \quad \text{Variance} \approx \dfrac{TOL^2}{C_\alpha^2} \end{cases}$$

Idea: Run pilots to estimate C_1, C_2 and

$Var[Q(u_{2h}(\omega))]$,

$Var[Q(u_h(\omega)) - Q(u_{2h}(\omega))]$

(!) Compare work required by MC and MC + CV to satisfy given TOL

Task 2.5 (3): Importance Sampling

$$\mathbb{P}[Q(u) < K] = \mathbb{E}[\mathbb{1}_{\{Q(u)<K\}}]$$

$$\approx \frac{1}{M} \sum_{m=1}^{M} \mathbb{1}_{\{Q(u_h(\omega_m)) < K\}}$$

$$u_h(\omega_m) \equiv u_h(Y^{(m)})$$

In Model 2, $Y^{(m)}$ – rdv with iid standard Gaussian

$\rho_Y(y)$ – joint pdf of Y

$$\rho_Y(y) \propto \prod_{n=1}^{N} \exp\left\{-\frac{1}{2} y_n^2\right\}$$

<u>Idea</u>: Mean-Shift measure change

$$\tilde{\rho}_Y(y) \propto \prod_{n=1}^{N} \exp\left\{-\frac{1}{2}(y_n - \mu_n)^2\right\}$$

Optimal mean shift can be found

$$\mu^* = \arg\max_{y} \left\{ \mathbb{1}_{\{Q(u) < k\}} \prod_{n=1}^{N} \exp\left\{-\frac{1}{2} y_n^2\right\} \right\}$$

⚠ N- dimensional optimization!!!

<u>Hint</u>: Use decaying nature of spectral contribution of KL coefficients to reduce dimensionality of above problem.