Computational Algebra. Lecture 2

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23 de febrero de 2022

Real Roots 1

In this session we study different methods to compute the number of real roots of a given polynomial with real coefficients.

- Descartes Rule of Signs (1636)
- Budan (1822)- Fourier (1831)
- Sturm (1829)

An important tool is given in the following definition:

Definition

Given a finite succession $(a_0, \ldots, a_n)$ of real numbers, we denote by $V(a_0, \ldots, a_n)$ the number of sign changes in this succession.

Example

$V(1, 3, -2, 3) = 2 = V(1, 0, 3, -2, 3)$

Descartes Rule of Signs (1636)

Theorem

Let $p(x) = a_0 + a_1 x + \ldots + a_n x^n \in \mathbb{R}[x]$ be a polynomial. The number of positive roots of $p(x)$ in less or equal to $V(a_0, \ldots, a_n)$. In all cases is congruent to $V(a_0, \ldots, a_n)$ mod 2.

(Note that to compute the number of negative roots we can apply Descartes to $p(-x)$.)

Demostración.

Homework. $\square$

Descartes Rule of Signs (1636)

Example

Let $p(x) = x^{11} + x^8 - 3x^5 + x^4 + x^3 - 2x^2 + x - 2$, then
$V(1, 0, 0, 1, 0, 0, -3, 1, 1, -2, 1, -2) = 5$,
$p(-x) = -x^{11} + x^8 + 3x^5 + x^4 - x^3 - 2x^2 - x - 2$,
$V(-1, 0, 0, 1, 0, 0, 3, 1, -1, -2, -1, -2) = 2$. Therefore:

- The number of positive roots of $p(x)$ is 1,3 or 5 and the number of negative roots is 0 or 2.
- $p(x)$ has at least 4 complex roots.

Budan-Fourier Theorem

The Budan-Fourier Theorem is a geneneralization of Descartes Rule.

Theorem

Let $p(x) = a_0 + a_1 x + \ldots + a_n x^n \in \mathbb{R}[x]$ be a polynomial. The number of real roots in (a, b) of $p(x)$, where $a < b$, $p(a) \neq 0, p(b) \neq 0$ is less or equal to $D(a, b) := V(p(a), p'(a), \ldots, p^{(n)}(a)) - V(p(b), p'(b), \ldots, p^{(n)}(b))$. In all cases is congruent to $D(a, b)$ mod 2.

Demostración.

Homework. □

Budan-Fourier

Example

Let $p(x) = x^5 - x^4 - x^3 + 4x^2 - x - 1$. We have that

$$V(D(-2)) = 5,\; V(D(-1)) = 4,\; V(D(0)) = 3,\; V(D(1)) = 0$$

Then

- In $(-2, -1) \longrightarrow$ one real zero.
- In $(-1, 0) \longrightarrow$ one real zero.
- In $(0, 1) \longrightarrow$ three real zeroes or one real zero.

With Mathematica: CountRoots, RootsIntervals

Sturm Theorem (1829)

Definition

The Sturm Succession of $p(x)$ is defined as:

- $p_0(x) := p(x)$.
- $p_1(x) := p'(x)$.
- For $i \geq 2$, $p_{i+1}(x) := -rem(p_{i-1}(x), p_i(x))$.

Theorem

Let $p(x) = a_0 + a_1 x + \ldots + a_n x^n \in \mathbb{R}[x]$ be a polynomial. The number of real roots of $p(x)$ in (a, b) (without multiplicities), where $a < b$, $p(a) \neq 0, p(b) \neq 0$ is equal to $S(a, b) := V(p_0(a), p_1(a), \ldots) - V(p_0(b), p_1(b), \ldots)$.

Sturm Theorem (1829)

Demostración.

- We can supose w.o.l.o.g than all the roots of $p(x)$ are simple roots. If not, then $p(x)$ and $p'(x)$ have a common factor $f(x)$ that will be a common factor of all $p_i(x)$ and

$$V(f(x)q_0(x), \dots) = f(x)V(q_0(x), \dots).$$

- It is straighforward to see that if $p(x)$ and $p'(x)$ don't have common roots, then $p_i(x)$ and $p_{i+1}(x)$ neither.

$\square$

Sturm. Sketch of the proof

Now, we want to study the variation of signs:

- If $\alpha \in (a, b)$ is such that $p(\alpha) = 0$, then $p'(\alpha)$ don't change the sign, and therefore we have a variation of sign between p and p'
- If $\exists i > 2$ such that $p_i(\alpha) = 0$, then $p_{i-1}(\alpha) = q(\alpha)p_i(\alpha) - p_{i+1}(\alpha) = -p_{i+1}(\alpha) \neq 0$ Therefore if for instance $p_i(x)$ swithches from being positive to negative, then our sign pattern here goes from either $-++$ to $--+$ or $++-$ to $+--$, depending on the signs of $p_{i-1}(\alpha), p_{i+1}(\alpha)$.

Sturm.

Example

- $p(x) = x^3 + 3x^2 - 1$
- $p(x) = x^5 + 2x^4 - 5x^3 + 8x^2 - 7x - 3$