Computational Algebra. Lecture 3

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Real Roots(II)

In this session we will follow with the study of different methods to compute the number of real roots of a given polynomial with real coefficients.

- Newton Sums
- Hermite Method

Newton Sums

Let $p(x) = a_n x^n + \ldots + a_0 \in \mathbb{F}[x]$ be a polynomial of degree n. Let $\alpha_1, \ldots, \alpha_n$ be the roots of $p(x)$. The following polynomials N_i are called **Newton sums**:

$$\begin{aligned}
N_1 &= \alpha_1 + \ldots + \alpha_n \\
N_2 &= \alpha_1^2 + \ldots + \alpha_n^2 \\
&\vdots \\
N_k &= \alpha_1^k + \ldots + \alpha_n^k \\
&\vdots
\end{aligned}$$

Newton Sums

Let us assume that p is monic, and
$p(x) = \Pi_{i=1}^{n}(x - \alpha_i) = S_0 x^n + \ldots + S_n$. Then,

$$S_i = (-1)^i \sum_{j_1 < \ldots < j_i} \alpha_{j_1} \ldots \alpha_{j_i}$$

and the polynomials $(-1)^i S_i$ are called **elementary symmetric polynomials**.

- The N_i's form a basis for tha symmetric polynomials.
- The S_i's form a basis for tha symmetric polynomials.

The transition formulas between these two basis of the symmetric polynomials are known as **Newton-Girard Formulae**(Newton 1673, Girard 1629):

In particular **Newton Identities** are:

$$
\begin{aligned}
a_n N_1 + a_{n-1} &= 0 \\
a_n N_2 + a_{n-1} N_1 + 2a_{n-2} &= 0 \\
a_n N_3 + a_{n-1} N_2 + a_{n-2} N_1 + 3a_{n-3} &= 0 \\
&\vdots \\
a_n N_{n-1} + a_{n-1} N_{n-2} + \ldots + a_1 N_1 + n a_0 &= 0 \\
&\vdots
\end{aligned}
$$

Define $a_j = 0$ for $j < 0$ and we can write

$$a_n N_k + a_{n-1} N_{k-1} + \ldots + a_{n-(k-1)} N_1 + k a_{n-k} = 0$$

Newton Sums

Proof.

Case $k = n$.

If $p(x) = a_n x^n + \ldots + a_1 x + a_0$, then

$$a_n\alpha_1^n + \ldots + a_1\alpha_1 + a_0 = 0$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$a_n\alpha_n^n + \ldots + a_1\alpha_n + a_0 = 0$$

and adding the above identities we have that

$$a_n N_n + \ldots + a_1 N_1 + n a_0 = 0$$

which is what we wanted to show. □

Newton Sums

Proof.

Case $k \geq n$. As above we have that:

$$\begin{aligned} a_n\alpha_1^n + \ldots + a_0 &= 0 \\ &\ldots\ldots\ldots\ldots \\ a_n\alpha_n^n + \ldots + a_0 &= 0 \end{aligned}$$

And the formula is obtained by multiplying each equation by $\alpha_1^{k-n}, \ldots, \alpha_n^{k-n}$ respectively and then sum these equations. $\square$

Newton Sums

Proof.

Case $k < n$, we want to show that

$$a_n N_k + a_{n-1} N_{k-1} + \ldots + a_{n-(k-1)} N_1 + k a_{n-k} = 0$$

that is

$$\sum_{i=1}^{k} S_{k-i} N_i + k S_k = 0$$

We can combine like terms and its suffice to prove that the coefficient of any term

$$\alpha_1^{e_1} \ldots \alpha_n^{e_n}$$

is zero $\square$

Newton Sums

Example

$p(x) = x^3 + 3x^2 + 4x - 8$, then

$$
\begin{aligned}
a_3 N_1 + a_2 = 0 \Rightarrow N_1 &= -\frac{a_2}{a_3} = -3 \\
a_3 N_2 + a_2 N_1 + 2a_1 = N_2 - 9 + 8 = 0 \Rightarrow N_2 &= 1 \\
a_3 N_3 + a_2 N_2 + a_1 N_1 + 3a_0 = 0 \Rightarrow N_3 &= 33
\end{aligned}
$$

In Mathematica lab 3 $\longrightarrow$ NewtonSum[f,k]

Newton Sums

Recall, let $p(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} + a_n x^n$

- If $k < n$, then

$$N_0 = n,\; k a_{n-k} + a_{n-k+1} N_1 + \ldots + a_{n-1} N_{k-1} + a_n N_k = 0$$

- If $k \geq n$, then

$$a_0 N_{k-n} + a_1 N_{k-n-1} + \ldots + a_{n-1} N_{k-1} + a_n N_k = 0$$

Hermite Method

Let $p(x) = a_n x^n + \ldots + a_0$ be a polynomial of degree n. Let $\alpha_1, \ldots, \alpha_n$ be the roots of $p(x)$. We recall that

$$N_k(p) = \alpha_1^k + \ldots + \alpha_n^k$$

Definition

The **Hermite Matrix** of $p(x)$ is the matrix:

$$H_p = \begin{pmatrix} N_0 & N_1 & N_2 & \ldots \\ N_1 & N_2 & \ldots & \ldots \\ N_2 & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix} \in M_n(\mathbb{R})$$

Hermite Method

Theorem (Jacobi)

- *The rank of $H(p)$ is equal to the number of distinct zeroes of $p(x)$.*
- *The signature of $H(p)$ (number of positive vaps-number of negative vaps) is equal to the number of distinct* **Real** *zeroes of $p(x)$.*

The above theorem can be considered as a corollary of the following.

Hermite Method

Let us consider the following matrix

$$H'_p = \begin{pmatrix} h_0 & h_1 & h_2 & \dots \\ h_1 & h_2 & \dots & \dots \\ h_2 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \in M_n(\mathbb{R})$$

where

$$h_k = \alpha_1^k g(\alpha_1) + \ldots + \alpha_n^k g(\alpha_n) =$$

$$= b_0 N_k(p) + \ldots + b_m N_{k+m}(p)$$

and $g(x) = b_0 + b_1 x + \ldots + b_m x^m$ is a polynomial of degree m such that p and g have not common roots.

Hermite Method

Theorem (Hermite, Sylvester, 1856)

Let $q = \frac{(rank(H) - \sigma(H))}{2}$*, that is, the number of distinct pairs of complex conjugate roots of* $p(x)$*. Then*

- $n_+(H') - q$ *is the number of distinct roots of* $p(x)$ *satisfying* $g(x) > 0$.
- $n_-(H') - q$ *is the number of distinct roots of* $p(x)$ *satisfying* $g(x) < 0$.

In Mathematica $\longrightarrow$ Hermite: Matrix H is s and Matrix H' is h.

Homework

1. Prove that

$$s[k] = -Res(x^k \frac{f'(x)}{f(x)}, \infty)$$

   Hint: Use that $Res(g(x), \infty) = Res(-\frac{1}{x^2} g(\frac{1}{x}), 0)$.

2. Modify the code Hermite[f,g] in such a way that it uses z[i] instead of s[i].
3. Modify the code Hermite[f,g] so that it computes the number of roots of f in the interval [a,b] (the input should be f, a, b).