Lab 3 : Real Roots II

Manuel Leclee

Exercise 1 :

Suppose we are given a polynomial $P(x)$ of degree n

$$P(x) = \sum_{i=0}^{n} a_i x^i .$$

Let $P(x)$ have the roots $x_1, \dots, x_n$.

Moreover, let us define $P_K = (x_1^k + \dots + x_n^k)$

We are asked to prove that $P_K = -\operatorname{Res}\left(x^K \frac{P'(x)}{P(x)}, \infty\right)$ given the hint that

$$\operatorname{Res}(g(x), \infty) = \operatorname{Res}\Big(\underbrace{-\frac{1}{x^2}\, g\left(\frac{1}{x}\right)}_{\tilde{g}(x)}, 0\Big).$$

We know that

$$P(x) = \prod_{i=1}^{n} (x - x_i) \text{ , therefore } \underbrace{P'(x)}_{"\tilde{g}(x)} = \sum_{i=1}^{n} \prod_{j \neq i} (x - x_j) = \sum_{i=1}^{n} \frac{P(x)}{(x - x_i)}$$

Let $g(x) = x^K \frac{P(x)'}{P(X)}$, then $\operatorname{Res}\left(-\frac{1}{x^2} g\left(\frac{1}{x}\right), 0\right) = \operatorname{Res}\left(-\frac{1}{x^2} \cdot \frac{1}{x^K} \cdot \frac{P'\left(\frac{1}{x}\right)}{P\left(\frac{1}{x}\right)}, 0\right)$

$$= \operatorname{Res}\left(-\frac{1}{x^{K+2}} \cdot \frac{\sum_{i=1}^{n} \frac{P(1/x)}{(1/x - x_i)}}{P(1/x)}, 0\right) = \operatorname{Res}\left(-\frac{1}{x^{K+2}} \cdot \sum_{i=1}^{n} \frac{1}{(1/x - x_i)}, 0\right) =$$

$$\operatorname{Res}\left(-\frac{1}{x^{K+1}} \cdot \sum_{i=1}^{n} \frac{1}{(1 - x x_i)}, 0\right).$$

Therefore, $\tilde{g}(x)$ has a pole of order $K+1$ in $x = 0$.

We know from complex analysis that the $\operatorname{Res}(\tilde{g}(x), 0)$ can be computed as follows:

$$\operatorname{Res}(\tilde{g}(x), 0) = \frac{1}{K!} \lim_{x \to 0} \left[\frac{\partial^k}{\partial x^k} (x^{K+1} \tilde{g}(x))\right] = -\frac{1}{K!} \lim_{x \to 0} \left[\frac{\partial^K}{\partial x^K} \left(\sum_{i=1}^{n} \frac{1}{(1 - x x_i)}\right)\right]$$

$$= -\frac{1}{k!} \lim_{x \to 0} \sum_{i=1}^{n} \frac{k!\, x_i^k}{(1 - x x_i)^{k+1}} = -\sum_{i=1}^{n} x_i^k = -P_K$$

Since $\operatorname{Res}(g(x), \infty) = \operatorname{Res}(\tilde{g}(x), 0) = -P_K$ we are done.

$\blacksquare$