Problem Session 2 Manuel Lecha

6) Prove that $\mathbb{Z}[x]$ is not an Euclidean domain.

Let R be an integral domain. A Euclidean function $f: R\setminus\{0\} \longrightarrow \mathbb{Z}_{\geq 0}$ is one such that satisfies the following property:

EF) If a,b are in R and b is nonzero, then there exist q and r in R such that $a = bq + r$ and either $r=0$ or $f(r) < f(b)$

Then, an euclidean domain is an integral domain which can be endowed with at least one Euclidean function.

Lemma: Every Euclidean domain is a PID

Let I be a nonzero proper ideal of D. Let $a \in I$ such that $d(a) := \min_{0 \neq x \in I} d(x)$

We claim that I is generated by a.

Suppose not, $\exists b \in I$ such that it is not a multiple of a. Hence, by the properties of a ED, $\exists r \in D$ such that $b = aq + r$ with $d(r) < d(a)$.

Therefore, $r = b - aq \in I$ and $d(a) := \min_{0 \neq x \in I} d(x)$ is contradicted.

$\boxed{\text{Class}}$

1) $\mathbb{Z}[x]$ is not a PID, since $\mathbb{Z}$ is not a field.

2) $\langle x, 2\rangle$ is a proper ideal, but not principal.

3) $\mathbb{Z}[x]/(x) \cong \mathbb{Z}$ which is not a field

irreducible so it should be maximal if PID but $\mathbb{Z}$ is not a field.

16) What is the least non-negative integer f with $f \equiv 2 \bmod 3$, $f \equiv 3 \bmod 5$, $f \equiv 2 \bmod 7$.

First of all, we can notice that 3, 5, 7 are pairwise coprime

Let $m = 105$ — Chinese Remainder Algorithm.

$M_1 = 105/3 = 35 \quad 35 \cdot 2 \equiv 1 \bmod(3)$

$M_2 = 105/5 = 21 \quad 21 \equiv 1 \bmod(5)$

$M_3 = 105/7 = 15 \quad 15 \equiv 1 \bmod(7)$

$x = 2 \cdot M_1 \cdot 2 + 3 \cdot M_2 \cdot 1 + 2 \cdot M_3 \cdot 1 = 233 \equiv 23 \bmod (105)$

3) Let R be an Euclidean Domain $f, g \in R$. Show that $\langle f, g\rangle$ and $\langle \gcd(f,g)\rangle$ coincide.

R. ED $\Rightarrow$ PID so it must exist $a \in R$ s.t $\langle f,g\rangle = \langle a\rangle$

$d = \gcd(f,g) \Leftrightarrow d \mid f,\ d \mid g$ (1) and $d = sf + tg$ (2) for $s,t \in R \Leftrightarrow \langle d\rangle \subseteq \langle f,g\rangle$ (2)

$\langle d\rangle \supseteq \langle f,g\rangle$ (1)